Generalized irregular sampling in shift-invariant spaces

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Abstract

This article concerns the problem of stable recovering of any function in a shift-invariant space from irregular samples of some filtered versions of the function itself. These samples arise as a perturbation of regular samples. The starting point is the generalized regular sampling theory which allows to recover any function f in a shift-invariant space from the samples at $\{rn\}_{n\in\mathbb{Z}}$ of s filtered versions $\mathcal{L}_1 f, \mathcal{L}_2 f, \dots, \mathcal{L}_s f$ of f, where the number of channels s is greater or equal than the sampling period r. These regular samples can be expressed as the frame coefficients of a related to f function in $L^2(0,1)$ with respect to certain frame for $L^2(0,1)$. The irregular samples are also obtained as a perturbation of the aforesaid frame. As a natural consequence, the irregular sampling results arise from the theory of perturbation of frames. The paper ends putting the theory to work in some spline examples where Kadec-type results are obtained.

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1 Statement of the problem

Suppose that s linear-time invariant systems (filters) $\mathcal{L}_j$, $j = 1, 2, \dots, s$, are defined on a shift-invariant space V_φ of $L^2(\mathbb{R})$

$$V_\varphi := \Big\{ f(t) = \sum_{n\in\mathbb{Z}} a_n \, \varphi(t-n) \, : \, \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} ,$$

where the function $\varphi \in L^2(\mathbb{R})$ is a stable generator for V_φ. In [6] it has been proved that any function $f \in V_\varphi$ is recovered by means of a stable sampling formula which involves the

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samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$, where the sampling period $r \in \mathbb{N}$ necessarily satisfies $r \le s$. Concretely, the sampling formula for $f \in V_\varphi$ reads

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn)\ S_j(t - rn)\,, \qquad t \in \mathbb{R}\,,$$

where the sequence $\{S_j(t-rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for V_φ. This frame is a Riesz basis for V_φ whenever $r = s$.

From a practical point of view it is of interest to have a similar result, but for a sequence of samples taken with a nonuniform distribution along the real line. A straightforward application of this result would be the recovering of signals affected by time-jitter error, i.e., taken at points $t_n = rn + \varepsilon_n$ with ε_n some measurement uncertainty. Very often in the mathematical literature, this problem has been solved as a perturbation problem of orthonormal (Riesz) bases or frames in a Hilbert space. A classical example where this methodology applies is the irregular sampling in Paley-Wiener spaces. Indeed, any function in the classical Paley-Wiener space

$$PW_\pi := \big\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \,:\, \operatorname{supp} \widehat{f} \subseteq [-\pi, \pi] \big\}\,,$$

i.e., bandlimited to $[-\pi,\pi]$, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ on the integers as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\ \operatorname{sinc}(t-n)\,, \qquad t \in \mathbb{R}\,, \tag{1}$$

where sinc denotes the cardinal sine function, $\operatorname{sinc}(t) = \sin \pi t/\pi t$. For any $f \in PW_\pi$ one has

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(w) e^{iwt} dw = \Big\langle \widehat{f}, \frac{e^{-iwt}}{\sqrt{2\pi}} \Big\rangle_{L^2[-\pi,\pi]}\,, \quad t \in \mathbb{R}\,,$$

so any value $f(t_n)$ of f is the inner product in $L^2[-\pi,\pi]$ of its Fourier transform $\widehat{f}$ and the complex exponential $e^{-it_n w}/\sqrt{2\pi}$. Thus, expanding $\widehat{f}$ with respect to the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ in $L^2[-\pi,\pi]$, and applying the inverse Fourier transform $\mathcal{F}^{-1}$ one deduces the Shannon sampling formula (1). An irregular sampling formula in PW_π at a sequence $\{t_n\}_{n\in\mathbb{Z}}$ of real points may be obtained by perturbating the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ in such a way that the sequence of complex exponentials $\{e^{-it_n w}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi,\pi]$. This is the case if, for instance, the sequence $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ verifies the Kadec condition: $\sup_{n\in\mathbb{Z}} |t_n - n| < 1/4$. Here, the classical Paley-Wiener criterion on stability of bases has been used (see [15, p. 38]). Moreover, the Paley-Wiener-Levinson sampling theorem states that any function $f \in PW_\pi$ can be recovered from its samples $\{f(t_n)\}_{n\in\mathbb{Z}}$ by means of the Lagrange-type interpolation series

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(t)}{G'(t_n)(t-t_n)}\,, \qquad t \in \mathbb{R}\,,$$

where G stands for the infinite product $G(t) := (t - t_0) \prod_{n=1}^{\infty} (1 - t/t_n)(1 - t/t_{-n})$ [15].

This perturbation technique has been successfully used for obtaining irregular sampling formulas in a general shift-invariant space V_φ. See, for instance, the papers of Chen et al. [1, 2], García et al. [7], Liu and Walter [12], Sun and Zhou [14] and Sun [13]. See also Liu [11].

The main aim in this paper is to recover any function $f \in V_\varphi$ from the perturbed sequence of samples $\{(\mathcal{L}_j f)(rn+\varepsilon_{j,n})\}$. It has been proved in [6] that the regular samples $\{(\mathcal{L}_j f)(rn)\}$ can be expressed as the frame coefficients of an appropriate function in $L^2(0,1)$, related to f, with respect to a particular frame in $L^2(0,1)$. Recall that a sequence $\{f_k\}$ is a frame for a separable Hilbert space $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|f\|^2 \le \sum_k |\langle f, f_k\rangle|^2 \le B\|f\|^2 \quad \text{for all } f \in \mathcal{H}\,.$$

Given a frame $\{f_k\}$ for $\mathcal{H}$ the representation property of any vector $f \in \mathcal{H}$ as a series $f = \sum_k c_k f_k$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for overcomplete frames) is sacrificed. For instance, any $f \in \mathcal{H}$ can be expanded as

$$f = \sum_k \langle f, S^{-1} f_k\rangle f_k = \sum_k \langle f, f_k\rangle S^{-1} f_k\,,$$

where S^{-1} denotes the inverse of the frame operator $Sf := \sum_k \langle f, f_k\rangle f_k$, which defines $\{S^{-1} f_k\}$, the canonical dual frame of $\{f_k\}$. For more details on the frame theory see the superb monograph [3] and references therein. As we prove in Section 3, the irregular samples $\{(\mathcal{L}_j f)(rn+\varepsilon_{j,n})\}$ are obtained as a perturbation of the aforesaid frame. Thus, the theory on perturbation of frames (see [3, Chapter 15]) yields generalized irregular sampling in the shift-invariant space V_φ for suitable error sequences $\{\varepsilon_{j,n}\}$. Moreover, for some important examples, the allowed sequences $\{\varepsilon_{j,n}\}$ will be given in terms of $\sup_{j,n} |\varepsilon_{j,n}|$.

2 Preliminaries on generalized sampling in shift-invariant spaces

In this section we introduce the preliminaries on shift-invariant spaces needed in the sequel. Also, we present the generalized regular sampling theory for these spaces as stated in [6].

2.1 Shift-invariant spaces

Let $\varphi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space

$$V_\varphi := \Big\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot - n) \,:\, \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R})\,,$$

i.e., the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for V_φ if and only if $0 < \|\Phi\|_0 \le \|\Phi\|_\infty < \infty$, where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2$ in $(0,1)$, and $\|\Phi\|_\infty$ its essential supremum ($\widehat{\varphi}$ stands for the Fourier transform $\widehat{\varphi}(w) := \int_{-\infty}^{\infty} \varphi(t)e^{-2\pi itw}dt)$. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds [3, p. 143].

We assume throughout the paper that the functions in the shift-invariant space V_φ are continuous on $\mathbb{R}$. Equivalently, that the generator φ is continuous on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ (see [16]). Thus, any $f \in V_\varphi$ is defined on $\mathbb{R}$ as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n\varphi(t-n)$.

Besides, V_φ is a reproducing kernel Hilbert space (RKHS) since the evaluation functionals are bounded in V_φ. Indeed, for each fixed $t \in \mathbb{R}$ we have

$$|f(t)|^2 \leq \frac{\|f\|^2}{\|\Phi\|_0} \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 \,, \quad f \in V_\varphi \,, \tag{2}$$

where we have used Cauchy-Schwartz's inequality in $f(t) = \sum_{n\in\mathbb{Z}} a_n \; \varphi(t-n)$, and the Riesz basis condition

$$\|\Phi\|_0 \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \|f\|^2 \leq \|\Phi\|_\infty \sum_{n\in\mathbb{Z}} |a_n|^2, \quad f \in V_\varphi \,.$$

Inequality (2) shows that convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence which is uniform on $\mathbb{R}$.

On the other hand, the space V_φ is the image of $L^2(0,1)$ by means of the isomorphism $\mathcal{T} : L^2(0,1) \to V_\varphi$ which maps the orthonormal basis $\{e^{-2\pi inw}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ for V_φ (see [7]), i.e.,

$$(\mathcal{T}F)(t) := \sum_{n\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi in\cdot} \big\rangle_{L^2(0,1)} \varphi(t-n), \qquad F \in L^2(0,1).$$

Notice that for each $F \in L^2(0,1)$ the function $f = \mathcal{T}F$ is given by

$$f(t) = \langle F, K_t \rangle_{L^2(0,1)} \,, \quad t \in \mathbb{R} \,.$$

The kernel transform $t \in \mathbb{R} \longrightarrow K_t \in L^2(0,1)$ is defined as $K_t(x) := \overline{Z\varphi}(t,x)$ where $Z\varphi$ denotes the Zak transform of φ. Recall that the Zak transform of $f \in L^2(\mathbb{R})$ is formally defined in $\mathbb{R}^2$ as $(Zf)(t,w) := \sum_{n\in\mathbb{Z}} f(t+n)e^{-2\pi inw}$. See [9, 10] for properties and uses of the Zak transform.

The following shifting property of $\mathcal{T}$ will be used later: For $F \in L^2(0,1)$, $r \in \mathbb{N}$ and $n \in \mathbb{Z}$ we have

$$\mathcal{T}\big[F(\cdot)e^{-2\pi irn\cdot}\big](t) = \mathcal{T}[F](t-rn) \,, \quad t \in \mathbb{R} \,. \tag{3}$$

2.2 Generalized regular sampling

In this section we introduce the generalized regular sampling theory in a shift-invariant space V_φ. These sampling formulas involve samples of filtered versions of the functions in V_φ. As in [6], we distinguish two types of linear-time invariant systems $\mathcal{L}_j$:

(a) The impulse response l_j of $\mathcal{L}_j$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, for any $f \in V_\varphi$ we have

$$\big(\mathcal{L}_j f\big)(t) := [f * l_j](t) = \int_{-\infty}^{\infty} f(x) l_j(t-x)dx = \langle f(\cdot), \phi_j(\cdot - t) \rangle_{L^2(\mathbb{R})} \,, \quad t \in \mathbb{R} \,,$$

where $\phi_j(t) := \overline{l_j(-t)}$. Notice that $\mathcal{L}_j f$ is a continuous and bounded function in $L^2(\mathbb{R})$.

(b) The impulse response l_j has the form $l_j = \sum_{k=0}^{N} c_k \delta^{(k)}(t+d_k)$ where $\delta^{(k)}$ denotes the k-th derivative of the Dirac delta and c_k, d_k are constants for $k = 0, 1, \dots, N$. For each $f \in V_\varphi$ we have

$$\big(\mathcal{L}_j f\big)(t) := \sum_{k=0}^{N} c_k f^{(k)}(t+d_k)\,, \quad t \in \mathbb{R}\,.$$

In this case we also assume that $\varphi^{(N)}$ exists on $\mathbb{R}$, and $\sum_{n\in\mathbb{Z}} |\varphi^{(k)}(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ for each $k = 0, 1, \dots, N$.

Whenever $\mathcal{L}_j$ is a filter of the type (a) or (b) above, for any $t \in \mathbb{R}$ the sequence $\{(\mathcal{L}_j\varphi)(t+n)\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ (see [6, Lemma 1]). Thus, for any fixed $t \in \mathbb{R}$, the Zak transform $(Z\mathcal{L}_j\varphi)(t,w) = \sum_{n\in\mathbb{Z}} \mathcal{L}_j\varphi(t+n)e^{-2\pi i n w}$ defines a function in $L^2(0,1)$. For notational ease we choose $t = 0$ without loss of generality. For $j = 1, 2, \dots, s$, the functions g_j in $L^2(0,1)$ defined by

$$g_j(w) := \sum_{n\in\mathbb{Z}} \mathcal{L}_j\varphi(n)e^{-2\pi i n w} = (Z\mathcal{L}_j\varphi)(0,w)\,, \quad j = 1, 2, \dots, s\,, \tag{4}$$

play an important role since they allow a representation of the samples $\{(\mathcal{L}_j f)(rn)\}$. Namely: Let f be a function in V_φ such that $f = \mathcal{T}F$ where $F \in L^2(0,1)$. For every $j = 1, 2, \dots, s$, we have

$$\big(\mathcal{L}_j f\big)(rn) = \big\langle F(\cdot), \overline{g}_j(\cdot)\, e^{-2\pi i r n \cdot} \big\rangle_{L^2(0,1)}\,, \quad n \in \mathbb{Z}\,. \tag{5}$$

Equation (5) leads us to study when a sequence $\{a_j(\cdot)\, e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ (or equivalently the sequence $\{\overline{a}_j(\cdot) e^{2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$), where $a_j \in L^2(0,1)$ for each $j = 1, 2, \dots, s$, is a Bessel sequence, a frame, or a Riesz basis for $L^2(0,1)$. To this end, associated with the functions a_j, $j = 1, 2, \dots, s$, we introduce the $s \times r$ matrix function defined for $w \in (0,1)$ as

$$\mathbf{A}(w) := \begin{bmatrix} a_1(w) & a_1(w+\frac{1}{r}) & \cdots & a_1(w+\frac{r-1}{r}) \\ a_2(w) & a_2(w+\frac{1}{r}) & \cdots & a_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ a_s(w) & a_s(w+\frac{1}{r}) & \cdots & a_s(w+\frac{r-1}{r}) \end{bmatrix} = \Big[a_j\Big(w + \frac{k-1}{r}\Big)\Big]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,r}}\,,$$

and its related constants

$$\alpha_{\mathbf{A}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbf{A}^*(w)\mathbf{A}(w)], \quad \beta_{\mathbf{A}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbf{A}^*(w)\mathbf{A}(w)]\,,$$

where $\mathbf{A}^*(w)$ denotes the transpose conjugate of the matrix $\mathbf{A}(w)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbf{A}^*(w)\mathbf{A}(w)$. Observe that $0 \le \alpha_{\mathbf{A}} \le \beta_{\mathbf{A}} \le \infty$. Notice that in the definition of the matrix $\mathbf{A}(w)$ we are considering the 1-periodic extensions of the involved functions a_j, $j = 1, 2, \dots, s$. Next Lemma gives the answer:

Lemma 1 *Let a_j be in $L^2(0,1)$ for $j=1,2,\dots,s$ and let $\mathbf{A}(w)$ be its associated matrix. Then,*

(i) The sequence $\{\overline{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Bessel sequence in $L^2(0,1)$ if and only if $a_j\in L^\infty(0,1)$ for $j=1,\dots,s$. In this case, the optimal Bessel bound is $\beta_{\mathbf{A}}/r$.

(ii) The sequence $\{\overline{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0<\alpha_{\mathbf{A}}\le\beta_{\mathbf{A}}<\infty$. In this case, the optimal frame bounds are $\alpha_{\mathbf{A}}/r$ and $\beta_{\mathbf{A}}/r$.

(iii) The sequence $\{\overline{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Riesz basis for $L^2(0,1)$ if and only if it is a frame for $L^2(0,1)$ and $r=s$.

For the proof of (i) and (ii) see [6]. Concerning (iii), it can be proved by using that a frame is a Riesz basis if and only if it has a biorthogonal sequence [3, p. 124]. In this case, this biorthogonal sequence is precisely $\{sb_j(\cdot)e^{2\pi isn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ where $[b_1,b_2,\dots,b_s]$ is the first row of the matrix $\mathbf{A}^{-1}$.

Assume that the function $g_j\in L^\infty(0,1)$ for $j=1,2,\dots,s$, and that there exists a vector $[a_1(w),\dots,a_s(w)]$ with entries in $L^\infty(0,1)$ such that

$$\big[a_1(w),\dots,a_s(w)\big]\ \mathbf{G}(w)=[1,0,\dots,0]\ \text{ a.e. in } (0,1)\,,$$

where $\mathbf{G}(w)$ denotes the corresponding $\mathbf{A}(w)$ matrix for the functions g_j, $j=1,2,\dots,s$, in (4). It has been proved in [6] that any $F\in L^2(0,1)$ can be expanded in terms of the dual frames $\{ra_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\ j=1,2,\dots,s}$ and $\{\overline{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\ j=1,2,\dots,s}$ for $L^2(0,1)$ as

$$\begin{aligned}F(w)&=r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\big\langle F(\cdot),\overline{g}_j(\cdot)\,e^{-2\pi irn\cdot}\big\rangle_{L^2(0,1)}a_j(w)e^{-2\pi inrw}\\&=r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\big(\mathcal{L}_jf\big)(rn)a_j(w)e^{-2\pi inrw}\qquad\text{in } L^2(0,1)\,.\end{aligned}$$

Thus, the isomorphism $\mathcal{T}$ gives the following sampling formula in V_φ

$$f(t)=r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\big(\mathcal{L}_jf\big)(rn)(\mathcal{T}a_j)(t-rn)\,,\qquad t\in\mathbb{R}\,,$$

where we have used (3). In fact, much more can be said about the above sampling formula in V_φ (see Theorem 1, Theorem 2 and Corollary 1 in [6]):

Theorem 1 *Assume that the function $g_j\in L^\infty(0,1)$ for each $j=1,2,\dots,s$. Then the following statements are equivalent*

(a) $\alpha_{\mathbf{G}}>0$.

(b) There exists a frame for V_φ having the form $\{S_j(t-rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ such that for any $f\in V_\varphi$,

$$f=\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\big(\mathcal{L}_jf\big)(rn)\ S_j(\cdot-rn)\quad \text{in } L^2(\mathbb{R})\,,\tag{6}$$

(c) There exist functions $a_j \in L^\infty(0,1)$, $j = 1, 2, \ldots, s$, such that

$$\big[a_1(w), \ldots, a_s(w)\big]\, \mathbf{G}(w) = [1, 0, \ldots, 0] \quad a.e.\ in\ (0,1)\,. \tag{7}$$

In case the equivalent conditions are satisfied, the reconstruction functions in (6) are given by $S_j = r\mathcal{T} a_j$, where the functions a_j, $j = 1, 2, \ldots, s$, satisfy (7). The convergence of the series in (6) is also absolute and uniform on $\mathbb{R}$. If $r = s$ then there exists a unique frame $\{S_j(t - sn)\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ for V_φ for which the sampling formula (6) holds. Moreover, this frame is a Riesz basis for V_φ, and the corresponding functions a_j, $j = 1, 2, \ldots, s$, form the first row of the matrix $\mathbf{G}^{-1}$.

3 Generalized irregular sampling

The key point for generalized regular sampling is Equation (5) which allows us to represent the regular samples $\{(\mathcal{L}_j f)(rn)\}$ by means of the frame $\{\overline{(Z\mathcal{L}_j\varphi)}(0,\cdot)e^{-2\pi i r n\cdot}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ for $L^2(0,1)$. In this section we prove that the perturbed samples $\{(\mathcal{L}_j f)(rn + \varepsilon_{j,n})\}$ are obtained from a perturbed version of the above frame. Thus, stable generalized irregular sampling may be studied from the perturbation theory of frames.

3.1 An expression for the irregular samples

Whenever the linear system $\mathcal{L}_j$ is of the type (a), the Minkowski inequality for integrals shows that the sequence $\{(\mathcal{L}_j\varphi)(t+n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for any fixed $t \in \mathbb{R}$ (see [6, Lemma 1]). Trivially, the same applies for $\mathcal{L}_j$ of the type (b). Therefore, for any fixed $t \in \mathbb{R}$, the function $(Z\mathcal{L}_j\varphi)(t,w) := \sum_{n\in\mathbb{Z}} (\mathcal{L}_j\varphi)(t+n)e^{-2\pi i n w}$ belongs to $L^2(0,1)$ and the following expression for the perturbed samples $(\mathcal{L}_j f)(rn + \varepsilon_{j,n})$ holds:

Lemma 2 *Let f be a function in V_φ such that $f = \mathcal{T}F$ where $F \in L^2(0,1)$. For $n \in \mathbb{Z}$, $j = 1, 2, \ldots, s$, and $\varepsilon_{j,n} \in \mathbb{R}$ we have*

$$\big(\mathcal{L}_j f\big)(rn + \varepsilon_{j,n}) = \big\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n}, \cdot)e^{-2\pi i r n\cdot}\big\rangle_{L^2(0,1)}\,.$$

Proof: Assume that $\mathcal{L}_j$ is a system of the type (a). For each $n \in \mathbb{Z}$ we have

$$\begin{aligned}
(\mathcal{L}_j f)(rn + \varepsilon_{j,n}) &= \big\langle f(\cdot), \phi_j(\cdot - rn - \varepsilon_{j,n})\big\rangle_{L^2(\mathbb{R})} \\
&= \Big\langle \sum_{k\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi i k\cdot}\big\rangle_{L^2(0,1)} \varphi(\cdot - k), \phi_j(\cdot - rn - \varepsilon_{j,n})\Big\rangle_{L^2(\mathbb{R})} \\
&= \sum_{k\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi i k\cdot}\big\rangle_{L^2(0,1)} \mathcal{L}_j\varphi(rn + \varepsilon_{j,n} - k)\,.
\end{aligned}$$

Parseval's equality and a change in the summation index gives

$$\begin{aligned}
(\mathcal{L}_j f)(rn + \varepsilon_{j,n}) &= \big\langle F(\cdot), \sum_{k\in\mathbb{Z}} \overline{\mathcal{L}_j\varphi}(rn + \varepsilon_{j,n} - k)\, e^{-2\pi i k\cdot}\big\rangle_{L^2(0,1)} \\
&= \big\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n}, \cdot)e^{-2\pi i r n\cdot}\big\rangle_{L^2(0,1)}\,.
\end{aligned}$$

Assume now that $\mathcal{L}_j$ is a system of the type (b). Under our hypotheses on $\mathcal{L}_j$, for each $k = 0, 1, \ldots, N$ we have that $f^{(k)}(t) = \sum_{l\in\mathbb{Z}} \left\langle F(\cdot), e^{-2\pi i l \cdot}\right\rangle_{L^2(0,1)} \varphi^{(k)}(t-l)$. Hence, for each $n \in \mathbb{Z}$, one gets

$$\begin{aligned}
(\mathcal{L}_j f)(rn + \varepsilon_{j,n}) &= \sum_{k=0}^{N} c_k f^{(k)}(rn + \varepsilon_{j,n} + d_k) \\
&= \sum_{k=0}^{N} c_k \sum_{l\in\mathbb{Z}} \left\langle F(\cdot), e^{-2\pi i l\cdot}\right\rangle_{L^2(0,1)} \varphi^{(k)}(rn + \varepsilon_{j,n} + d_k - l) \\
&= \left\langle F(\cdot), \sum_{k=0}^{N} \overline{c}_k \sum_{l\in\mathbb{Z}} \overline{\varphi}^{(k)}(rn + \varepsilon_{j,n} + d_k - l) e^{-2\pi i l\cdot}\right\rangle_{L^2(0,1)} \\
&= \left\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n}, \cdot) e^{-2\pi i rn\cdot}\right\rangle_{L^2(0,1)} .
\end{aligned}$$

■

It is worth to point out that the same proof in Lemma 2 shows that

$$\mathcal{L}_j f(t) = \langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(t, \cdot)\rangle_{L^2(0,1)} \quad t \in \mathbb{R}$$

for any linear system $\mathcal{L}_j$ of the type (a) or (b).

Lemma 2 leads us to study when the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n}, \cdot)e^{-2\pi i rn\cdot}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ is a frame for $L^2(0,1)$. On the other hand, we know that, whenever $0 < \alpha_{\mathbf{G}} \le \beta_{\mathbf{G}} < \infty$, the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(0, \cdot)e^{-2\pi i rn\cdot}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ is a frame for $L^2(0,1)$ with optimal frame bounds $\alpha_{\mathbf{G}}/r$ and $\beta_{\mathbf{G}}/r$. In the case of $r = s$, the above sequence is a Riesz basis for $L^2(0,1)$. One possibility is to use the theory of perturbation of frames in order to find the suitable error sequences for which the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n}, \cdot)e^{-2\pi i rn\cdot}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ is a frame for $L^2(0,1)$. The following result on perturbation of frames which proof is in [3, p. 354] will be used later:

Lemma 3 *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A, B, and let $\{g_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$. If there exists a constant $R < A$ such that*

$$\sum_{k=1}^{\infty} |\langle f_k - g_k, f\rangle|^2 \le R\|f\|^2 \quad \textit{for each } f \in \mathcal{H},$$

then $\{g_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds

$$A\left(1 - \sqrt{\frac{R}{A}}\right)^2, \quad B\left(1 + \sqrt{\frac{R}{B}}\right)^2.$$

If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis, then $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis.

3.2 The resulting sampling theory

Given an error sequence $\varepsilon := \{\varepsilon_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ we define on $\ell^2(\mathbb{Z})$ the operator $D_\varepsilon = \big[D_{\varepsilon,1}, \ldots, D_{\varepsilon,s}\big]$, where

$$D_{\varepsilon,j}\ c := \Big\{\sum_{k\in\mathbb{Z}} \big[\mathcal{L}_j\varphi(rn - k + \varepsilon_{j,n}) - \mathcal{L}_j\varphi(rn - k)\big] c_k\Big\}_{n\in\mathbb{Z}}$$

for each $c = \{c_l\}_{l\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. The operator norm is defined as usual

$$\|D_\varepsilon\| := \sup_{c\in\ell^2(\mathbb{Z})\setminus\{0\}} \frac{\|D_\varepsilon\, c\,\|_{\ell^2_s(\mathbb{Z})}}{\|c\,\|_{\ell^2(\mathbb{Z})}}\,,$$

where $\|D_\varepsilon\, c\,\|^2_{\ell^2_s(\mathbb{Z})} := \sum_{j=1}^s \|D_{\varepsilon,j}\, c\,\|^2_{\ell^2(\mathbb{Z})}$ for each $c \in \ell^2(\mathbb{Z})$.

Theorem 2 *Assume that $g_j \in L^\infty(0,1)$ for $j = 1,2,\dots,s$ with $\alpha_{\mathbf{G}} > 0$. If the error sequence $\varepsilon := \{\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,\dots,s}$ satisfies the inequality $\|D_\varepsilon\|^2 < \alpha_{\mathbf{G}}/r$, then there exists a frame $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ for V_φ such that, for any $f \in V_\varphi$*

$$f(t) = \sum_{n\in\mathbb{Z}}\sum_{j=1}^s \big(\mathcal{L}_j f\big)(rn+\varepsilon_{j,n}) S^\varepsilon_{j,n}(t) \quad t\in\mathbb{R}\,, \tag{8}$$

where the convergence of the series is in the $L^2(\mathbb{R})$-sense, absolute and uniform on $\mathbb{R}$. Moreover, when $r = s$ the sequence $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Riesz basis for V_φ, and the interpolation property $(\mathcal{L}_l S^\varepsilon_{j,n})(sm+\varepsilon_{j,m}) = \delta_{j,l}\delta_{n,m}$ holds.

Proof: The sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(0,\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame (a Riesz basis if $r = s$) for $L^2(0,1)$ with frame (Riesz) bounds $\alpha_{\mathbf{G}}/r$ and $\beta_{\mathbf{G}}/r$. For any $F(w) = \sum_{l\in\mathbb{Z}} c_l e^{-2\pi ilw}$ in $L^2(0,1)$ we have

$$\begin{aligned}
&\sum_{j=1}^s\sum_{n\in\mathbb{Z}} \big|\big\langle \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot} - \overline{(Z\mathcal{L}_j\varphi)}(0,\cdot)e^{-2\pi irn\cdot}\,,\, F(\cdot)\big\rangle_{L^2(0,1)}\big|^2\\
&= \sum_{j=1}^s\sum_{n\in\mathbb{Z}} \big|\big\langle \sum_{k\in\mathbb{Z}}[\overline{\mathcal{L}_j\varphi}(k+\varepsilon_{j,n}) - \overline{\mathcal{L}_j\varphi}(k)]e^{-2\pi i(rn-k)\cdot}\,,\, \sum_{l\in\mathbb{Z}} c_l e^{-2\pi il\cdot}\big\rangle_{L^2(0,1)}\big|^2\\
&= \sum_{j=1}^s\sum_{n\in\mathbb{Z}} \big|\sum_{k\in\mathbb{Z}}[\overline{\mathcal{L}_j\varphi}(k+\varepsilon_{j,n}) - \overline{\mathcal{L}_j\varphi}(k)]\,\overline{c}_{rn-k}\big|^2 \qquad (9)\\
&= \sum_{j=1}^s\sum_{n\in\mathbb{Z}} \big|\sum_{k\in\mathbb{Z}}[\mathcal{L}_j\varphi(rn-k+\varepsilon_{j,n}) - \mathcal{L}_j\varphi(rn-k)]c_k\big|^2\\
&= \sum_{j=1}^s \|D_{\varepsilon,j}\{c_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})} \le \|D_\varepsilon\|^2\|\{c_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})} = \|D_\varepsilon\|^2\|F\|^2_{L^2(0,1)}\,.
\end{aligned}$$

By using Lemma 3 we obtain that the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ (a Riesz basis if $r = s$). Let $\{h^\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ be its canonical dual frame. Hence, for any $F \in L^2(0,1)$

$$\begin{aligned}
F &= \sum_{n\in\mathbb{Z}}\sum_{j=1}^s \big\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\big\rangle_{L^2(0,1)} h^\varepsilon_{j,n}\\
&= \sum_{n\in\mathbb{Z}}\sum_{j=1}^s \big(\mathcal{L}_j f\big)(rn+\varepsilon_{j,n}) h^\varepsilon_{j,n}\,.
\end{aligned}$$

Applying the isomorphism $\mathcal{T}$, one gets (8) in $L^2(\mathbb{R})$ where $S^{\varepsilon}_{j,n} = \mathcal{T} h^{\varepsilon}_{j,n}$. Since $\mathcal{T}$ is an isomorphism between $L^2(0,1)$ and V_φ, $\{S^{\varepsilon}_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for V_φ (a Riesz basis if $r = s$).

Pointwise convergence in the sampling series is absolute due to the unconditional character of a frame. The uniform convergence on $\mathbb{R}$ is a consequence of (2). The interpolatory property in the case $r = s$ follows from the uniqueness of the coefficients with respect to a Riesz basis.

■

The next result yields a uniform bound of the norm $\|D_\varepsilon\|$ regardless the sequence ε such that $\{\varepsilon_{j,n}\}_{n\in\mathbb{Z}}$ is in $[\alpha_j, \beta_j] \subset [-r, r]$, for each $j = 1, 2, \dots, s$.

Theorem 3 *For any sequence ε such that $\{\varepsilon_{j,n}\}_{n\in\mathbb{Z}} \subset [\alpha_j, \beta_j] \subset [-r, r]$ for each $j = 1, 2, \dots, s$ the following inequality holds*

$$\|D_\varepsilon\|^2 \le \sum_{j=1}^{s} \Lambda_j \Gamma_j \,, \tag{10}$$

where, for each $j = 1, 2, \dots, s$, the constants Λ_j and Γ_j are given by

$$\Lambda_j := \sup_{\substack{l=0,1,\dots,r-1 \\ \{d_k\}\subset[\alpha_j,\beta_j]}} \sum_{k\in\mathbb{Z}} |\mathcal{L}_j\varphi(rk + l + d_k) - \mathcal{L}_j\varphi(rk + l)| \,,$$

$$\Gamma_j := \sup_{d\in[\alpha_j,\beta_j]} \sum_{k\in\mathbb{Z}} |\mathcal{L}_j\varphi(k + d) - \mathcal{L}_j\varphi(k)| \,.$$

Proof: Suppose that $\sum_{j=1}^{s} \Lambda_j \Gamma_j < \infty$; otherwise the result obviously holds. Denoting

$$d^{(j)}_{n,k} := \mathcal{L}_j\varphi(rn - k + \varepsilon_{j,n}) - \mathcal{L}_j\varphi(rn - k) \,,$$

for $k, n \in \mathbb{Z}$, the inequalities

$$\sum_{l\in\mathbb{Z}} |d^{(j)}_{l,k}| \le \Lambda_j \quad \text{and} \quad \sum_{l\in\mathbb{Z}} |d^{(j)}_{n,l}| \le \Gamma_j$$

hold. For any sequence $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have

$$\begin{aligned}
\|D_\varepsilon c\|^2_{\ell^2_s(\mathbb{Z})} &= \sum_{j=1}^{s} \sum_{n\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} d^{(j)}_{n,k}\; c_k \Big|^2 \le \sum_{j=1}^{s} \sum_{n\in\mathbb{Z}} \sum_{l,k\in\mathbb{Z}} |d^{(j)}_{n,l} c_l \overline{d}^{(j)}_{n,k} \overline{c}_k| \\
&= \sum_{j=1}^{s} \sum_{l,k\in\mathbb{Z}} |c_l|\; |c_k| \sum_{n\in\mathbb{Z}} |d^{(j)}_{n,l} d^{(j)}_{n,k}| \le \sum_{j=1}^{s} \sum_{l,k\in\mathbb{Z}} \frac{|c_l|^2 + |c_k|^2}{2} \sum_{n\in\mathbb{Z}} |d^{(j)}_{n,l} d^{(j)}_{n,k}| \\
&= \sum_{j=1}^{s} \sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{k,n\in\mathbb{Z}} |d^{(j)}_{n,l} d^{(j)}_{n,k}| \le \sum_{j=1}^{s} \sum_{l\in\mathbb{Z}} |c_l|^2 \Gamma_j \sum_{n\in\mathbb{Z}} |d^{(j)}_{n,l}| \\
&\le \sum_{j=1}^{s} \sum_{l\in\mathbb{Z}} |c_l|^2 \Gamma_j \Lambda_j = \Big(\sum_{j=1}^{s} \Lambda_j \Gamma_j \Big) \|c\|^2_{\ell^2(\mathbb{Z})} \,,
\end{aligned}$$

which concludes the proof.

■

Next we give a particular example when Theorem 3 works. Namely, suppose that, for each $j = 1, 2, \ldots, s$, the function $\mathcal{L}_j\varphi \in \mathcal{C}^1(\mathbb{R})$, and there exists $\eta_j > 0$ such that $(\mathcal{L}_j\varphi)'(t) = O(|t|^{-(1+\eta_j)})$ whenever $|t| \to \infty$. Then, it is easy to check that, for $\delta_j > 0$,

$$M_{(\mathcal{L}_j\varphi)'}(\delta_j) := \sum_k \max_{t\in[k-\delta_j, k+\delta_j]} |(\mathcal{L}_j\varphi)'(t)| < \infty\,.$$

Corollary 1 *Suppose that, for each $j = 1, 2, \ldots, s$, the function $\mathcal{L}_j\varphi \in \mathcal{C}^1(\mathbb{R})$ and $M_{(\mathcal{L}_j\varphi)'}(\delta_j) < \infty$, where $\delta_j := \sup_{n\in\mathbb{Z}} |\varepsilon_{j,n}|$. Consider*

$$N_{(\mathcal{L}_j\varphi)'}(\delta_j) := \sup_{l=0,1,\ldots,r-1} \sum_k \max_{t\in[rk+l-\delta_j, rk+l+\delta_j]} |(\mathcal{L}_j\varphi)'(t)|\,.$$

Then, the condition

$$\sum_{j=1}^{s} \delta_j^2\, N_{(\mathcal{L}_j\varphi)'}(\delta_j) M_{(\mathcal{L}_j\varphi)'}(\delta_j) < \frac{\alpha_{\mathbf{G}}}{r}$$

implies the existence of a frame $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ for V_φ such that, for any $f \in V_\varphi$

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \big(\mathcal{L}_j f\big)(rn + \varepsilon_{j,n}) S^\varepsilon_{j,n}(t)\,, \quad t \in \mathbb{R}\,,$$

uniformly on $\mathbb{R}$. If $r = s$, the sequence $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ is a Riesz basis for V_φ.

Proof: For each $j = 1, 2, \ldots, s$, the mean value theorem gives

$$\sup_{d\in[-\delta_j,\delta_j]} \sum_{k\in\mathbb{Z}} |\mathcal{L}_j\varphi(k+d) - \mathcal{L}_j\varphi(k)| \le \delta_j\ M_{(\mathcal{L}_j\varphi)'}(\delta_j)\,,$$

and

$$\sup_{\substack{l=0,1,\ldots,r-1\\ \{d_k\}\subset[-\delta_j,\delta_j]}} \sum_{k\in\mathbb{Z}} |\mathcal{L}_j\varphi(rk+l+d_k) - \mathcal{L}_j\varphi(rk+l)| \le \delta_j\ N_{(\mathcal{L}_j\varphi)'}(\delta_j)\,.$$

Theorem 2 concludes the proof.

■

Notice that $N_{(\mathcal{L}_j\varphi)'}(\delta_j) \le M_{(\mathcal{L}_j\varphi)'}(\delta_j)$. For $r = 1$ the equality holds.

3.3 The frame algorithm

Formula (8) in Theorem 2 is useless from a practical point of view. The involved frame $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$, which depends on the error sequence $\{\varepsilon_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$, is impossible to determine. As a consequence, in order to recover any function $f \in V_\varphi$ from the samples $\{\mathcal{L}_j f(rn + \varepsilon_{j,n})\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ we should use the frame algorithm (see [5]). We are going to implement this algorithm in the $\ell^2(\mathbb{Z})$ setting. To this end, consider the canonical isometry

$$\mathcal{U} : \ell^2(\mathbb{Z}) \to L^2(0,1)\,, \quad \mathcal{U}\,\{a_l\}_{l\in\mathbb{Z}} := \sum_{l\in\mathbb{Z}} a_l e^{-2\pi i l w}\,.$$

For $f(t)=\sum_{l\in\mathbb{Z}} a_l\varphi(t-l)\in V_\varphi$, denote by $\mathbb{F}$ the sequence

$$\mathbb{F} := \mathcal{U}^{-1}F = \mathcal{U}^{-1}\mathcal{T}^{-1}f = \{a_l\}_{l\in\mathbb{Z}}\,.$$

The samples $\{(\mathcal{L}_j f)(rn+\varepsilon_{j,n})\}$ can be written as

$$\big(\mathcal{L}_j f\big)(rn+\varepsilon_{j,n}) = \big\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\big\rangle_{L^2(0,1)} = \big\langle \mathbb{F}, \mathbb{L}_{j,n}\big\rangle_{\ell^2(\mathbb{Z})}$$

where, for $j=1,2,\dots,s$ and $n\in\mathbb{Z}$,

$$\mathbb{L}_{j,n} := \mathcal{U}^{-1}\big(\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\big) = \big\{\overline{(\mathcal{L}_j\varphi)}(rn-l+\varepsilon_{j,n})\big\}_{l\in\mathbb{Z}}\,.$$

The sequence $\{\mathbb{L}_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $\ell^2(\mathbb{Z})$. Indeed, assume that $\{\varepsilon_{j,n}\}_{n\in\mathbb{Z}}\subset[\alpha_j,\beta_j]\subset[-r,r]$ for each $j=1,2,\dots,s$, and that $\sum_{j=1}^s\Lambda_j\Gamma_j<\alpha_{\mathbf{G}}/r$. According to the proof of Theorem 2 and Lemma 3, the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ with bounds

$$A := \frac{\alpha_{\mathbf{G}}}{r}\left(1-\sqrt{\frac{r}{\alpha_{\mathbf{G}}}\sum_{j=1}^s\Lambda_j\Gamma_j}\right)^2,\quad B := \frac{\beta_{\mathbf{G}}}{r}\left(1+\sqrt{\frac{r}{\beta_{\mathbf{G}}}\sum_{j=1}^s\Lambda_j\Gamma_j}\right)^2 \tag{11}$$

Since $\mathcal{U}^{-1}$ is an isometry, the sequence $\{\mathbb{L}_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $\ell^2(\mathbb{Z})$ with the same bounds.

Hence, the recovering of the function $f=\mathcal{T}\,\mathcal{U}\,\mathbb{F}\in V_\varphi$ from the samples $\{(\mathcal{L}_j f)(rn+\varepsilon_{j,n})\}$ is reduced to recover the sequence $\mathbb{F}$ from the sequence $\big\{\langle\mathbb{F},\mathbb{L}_{j,n}\rangle_{\ell^2(\mathbb{Z})}\big\}_{n\in\mathbb{Z},j=1,2,\dots,s}$. In so doing, the classical frame algorithm reads:

Consider

$$\mathbb{F}_0=\mathcal{A}\,\mathbb{F} := \frac{2}{A+B}\sum_{n\in\mathbb{Z}}\sum_{j=1}^s\langle\mathbb{F},\mathbb{L}_{j,n}\rangle_{\ell^2(\mathbb{Z})}\mathbb{L}_{j,n} = \frac{2}{A+B}\sum_{n\in\mathbb{Z}}\sum_{j=1}^s\big(\mathcal{L}_j f\big)(rn+\varepsilon_{j,n})\mathbb{L}_{j,n}$$

and define recursively $\mathbb{F}_{k+1}=\mathbb{F}_k+\mathcal{A}(\mathbb{F}-\mathbb{F}_k)\,,\quad k\in\mathbb{N}$. Then, the sequence $\{f_k\}_{k\in\mathbb{N}}$ in V_φ given by $f_k(t)=\sum_{l\in\mathbb{Z}}a_l^{(k)}\varphi(t-l)$ where $\mathbb{F}_k=\{a_l^{(k)}\}_{l\in\mathbb{Z}}$, satisfies

$$\begin{aligned}\|f-f_k\|_{L^2(\mathbb{R})} &\le \|\mathcal{T}\|\,\|\mathbb{F}-\mathbb{F}_k\|_{\ell^2(\mathbb{Z})}\le\|\mathcal{T}\|\,\gamma^{k+1}\|\mathbb{F}\|_{\ell^2(\mathbb{Z})}\\ &\le\|\mathcal{T}\|\,\|\mathcal{T}^{-1}\|\,\gamma^{k+1}\|f\|_{L^2(\mathbb{R})}=\sqrt{\frac{\|\Phi\|_\infty}{\|\Phi\|_0}}\,\gamma^{k+1}\|f\|_{L^2(\mathbb{R})}\,,\end{aligned}$$

where $\gamma:=(B-A)/(B+A)$, and we have used that $\|\mathcal{T}^{-1}\|^{-2}=\|\Phi\|_0$ and $\|\mathcal{T}\|^2=\|\Phi\|_\infty$ [3, Prop. 3.6.8]. It is worth to mention that, in some important examples, the value of $\sum_{j=1}^s\Lambda_j\Gamma_j$ in (11) can be explicitely computed in terms of $\delta:=\sup_{j,n}|\varepsilon_{j,n}|$.

In order to improve this algorithm, specially when the ratio B/A is large, we can use the methods of acceleration of the frame algorithm proposed by Gröchenig in [8].

3.4 Examples in Spline spaces

For each fixed $m \in \mathbb{N}$, the B-spline N_m is defined as $N_m := N_1 * N_1 * \ldots * N_1$ (m times) where N_1 denotes the characteristic function of the interval $(0,1)$. It is known [4] that $\{N_m(\cdot - k\}_{k\in\mathbb{Z}}$ is a Riesz sequence in $L^2(\mathbb{R})$. The corresponding shift-invariant space V_{N_m} is the space of splines of degree $m-1$ in $L^2(\mathbb{R})$ with nodes at the integers. For these shift-invariant spaces it is not difficult to obtain the sum $\sum_{j=1}^{s} \Lambda_j \Gamma_j$ in terms of $\delta := \sup_{j,n} |\varepsilon_{j,n}|$. Thus, the largest δ satisfying the condition $\sum_{j=1}^{s} \Lambda_j \Gamma_j < \alpha_{\mathbf{G}}/r$ can be explicitely computed. Notice that, having in mind (11), the rate of convergence γ can also be computed in terms of $\delta := \sup_{j,n} |\varepsilon_{j,n}|$. We present here some examples:

3.4.1 Recovering linear, cuadratic and cubic Splines from irregular samples

For $r = s = 1$ and $(\mathcal{L}_1 f)(t) := f(t+a)$, Theorem 2 gives the irregular sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n + a + \varepsilon_n)\ S_n^{\varepsilon}(t) \quad t \in \mathbb{R}\,, \tag{12}$$

where $\{S_n^{\varepsilon}\}_{n\in\mathbb{Z}}$ denotes the Riesz basis associated with the perturbation $\{\varepsilon_n\}_{n\in\mathbb{Z}}$. A challenge problem is to measure how large should be $\delta := \sup_n |\varepsilon_n|$ in order to have a sampling formula like (12) in the shift-invariant space spanned by the B-spline N_m.

• **Linear Splines:** Consider the linear B-spline $N_2(t) := t\mathcal{X}_{[0,1)} + (2-t)\mathcal{X}_{[1,2)}$. For $d \in [0,1]$ we have

$$\sum_{k\in\mathbb{Z}} |N_2(k+d) - N_2(k)| = |N_2(d)| + |N_2(1+d) - N_2(1)| = d + d = 2d$$

By symmetry also we have $\sum_{k\in\mathbb{Z}} |N_2(k-d) - N_2(k)| = 2d$. Thus, for $a = 0$ and $[\alpha_1, \beta_1] = [-\delta, \delta]$, where $\delta \le 1$, we obtain $\Gamma_1 = 2\delta$. Besides,

$$\Lambda_1 = \sup_{d\in[-\delta,\delta]} |N_2(d)| + \sup_{d\in[-\delta,\delta]} |N_2(1+d) - N_2(1)| + \sup_{d\in[-\delta,\delta]} |N_2(2+d)| = 3\delta$$

Hence, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta, \delta]$, $\delta \le 1$, we have that $\|D_\varepsilon\|^2 \le \Lambda_1\Gamma_1 = 6\,\delta^2$. Since

$$\alpha_{\mathbf{G}} = \inf_{w\in(0,1)} |(ZN_1)(0,w)|^2 = \inf_{w\in(0,1)} \left|e^{-2\pi i w}\right|^2 = 1\,,$$

whenever

$$\sup_{n\in\mathbb{Z}} |\varepsilon_n| < C := \frac{1}{\sqrt{6}} \approx 0.408\,,$$

a sampling formula like (12) with $a = 0$ holds. This bound improves the value 1/3 obtained by Chen et al. in [2]. Assuming that $\varepsilon_n \ge 0$ (or $\varepsilon_n \le 0$) for all $n \in \mathbb{Z}$ the bound $1/\sqrt{6}$ can be improved up to 1/2 since $\Lambda_1 = 2\delta$ in this case. This result appeared first time in [12].

Since $\beta_{\mathbf{G}} = 1$ we obtain the convergence rate $\gamma = 2\sqrt{6}\ \delta/(1 + 6\ \delta^2)$ where $\delta := \sup_{n\in\mathbb{Z}} |\varepsilon_n|$.

• **Cubic Splines:** Consider the cubic B-spline

$$N_4(t) := \frac{t^3}{6}\mathcal{X}_{[0,1)}(t) + \Big(\frac{2}{3} - 2t + 2t^2 - \frac{t^3}{2}\Big)\mathcal{X}_{[1,2)}(t) + \Big(-\frac{22}{3} + 10t - 4t^2 + \frac{t^3}{2}\Big)\mathcal{X}_{[2,3)}(t) + \Big(\frac{32}{3} - 8t + 2t^2 - \frac{t^3}{6}\Big)\mathcal{X}_{[3,4)}(t)$$

Fix $a = 0$. For $d \in [0,1)$ the symmetry of N_4 gives

$$\sum_{k=0}^{4} |N_4(k-d) - N_4(k)| = \sum_{k=0}^{4} |N_4(k+d) - N_4(k)| = d + d^2 - \frac{2d^3}{3}$$

Hence, for $[\alpha_1, \beta_1] = [-\delta, \delta]$, where $\delta < 1$, one gets $\Gamma_1 = \delta + \delta^2 - 2\delta^3/3$. Having in mind the symmetry and monotony of N_4 and that $N_4(1+\delta) - N_4(1) > N_4(1) - N_4(1-\delta)$, we obtain

$$\Lambda_1 = 2N_4(\delta) + 2[N_4(1+\delta) - N_4(1)] + N_4(2) - N_4(2+\delta) = \delta + 2\delta^2 - \frac{7\delta^3}{6}$$

Thus, for any $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta, \delta]$ we have

$$\|D_\varepsilon\|^2 \le \Lambda_1\Gamma_1 = \frac{7\delta^6}{9} - \frac{5\delta^5}{2} + \frac{\delta^4}{6} + 3\delta^3 + \delta^2$$

Since

$$\alpha_{\mathbf{G}} = \inf_{w\in(0,1)} |(ZN_4)(0,\cdot)|^2 = \inf_{w\in(0,1)} \Big|\frac{1}{6}e^{-2\pi i w} + \frac{2}{3}e^{-4\pi i w} + \frac{1}{6}e^{-6\pi i w}\Big|^2 = \frac{1}{9}\,,$$

for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta, \delta]$ such that $\sup_{n\in\mathbb{Z}} |\varepsilon_n| < C \approx 0.253$, where C is the root of the polynomial $7\delta^6/9 - 5\delta^5/2 + \delta^4/6 + 3\delta^3 + \delta^2 - 1/9$ in $(0, 1/2)$, we can recover any cubic spline in $L^2(\mathbb{R})$ from its samples at $\{n + \varepsilon_n\}_{n\in\mathbb{Z}}$ by means of a sampling expansion expansion like (12).

• **Cuadratic Splines:** Here the generator is the cuadratic B-spline

$$N_3(t) := \frac{t^2}{2}\mathcal{X}_{[0,1)}(t) + \big(3t - t^2 - \frac{3}{2}\big)\mathcal{X}_{[1,2)}(t) + \frac{(3-t)^2}{2}\mathcal{X}_{[2,3)}(t)$$

For $a = 0$ we have that

$$\alpha_{\mathbf{G}} := \inf_{w\in(0,1)} |(ZN_3)(0,\cdot)|^2 = \frac{1}{4}\inf_{w\in(0,1)} |e^{-2\pi i w} + e^{-4\pi i w}|^2 = 0\,,$$

and, consequently, we cannot consider $a = 0$. However, for $a = 1/2$

$$\alpha_{\mathbf{G}} = \inf_{w\in(0,1)} |(ZN_3)(1/2,\cdot)|^2 = \inf_{w\in(0,1)} \Big|\frac{1}{8} + \frac{3}{4}e^{-2\pi i w} + \frac{1}{8}e^{-4\pi i w}\Big|^2 = \frac{1}{4}$$

For $d \in [0, 1/2)$ we have

$$\sum_{k=0}^{2} |N_3(k + 1/2 + d) - N_3(k)| = \sum_{k=0}^{2} |N_3(k + 1/2 - d) - N_3(k)| = d + d^2$$

Then, for $[\alpha_1,\beta_1]=[-\delta,\delta]$, where $\delta<1/2$, we obtain that $\Gamma_1=\delta+\delta^2$. Having in mind the symmetry and monotony of N_3 and that $N_3(1/2+\delta)-N_3(1/2)>N_3(1/2)-N_3(1/2-\delta)$ we get

$$\Lambda_1=2[N_3(1/2+\delta)-N_3(1/2)]+N_4(3/2)-N_4(3/2+\delta)=\delta+2\delta^2$$

Thus, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}}\subset[-\delta,\delta]$, where $\delta<1/2$, we have that

$$\|D_\varepsilon\|^2\le\Lambda_1\Gamma_1=\delta^2+3\delta^3+2\delta^4$$

As a consequence, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ satisfying $\sup_{n\in\mathbb{Z}}|\varepsilon_n|<C\approx 0.334$, where C is the root of $\delta^2+3\delta^3+2\delta^4-1/4$ in $(0,1/2)$, a sampling formula like (12) with $a=1/2$ holds. This bound improves the value 1/4 obtained by Chen et al. in [2].

3.4.2 Recovering cubic Splines from the derivative sampling

For $r=s=2$, consider the systems $(\mathcal{L}_1f)(t):=f(t+a)$ and $(\mathcal{L}_2f)(t):=f'(t+a)$. For $a=0$ we have

$$\mathbf{G}(w)=\begin{pmatrix}ZN_4(0,w) & ZN_4(0,w+1/2)\\ ZN_4'(0,w) & ZN_4'(0,w+1/2)\end{pmatrix}=\frac{1}{6}\begin{pmatrix}z+4z+z^3 & -z+4z^2-z^3\\ 3z-3z^3 & -3z+3z^3\end{pmatrix}$$

where $z=e^{-2\pi iw}$. Since $\det\mathbf{G}=2(z^5-z^3)/3$ vanishes at $w=0$, it follows that $\alpha_{\mathbf{G}}=0$. Hence, Theorem 2 does not apply. However, taking $a=1/2$ we obtain

$$g_1(w)=(ZN_4)(1/2,w)=\frac{1}{48}+\frac{23}{48}e^{-2\pi iw}+\frac{23}{48}e^{-4\pi iw}+\frac{1}{48}e^{-6\pi iw}$$
$$g_2(w)=(ZN_4')(1/2,w)=\frac{1}{8}+\frac{5}{8}e^{-2\pi iw}-\frac{5}{8}e^{-4\pi iw}-\frac{1}{8}e^{-6\pi iw}$$

The eigenvalues of the matrix $\mathbf{G}^*(w)\mathbf{G}(w)$ are

$$1+\frac{157}{288}\sin^2 2\pi w\pm\frac{7}{288}\sqrt{576\sin^2 2\pi w+265\sin^4 2\pi w}$$

The minimum on $(0,1/2)$ of the smallest eigenvalue is attained at $w=\frac{1}{2\pi}\arctan\sqrt{\frac{392}{403}}$ and takes the value $\alpha_{\mathbf{G}}=\frac{216}{265}$. Besides, the maximum on $(0,1/2)$ of the largest eigenvalue is $\beta_{\mathbf{G}}=9/4$ attained at $w=1/4$.

For $d\in[0,1/2]$, we have

$$\sum_{k=0}^{3}|N_4(k+1/2-d)-N_4(k)|=\sum_{k=0}^{3}|N_4(k+1/2+d)-N_4(k)|=\frac{3}{2}d-\frac{2}{3}d^3$$

For $d\in(0,1/3)$ the inequality $N_4'(5/2)>N_4'(5/2+d)$ holds. Thus, for $d\in[0,1/3)$ we get

$$\sum_{k=0}^{3}|N_4'(k+1/2-d)-N_4'(k)|=\sum_{k=0}^{3}|N_4'(k+1/2+d)-N_4'(k)|=2d$$

Therefore, whenever $[\alpha_1,\beta_1]=[\alpha_2,\beta_2]=[-\delta,\delta]$, with $0<\delta<1/3$, we obtain that $\Gamma_1=(3/2)\delta-(2/3)\delta^3$ and $\Gamma_2=2\delta$.

Now, having in mind the symmetry of N_4 and the inequalities $N_4(1/2+\delta)-N_4(1/2) > N_4(1/2)-N_4(1/2-\delta)$ and $N_4(5/2)-N_4(5/2+d) > N_4(5/2-d)-N_4(5/2)$, we get

$$\begin{aligned}&\sup_{d\in[-\delta,\delta]} |N_4(3/2+\delta)-N_4(3/2)| + \sup_{d\in[-\delta,\delta]} |N_4(7/2+\delta)-N_4(7/2)| \\ &= \sup_{d\in[-\delta,\delta]} |N_4(1/2+\delta)-N_4(1/2)| + \sup_{d\in[-\delta,\delta]} |N_4(5/2+\delta)-N_4(5/2)| \\ &= \frac{3\delta}{4}+\frac{\delta^2}{2}-\frac{\delta^3}{3}\end{aligned}$$

Analogously, using the symmetry of N_4', the inequality $N_4'(1/2+\delta)-N_4'(1/2) > N_4'(1/2)-N_4'(1/2-\delta)$, and that $\sup_{d\in[-\delta,\delta]} |N_4'(5/2+d)-N_4(5/2)| = N_4'(5/2-\delta)-N_4(5/2)$, we get

$$\begin{aligned}&\sup_{d\in[-\delta,\delta]} |N_4'(3/2+\delta)-N_4'(3/2)| + \sup_{d\in[-\delta,\delta]} |N_4'(7/2+\delta)-N_4'(7/2)| \\ &= \sup_{d\in[-\delta,\delta]} |N_4'(1/2+\delta)-N_4'(1/2)| + \sup_{d\in[-\delta,\delta]} |N_4'(5/2+\delta)-N_4'(5/2)| \\ &= \delta+2\delta^2\end{aligned}$$

Hence, $\Lambda_1 = (3/4)\,\delta + (1/2)\,\delta^2 - (1/3)\,\delta^3$ and $\Lambda_2 = \delta + 2\delta^2$. Thus, for any sequence $\varepsilon = \{\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2} \subset [\delta,\delta]$, where $\delta < 1/3$, we have that

$$\|D_\varepsilon\|^2 \le \Lambda_1\Gamma_1+\Lambda_2\Gamma_2 = \frac{25\,\delta^2}{8}+\frac{19\,\delta^3}{4}-\delta^4-\frac{\delta^5}{3}+\frac{2\,\delta^6}{9}$$

From Theorem 2, whenever $\sup_{j,n}|\varepsilon_{j,n}| < C \approx 0.3022$, where C is the root of $25\delta^2/8+19\delta^3/4-\delta^4-\delta^5/3+2\delta^6/9-108/265=0$ in $(0,1/3)$, there exists a Riesz basis $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2}$ for V_{N_4} such that the expansion

$$f(t) = \sum_{n\in\mathbb{Z}} \big[f(2n+1/2+\varepsilon_{1,n})\ S^\varepsilon_{1,n}(t) + f'(2n+1/2+\varepsilon_{2,n})\ S^\varepsilon_{2,n}(t)\big]\,, \quad t\in\mathbb{R}\,,$$

holds.

3.4.3 Recovering cubic Splines from average sampling

For each $f\in V_{N_4}$ consider the system defined as $(\mathcal{L}f)(t) := \int_{t-1/2}^{t+1/2} f(x)dx$. For $d\in[0,1/2]$ we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} |(\mathcal{L}_1N_4)(k-d)-(\mathcal{L}_1N_4)(k)| = \sum_{k\in\mathbb{Z}} |(\mathcal{L}_1N_4)(k+d)-(\mathcal{L}_1N_4)(k)| \\ &= \frac{23d}{24}+\frac{5d^2}{8}-\frac{d^3}{6}-\frac{d^4}{4}\end{aligned}$$

where we have used the symmetry of $\mathcal{L}N_4$ with respect the line $t=2$. Thus, for $[\alpha_1,\beta_1]=[-\delta,\delta]$, where $\delta\le 1/2$, we obtain $\Gamma_1 = 23\delta/24+5\delta^2/8-\delta^3/6-\delta^4/4$. Besides,

$$\begin{aligned}\Lambda_1 = 2\sup_{d\in[-\delta,\delta]} \big|(\mathcal{L}_1N_4)(d)-(\mathcal{L}_1N_4)(0)\big| + 2\sup_{d\in[-\delta,\delta]} \big|(\mathcal{L}_1N_4)(1+d)-(\mathcal{L}_1N_4)(1)\big| \\ + \sup_{d\in[-\delta,\delta]} \big|(\mathcal{L}_1N_4)(2)-(\mathcal{L}_1N_4)(2+d)\big| = \frac{5\delta^2}{4}-\frac{\delta^3}{6}-\frac{\delta^4}{2}+\frac{23\delta}{24}\end{aligned}$$

Hence, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta, \delta]$, $\delta \le 1/2$, we have that

$$\|D_\varepsilon\|^2 \le \Lambda_1\Gamma_1 = \frac{529\delta^2}{576} + \frac{115\delta^3}{64} + \frac{133\delta^4}{288} - \frac{33\delta^5}{32} - \frac{43\delta^6}{72} + \frac{\delta^7}{8} + \frac{\delta^8}{8}$$

Moreover

$$\alpha_{\mathbf{G}} = \inf_{w\in(0,1)} |g_1(w)|^2 = \inf_{w\in(0,1)} \left| \frac{1 + 76e^{-2\pi iw} + 230e^{-4\pi iw} + 76e^{-6\pi inw} + e^{-8\pi inw}}{384} \right|^2 = \frac{25}{576}$$

From Theorem 2, whenever $\sup_n |\varepsilon_n| < C \approx 0.185$, where C is the root of $529\delta^2/576 + 115\delta^3/64 + 133\delta^4/288 - 33\delta^5/32 - 43\delta^6/72 + \delta^7/8 + \delta^8/8 - 25/576 = 0$ in $(0, 1/2)$, there exists a Riesz basis $\{S_n^\varepsilon\}_{n\in\mathbb{Z}}$ for V_{N_4} such that the expansion

$$f(t) = \sum_{n\in\mathbb{Z}} (\mathcal{L}f)(n + \varepsilon_n)\ S_n^\varepsilon(t)\,, \qquad t \in \mathbb{R}\,,$$

holds for each $f \in V_{N_4}$.

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