The discrete Kramer sampling theorem and indeterminate moment problems

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Abstract

In this paper we propose candidates to be the kernel appearing in the discrete Kramer sampling theorem. These kernels arise either from orthonormal polynomials associated with indeterminate Hamburger or Stieltjes moment problems, or from the second kind orthogonal polynomials associated with the former ones. The sampling points are given by the zeros of the denominator in the Nevanlinna parametrization of the N–extremal measures. Explicit formulae are given associated with some cases where the Nevanlinna parametrization is known explicitly.

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1 Introduction

The classical Kramer sampling theorem provides a method for obtaining sampling theorems [11,13,17]. The statement of this result is as follows: *Let $K(\omega,\lambda)$ be a function, continuous in λ such that, as a function of ω, $K(\omega,\lambda)\in L^2(I)$ for every real number λ, where I is an interval of the real line. Assume that there exists a sequence of real numbers $\{\lambda_n\}$, with n belonging to an indexing*

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set contained in $\mathbb{Z}$, such that $\{K(\omega, \lambda_n)\}$ is a complete orthogonal sequence of functions of $L^2(I)$. Then for any f of the form

$$f(\lambda) = \int_I F(\omega) K(\omega, \lambda) \, d\omega \,,$$

where $F \in L^2(I)$, we have

$$f(\lambda) = \lim_{N \to \infty} \sum_{|n| \leq N} f(\lambda_n) S_n(\lambda) \,, \tag{1}$$

with

$$S_n(\lambda) = \frac{\int_I K(\omega, \lambda) \overline{K(\omega, \lambda_n)} \, d\omega}{\int_I |K(\omega, \lambda_n)|^2 \, d\omega} \,.$$

The series (1) converges uniformly wherever $\|K(\cdot, \lambda)\|_{L^2(I)}$ is bounded.

Taking $I = [-\pi, \pi]$, $K(\omega, \lambda) = e^{i\lambda\omega}$ and $\{\lambda_n = n\}_{n \in \mathbb{Z}}$, we get the well–known Whittaker–Shannon–Kotel'nikov sampling formula

$$f(\lambda) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(\lambda - n)}{\pi(\lambda - n)}$$

for functions in $L^2(\mathbb{R})$ whose Fourier transform has support in $[-\pi, \pi]$.

Now, if we take $I = [0, 1]$, $K(x, t) = \sqrt{xt} J_\nu(xt)$ and $\{x_n\}$, the sequence of the positive zeros of the Bessel function J_ν of ν-th order with $\nu > -1$, then

$$f(t) = \sum_n f(x_n) \frac{2\sqrt{x_n t} J_\nu(t)}{J'_\nu(x_n)(t^2 - x_n^2)}$$

for every f of the form $f(t) = \int_0^1 g(x) \sqrt{xt} J_\nu(xt) dx$, where $g \in L^2(0, 1)$.

One way to generate the kernel $K(\omega, \lambda)$ and the sampling points $\{\lambda_n\}$ is to consider certain Sturm–Liouville boundary–value problems. See [17] for details.

A straightforward discrete version of Kramer's theorem can be obtained [4,5]. To this end, let a kernel $K(n, z)$ such that, as function of n, $K(n, z) \in \ell^2(\mathbb{I})$ for any $z \in \mathbb{C}$, where $\mathbb{I}$ is a countable index set. Assume that, for a suitable sequence $\{z_n\}$, $\{K(\cdot, z_n)\}$ is an orthogonal basis for $\ell^2(\mathbb{I})$. Then, we show in Section 2 that any function of the form $f(z) = \sum_{n \in \mathbb{I}} c_n K(n, z)$, where $\{c_n\} \in \ell^2(\mathbb{I})$, can be expanded by means of a sampling series like (1).

This raises the question of whether there are examples of the discrete Kramer sampling theorem arising from an infinite difference Sturm–Liouville problem as

$$\begin{aligned} &(\tau y)(n) \doteq \nabla[p(n)\Delta y(n)] + q(n)y(n) = \lambda y(n)\,,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \\ &y(-1) = 0 \end{aligned}$$

where $p(m) > 0$ if $m \in \mathbb{N}_0$ and $p(-1) \geq 0$.

In [10], under appropriate conditions, kernels which fulfill the discrete Kramer theorem are found, and the sampling series obtained is nothing but a Lagrange interpolation type series. When $p(-1) > 0$, the classical Weyl–Titchmarsh theory about *limit point* and *limit circle* cases is used. As one can see in [3,15,16], this theory is intimately related with the *classical Hamburger or Stieltjes moment problem* and with the spectral theory of semi–infinite *Jacobi matrices.*

The statement of the Hamburger moment problem is:

Given a sequence $\{s_n\}_{n=0}^{\infty}$ of real numbers, is there a measure $d\rho$ on $(-\infty, \infty)$ so that

$$s_n = \int_{-\infty}^{\infty} x^n d\rho(x)\,? \tag{2}$$

And, if such a ρ exists, is it unique? If we look for a measure supported in $(0, \infty)$, we are dealing with the Stieltjes moment problem.

Necessary and sufficient conditions for the existence of ρ are given in [16, p.86]. We will call a sequence of moments $\{s_n\}_{n=0}^{\infty}$ a set of *Hamburger moments* (resp. *Stieltjes moments*) if a measure ρ supported in $(-\infty, \infty)$ (resp. $(0, \infty)$) satisfying (2) exists. If there is a unique solution of (2), the moment problem is called *determinate.* Otherwise, it is called *indeterminate.*

Now, given a sequence $\{s_n\}_{n=0}^{\infty}$ of Hamburger moments, we can find two sequences $\{b_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ of real and positive numbers respectively, so that the moment problem is associated to self–adjoint extensions of the semi–infinite Jacobi matrix

$$\mathcal{A} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \ddots \\ 0 & 0 & a_2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{3}$$

whose domain $D(\mathcal{A})$ is the set of sequences of finite support. There are explicit formulae for the elements of the matrix $\mathcal{A}$ in terms of the s_n's [16]. Associated with the Jacobi matrix $\mathcal{A}$ there is a three–term recurrence relation

$$xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x) \quad n \geq 0 \,. \tag{4}$$

If we take the initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, we obtain a sequence of orthogonal polynomials $\{P_n(x)\}_{n\in\mathbb{N}_0}$ with respect to any solution $d\rho$ of the Hamburger moment problem (2).

The uniqueness of the Hamburger problem depends on the existence or not of a unique self–adjoint extension of the operator defined by $\mathcal{A}$ [16, p.86]. These results rely on the search of the von Neumann solutions, i.e., the solutions of the moment problem (2) coming from self–adjoint extensions of the operator $\mathcal{A}$. In the Stieltjes case, we are concerned with the positive self–adjoint extensions.

The indeterminacy of the Hamburger moment problem can be thought as the discrete Weyl's limit point/limit circle theory for the self-adjoint extensions of the discrete Sturm–Liouville problem given by (4). Indeed, the Hamburger moment problem is indeterminate if and only if the difference problem (4) belongs to the limit circle case. Therefore, we can apply the known theory of the indeterminate Hamburger moment problem to the limit circle case for the difference Sturm–Liouville problem (4). In particular, this provides us with criteria to decide when a difference Sturm–Liouville problem belongs to the limit circle case and, for practical purposes, which are its eigenvalues and the candidates to be the kernel function $K(n, z)$ appearing in the discrete Kramer sampling theorem.

The aim of the present paper is to obtain examples in which the discrete Kramer theorem can be applied. In these examples, the kernel function $K(n, z)$ is given either by $P_n(z)$ where $\{P_n\}$ is the sequence of orthonormal polynomials associated with indeterminate Hamburger or Stieltjes moment problems, or $Q_n(z)$ where $\{Q_n\}$ is the sequence of second kind orthogonal polynomials associated with $\{P_n\}$.

The paper is organized as follows: Section 2 is devoted to the statement and proof of the discrete Kramer sampling theorem. In Section 3 we obtain a general example of the Kramer theorem for orthogonal polynomials associated with indeterminate moment problems. In this derivation we use the Nevanlinna parametrization of the so–called N–extremal measures of the moment problem. Finally, we present in Section 4 some particular examples of the sampling theorem developed in Section 3.

2 The discrete Kramer sampling theorem

For the sake of completeness we give a proof of the discrete Kramer sampling theorem, which exhibits the most important features of this simple result

Theorem 1 *Let $K(n,z) \in \ell^2(\mathbb{N}_0)$ for each $z \in \Omega \subset \mathbb{C}$ and assume there is a sequence $\{z_n\}$, n in $\mathbb{N}_0$, such that $\{K(\cdot, z_n)\}$ is an orthogonal basis of $\ell^2(\mathbb{N}_0)$. Then, for any F of the form*

$$F(z) = \sum_{n=0}^{\infty} c_n K(n,z)\,, \tag{5}$$

where $\{c_n\} \in \ell^2(\mathbb{N}_0)$, we have the following sampling series for F

$$F(z) = \sum_{n=0}^{\infty} F(z_n) S_n(z)\,, \tag{6}$$

where

$$S_n(z) = \frac{1}{\|K(\cdot, z_n)\|^2} \sum_{m=0}^{\infty} \overline{K(m,z_n)} K(m,z)\,. \tag{7}$$

The sampling series (6) converges absolutely, and uniformly on subsets of Ω for which $\|K(\cdot, z)\|$ is bounded.

Proof: We define the space $\mathcal{H}$ as

$$\mathcal{H} = \left\{ F : \Omega \to \mathbb{C} \mid F(z) = \sum_{n=0}^{\infty} c_n K(n,z) \text{ with } \{c_n\} \in \ell^2(\mathbb{N}_0) \text{ and } z \in \Omega \right\}.$$

Now, we consider the map $\Phi : \mathcal{H} \longrightarrow \ell^2(\mathbb{N}_0)$ such that $\Phi(F) = \{c_n\}$ where $F(z) = \sum_{n=0}^{\infty} c_n K(n,z)$. This map is well defined and one–to–one since the sequence $\{K(\cdot, z_n)\}$ is a complete set for $\ell^2(\mathbb{N}_0)$. Furthermore, Φ is an isometry if we endow $\mathcal{H}$ with the norm $\|F\|_{\mathcal{H}} = \|\{c_n\}\|_{\ell^2}$. It is straightforward to prove that $\mathcal{H}$ is a Reproducing Kernel Hilbert Space. Indeed, using the Cauchy–Schwarz inequality, for a fixed $z \in \Omega$,

$$|F(z)| \le \|c_n\|_{\ell^2} \|K(\cdot, z)\|_{\ell^2} = \|K(\cdot, z)\|_{\ell^2} \|F\|_{\mathcal{H}} \tag{8}$$

for each $F \in \mathcal{H}$ and hence, the evaluation functional $T_z(F) = F(z)$ is continuous [2].

Since $\mathcal{B} = \left\{ \frac{\overline{K(\cdot, z_n)}}{\|K(\cdot, z_n)\|} \right\}_{n \in \mathbb{N}_0}$ is an orthonormal basis for $\ell^2(\mathbb{N}_0)$ it follows that $\Phi^{-1}(\mathcal{B})$ is an orthonormal basis for the Hilbert space $\mathcal{H}$. Taking into account that

$$\Phi^{-1}\left(\frac{\overline{K(\cdot,z_n)}}{\|K(\cdot,z_n)\|}\right)=\|K(\cdot,z_n)\|S_n(z)\,,$$

we have, for each F in $\mathcal{H}$

$$F(z)=\sum_{n=0}^{\infty}\langle F,\|K(\cdot,z_n)\|S_n(z)\rangle\,\|K(\cdot,z_n)\|S_n(z) \tag{9}$$

with convergence in the $\mathcal{H}$ norm. Now, applying the isometry Φ we obtain

$$F(z)=\sum_{n=0}^{\infty}\langle c_n,\frac{\overline{K(\cdot,z_n)}}{\|K(\cdot,z_n)\|}\rangle\,\|K(\cdot,z_n)\|S_n(z)=\sum_{n=0}^{\infty}F(z_n)S_n(z)\,,$$

which converges in the $\mathcal{H}-$norm, and hence pointwise in Ω from (8). The absolute convergence comes from the fact that an orthonormal basis is an unconditional basis, as one can derive from Parseval's equality. Finally, the uniform convergence in subsets of Ω where $\|K(\cdot,z)\|_{\ell^2}$ is uniformly bounded comes from inequality (8) again □

3 A Sampling Theorem for indeterminate moment problems

Let $s=\{s_n\}_{n\in\mathbb{N}_0}$ be an indeterminate Hamburger moment sequence and let V_s be the set of positive Borel measures μ on $\mathbb{R}$ satifying $\int_{-\infty}^{\infty}x^n d\mu(x)=s_n$, $n\geq 0$. The functional $\mathcal{L}$ defined on the vector space $\mathbb{C}[x]$ of polynomials $p(x)=\sum_{k=0}^{n}p_k x^k$ by

$$\mathcal{L}(p)=\sum_{k=0}^{n}p_k s_k=\int_{-\infty}^{\infty}p(x)d\mu(x)$$

is independent of $\mu\in V_s$. Let $\{P_n\}_{n\in\mathbb{N}_0}$ be the corresponding orthonormal polynomials satisfying

$$\int_{-\infty}^{\infty}P_n(x)P_m(x)d\mu(x)=\delta_{nm}\,,\quad \text{for each } \mu\in V_s\,.$$

We assume that P_n is of degree n with positive leading coefficient. Recall that $\{P_n(x)\}$ satisfy the three–term recurrence relation

$$\begin{aligned} xP_n(x)&=a_nP_{n+1}(x)+b_nP_n(x)+a_{n-1}P_{n-1}(x)\quad n\geq 0\\ P_{-1}(x)&=0\,;\quad P_0(x)=1\end{aligned}$$

The set V_s of solutions to an indeterminate moment problem can be parametrized with the one–point compactification $\mathcal{P}\cup\{\infty\}$ of the Pick (or Herglotz) functions set $\mathcal{P}$ [3,16]. When the parameter is restricted to constant functions taking values in $\mathbb{R} \cup \{\infty\}$, we obtain the set of N–extremal measures $\{\mu_t\}$ which satisfy

$$\int_{-\infty}^{\infty} \frac{d\mu_t(x)}{x-z} = -\frac{A(z)t - C(z)}{B(z)t - D(z)}\,, \quad z \in \mathbb{C} \setminus \mathbb{R}\,,$$

where A, B, C, D are the entire functions forming the so–called Nevanlinna matrix associated with the moment problem. See [7,16] for explicit formulas of the Nevanlinna matrix. It is known that, for each $t \in \mathbb{R} \cup \{\infty\}$, μ_t is the discrete measure $\mu_t = \sum_{z \in \Lambda_t} m_z \delta_z$ where

$$\Lambda_t = \begin{cases} \{z \in \mathbb{C} \mid B(z)t - D(z) = 0\} & \text{if } t \in \mathbb{R}\,, \\ \{z \in \mathbb{C} \mid B(z) = 0\} & \text{if } t = \infty\,, \end{cases}$$

and

$$m_z = \frac{A(z)t - C(z)}{B'(z)t - D'(z)}\,, \quad \text{for } z \in \Lambda_t\,.$$

Recall that the zeros of the entire function $B(z)t - D(z)$ or $B(z)$ are real and simple, and they form a sequence $\{z_m^t\}$. The N–extremal measures are characterized as those measures $\mu \in V_s$ for which the polynomials are dense in $L^2(\mu)$. It is worth noting that those N–extremal measures supported in $(0, \infty)$, i.e., the N–extremal measures solutions of the corresponding Stieltjes moment problem, are parametrized by t in a suitable interval $[\alpha, 0]$ [7].

In [6,10] it is proved that the sequence in $\ell^2(\mathbb{N}_0)$ given by

$$\{P_0(z_m^t),\, P_1(z_m^t),\, P_2(z_m^t),\, P_3(z_m^t), \cdots\cdots\}_{m \in \mathbb{N}_0}$$

is an orthogonal basis for $\ell^2(\mathbb{N}_0)$.

At this point, we are able to prove that $K(n, z) = P_n(z)$ is a suitable candidate to be the kernel of the discrete transform appearing in Kramer's theorem.

Theorem 2 *Let μ_t be an N–extremal measure for an indeterminate moment problem and let $\{P_n\}$ be the sequence of associated orthonormal polynomials. Assume that $\{z_m^t\}$ are the zeros of $B(z)t - D(z)$ if $t \in \mathbb{R}$, or the zeros of $B(z)$ if $t = \infty$. Then, $F(z) = \sum_{n=0}^{\infty} c_n P_n(z)$ where $\{c_n\} \in \ell^2(\mathbb{N}_0)$ can be recovered from its samples $\{F(z_m^t)\}$ through the Lagrange–type interpolatory series*

$$F(z) = \sum_{m=0}^{\infty} F(z_m^t) \frac{G_t(z)}{G_t'(z_m^t)(z - z_m^t)}\,, \tag{10}$$

where

$$G_t(z) = \begin{cases} B(z)t - D(z) & \textit{if } t \in \mathbb{R} \\ B(z) & \textit{if } t = \infty\,. \end{cases}$$

The series (10) *converges absolutely and uniformly on compact subsets of* $\mathbb{C}$.

Proof: The indeterminacy of the moment problem implies that $\sum_{n=0}^{\infty} |P_n(z)|^2$ converges uniformly on compact subsets of $\mathbb{C}$ [3]. Taking $K(n,z) = P_n(z)$ in the discrete Kramer sampling theorem and using that

$$\{P_0(z_m^t),\ P_1(z_m^t),\ P_2(z_m^t),\ P_3(z_m^t), \cdots\cdots\}_{m\in\mathbb{N}_0}$$

is an orthogonal basis for $\ell^2(\mathbb{N}_0)$, we obtain

$$F(z) = \sum_{m=0}^{\infty} F(z_m^t) \frac{1}{\|K(\cdot, z_m^t)\|^2} \sum_{n=0}^{\infty} K(n, z_m^t) K(n, z)\,,$$

where for notational convenience we keep $K(n,z)$ as $P_n(z)$. To end we have to prove that the latter series can be written as (10). Indeed, let

$$\mathcal{U}_m^t(z) = \sum_{n=0}^{\infty} K(n, z_m^t) K(n, z)\,. \tag{11}$$

Since $K(\cdot, z_m^t) \in \ell^2(\mathbb{N}_0)$ for each $m \in \mathbb{N}_0$ and $t \in \mathbb{R} \cup \{\infty\}$, $\mathcal{U}_m^t$ is an entire function of minimal exponential type. Therefore, it is determined up to a constant factor by their zeros [7].

The zeros of $\mathcal{U}_m^t$ are $\{z_j^t\}_{j\neq m}$, hence there exists a constant $C_m^t \in \mathbb{C}$ such that

$$(z - z_m^t)\mathcal{U}_m^t(z) = C_m^t G_t(z)\,,\ \ z \in \mathbb{C}\,. \tag{12}$$

From (12) we obtain that $\mathcal{U}_m^t(z_m^t) = C_m^t G_t'(z_m^t)$. On the other hand, from (11) we have that $C_m^t G_t'(z_m^t) = \|K(\cdot, z_m^t)\|_{\ell^2}^2$, therefore

$$\sum_{n=0}^{\infty} \frac{K(n, z_m^t) K(n, z)}{\|K(\cdot, z_m^t)\|^2} = \frac{G_t(z)}{(z - z_m^t) G_t'(z_m^t)}\,.$$

□

This theorem has been proved in [9] using a different technique.

As a byproduct of the Kramer theorem we obtain that $\mathcal{H} = L^2(\mu_t)$. Indeed, the series $F(z) = \sum_{n=0}^{\infty} c_n P_n(z)$ converges to F in $L^2(\mu_t)$

$$\int_{-M}^{M} \Big|\sum_{n=0}^{N} c_n P_n(x)\Big|^2 d\mu_t(x) \le \int_{-\infty}^{\infty} \Big|\sum_{n=0}^{N} c_n P_n(x)\Big|^2 d\mu_t(x) = \sum_{n=0}^{N} |c_n|^2\,.$$

Letting $N \to \infty$ first, and then $M \to \infty$, we have $F \in L^2(\mu_t)$. Now, since $\{P_n\}$ is an orthonormal basis for $L^2(\mu_t)$ we obtain that $\mathcal{H} = L^2(\mu_t)$. Moreover, if $F(z) = \sum_{n=0}^{\infty} c_n P_n(z)$ and $H(z) = \sum_{n=0}^{\infty} d_n P_n(z)$ then

$$\langle F, H\rangle_{\mathcal{H}} = \sum_{n=0}^{\infty} c_n \bar{d}_n = \langle F, H\rangle_{L^2(\mu_t)}\,.$$

Finally, since $\{P_n\}$ is an orthonormal basis for $L^2(\mu_t)$, its reproducing kernel will be $k(s,t)=\sum_{n=0}^{\infty} P_n(s)P_n(t)$ [2], and the formula $\int F(x)k(x,y)d\mu_t = F(y)$ holds for every $F\in L^2(\mu_t)$.

Another good candidate to be the kernel in the discrete Kramer theorem is $K(n,z)=Q_n(z)$, where $\{Q_n\}$ is the sequence of the second kind orthogonal polynomials associated to $\{P_n\}$. These polynomials are given in terms of the measure μ_t by

$$Q_n(z)=\int \frac{P_n(z)-P_n(u)}{z-u}\,d\mu_t(u)\,.$$

Indeed, the Q_n polynomials are related with the first kind orthogonal polynomials $\{\widetilde{P}_n\}$ associated with the so–called shifted moment problem by the relationship $\widetilde{P}_n(z)=a_0Q_{n+1}(z)$.

It is known that the sequence $\{\widetilde{P}_n\}$ satisfies the three–term recurrence relation

$$x\widetilde{P}_n(x)=a_{n+1}\widetilde{P}_{n+1}(x)+b_{n+1}\widetilde{P}_n(x)+a_n\widetilde{P}_{n-1}(x)\quad n\geq 0\,, \tag{13}$$

i.e., relation (4) with shifted coefficients, and initial conditions $\widetilde{P}_{-1}=0$ and $\widetilde{P}_0=1$. The Jacobi matrix associated with the shifted moment problem is obtained from (3) by deleting the first row and column. If the original Hamburger moment problem is indeterminate, the same occurs for the shifted one corresponding to (13). Moreover, the Nevanlinna matrix associated with the shifted moment problem can be expressed in terms of the Nevanlinna matrix of the original problem. In particular, the following relations hold: $\widetilde{B}(z)=-C(z)-b_0A(z)$ and $\widetilde{D}(z)=a_0^2A(z)$ (see [14] for more details). Therefore, adapting Theorem 2 to the shifted moment problem we obtain the following sampling theorem, where the kernel in (5) is the sequence of second kind orthogonal polynomials Q_n.

Corollary 3 *Assume that $\{\omega_m^t\}$ are the zeros of $\widetilde{B}(z)t-\widetilde{D}(z)$ if $t\in\mathbb{R}$, or the zeros of $\widetilde{B}(z)$ if $t=\infty$. Then, any function of the form $F(z)=\sum_{n=1}^{\infty}c_nQ_n(z)$ where $\{c_n\}\in\ell^2(\mathbb{N})$ can be recovered from its samples $\{F(\omega_m^t)\}$ through the Lagrange–type interpolatory series*

$$F(z)=\sum_{m=0}^{\infty}F(z_m^t)\frac{G_t(z)}{G_t'(\omega_m^t)(z-\omega_m^t)}\,, \tag{14}$$

where

$$G_t(z)=\begin{cases}\widetilde{B}(z)t-\widetilde{D}(z) & \text{if } t\in\mathbb{R}\\ \widetilde{B}(z) & \text{if } t=\infty\,.\end{cases}$$

The series (14) *converges absolutely and uniformly on compact subsets of* $\mathbb{C}$.

4 Some particular sampling formulae

In this section we illustrate the result in Section 3 with some particular examples.

q^{-1}–Hermite polynomials $(0 < q < 1)$

These polynomials have the explicit representation [12]

$$h_n(x|q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{k(k-n)} (x + \sqrt{x^2+1})^{n-2k} \, ,$$

where the q-shifted factorial notation is used

$$(c_1, c_2, \cdots, c_p; q)_n = \prod_{k=1}^{n} \prod_{j=1}^{p} (1 - c_j q^{k-1}) \, ,$$

where $n = 0, 1, \cdots, \infty$. The moment problem associated with $\{h_n(x|q)\}$ is indeterminate, and their norms are given by $\|h_n\| = \sqrt{\frac{(q;q)_n}{q^{n(n+1)/2}}}$ [12]. The Nevanlinna parametrization of the N–extremal measures are computed in [12]. In particular, $G_0(z) = D(z)$ is given by

$$D(z) = -\frac{(qe^{2\xi}, qe^{-2\xi}; q^2)_\infty}{(q;q^2)_\infty^2}$$

where $z = \sinh \xi$. Its zeros are $\pm\lambda_n$ with $n = 0, 1, \ldots$ where $\lambda_n = \frac{1}{2}(q^{-n-1/2} - q^{n+1/2})$. Then, for any function

$$F(z) = \sum_{n=0}^{\infty} c_n \frac{h_n(x|q)}{\|h_n\|} \text{ with } \{c_n\} \in \ell^2 \, ,$$

the following sampling formula

$$F(z) = \sum_{n=0}^{\infty} F(-z_n) \frac{D(z)}{(z+z_n)D'(-z_n)} + \sum_{n=0}^{\infty} F(z_n) \frac{D(z)}{(z-z_n)D'(z_n)} \, .$$

holds.

Birth and death polynomials with quartic rates

These polynomials, appearing in [8], satisfy the three–term recurrence relation

$$\begin{aligned} xP_n(x) &= a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x) \quad n \geq 0 \\ P_{-1}(x) &= 0 \, ; \quad P_0(x) = 1 \end{aligned}$$

where for $n \geq 0$

$$a_n = \sqrt{\lambda_n \mu_{n+1}} \qquad b_n = \lambda_n + \mu_n$$
$$\lambda_n = (4n+1)(4n+2)^2(4n+3) \qquad \mu_n = (4n-1)(4n)^2(4n+1)$$

By Favard's theorem $\{P_n\}_{n \in \mathbb{N}_0}$ are the orthonormal polynomials with respect to a normalized Hamburger moment sequence s which is indeterminate [8]. It is known that s is an indeterminate Stieltjes moment sequence as well [8]. The entire function D in the Nevanlinna parametrization is given by

$$D(z) = \frac{4}{\pi}\sqrt{z}\sin\left(\frac{\sqrt[4]{z}}{2}K_0\right)\sinh\left(\frac{\sqrt[4]{z}}{2}K_0\right)$$

whose zeros are $z_n = (2\pi n/K_0)^4$, $n \in \mathbb{N}_0$ and K_0 denotes the constant $\dfrac{\Gamma^2(1/4)}{4\sqrt{\pi}}$ [8].

Using Theorem 2 for $t = 0$, any function of the form $F(z) = \sum_{n=0}^{\infty} c_n P_n(z)$ with $\{c_n\} \in \ell^2$ can be written as

$$F(z) = F(0)\frac{4\sqrt{z}\sin\left(\frac{\sqrt[4]{z}}{2}K_0\right)\sinh\left(\frac{\sqrt[4]{z}}{2}K_0\right)}{zK_0^2} + \\ + \frac{16\pi}{K_0^2}\sum_{n=1}^{\infty} F\left[\left(\frac{2\pi n}{K_0}\right)^4\right]\frac{(-1)^n n}{\sinh(n\pi)}\frac{\sqrt{z}\sin\left(\frac{\sqrt[4]{z}}{2}K_0\right)\sinh\left(\frac{\sqrt[4]{z}}{2}K_0\right)}{\left(z - \left(\frac{2\pi n}{K_0}\right)^4\right)}.$$

We have used that

$$D'(0) = \frac{K_0^2}{\pi} \quad \text{and} \quad D'\left[\left(\frac{2\pi n}{K_0}\right)^4\right] = \frac{K_0^2}{4n\pi^2}(-1)^n\sinh(n\pi).$$

Al-Salam–Carlitz q–polynomials

These monic polynomials $V_n(x) = V_n^{(a)}(x;q)$ were introduced in [1], and are determined by the three–term recurrence formula

$$V_{n+1}(z) = (z - (1+a)q^{-n})V_n(z) - aq^{-(2n-1)}(1-q^n)V_{n-1}(z)$$
$$V_{-1}(z) = 0\,, \quad V_0(z) = 1\,.$$

They are connected with the orthonormal polynomials $\{P_n\}$ associated with a birth and death process with rates

$$\lambda_n = aq^{-n}\,, \qquad \mu_n = q^{-n} - 1\,, \qquad n \in \mathbb{N}_0\,,$$

by the formula

$$V_n(z+1) = \frac{\sqrt{(q;q)_n (aq)^n}}{q^{n(n+1)/2}} P_n(z)\,.$$

Within the domain $a > 0$, $0 < q < 1$, the associated Hamburger problem is indeterminate if $q < a < q^{-1}$. In the Nevanlinna parametrization, the entire function $D(z)$ is given by [8]

$$D(z) = -\frac{(1+z;q)_\infty}{(q;q)_\infty (aq;q)_\infty}\,,$$

which has its zeros located at $z_n = q^{-n} - 1$, $n \in \mathbb{N}_0$. Furthermore, it is known [8] that

$$D'(z_n) = \frac{(-1)^n (q;q)_n}{(aq;q)_\infty} q^{-n(n-1)/2}\,.$$

Hence, any function defined by $F(z) = \sum_{n=0}^\infty c_n P_n(z)$ with $\{c_n\} \in \ell^2(\mathbb{N}_0)$ verifies the sampling formula

$$F(z) = \sum_{n=0}^\infty F(q^{-n} - 1) \frac{(-1)^{n+1} (1+z;q)_\infty\, q^{n(n-1)/2}}{(z - q^{-n} + 1)\,(q;q)_\infty\,(q;q)_n}\,.$$

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