Discrete Sturm–Liouville problems, Jacobi matrices and Lagrange interpolation series

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Abstract

The close relationship between discrete Sturm–Liouville problems belonging to the so–called limit–circle case, the indeterminate Hamburger moment problem and the search of self–adjoint extensions of the associated semi–infinite Jacobi matrix is well–known. In this paper, all these important topics are also related with associated sampling expansions involving analytic Lagrange–type interpolation series.

KEY WORDS: Indeterminate moment problems; Difference operators; Lagrange interpolation series

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1 Introduction

In [13] and the references cited therein, a sampling theorem associated with a singular Sturm–Liouville problem on the halfline $[0, \infty)$ is proved. Namely, consider the singular Sturm–Liouville boundary value problem:

$$\begin{cases} -y'' + q(x)y = \lambda y\,, & x \in [0,\infty)\,, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0\,, \end{cases} \tag{1}$$

where $q(x)$ is a continuous function on $[0, \infty)$. Let $\phi(x, \lambda)$, $\theta(x, \lambda)$ be the solutions of the differential equation in (1) such that

$$\begin{aligned} \phi(0) &= \sin\alpha\,, & \phi'(0) &= -\cos\alpha\,, \\ \theta(0) &= \cos\alpha\,, & \theta'(0) &= \sin\alpha\,. \end{aligned}$$

From the Weyl–Titchmarsh theory [12] it follows that there exists a complex valued function $m_\infty(\lambda)$, the so–called *Weyl–Titchmarsh function*, such that for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$

the equation in (1) has a solution $\psi_\infty(x,\lambda) = \theta(x,\lambda) + m_\infty(\lambda)\,\phi(x,\lambda)$ belonging to $L^2(0,\infty)$. In the so–called *limit–point case* $m_\infty(\lambda)$ is unique, while in the *limit–circle case* there are uncountably many such functions (see [12] for the details). For each $\lambda \in \mathbb{C}\setminus\mathbb{R}$, the values of these functions at λ belong to a geometrical circle $\mathcal{C}_\infty(\lambda)$ (in the limit–point case, this circle collapses into a point). In the case when a pure point spectrum associated with (1) exists, the associated sampling theorem reads:
Let f be defined as

$$f(\lambda) = \int_0^\infty F(s)\Phi(s,\lambda)\,ds\,,$$

where $F \in L^2(0,\infty)$, $\Phi(x,\lambda) = P(\lambda)\psi_\infty(x,\lambda)$ and $P(\lambda)$ is the canonical product associated with the eigenvalues $\{\lambda_n\}_{n=0}^\infty$ of the boundary value problem (1)*. Then, f is an entire function that can be recovered through the Lagrange–type interpolation series*

$$f(\lambda) = \sum_{n=0}^\infty f(\lambda_n)\frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}\,.$$

The convergence of the series is absolute and uniform on compact subsets of $\mathbb{C}$.
A more general result has been obtained by Everitt et al. [5] without assuming the existence of the canonical product of the eigenvalues. Explicit examples of this sampling result can be found in [5, 13].

In the same way we may consider the Sturm–Liouville difference equation

$$a_n z_{n+1} + b_n z_n + a_{n-1}z_{n-1} = \lambda z_n\,, \quad n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}\,, \tag{2}$$

where $a_{-1} = 1$, $a_n > 0$ and $b_n \in \mathbb{R}$ for each $n \geq 0$ along with the boundary condition $z_{-1} = 0$. Notice that equation (2) can be written in the form

$$\nabla[a_n \Delta z_n] + (b_n + a_n + a_{n-1})z_n = \lambda z_n\,,$$

where ∇ and Δ denote respectively the usual backward and forward operators.

One can ask for the existence of eigenvalues and eigenfunctions associated with (2) and the boundary condition $z_{-1} = 0$. In general, to obtain an eigenvalue problem, an additional boundary condition at ∞ will be needed, as we will see later in Section 2. We can proceed as in the continuous case by considering the following solutions of the difference equation (2)

$$\Pi(\lambda) = (P_0(\lambda), P_1(\lambda), P_2(\lambda), \dots) \quad\text{and}\quad \Theta(\lambda) = (Q_0(\lambda), Q_1(\lambda), Q_2(\lambda), \dots)\,,$$

corresponding to the initial data $z_{-1} = 0$, $z_0 = 1$ and $z_{-1} = -1$, $z_0 = 0$ respectively, and searching for the Weyl–Titchmarsh functions $m_\infty(\lambda)$ in such a way that

$$\psi_\infty(\lambda, n) = \Theta(\lambda) + m_\infty(\lambda)\,\Pi(\lambda) \in \ell^2(\mathbb{N}_0)\,.$$

Analogous to the continuous case, if there exists a unique function $m_\infty(\lambda)$, we are in the limit–point case, while in the limit–circle case there are uncountably many such functions. Whenever the discrete Sturm–Liouville problem has a pure point spectrum $\{\lambda_i\}$ (in the limit–circle case this always holds), these points are precisely the poles of the resulting meromorphic function $m_\infty(\lambda)$. The corresponding eigenfunctions are the sequences $\{\Pi_i = \Pi(\lambda_i)\}$. See [2, 6] for the details.

On the other hand, one may consider the semi-infinite Jacobi matrix $\mathcal{A}$ associated with the difference equation in (2):

$$\mathcal{A} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \ddots \\ 0 & 0 & a_2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

This matrix defines a densely defined operator in ℓ^2. It is well–known that there is a close relationship between problem (2) being in the limit–point or in the limit–circle case and the search of self–adjoint extensions of $\mathcal{A}$. Furthermore, both problems are equivalent to deciding the determinacy of the *Hamburger moment problem* associated with $\mathcal{A}$. This latter problem reads as follows: Given the real numbers $s_n = \langle \delta_0, \mathcal{A}^n \delta_0 \rangle_{\ell^2}$, $n \geq 0$, where δ_0 stands for the sequence $(1, 0, 0, \dots)$, we are interested in the search of positive Borel measures μ supported on $(-\infty, \infty)$ satisfying

$$s_n = \int_{-\infty}^{\infty} x^n d\mu(x)\,, \;\; n \geq 0\,.$$

If such a measure exists and is unique, the moment problem is *determinate*. If a measure μ exists, but it is not unique, the moment problem is called *indeterminate*. One can find the relations between all these topics in the superb article by B. Simon [10] or in the classical reference [1].

In this paper we deal with the equivalent of Zayed's sampling result associated with the singular discrete Sturm-Liouville problem (2) in the limit–circle case. For the limit–point case we refer the reader to [6]. In our study we will use the different points of view involved in this problem, namely: operator theory (search of self–adjoint extensions of $\mathcal{A}$) and the classical Hamburger indeterminate moment problem theory (Cauchy or Stieltjes transform of the von Neumann solutions and their Nevanlinna parametrization).

The paper is organized as follows: In the next section we deal with the self–adjoint extensions of the operator defined by means of the Jacobi matrix $\mathcal{A}$. This approach will be used in Section 3 in order to obtain a sampling expansion given by a Lagrange–type interpolation series. In Section 4, we relate the sampling result obtained with the Weyl–Titchmarsh functions associated with (2). Indeed, these functions are precisely the Cauchy (Stieltjes) transforms of the von Neumann measures which are solutions of the moment problem. The Nevanlinna parametrization of the former transforms allows us to obtain the sequence of sampling points (the eigenvalues of the associated self–adjoint extension). Finally, in Section 5, we put to use the sampling result in the case of the q^{-1}-Hermite polynomials.

2 The operator theory approach

Given two sequences $\{b_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ of, respectively, real and positive numbers consider the semi-infinite Jacobi matrix

$$\mathcal{A} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \ddots \\ 0 & 0 & a_2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{3}$$

whose domain $D(\mathcal{A})$ is the set of sequences of finite support. This operator is closable since it is symmetric and densely defined. We denote again by $\mathcal{A}$ its closure. The domain of the adjoint of $\mathcal{A}$ is given by $D(\mathcal{A}^*) = \{z \in \ell^2(\mathbb{N}_0) \mid \mathcal{A}z \in \ell^2(\mathbb{N}_0)\}$ [10, p. 105]. If $\mathcal{A}$ is not a self-adjoint operator (i.e., the associated Hamburger moment problem is indeterminate) its self-adjoint extensions, $\mathcal{A} \subset \mathcal{S}_t \subset \mathcal{A}^*$, can be parametrized by $t \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and their domains are [10, p. 125]

$$\mathcal{D}(S_t) = \begin{cases} \mathcal{D}(\mathcal{A}) + \operatorname{span}\{t\Pi(0) + \Theta(0)\} & \text{if } t \in \mathbb{R}\,, \\ \mathcal{D}(\mathcal{A}) + \operatorname{span}\{\Pi(0)\} & \text{if } t = \infty\,. \end{cases}$$

Equivalently (see [10, p. 126]),

$$z \in D(\mathcal{S}_t) \Leftrightarrow \begin{cases} \lim\limits_{n\to\infty} W(z, t\Pi(0) + \Theta(0))(n) = 0 & \text{if } t \in \mathbb{R}, \\ \lim\limits_{n\to\infty} W(z, \Pi(0))(n) = 0 & \text{if } t = \infty. \end{cases}$$

where $W(z, z')(n) = a_n(z_{n+1}z'_n - z_n z'_{n+1})$ denotes the Wronskian of the sequences $z = \{z_n\}$ and $z' = \{z'_n\}$.

The eigenvalue problem $(\lambda I - \mathcal{S}_t)x = 0$ is equivalent to the discrete Sturm-Liouville problem

$$\begin{cases} a_n z_{n+1} + b_n z_n + a_{n-1} z_{n-1} = \lambda z_n\,, & n \in \mathbb{N}_0 \\ z_{-1} = 0\,,\ \lim_{n\to\infty} W(z, t\Pi(0) + \Theta(0))(n) = 0. \end{cases}$$

whenever $t \in \mathbb{R}$, or

$$\begin{cases} a_n z_{n+1} + b_n z_n + a_{n-1} z_{n-1} = \lambda z_n\,, & n \in \mathbb{N}_0 \\ z_{-1} = 0\,,\ \lim_{n\to\infty} W(z, \Pi(0))(n) = 0. \end{cases}$$

in the case $t = \infty$. As a consequence, λ will be an eigenvalue of $\mathcal{S}_t$ if and only if

$$\begin{cases} \lim\limits_{n\to\infty} W(\Pi(\lambda), t\Pi(0) + \Theta(0))(n) = 0\,, & \text{if } t \in \mathbb{R}\,, \\ \lim\limits_{n\to\infty} W(\Pi(\lambda), \Pi(0))(n) = 0\,, & \text{if } t = \infty\,. \end{cases}$$

It is known that each self–adjoint extension $\mathcal{S}_t$ of $\mathcal{A}$ has a pure point spectrum $\{\lambda_i = \lambda_i(S_t)\}_{i=0}^{\infty}$ [10, p. 127]. The corresponding eigenfunctions $\{\Pi_i = \Pi(\lambda_i)\}_{i=0}^{\infty}$ are given by

$$\Pi_i = (P_0(\lambda_i), P_1(\lambda_i), \ldots, P_n(\lambda_i), \ldots), \quad i \in \mathbb{N}_0\,,$$

and they form an orthogonal basis in $\ell^2(\mathbb{N}_0)$ [2, 6]. Consequently, the resolvent operator $R_\lambda = (\lambda I - S_t)^{-1}$, where $\lambda \in \rho(S_t)$, is a compact operator [4, p. 423]. Moreover, since $\sum_i |\lambda_i|^{-p} < \infty$ for any $p > 1$ [10, p. 128], we obtain that R_λ is a Hilbert–Schmidt operator [3, p. 262]. Therefore, there exists a kernel $K_\lambda \in \ell^2(\mathbb{N}_0 \times \mathbb{N}_0)$ such that

$$[R_\lambda x](m) = \sum_{n=0}^{\infty} K_\lambda(m,n)x(n)\,, \quad m \in \mathbb{N}_0\,,$$

for each $x \in \ell^2(\mathbb{N}_0)$ [3, p. 277]. Expanding the kernel K_λ in Fourier series of eigenfunctions in $\ell^2(\mathbb{N}_0 \times \mathbb{N}_0)$, we obtain

$$K_\lambda(m,n) = \sum_{i=0}^{\infty} \frac{1}{\lambda - \lambda_i} \frac{\Pi_i(m)}{\|\Pi_i\|} \frac{\Pi_i(n)}{\|\Pi_i\|} = \sum_{i=0}^{\infty} \frac{1}{\lambda - \lambda_i} \frac{P_m(\lambda_i)}{\|\Pi_i\|} \frac{P_n(\lambda_i)}{\|\Pi_i\|}\,. \tag{4}$$

3 The Lagrange–type interpolation formula

We define the sampling kernel

$$\Psi(\lambda, m) = P(\lambda)K_\lambda(m,0), \quad m \in \mathbb{N}_0,\ \lambda \in \mathbb{C}, \tag{5}$$

where $P(\lambda)$ is the canonical product of the sequence of eigenvalues $\{\lambda_i\}_{i=0}^{\infty}$. This canonical product always exists because, in particular, $\sum_{i=0}^{\infty} |\lambda_i|^{-2} < \infty$ [10]. Specifically, the canonical product is given by

$$P(\lambda) = \begin{cases} \prod_{n=0}^{\infty}(1 - \frac{\lambda}{\lambda_n}) \exp(\lambda/\lambda_n) & \text{if } \sum_{n=0}^{\infty} |\lambda_n|^{-1} = \infty \\ \prod_{n=0}^{\infty}(1 - \frac{\lambda}{\lambda_n}) & \text{if } \sum_{n=0}^{\infty} |\lambda_n|^{-1} < \infty \end{cases}$$

whenever $\lambda_0 \neq 0$, and

$$P(\lambda) = \begin{cases} \lambda \prod_{n=1}^{\infty}(1 - \frac{\lambda}{\lambda_n}) \exp(\lambda/\lambda_n) & \text{if } \sum_{n=0}^{\infty} |\lambda_n|^{-1} = \infty \\ \lambda \prod_{n=1}^{\infty}(1 - \frac{\lambda}{\lambda_n}) & \text{if } \sum_{n=0}^{\infty} |\lambda_n|^{-1} < \infty \end{cases}$$

in the case $\lambda_0 = 0$.

In Section 4 we will list the most important properties of the kernel (5). In the definition of the sampling kernel we can choose any $n_0 \in \mathbb{N}_0$ instead of 0. The convenience of this particular choice is that $P_0(\lambda_i) = 1$ for each $i \in \mathbb{N}_0$. The following sampling theorem holds.

Theorem 1 *Let f be the function defined as*

$$f(\lambda) = \sum_{n=0}^{\infty} c_n \Psi(\lambda, n),$$

where $\{c_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)$. Then f is an entire function which can be recovered through the Lagrange-type interpolation series

$$f(\lambda) = \sum_{i=0}^{\infty} f(\lambda_i) \frac{P(\lambda)}{(\lambda - \lambda_i)P'(\lambda_i)}. \tag{6}$$

The convergence of (6) is absolute and uniform on compact subsets of $\mathbb{C}$.

Proof: Since $P_0(\lambda_i) = 1$ for each $i \in \mathbb{N}_0$, from (4) we have

$$K_\lambda(m, 0) = \sum_{i=0}^{\infty} \frac{1}{\lambda - \lambda_i} \frac{P_m(\lambda_i)}{\|\Pi_i\|^2} \,. \tag{7}$$

Expanding the sequences $c = \{c_m\}_{m=0}^{\infty}$ and $\{\Psi(\lambda, m)\}_{m=0}^{\infty}$ in Fourier series of the eigenfunctions $\{\Pi_i\}$, we obtain

$$c_m = \sum_{i=0}^{\infty} \langle c, \Pi_i \rangle_{\ell^2(\mathbb{N}_0)} \frac{\Pi_i(m)}{\|\Pi_i\|^2}$$
$$\Psi(\lambda, m) = \sum_{i=0}^{\infty} \frac{P(\lambda)}{\lambda - \lambda_i} \frac{\Pi_i(m)}{\|\Pi_i\|^2} .$$

Using Parseval's identity, we obtain

$$f(\lambda) = \langle c_m, \overline{\Psi(\lambda, m)} \rangle_{\ell^2(\mathbb{N}_0)} = \sum_{i=0}^{\infty} \frac{P(\lambda)}{\lambda - \lambda_i} \frac{\langle c, \Pi_i \rangle_{\ell^2(\mathbb{N}_0)}}{\|\Pi_i\|^2} \,.$$

Now, it can be easily proved that

$$f(\lambda_k) = \lim_{\lambda \to \lambda_k} f(\lambda) = P'(\lambda_k) \frac{\langle c, \Pi_k \rangle_{\ell^2(\mathbb{N}_0)}}{\|\Pi_k\|^2} \,.$$

Therefore,

$$f(\lambda) = \sum_{i=0}^{\infty} f(\lambda_i) \frac{P(\lambda)}{P'(\lambda_i)(\lambda - \lambda_i)} \,.$$

Let $\Omega \subset \mathbb{C}$ be a compact subset of $\mathbb{C}$. There exists $R > 0$ such that $\Omega \subset \{z \in \mathbb{C} \mid |z| \leq R\}$ and there exists $n_0 \in \mathbb{N}_0$ such that $|\lambda_i| \geq 2R$ for every $i \geq n_0 + 1$. Then, for $N \geq N_0$

$$\sum_{i=N+1}^{\infty} \frac{|P(\lambda)|^2}{|\lambda - \lambda_i|^2 \|\Pi_i\|^2} \leq |P(\lambda)|^2 \sum_{i=N+1}^{\infty} \frac{1}{(|\lambda_i| - R)^2 \|\Pi_i\|^2} \,.$$

The last series converges since it has the same character as the series

$$\sum_{i=0}^{\infty} \frac{|P(\lambda)|^2}{|\lambda - \lambda_i|^2 \|\Pi_i\|^2} = \|\Psi(\lambda, \cdot)\|^2 \,.$$

As a consequence, there exists a constant $C_\Omega > 0$, independent of $\lambda \in \Omega$, such that

$$\sum_{i=N+1}^{\infty} \frac{|P(\lambda)|^2}{|\lambda - \lambda_i|^2 \|\Pi_i\|^2} \leq C_\Omega \,.$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left| f(\lambda) - \sum_{i=0}^{N} f(\lambda_i) \frac{P(\lambda)}{(\lambda-\lambda_i)P'(\lambda_i)} \right|^2 = \\
&= \left| \sum_{i=N+1}^{\infty} \frac{f(\lambda_i)}{P'(\lambda_i)} \frac{P(\lambda)}{(\lambda-\lambda_i)} \right|^2 \le \left(\sum_{i=N+1}^{\infty} \left| \frac{f(\lambda_i)}{P'(\lambda_i)} \frac{P(\lambda)}{(\lambda-\lambda_i)} \right| \right)^2 \\
&\le \left(\sum_{i=N+1}^{\infty} \frac{|\langle c, \Pi_i \rangle|^2}{\|\Pi_i\|^2} \right) \left(\sum_{i=N+1}^{\infty} \frac{|P(\lambda)|^2}{|\lambda-\lambda_i|^2 \|\Pi_i\|^2} \right) \\
&\le C_\Omega \left(\sum_{i=N+1}^{\infty} \frac{|\langle c, \Pi_i \rangle|^2}{\|\Pi_i\|^2} \right),
\end{aligned}$$

which goes to zero as $N \to \infty$ regardless of $\lambda \in \Omega$, since $\{c_n\}_{n=0}^\infty \in \ell^2(\mathbb{N}_0)$. The proof of the absolute convergence goes much in the same manner. The uniform convergence of (6) on compact subsets of $\mathbb{C}$ implies that f is an entire function. ■

As a straightforward consequence of this theorem, we derive the following interpolation result:

Corollary 1 *Under the conditions of Theorem 1, let $\{a_i\}_{i=0}^\infty$ be a sequence of complex numbers such that*

$$\sum_{i=0}^{\infty} \frac{|a_i|^2}{\|\Psi(\lambda_i, \cdot)\|^2} < \infty \,. \tag{8}$$

Then there exists a unique sequence $\{c_n\}_{n=0}^\infty \in \ell^2(\mathbb{N}_0)$ such that the corresponding function given by $f(\lambda) = \sum_{n=0}^\infty c_n \Psi(\lambda, n)$ satisfies $f(\lambda_i) = a_i$ for each $i \in \mathbb{N}_0$.

Proof: Taking into account (8), by the Riesz–Fischer theorem there exists a unique sequence $\{d_n\}_{n=0}^\infty$ in $\ell^2(\mathbb{N}_0)$ such that

$$\overline{a_i} = \langle \{d_n\}, \{\Psi(\lambda_i, n)\} \rangle_{\ell^2(\mathbb{N}_0)} = \sum_{n=0}^{\infty} d_n \overline{\Psi(\lambda_i, n)} \,.$$

Therefore, taking $c_n = \overline{d_n}$, $n \in \mathbb{N}_0$, the corresponding function f satisfies

$$f(\lambda_i) = \sum_{n=0}^{\infty} c_n \Psi(\lambda_i, n) = \langle \{\overline{d_n}\}, \{\overline{\Psi(\lambda_i, n)}\} \rangle_{\ell^2(\mathbb{N}_0)} = a_i \,,$$

for each $i \in \mathbb{N}_0$. ■

4 The Weyl–Titchmarsh functions

Let $\{s_n\}_{n=0}^\infty$ be our indeterminate Hamburger moment sequence. We denote by V the set of positive Borel measures satisfying it, i.e.,

$$V = \left\{ \mu \ge 0 \,:\, s_n = \int_{-\infty}^{\infty} x^n d\mu(x) \,,\ n \ge 0 \right\} \,.$$

As we said in the introduction, the moment problem is related with the self–adjoint extensions of the semi–infinite Jacobi matrix $\mathcal{A}$. In this context, the existence of the sequence $\{\lambda_i^t = \lambda_i(S_t)\}_{i=0}^{\infty}$ of eigenvalues associated with the self–adjoint extension S_t of $\mathcal{A}$ can be explained as follows: The set V can be parametrized with the one–point compactification $\mathcal{P} \cup \{\infty\}$ of the Pick (or Herglotz) function set $\mathcal{P}$ [1, 10]. When the parameter is restricted to constant functions taking values in $\overline{\mathbb{R}}$, we obtain the set of von Neumann measures $\{\mu_t\}$ which satisfy

$$\int_{-\infty}^{\infty} \frac{d\mu_t(x)}{\lambda - x} = \frac{A(\lambda) + tC(\lambda)}{B(\lambda) + tD(\lambda)}\,, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}\,,$$

where A, B, C, D are the entire functions forming the so–called Nevanlinna matrix associated with the moment problem. See [10] for explicit formulas of the Nevanlinna matrix. It is known that, for each $t \in \overline{\mathbb{R}}$, μ_t is the discrete measure $\mu_t = \sum_{\lambda \in Z_t} a_\lambda \delta_\lambda$, where

$$Z_t = \begin{cases} \{\lambda \in \mathbb{C} \mid B(\lambda) + tD(\lambda) = 0\} & \text{if } t \in \mathbb{R}\,, \\ \{\lambda \in \mathbb{C} \mid D(\lambda) = 0\} & \text{if } t = \infty\,, \end{cases}$$

and

$$a_\lambda = \frac{A(\lambda) + tC(\lambda)}{B'(\lambda) + tD'(\lambda)}\,, \quad \text{for } \lambda \in Z_t\,.$$

Notice that μ_t is the spectral measure of S_t. The zeros of the entire function $B(\lambda) + tD(\lambda)$ (or of $D(\lambda)$) which are real and simple, are precisely the sequence $\{\lambda_i^t\}_{i=0}^{\infty}$ and moreover, $\mu_t(\{\lambda_i^t\}) = 1/\|\Pi_i\|^2$ [10, p. 127]. The von Neumann measures are precisely those measures $\mu \in V$ for which the polynomials $\{P_n\}_{n=0}^{\infty}$ are dense in $L^2(\mu)$.

As mentioned in the introduction, the indeterminacy of the Hamburger moment problem can be thought as the discrete analogue of the Weyl's limit–point/limit–circle theory for the self-adjoint extensions of the discrete Sturm–Liouville problem given by (2). Indeed, the Hamburger moment problem is indeterminate if and only if the difference problem (2) belongs to the limit–circle case. Therefore, we can apply the known theory of the indeterminate Hamburger moment problem to the limit–circle case for the difference Sturm–Liouville problem (2). In particular, this provides us with criteria to decide when a difference Sturm–Liouville problem (2) belongs to the limit–circle case.

In fact, if $\mathcal{C}_\infty(\lambda)$ is the limit circle associated to $\lambda \in \mathbb{C} \setminus \mathbb{R}$, its points admit a parametrization through $\overline{\mathbb{R}}$ as follows: $m_\infty^t(\lambda) \in \mathcal{C}_\infty(\lambda)$ if and only if there exists $t \in \overline{\mathbb{R}}$ such that

$$m_\infty^t(\lambda) = \frac{A(\lambda) + tC(\lambda)}{B(\lambda) + tD(\lambda)} = \int_{-\infty}^{\infty} \frac{d\mu_t(x)}{\lambda - x}$$

where A, B, C and D are the components of the Nevanlinna matrix and μ_t is the von Neumann spectral measure associated with S_t [10, p. 128]. Furthermore, to close the relationships involved in our study, we point out that, for each $t \in \overline{\mathbb{R}}$, the Weyl–Titchmarsh function $m_\infty^t(\lambda)$ admits the representation

$$m_\infty^t(\lambda) = \langle \delta_0, (\lambda I - S_t)^{-1} \delta_0 \rangle_{\ell^2}\,.$$

Now we can derive that the sampling kernel used in Theorem 1 involves the function

$$\psi_\infty^t(\lambda, n) = Q_n(\lambda) + m_\infty^t(\lambda) P_n(\lambda)\,.$$

Indeed, in [6] it is proved that

$$\sum_{n=0}^{\infty} \psi_\infty^t(\lambda, n) P_n(\lambda_i^t) = \frac{1}{\lambda - \lambda_i^t}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} .$$

As a consequence, taking into account (7), we deduce that $\psi_\infty^t(\lambda, n) = K_\lambda(n, 0)$. Thus we have obtained a discrete analogue of the Zayed's sampling expansion for the limit–circle case.

Finally, we conclude the section listing the most important properties of the sampling kernel $\Psi(\lambda, n)$ (5):

a) For each $\lambda \in \mathbb{C}$, the sequence $\{\Psi(\lambda, n)\}_{n=0}^\infty$ is in $\ell^2(\mathbb{N}_0)$, and satisfies the difference equation in (2).

b) For each fixed $n \in \mathbb{N}_0$, $\Psi(\lambda, n)$ is an entire function.

c) $\Psi(\lambda_i, n) = k_i \Pi_i(n) = k_i P_n(\lambda_i)$, where $k_i = \frac{P'(\lambda_i)}{\|\Pi_i\|^2} \in \mathbb{R} \setminus \{0\}$.

d) The sequence $\{\Psi(\lambda_i, n)\}_{n=0}^\infty$ satisfies the boundary conditions at -1 and ∞.

Note that the summation kernel $\Psi(\lambda, n)$ fulfills all the requirements in the discrete version of the Kramer sampling theorem [7].

In the next section, we put to use the obtained sampling expansion in the case of q^{-1}-Hermite polynomials, a sequence of orthogonal polynomials associated with an indeterminate Hamburger moment problem [8].

5 Sampling formulas associated with the q^{-1}-Hermite polynomials

The q^{-1}-Hermite polynomials $\{h_n(x|q)\}_{n=0}^\infty$ where $0 < q < 1$ satisfy the three–term recurrence relation

$$h_{n+1}(x|q) = 2x h_n(x|q) - q^{-n}(1 - q^n) h_{n-1}(x|q), \quad n > 0 \tag{9}$$

and $h_0(x|q) = 1\,,\ \ h_1(x|q) = 2x$.

These polynomials have the explicit representation [8]

$$h_n(x|q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{k(k-n)} (x + \sqrt{x^2+1})^{n-2k} ,$$

where $(q;q)_j$ denotes the q-shifted factorial

$$(q;q)_j = \prod_{k=1}^{j} (1 - q^k) .$$

The recurrence (9) can be written in the self–adjoint form where multiplying by $\omega(n) = \frac{q^{n(n+1)/2}}{(q;q)_n}$ [9]. This gives

$$\frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n} h_{n+1}(x|q) = 2x \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n} h_n(x|q) - \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_{n-1}} h_{n-1}(x|q) . \tag{10}$$

Having in mind that $\|h_n\| = \sqrt{\frac{(q;q)_n}{q^{n(n+1)/2}}}$ [8], we can write (10) as

$$a_n \widetilde{h}_{n+1}(x|q) - x\widetilde{h}_n(x|q) + a_{n-1}\widetilde{h}_{n-1}(x|q) = 0 \tag{11}$$

where $a_n = \frac{1}{2}\sqrt{\frac{1-q^{n+1}}{q^{n+1}}}$, and $\widetilde{h}_n = \frac{h_n}{\|h_n\|}$ is the corresponding sequence of orthonormal polynomials.

The moment problem associated with $\{\widetilde{h}_n(x|q)\}_{n=0}^{\infty}$ is indeterminate [8]. Thus, the $\widetilde{h}_n$'s are orthonormal with respect to infinitely many measures [10], or equivalently, the infinite Sturm-Liouville problem associated with (11) belongs to the limit–circle case.

Let $\{\widetilde{h}_n^*\}_{n=0}^{\infty}$ be the sequence of second kind polynomials associated with (11), i.e., the $\{Q_n\}$ sequence associated with (11). We derive the sampling formulas associated with the special Weyl functions $m_{\infty}^{0}(\lambda)$ and $m_{\infty}^{\infty}(\lambda)$.

The $m_{\infty}^{0}(\lambda)$ case: In [8] it is proven that $m_{\infty}^{0}(\lambda)$ can be written in terms of a hypergeometric function as

$$m_{\infty}^{0}(\lambda) = \frac{-4\lambda q(q^2;q)_{\infty}}{(qe^{2\xi}, qe^{-2\xi}; q^2)_{\infty}} \ {}_2\phi_1(qe^{2\xi}, qe^{-2\xi}; q^3; q^2, q^2)\,,$$

where $\sinh\xi = \lambda$. The poles of $m_{\infty}^{0}(\lambda)$ are $\pm\lambda_n$ with $n = 0, 1, \ldots$ where $\lambda_n = \frac{1}{2}(q^{-n-\frac{1}{2}} - q^{n+\frac{1}{2}})$. In this case, the infinite product of the eigenvalues is given by $P(\lambda) = \prod_{n=0}^{\infty}\left(1 - \frac{\lambda^2}{\lambda_n^2}\right)$, and we can state the following sampling result:

Let $\Phi(n,\lambda) = P(\lambda)\psi_{\infty}^{0}(n,\lambda)$ with $\psi_{\infty}^{0}(n,\lambda) = \widetilde{h}_n^*(\lambda) + m_{\infty}^{0}(\lambda)\widetilde{h}_n(\lambda)$ and $\{c_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)$. If

$$f(\lambda) = \sum_{n=0}^{\infty} c_n \Phi(n,\lambda)\,,$$

then the function f is entire and it can be recovered from its values on the eigenvalues $\{\pm\lambda_n\}_{n=0}^{\infty}$ through the formula

$$f(\lambda) = \sum_{n=0}^{\infty} f(-\lambda_n)\frac{P(\lambda)}{(\lambda+\lambda_n)P'(-\lambda_n)} + \sum_{n=0}^{\infty} f(\lambda_n)\frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}\,.$$

The sampling series converges absolutely and uniformly on compact sets of $\mathbb{C}$.

The $m_{\infty}^{\infty}(\lambda)$ case: In this case we have an explicit formula for $m_{\infty}^{\infty}(\lambda)$ [8]

$$m_{\infty}^{\infty}(\lambda) = \frac{(q;q)_{\infty}}{\lambda(q^2e^{2\xi}, q^2e^{-2\xi}; q^2)_{\infty}} \ {}_2\phi_1(e^{-2\xi}, e^{2\xi}; q; q^2, q^2)\,.$$

The poles are 0 and $\pm\lambda_n$ with $n = 0, 1, \cdots$, where $\lambda_n = \frac{1}{2}\left(q^{-(n+1)} - q^{(n+1)}\right)$. Now the infinite product is $P(\lambda) = \lambda\prod_{n=0}^{\infty}\left(1 - \frac{\lambda^2}{\lambda_n^2}\right)$ and the associated sampling result is:

Let $\Phi(n,\lambda) = P(\lambda)\psi_{\infty}^{\infty}(n,\lambda)$ with $\psi_{\infty}^{\infty}(n,\lambda) = \widetilde{h}_n^*(\lambda) + m_{\infty}^{\infty}(\lambda)\widetilde{h}_n(\lambda)$ and $\{c_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)$. If

$$f(\lambda) = \sum_{n=0}^{\infty} c_n \Phi(n,\lambda)\,,$$

then the function f is entire and can be recovered from its values on the eigenvalues $\{\lambda_n\}_{n=0}^{\infty} \cup \{0\}$ through the formula

$$f(\lambda) = \sum_{k=-\infty}^{\infty} f(\mu_k) \frac{P(\lambda)}{P'(\mu_k)(\lambda - \mu_k)} \,,$$

where $\mu_k = \lambda_{k-1}$ for $k = 1, 2, \cdots$, $\mu_0 = 0$ and $\mu_k = -\lambda_{-k-1}$ for $k = -1, -2, \cdots$. The sampling series converges absolutely and uniformly on compact sets of $\mathbb{C}$.

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