Riesz bases in $L^2(0,1)$ related with sampling in shift-invariant spaces

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Abstract

The Fourier duality is an elegant technique to obtain sampling formulas in Paley-Wiener spaces. In this paper it is proved that there exists an analogous of the Fourier duality technique in the setting of shift-invariant spaces. In fact, any shift-invariant space V_φ with a stable generator φ is the range space of a bounded one-to-one linear operator T between $L^2(0,1)$ and $L^2(\mathbb{R})$. Thus, regular and irregular sampling formulas in V_φ are obtained by transforming, via T, expansions in $L^2(0,1)$ with respect to some appropriate Riesz bases.

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1 Introduction

The Whittaker-Shannon-Kotel'nikov sampling theorem states that any function f in the classical Paley-Wiener space PW_π

$$PW_\pi := \left\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \,:\, \operatorname{supp} \widehat{f} \subset [-\pi,\pi] \right\},$$

i.e., bandlimited to $[-\pi,\pi]$, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ on the integers as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\, \operatorname{sinc}(t-n)\,, \tag{1}$$

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where sinc denotes the cardinal sine function, $\operatorname{sinc}(t) = \sin \pi t/\pi t$.

This theorem and its numerous offspring have been proved in many different ways, e.g. using Fourier expansions, the Poisson summation formula, contour integrals, etc. (see, for instance, [9, 17]). But the most elegant proof is probably the one due to Hardy [8], using that the Fourier transform $\mathcal{F}$ is an isometry between PW_π and $L^2[-\pi, \pi]$. For any $f \in PW_\pi$ one has

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(w) e^{iwt} dw = \langle \widehat{f}, \frac{e^{-iwt}}{\sqrt{2\pi}} \rangle_{L^2[-\pi,\pi]}, \quad t \in \mathbb{R},$$

so any value $f(t_n)$ of f is the inner product in $L^2[-\pi, \pi]$ of $\widehat{f}$ and the complex exponential $e^{-it_n w}/\sqrt{2\pi}$. The key point in Hardy's proof is that an expansion converging in $L^2[-\pi, \pi]$ is transformed by $\mathcal{F}^{-1}$ into another expansion which converges in the topology of PW_π. This implies, in particular, that it converges absolutely and uniformly on $\mathbb{R}$. Recall that the Paley-Wiener space PW_π is a reproducing kernel Hilbert space (RKHS) whose reproducing kernel is $k(t, s) = \operatorname{sinc}(t - s)$. This technique has been coined in [9, p. 56] as the *Fourier duality* in Paley-Wiener spaces. Thus, expanding $\widehat{f}$ with respect to the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ and transforming by $\mathcal{F}^{-1}$ we obtain the Shannon sampling formula (1). Besides, an irregular sampling formula in PW_π at a sequence $\{t_n\}_{n\in\mathbb{Z}}$ of real points may be obtained by perturbating the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ in such a way that the sequence of complex exponentials $\{e^{-it_n w}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for PW_π. This is the case if, for instance, the sequence $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ verifies the Kadec condition: $\sup_{n\in\mathbb{Z}} |t_n - n| < 1/4$. Moreover, the Paley-Wiener-Levinson sampling theorem states that any function $f \in PW_\pi$ can be recovered from its samples $\{f(t_n)\}_{n\in\mathbb{Z}}$ by means of the Lagrange-type interpolation series

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(t)}{G'(t_n)(t - t_n)},$$

where G stands for the infinite product $G(t) := (t - t_0) \prod_{n=1}^{\infty} (1 - t/t_n)(1 - t/t_{-n})$ [16]. On the other hand, the Paley-Wiener space PW_π is a particular case of a shift-invariant space, i.e., a closed subspace in $L^2(\mathbb{R})$ generated by the integer shifts of a single function $\varphi \in L^2(\mathbb{R})$. Whenever the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ forms, at least, a frame sequence in $L^2(\mathbb{R})$, the corresponding shift-invariant space can be described as

$$V_\varphi := \Big\{ \sum_{n\in\mathbb{Z}} a_n \, \varphi(\cdot - n) \,:\, \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}.$$

The generator φ is stable if the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. For PW_π, a stable generator is $\varphi = \operatorname{sinc}$. Wavelet subspaces are important examples of shift-invariant spaces generated by the scaling function of the corresponding multiresolution analysis. See [3, 4, 13] for the general theory of shift-invariant spaces and their applications. In addition, sampling theory in shift-invariant spaces and, in particular, in wavelet subspaces has been largely studied in the recent years. Let us to cite, for instance, the works of Aldroubi and Gröchenig [1], Aldroubi and Unser [2], Chen, Itoh and Shiki [5, 6], Janssen [11], Sun and Zhou [14, 18], or Walter [12, 15] among others.

The main aim in this paper is to show that the Fourier duality for Paley-Wiener spaces can be generalized to the case of a shift-invariant space V_φ with a stable generator φ. To this end, we define a bounded one-to-one linear operator T between $L^2(0,1)$ and $L^2(\mathbb{R})$ as

$$\begin{aligned} T:\ \ L^2(0,1) &\longrightarrow\ \ L^2(\mathbb{R}) \\ F &\longrightarrow\ \ f \qquad \text{such that}\ \ f(t) := \langle F, K_t\rangle_{L^2(0,1)}\,, \end{aligned}$$

where the kernel transform $t \in \mathbb{R} \mapsto K_t \in L^2(0,1)$ is given by the Zak transform of $\overline{\varphi}$ namely, $K_t(x) := Z\overline{\varphi}(t,x)$, a.e. $x \in (0,1)$. The shift-invariant space V_φ coincides with the range space of T. Thus, sampling expansions in V_φ can be seen as transformed expansions via T of expansions in $L^2(0,1)$ with respect to appropriate Riesz bases. Taking into account the definition of T, these bases should have the particular form $\{K_{t_n}\}_{n\in\mathbb{Z}}$. Taking the sampling points $\{t_n = a+n\}_{n\in\mathbb{Z}}$ we obtain the regular sampling in V_φ, whereas perturbing this sequence as $\{t_n = a+n+\delta_n\}_{n\in\mathbb{Z}}$ we obtain the irregular sampling. These steps will be carried out throughout the remaining sections.

2 Preliminaries on shift-invariant spaces

Let $\varphi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space

$$V_\varphi := \Big\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot\ - n)\ :\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R})\,,$$

i.e., the sequence $\{\varphi(\cdot\ - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. Recall that the sequence $\{\varphi(\cdot\ - n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for V_φ if and only if

$$0 < \|\Phi\|_0 \le \|\Phi\|_\infty < \infty\,,$$

where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2$ in $[0,1]$, and $\|\Phi\|_\infty$ its essential supremum. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds [7, p. 143].

We assume along the paper that, for each $t \in \mathbb{R}$, the series $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ converges. Thus, by using the Riesz' subsequence theorem [7, p. 39] we can choose the pointwise limit $f(t) := \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n)$ for each $t \in \mathbb{R}$, as the representative element of the class $\sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot\ - n)$ in $L^2(\mathbb{R})$. Moreover, if φ is a continuous function and the series $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ converges uniformly in compact subsets of $\mathbb{R}$, we can take any $f \in V_\varphi$ as a continuous function in $\mathbb{R}$.

Besides, V_φ is a RKHS since the evaluation functionals are bounded in V_φ. Indeed, for each fixed $t \in \mathbb{R}$ we have

$$|f(t)|^2 \leq \frac{1}{\|\Phi\|_0} \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 \; \|f\|^2, \quad f \in V_\varphi\,, \tag{2}$$

where we have used Cauchy-Schwartz's inequality in $f(t) = \sum_{n\in\mathbb{Z}} a_n \; \varphi(t-n)$, and the Riesz basis condition

$$\|\Phi\|_0 \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \|f\|^2 \leq \|\Phi\|_\infty \sum_{n\in\mathbb{Z}} |a_n|^2, \quad f \in V_\varphi\,.$$

Inequality (2) shows that convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence in $\mathbb{R}$. The convergence is uniform in subsets of the real line where $\|K_t\|^2_{L^2(0,1)} = \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is bounded.

The reproducing kernel of V_φ is given by $k(t,s) = \sum_{n\in\mathbb{Z}} \varphi(t-n)\, \overline{\varphi^*(s-n)}$ where the sequence $\{\varphi^*(\cdot\; - n)\}_{n\in\mathbb{Z}}$ denotes the dual Riesz basis of $\{\varphi(\cdot\; - n)\}_{n\in\mathbb{Z}}$. Recall that the function φ^* has Fourier transform $\widehat{\varphi^*} = \widehat{\varphi}/\Phi$ [2].

3 A linear transform defining a shift-invariant space

For each $t \in \mathbb{R}$, consider the function $K_t \in L^2(0,1)$ defined by the Fourier series

$$K_t := \sum_{n\in\mathbb{Z}} \overline{\varphi(t+n)} e^{-2\pi i n x}\,.$$

Notice that $K_t(x) = Z\overline{\varphi}(t,x)$ a.e. $x \in (0,1)$, where Z denotes the Zak transform of $\overline{\varphi}$. See [10] for properties and uses of the Zak transform.

Thus, for each $F \in L^2(0,1)$ we can define the function

$$\begin{array}{rrcl} f: & \mathbb{R} & \longrightarrow & \mathbb{C} \\ & t & \longrightarrow & f(t) := \langle F, K_t\rangle_{L^2(0,1)}\,. \end{array}$$

If we denote by T the linear transform which maps $F \in L^2(0,1)$ into f, i.e., $T(F) = f$, then we can identify the range space of T as the shift-invariant V_φ, i.e., $T(L^2(0,1)) = V_\varphi$. Indeed, for $F \in L^2(0,1)$ we have that

$$[T(F)](t) = \langle F, K_t\rangle_{L^2(0,1)} = \sum_{n\in\mathbb{Z}} \langle F, e^{-2\pi i n x}\rangle_{L^2(0,1)} \varphi(t+n)\,, \quad t \in \mathbb{R}\,,$$

which belongs to V_φ. Furthermore, for each $f \in V_\varphi$ there exists a sequence $\{a_n\} \in \ell^2(\mathbb{Z})$ such that $f = \sum_{n\in\mathbb{Z}} a_n \varphi(\cdot + n)$ in $L^2(\mathbb{R})$. Since $\{e^{-2\pi i n x}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(0,1)$, there exists a function $F \in L^2(0,1)$ such that $\langle F, e^{-2\pi i n x}\rangle_{L^2(0,1)} = a_n$ for every $n \in \mathbb{Z}$. Hence, $T(F) = f$. Moreover, the following result holds

Theorem 1 *The mapping T is an invertible bounded operator between $L^2(0,1)$ and V_φ.*

Proof: The operator T is bijective since it applies the orthonormal basis $\{e^{-2\pi inx}\}_{n\in\mathbb{Z}}$ in $L^2(0,1)$ into the Riesz basis $\{\varphi(t+n)\}_{n\in\mathbb{Z}}$ in V_φ. Concerning the continuity, for $F\in L^2(0,1)$ we have

$$\begin{aligned}\|T(F)\|^2_{L^2(\mathbb{R})} &= \Big\|\sum_{n\in\mathbb{Z}}\langle F,e^{-2\pi inx}\rangle_{L^2(0,1)}\varphi(t+n)\Big\|^2_{L^2(\mathbb{R})} \le \|\Phi\|_\infty \sum_{n\in\mathbb{Z}}|\langle F,e^{-2\pi inx}\rangle|^2\\ &= \|\Phi\|_\infty\,\|F\|^2_{L^2(0,1)}\,,\end{aligned}$$

where we have used the upper Riesz basis condition for $\{\varphi(\cdot+n)\}_{n\in\mathbb{Z}}$.

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Having in mind the periodicity relations of the Zak transform, the function K_t satisfies for $t\in\mathbb{R}$ and $m\in\mathbb{Z}$

$$K_{t+m}(x)=e^{2\pi imx}K_t(x)\quad\text{in } L^2(0,1)\,.$$

4 Riesz bases in $L^2(0,1)$ related with sampling in shift-invariant spaces

It is a straightforward result that a sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$, with $F\in L^2(0,1)$, is an orthonormal basis in $L^2(0,1)$ if and only $|F(x)|=1$ a.e. in $(0,1)$. The aim in this section is to characterize Bessel sequences, Riesz bases and frames in $L^2(0,1)$ having this particular form. From now on, $\|F\|_\infty$ (respect. $\|F\|_0$) will denote the essential supremum (respect. infimum) of $|F|$ in $(0,1)$.

Theorem 2 *Given a function $F\in L^2(0,1)$, the following results hold:*

(a) The sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(0,1)$ if and only if the function F satisfies $\|F\|_\infty<\infty$.

(b) The sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$ if and only if the function F satisfies $0<\|F\|_0\le\|F\|_\infty<\infty$. In this case, the optimal Riesz bounds of $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ are $\|F\|_0^2$ and $\|F\|_\infty^2$.

(c) The sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a frame in $L^2(0,1)$ if and only if is a Riesz basis for $L^2(0,1)$.

Proof: The proof of the theorem relies on the operator

$$\begin{aligned}\mathcal{G}:\ L^2(0,1)&\longrightarrow\ L^2(0,1)\\ f&\longrightarrow\ fF\end{aligned}$$

It is a well-defined bounded operator if and only if $\|F\|_\infty < \infty$. Following the characterization of Bessel sequences [7, Th. 3.2.3], this is equivalent to be $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ a Bessel sequence in $L^2(0,1)$, which proves (a). Moreover, the operator norm is $\|\mathcal{G}\| = \|F\|_\infty$.

Since $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(0,1)$, the sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$ if and only if the operator $\mathcal{G}$ is a bounded invertible operator. The continuity of the inverse operator, $\mathcal{G}^{-1} : f \mapsto f/F$, is equivalent to $0 < \|F\|_0$, and $\|\mathcal{G}^{-1}\| = \|F\|_0^{-1}$. In this case, the optimal Riesz bounds are $\|\mathcal{G}^{-1}\|^{-2} = \|F\|_0^2$ and $\|\mathcal{G}\|^2 = \|F\|_\infty^2$ [7, p. 68]. Thus, we have proved (b).

Finally, having in mind the characterization of frames [7, Th. 5.5.5], the sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a frame in $L^2(0,1)$ if and only if operator $\mathcal{G}$ is a bounded surjective operator. But in this particular case, $\mathcal{G}$ surjective obviously implies that $\mathcal{G}$ is one-to-one, and $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.

■

The above theorem has its counterpart in V_φ

Corollary 1 *Given a function $g \in V_\varphi$, consider $G = T^{-1}(g) \in L^2(0,1)$. Then, the sequence $\{g(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ if and only if $0 < \|G\|_0 \le \|G\|_\infty < \infty$.*

Proof: For each $n \in \mathbb{Z}$ we have

$$\begin{aligned} T\big[G(x)e^{2\pi inx}\big](t) &= \langle G(\cdot)e^{2\pi in\cdot}, K_t(\cdot)\rangle_{L^2(0,1)} = \langle G(\cdot), e^{-2\pi in\cdot}K_t(\cdot)\rangle_{L^2(0,1)} \\ &= \langle G, K_{t-n}\rangle_{L^2(0,1)} = g(t-n)\,, \quad t \in \mathbb{R}\,. \end{aligned}$$

Since T is a bounded invertible operator, the sequence $\{G(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$ if and only if $\{g(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ, and the result follows from Theorem 2.

■

5 Regular sampling in shift-invariant spaces

Regular sampling in V_φ arises by considering appropriate Riesz bases in $L^2(0,1)$. Namely, for a fixed $a \in [0,1)$, the regular samples at $\{a+n\}_{n\in\mathbb{Z}}$ of $f \in V_\varphi$ are given by

$$f(a+n) = \langle F, K_{a+n}\rangle_{L^2(0,1)} = \langle F, K_a e^{2\pi inx}\rangle_{L^2(0,1)}\,, \quad n \in \mathbb{Z}\,,$$

where $F = T^{-1}(f)$. The sequence $\{K_a(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ in $L^2(0,1)$ has the biorthonormal sequence $\{e^{2\pi inx}/\overline{K}_a(x)\}_{n\in\mathbb{Z}}$ provided $1/K_a \in L^2(0,1)$. Hence, stable regular sampling in V_φ reduces to study whenever the sequence $\{K_a(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$, and this depends on the function K_a as we have proved in Theorem 2. Expanding $F = T^{-1}(f)$ with respect to the Riesz basis $\{e^{2\pi inx}/\overline{K}_a(x)\}_{n\in\mathbb{Z}}$, via the invertible bounded operator T we obtain a regular sampling formula for f.

Lemma 1 *Given $a \in [0,1)$, there exists a function $S_a \in V_\varphi$ satisfying the interpolation condition $S_a(a+n) = \delta_{n,0}$, where $n \in \mathbb{Z}$, if and only if the function $1/K_a$ belongs to $L^2(0,1)$. In this case $S_a = T(1/\overline{K}_a)$.*

Proof: Assume that there exists a function $S_a \in V_\varphi$ satisfying the interpolation condition $S_a(a+n) = \delta_{n,0}$, where $n \in \mathbb{Z}$. For $F_a = T^{-1}(S_a)$ we have

$$\begin{aligned} S_a(a+n) &= \langle F_a, K_{a+n}\rangle_{L^2(0,1)} = \langle F_a, e^{2\pi i n x} K_a\rangle_{L^2(0,1)} \\ &= \int_0^1 F_a(x)\overline{K_a(x)} e^{-2\pi i n x} dx = \delta_{n,0}\,, \end{aligned}$$

which implies that $F_a(x)\overline{K_a(x)} = 1$ a.e. in $(0,1)$, and consequently the function $1/K_a$ belongs to $L^2(0,1)$.

Conversely, if $1/K_a$ is in $L^2(0,1)$ we define $S_a = T(1/\overline{K}_a)$. For $n \in \mathbb{Z}$ it satisfies

$$S_a(a+n) = \langle \frac{1}{\overline{K}_a}, K_{a+n}\rangle_{L^2(0,1)} = \langle 1, e^{2\pi i n x}\rangle_{L^2(0,1)} = \delta_{n,0}\,.$$

■

Thus we can characterize stable regular sampling in V_φ

Theorem 3 *Consider $a \in [0,1)$ such that the function $1/K_a \in L^2(0,1)$. The following conditions are equivalent:*

(a) $0 < \|K_a\|_0 \le \|K_a\|_\infty < \infty$.

(b) There exists a Riesz basis $\{S_n\}_{n\in\mathbb{Z}}$ for V_φ such that, for each $f \in V_\varphi$, we have the pointwise expansion

$$f(t) = \sum_{n\in\mathbb{Z}} f(a+n)\, S_n(t)\,, \quad t \in \mathbb{R}\,.$$

Furthermore, in this case the sampling functions are $S_n(t) = S_a(t-n)$, where $S_a = T(1/\overline{K}_a)$. The sampling series converges in the $L^2(\mathbb{R})$-norm sense, absolutely and uniformly in subsets of $\mathbb{R}$ where $\|K_t\|$ is bounded.

Proof: First we prove that (a) implies (b). Consider $S_a = T(1/\overline{K}_a)$. Condition (a) implies that $0 < \|1/\overline{K}_a\|_0 \le \|1/\overline{K}_a\|_\infty < \infty$ and, as a consequence, Corollary 1 gives that $\{S_a(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. For each $f \in V_\varphi$, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ such that $f(t) = \sum_{n\in\mathbb{Z}} a_n S_a(t-n)$ where the convergence is also pointwise for each $t \in \mathbb{R}$ since V_φ is a RKHS. Taking $t = a+m$, and using the interpolatory condition $S_a(a+n) = \delta_{n,0}$, we obtain that $a_m = f(a+m)$ for any $m \in \mathbb{Z}$.

Conversely, assume that the condition (b) holds. Taking $f(t) = S_a(t-m)$, $m \in \mathbb{Z}$, we obtain that $S_m(t) = S_a(t-m)$ and , as a consequence, $\{S_a(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. Since $S_a = T(1/\overline{K}_a)$, Corollary 1 gives condition (a).

Absolute convergence comes from the inconditional character of a Riesz basis. The uniform convergence is a standard result in the setting of the RKHS theory.

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A straightforward calculation gives the Fourier transform of S_a. Indeed,

$$\widehat{S_a}(w) = \widehat{T(1/\overline{K}_a)}(w) = \frac{\widehat{\varphi}(w)}{Z\varphi(a, w/2\pi)} \quad a.e. \text{ in } \mathbb{R}\,.$$

6 Irregular sampling in shift-invariant spaces

Usually, one may consider irregular sampling as a perturbation of the regular sampling. In the present setting, we can try to recover any function $f \in V_\varphi$ from its perturbed samples $\{f(a+n+\delta_n)\}_{n\in\mathbb{Z}}$, where $a \in [0,1)$ and $\{\delta_n\}_{n\in\mathbb{Z}}$ is a sequence in $(-1,1)$. Since

$$f(a+n+\delta_n) = \langle F, K_{a+n+\delta_n}\rangle_{L^2(0,1)}\,, \quad n \in \mathbb{Z}\,, \text{ where } F = T^{-1}(f) \in L^2(0,1)\,,$$

a challenge problem is to prove that $\{K_{a+n+\delta_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$. For, a possibility is to use a perturbation technique on the Riesz basis $\{K_{a+n}\}_{n\in\mathbb{Z}} = \{K_a e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ which gives the sequence of regular samples $\{f(a+n)\}_{n\in\mathbb{Z}}$. As a consequence, we need a perturbation result for those Riesz bases in $L^2(0,1)$ appearing in Theorem 2.

For an infinite matrix $M = \{m_{n,k}\}_{n,k\in\mathbb{Z}}$ defining a bounded operator in $\ell^2(\mathbb{Z})$ we denote its operator norm as $\|M\|_2 := \sup_{\|c\|_{\ell^2(\mathbb{Z})}=1} \|Mc\|_{\ell^2(\mathbb{Z})}$.

Theorem 4 *Let $F = \sum_{k\in\mathbb{Z}} a_k e^{-2\pi i k x}$ be in $L^2(0,1)$ such that $0 < \|F\|_0 \le \|F\|_\infty < \infty$. Let $\{F_n\}_{n\in\mathbb{Z}}$ be a sequence of functions in $L^2(0,1)$ with Fourier expansions $F_n = \sum_{k\in\mathbb{Z}} a_k(n) e^{-2\pi i k x}$, $n \in \mathbb{Z}$. Suppose that the infinite matrix $D = \{d_{n,k}\}_{n,k\in\mathbb{Z}}$ with entries $d_{n,k} := a_{n-k}(n) - a_{n-k}$, $n,k \in \mathbb{Z}$, satisfies the condition $\|D\|_2 < \|F\|_0$. Then, the sequence $\{F_n(x)e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.*

Proof: To this end we use the following result on perturbation of Riesz bases in a Hilbert space $\mathcal{H}$ which proof can be found in [7, p. 354]: Let $\{f_k\}_{k=1}^\infty$ be a Riesz basis for $\mathcal{H}$ with Riesz bounds A, B, and let $\{g_k\}_{k=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R < A$ such that

$$\sum_{k=1}^{\infty} |\langle f_k - g_k, f\rangle|^2 \le R\,\|f\|^2\,, \quad \text{for each} \quad f \in \mathcal{H}\,,$$

then $\{g_k\}_{k=1}^\infty$ is a Riesz basis for $\mathcal{H}$.

For any $f=\sum_{j\in\mathbb{Z}}\overline{c_j}e^{2\pi ijx}$ in $L^2(0,1)$ we have

$$\sum_{n\in\mathbb{Z}}|\langle F_n(x)e^{2\pi inx}-F(x)e^{2\pi inx},f\rangle|^2=\sum_{n\in\mathbb{Z}}|\langle\sum_{k\in\mathbb{Z}}(a_k(n)-a_k)e^{2\pi i(n-k)x},\sum_{j\in\mathbb{Z}}\overline{c_j}e^{2\pi ijx}\rangle|^2$$
$$=\sum_{n\in\mathbb{Z}}\Big|\sum_{k\in\mathbb{Z}}\big(a_{n-k}(n)-a_{n-k}\big)\ c_k\Big|^2=\sum_{n\in\mathbb{Z}}\Big|\sum_{k\in\mathbb{Z}}d_{n,k}\ c_k\Big|^2=\|D\ c\|^2_{\ell^2(\mathbb{Z})}\leq\|D\|_2^2\ \|f\|^2\,.$$

Taking into account that in our case $A=\|F\|_0^2$, we obtain that $\{F_n(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.

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As a consequence of the above perturbation theorem in $L^2(0,1)$, we obtain an irregular sampling result in V_φ

Theorem 5 *Given $a\in[0,1)$ such that $0<\|K_a\|_0\leq\|K_a\|_\infty<\infty$. Let $\Delta=\{\delta_n\}_{n\in\mathbb{Z}}$ be a sequence in $(-1,1)$ such that the infinite matrix $D_\Delta=\{d_{n,k}\}_{n,k\in\mathbb{Z}}$ whose entries are given by*

$$d_{n,k}:=\overline{\varphi(a+n-k+\delta_n)}-\overline{\varphi(a+n-k)}\,,\quad n,k\in\mathbb{Z}$$

satisfies $\|D_\Delta\|_2<\|K_a\|_0$. Then, there exists a Riesz basis $\{S_n\}_{n\in\mathbb{Z}}$ for V_φ such that any function $f\in V_\varphi$ can be expanded as

$$f(t)=\sum_{n\in\mathbb{Z}}f(a+n+\delta_n)\,S_n(t)\,,\quad t\in\mathbb{R}.$$

The convergence of the series is absolute and uniform in subsets of $\mathbb{R}$ where $\|K_t\|$ is bounded. Also, it converges in the $L^2(\mathbb{R})$-norm sense.

Proof: Applying Theorem 4 to

$$K_a(x)=\sum_{k\in\mathbb{Z}}\overline{\varphi(a+k)}e^{-2\pi ikx}\quad\text{and}\quad K_{a+\delta_n}(x)=\sum_{k\in\mathbb{Z}}\overline{\varphi(a+k+\delta_n)}e^{-2\pi ikx}\,,n\in\mathbb{Z}\,,$$

we obtain that $\{K_{a+\delta_n}e^{2\pi inx}\}_{n\in\mathbb{Z}}=\{K_{a+n+\delta_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$. Denote by $\{G_n\}_{n\in\mathbb{Z}}$ its dual Riesz basis. Now, given $f\in V_\varphi$, we expand the function $F=T^{-1}(f)\in L^2(0,1)$ with respect to $\{G_n\}_{n\in\mathbb{Z}}$. Thus,

$$F=\sum_{n\in\mathbb{Z}}\langle F,K_{a+n+\delta_n}\rangle_{L^2(0,1)}\,G_n=\sum_{n\in\mathbb{Z}}f(a+n+\delta_n)\ G_n\,,\ \text{ in }\ L^2(0,1)\,.$$

Applying the operator T we get

$$f=\sum_{n\in\mathbb{Z}}f(a+n+\delta_n)\ T(G_n)\,,\ \text{ in }\ L^2(\mathbb{R})\,.$$

Furthermore, since T is an invertible bounded operator, the sequence $\{S_n := T(G_n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. The pointwise convergence properties of the series come out as in Theorem 3.

■

The next result yields a uniform bound of the norm $\|D_\Delta\|_2$ regardless the sequence $\Delta = \{\delta_n\}_{n\in\mathbb{Z}}$ in $[\alpha,\beta] \subset [-1,1]$.

Theorem 6 *For any sequence $\Delta = \{\delta_n\}_{n\in\mathbb{Z}}$ in $[\alpha,\beta]$ the following inequality holds*

$$\|D_\Delta\|_2 \le \sup_{\{d_n\}\subset[\alpha,\beta]} \sum_{n\in\mathbb{Z}} |\varphi(a+n+d_n) - \varphi(a+n)| \,. \tag{3}$$

Proof: Assume that the second term in the above inequality is finite. Otherwise, the inequality trivially holds. For any $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have

$$\begin{aligned}
\|D_\Delta c\|^2_{\ell^2(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} d_{n,k}\ c_k \Big|^2 \le \sum_{n\in\mathbb{Z}} \sum_{l,j\in\mathbb{Z}} |d_{n,l}|\ |c_l|\ |\overline{d_{n,j}}|\ |\overline{c_j}| \\
&= \sum_{l,j\in\mathbb{Z}} |c_l|\ |c_j| \sum_{n\in\mathbb{Z}} |d_{n,l}|\ |d_{n,j}| \le \sum_{l,j\in\mathbb{Z}} \frac{|c_l|^2 + |c_j|^2}{2} \sum_{n\in\mathbb{Z}} |d_{n,l}|\ |d_{n,j}| \\
&= \sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{j,n\in\mathbb{Z}} |d_{n,l}|\ |d_{n,j}| \le \sup_{l\in\mathbb{Z}} \Big(\sum_{j,n\in\mathbb{Z}} |d_{n,l}|\ |d_{n,j}| \Big) \|c\|^2_{\ell^2(\mathbb{Z})} \\
&\le \sup_{l\in\mathbb{Z}} \big(\sum_{n\in\mathbb{Z}} |d_{n,l}| \big) \sum_{j\in\mathbb{Z}} |d_{n,j}|\ \|c\|^2_{\ell^2(\mathbb{Z})} \,.
\end{aligned}$$

Having in mind that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |d_{n,j}| &= \sum_{j\in\mathbb{Z}} |\varphi(a+j-k+\delta_j) - \varphi(a+j-k)| \\
&= \sum_{n\in\mathbb{Z}} |\varphi(a+n+\delta_{n+k}) - \varphi(a+n)| \,,
\end{aligned}$$

we obtain the desired result.

■

A comment about the second term in (3) is in order. Namely, looking for an estimation of the ratio between $\sum_{n\in\mathbb{Z}} |\varphi(a+n+d_n) - \varphi(a+n)|$ and $(\sup_n |d_n|)^\lambda$ for a fixed $\lambda > 0$, led Chen et al. to introduce in [5] the classes of functions $L_a^\lambda[\alpha,\beta]$.

Next we give a particular example when Theorem 6 works. Namely, suppose that the stable generator $\varphi \in \mathcal{C}^1(\mathbb{R})$ and for some $\epsilon > 0$ it satisfies $\varphi'(t) = O(|t|^{-(1+\epsilon)})$ as $|t| \to \infty$. Then, it is easy to prove that, for $\delta \in (0,1]$

$$M_{\varphi'}(\delta) := \sum_k \max_{I_k(\delta)} |\varphi'(t)| \le M_{\varphi'}(1) < \infty \,,$$

where $I_k(\delta)$ denotes the interval $[a+k-\delta, a+k+\delta]$.

Corollary 2 *Let $\varphi \in \mathcal{C}^1(\mathbb{R})$ be a stable generator such that $M_{\varphi'}(\delta) < \infty$, where $\delta := \sup_{n\in\mathbb{Z}} |\delta_n|$. Then, the condition $\delta\ M_{\varphi'}(\delta) < \|K_a\|_0$ implies the existence of a Riesz basis $\{S_n\}_{n\in\mathbb{Z}}$ for V_φ such that any function in this space can be expanded as*

$$f(t) = \sum_{n\in\mathbb{Z}} f(a+n+\delta_n)\, S_n(t)\,, \quad t \in \mathbb{R}.$$

The convergence in the series is absolute and uniform in subsets of $\mathbb{R}$ where $\|K_t\|$ is bounded. It converges also in the $L^2(\mathbb{R})$-norm sense.

Proof: The mean value theorem gives

$$\sup_{\{d_n\}\subset[-\delta,\delta]} \sum_{n\in\mathbb{Z}} |\varphi(a+n+d_n) - \varphi(a+n)| \leq \delta\ M_{\varphi'}(\delta)\,.$$

Theorem 5 concludes the proof.

■

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