Approximation from shift-invariant spaces by generalized sampling formulas

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Abstract

In this paper we investigate approximation from shift-invariant spaces by using generalized sampling formulas. These sampling formulas involve samples of filtered versions of the function itself. The considered systems include averaging samplers, and classical sampling of the function and its derivatives. Under appropriate hypotheses on the generator φ of the shift-invariant space and on the involved systems, we derive stable generalized sampling formulas in a shift-invariant subspace of $L^p(\mathbb{R}^n)$. From these generalized sampling formulas we construct approximation schemes valid for smooth functions. The approximation order depends both on the order for which the Strang-Fix conditions are satisfied by φ, and on the largest order of the derivatives appearing in the systems if any.

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1 Statement of the problem

As pointed out in [19] by Lei et al., there are many ways to construct approximation schemes with shift-invariant spaces. Among them, they cite cardinal interpolation, quasi-interpolation, projection and convolution (see also Refs. [1, 7, 10, 12, 15, 16, 23]). In this paper, an approximation scheme is proposed by using generalized sampling formulas. Our aim is to provide a method to approximate a smooth function from samples of some filtered versions of the function itself. For example, approximating a function f from a sequence consisting of samples of f, its derivatives, and/or of the convolution $f * h$, where the function h reflects the characteristics of the sampling devices.

For $1 \le p \le \infty$, the shift-invariant space V_φ^p is the closed subspace in $L^p(\mathbb{R}^n)$ generated by the integer shifts of a single function $\varphi \in L^p(\mathbb{R}^n)$, i.e., $V_\varphi^p := \overline{\operatorname{span}}_{L^p(\mathbb{R}^n)}\{\varphi(\cdot - \alpha) : \alpha \in \mathbb{Z}^n\}$. Shift-invariant spaces are important in a number of areas of analysis. Many

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spaces, encountered in approximation theory and in finite element analysis, are generated by the integer shifts of a function φ (see, for instance, Refs. [8, 9, 17, 21]). Shift-invariant spaces also play a key role in the construction of wavelets [18].

An important particular case is the shift-invariant space V_φ^2 with stable generator $\varphi \in L^2(\mathbb{R}^n)$, i.e., the sequence $\{\varphi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^n}$ is a Riesz sequence in $L^2(\mathbb{R}^n)$. In this case, $V_\varphi^2 := \{f(t) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha \varphi(t-\alpha) \,:\, \{a_\alpha\} \in \ell^2(\mathbb{Z}^n)\}$. Nowadays, sampling theory in shift-invariant spaces is a very active research topic since an appropriate choice for the generator φ (for instance a B-spline) eliminates most of the problems associated with the classical Shannon's sampling theory, where the generator φ is the sinc function (see, among others, Refs. [2, 3, 4, 26]).

Concerning generalized sampling, suppose that s linear time-invariant (convolution) systems Υ_j, $j = 1, 2, \ldots, s$, are defined on the shift-invariant space V_φ^p. A generalized sampling formula in V_φ^p is a sampling expansion which allows to recover any function $f \in V_\varphi^p$ from the generalized samples $\{\Upsilon_j f(r\alpha\}_{\alpha\in\mathbb{Z}^n, j=1,2,\ldots,s}$, i.e.,

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^n} (\Upsilon_j f)(r\alpha) S_j(t - r\alpha)\,, \quad t \in \mathbb{R}^n\,, \tag{1}$$

where the sampling period $r \in \mathbb{N}$ necessarily satisfies $r^n \le s$, and $S_j \in V_\varphi^p$, $j = 1, 2, \ldots, s$, are the reconstruction functions. For generalized sampling see Refs. [6, 11, 14, 22, 24, 25]. Under suitable hypotheses on the generator φ and on the systems, we derive in this work a stable generalized regular sampling formula in V_φ^p. The involved samples $\{\Upsilon_j f(r\alpha\}_{\alpha\in\mathbb{Z}^n, j=1,2,\ldots,s}$ in formula (1) could be averaging samples and/or samples of f or its derivatives. Whenever $s = r^n$, there exists a unique such a formula, whilst $s > r^n$ implies the existence of many such formulas. This flexibility can be used to obtain reconstruction functions with prescribed properties; for instance, with compact support (see Refs. [11, 13, 14]).

From a generalized sampling formula like (1) we can construct an approximation scheme as follows: For a suitable smooth function f, consider the operator Γ, formally defined as

$$\big(\Gamma f\big)(t) := \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^n} (\Upsilon_j f)(r\alpha) S_j(t - r\alpha)\,, \quad t \in \mathbb{R}^n\,.$$

The aim is to obtain a good approximation of f by means of the scaled operator Γ^h given by $\Gamma^h := \sigma_h \Gamma \sigma_{1/h}$, where $\sigma_h f := f(\cdot/h)$, $h > 0$. Thus, the goal is to obtain an estimation of the type $\|\Gamma^h f - f\|_p = \mathcal{O}(h^k)$ as $h \to 0^+$, where $1 \le p \le \infty$, and $k \in \mathbb{N}$ denotes the approximation order. This constant k depends both on the order for which the Strang-Fix conditions are satisfied by φ, and on the greatest order of the derivatives appearing in the systems Υ_j, if any. All these steps will be carried out in the remaining sections.

2 The shift-invariant spaces V_φ^p $(1 \le p \le \infty)$

We start this Section by introducing some notations and preliminaries used in the sequel. We denote by $C(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ and $C_b(\mathbb{R}^n)$ the space of continuous functions on $\mathbb{R}^n$, the space of continuous functions on $\mathbb{R}^n$ vanishing at ∞ and the space of continuous bounded functions on $\mathbb{R}^n$ respectively. For $1 \le p \le \infty$, $L^p(\mathbb{R}^n)$ denotes the classical Lebesgue space. We denote by $\ell^p(\mathbb{Z}^n)$ $(1 \le p < \infty)$ the space of pth power summable sequences on $\mathbb{Z}^n$, by $\ell^\infty(\mathbb{Z}^n)$ the bounded sequences, and by $\mathsf{c}_0(\mathbb{Z}^n)$ the space of sequences on $\mathbb{Z}^n$ vanishing at ∞.

Given a Lebesgue measurable function $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$, set

$$|\phi|_p := \Big(\int_{[0,1)^n} \Big(\sum_{\alpha \in \mathbb{Z}^n} |\phi(t-\alpha)| \Big)^p dt \Big)^{1/p} \quad \text{when } 1 \le p < \infty\,,$$
$$|\phi|_\infty := \sup_{t\in[0,1)^n} \sum_{\alpha \in \mathbb{Z}^n} |\phi(t-\alpha)|\,.$$

For $1 \le p \le \infty$, let

$$\mathcal{L}^p(\mathbb{R}^n) := \big\{ f : \mathbb{R}^n \longrightarrow \mathbb{C} \text{ measurable} : |f|_p < \infty \big\}\,.$$

Equipped with the norm $|\cdot|_p$, $\mathcal{L}^p(\mathbb{R}^n)$ becomes a Banach space. These spaces were introduced by Jia and Micchelli in [18]. Clearly $|\phi|_1 = \|\phi\|_1$, $\|\phi\|_p \le |\phi|_p$ and $|\phi|_{p'} \le |\phi|_p$ for $1 \le p' \le p \le \infty$. This shows that $\mathcal{L}^p(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and $\mathcal{L}^\infty(\mathbb{R}^n) \subset \mathcal{L}^p(\mathbb{R}^n) \subset \mathcal{L}^{p'}(\mathbb{R}^n) \subset \mathcal{L}^1(\mathbb{R}^n) = L^1(\mathbb{R}^n)$ for $1 \le p' \le p \le \infty$. Observe that if there are constants $C > 0$ and $\delta > 0$ such that

$$|\phi(t)| \le \frac{C}{1 + |t|^{n+\delta}}\,, \quad t \in \mathbb{R}^n\,,$$

then $\phi \in \mathcal{L}^\infty(\mathbb{R}^n)$.

Given a function $\phi \in \mathcal{L}^p(\mathbb{R}^n)$ and a sequence $a \in \ell^\infty(\mathbb{Z}^n)$, the semi-discrete convolution product is defined by

$$\phi *' a := \sum_{\alpha\in\mathbb{Z}^n} a(\alpha)\phi(\cdot - \alpha)\,.$$

In [18] we can find the following useful inequalities (see Theorem 2.1 and Theorem 3.1 respectively):

- If $\phi \in \mathcal{L}^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$ then

$$|\phi *' a|_p \le |\phi|_p\, \|a\|_1 \qquad \text{and} \qquad \|\phi *' a\|_p \le |\phi|_p\, \|a\|_p. \tag{2}$$

- If $f \in L^p(\mathbb{R}^n)$ and $\mathsf{h} \in L^q(\mathbb{R}^n)$ $(1 \le p \le \infty,\ 1/p + 1/q = 1)$ then

$$\big\| \{\mathsf{h} * f(\alpha)\}_{\alpha\in\mathbb{Z}^n} \big\|_1 \le |\mathsf{h}|_q\, |f|_p \qquad \text{and} \qquad \big\| \{\mathsf{h} * f(\alpha)\}_{\alpha\in\mathbb{Z}^n} \big\|_p \le |\mathsf{h}|_q\, \|f\|_p\,, \tag{3}$$

  where, as usual, the convolution is given by $\mathsf{h} * f := \int_{\mathbb{R}^n} f(x)\mathsf{h}(\cdot - x)dx$.

We denote the Fourier transform by $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(t) e^{-2\pi i \xi t} dt$.

Throughout this section p denotes a fixed number satisfying $1 \le p \le \infty$. First, we state the ℓ^p-stable concept as established in [18]:

Definition 1 *We say that the integer translates of $\varphi \in \mathcal{L}^p(\mathbb{R}^n)$ are ℓ^p-stable if there exists a constant $A_p > 0$ such that*

$$A_p \|a\|_p \le \|\varphi *' a\|_p \quad \textit{for all } a \in \ell^p(\mathbb{Z}^n)\,. \tag{4}$$

A necessary and sufficient condition for $\varphi \in \mathcal{L}^p(\mathbb{R}^n)$ to have ℓ^p-stable integer translates is that it satisfies

$$\sup_{\alpha\in\mathbb{Z}^n} |\widehat{\varphi}(w+\alpha)| > 0 \quad \text{for all } \ w \in \mathbb{R}^n \,.$$

(See [18, Theorem 3.5]). Notice that this last condition does not depend on p.

Let V_φ^p be the L^p-closure of the linear span of the integer translates of φ, i.e.,

$$V_\varphi^p := \overline{\text{span}}_{L^p(\mathbb{R}^n)}\{\varphi(t-\alpha) \,:\, \alpha\in\mathbb{Z}^n\}\,.$$

If the integer translates of $\varphi \in \mathcal{L}^p(\mathbb{R}^n)$ are ℓ^p-stable, then this space can be expressed as

$$V_\varphi^p = \big\{\varphi *' a :\, a \in \ell^p(\mathbb{Z}^n)\big\} \quad \text{if } 1\le p<\infty\,,$$

or

$$V_\varphi^\infty = \big\{\varphi *' a :\, a \in \mathsf{c}_0(\mathbb{Z}^n)\big\} \quad \text{if } p=\infty\,.$$

(See the proof of Lemma 5.1 in [19]). As a consequence, for $1 \le p' < p \le \infty$ we have the set inclusion $V_\varphi^{p'} \subset V_\varphi^p$. For more details and properties on shift-invariant spaces in $L^p(\mathbb{R}^n)$ see [5].

On the other hand, Zhou and Sun have characterized in [26] when the shift-invariant space V_φ^2 is a space of continuous functions. In the following theorem we generalize their result for the spaces V_φ^p.

Theorem 1 *Let $1\le p,q\le\infty$ with $1/p+1/q=1$. The following assertions are equivalent:*

(a) For each sequence $a\in\ell^p(\mathbb{Z}^n)$ ($a\in\mathsf{c}_0(\mathbb{Z}^n)$ if $p=\infty$), the series $\sum_{\alpha\in\mathbb{Z}^n} a(\alpha)\varphi(t-\alpha)$ converges to a continuous function on $\mathbb{R}^n$.

(b) The function φ is continuous on $\mathbb{R}^n$ and $\sup\limits_{t\in[0,1)^n} \big\|\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^n}\big\|_q < \infty\,.$

Assuming that the equivalent conditions hold and that the integer translates of $\varphi \in \mathcal{L}^p(\mathbb{R}^n)$ are ℓ^p-stable, the convergence in the L^p-sense of a sequence in V_φ^p implies uniform convergence on $\mathbb{R}^n$.

Proof. Assume first that (a) holds. For any fixed $t\in[0,1)^n$ we have that the series $\sum_{\alpha\in\mathbb{Z}^n}|a(\alpha)\varphi(t-\alpha)|$ converges for all $a\in\ell^p(\mathbb{Z}^n)$ ($a\in\mathsf{c}_0(\mathbb{Z}^n)$ if $p=\infty$). For each $N\in\mathbb{N}$ we consider the bounded linear functional

$$\begin{array}{rrcl} \Lambda_{t,N}: & \ell^p(\mathbb{Z}^n) & \longrightarrow & \mathbb{C} \\ & a & \longrightarrow & \sum_{|\alpha|<N} a(\alpha)\varphi(t-\alpha)\,. \end{array}$$

($\Lambda_{t,N}:\mathsf{c}_0(\mathbb{Z}^n)\to\mathbb{C}$ if $p=\infty$). For all $a\in\ell^p(\mathbb{Z}^n)$, we have that

$$\sup_{N\in\mathbb{N}}|\Lambda_{t,N}(a)| < \sum_{\alpha\in\mathbb{Z}^n}|a(\alpha)\varphi(t-\alpha)| < \infty\,.$$

Using the Banach-Steinhaus Theorem we obtain that, for any fixed $t\in[0,1)^n$, the norm $\|\Lambda_{t,N}\|$ is uniformly bounded on $N\in\mathbb{N}$. Since $\|\Lambda_{t,N}\| = \big(\sum_{|\alpha|<N}|\varphi(t-\alpha)|^q\big)^{1/q}$ for $q<\infty$, and $\|\Lambda_{t,N}\| = \sup_{|\alpha|<N}|\varphi(t-\alpha)|$ for $q=\infty$, we have that $\big\|\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^n}\big\|_q<\infty$.

Now, for each $t \in [0,1)^n$, we consider the bounded linear operator

$$\begin{array}{rccl} \Lambda_t : & \ell^p(\mathbb{Z}^n) & \longrightarrow & \mathbb{C} \\ & a & \longrightarrow & \sum_{\alpha\in\mathbb{Z}^n} a(\alpha)\varphi(t-\alpha) \end{array}$$

($\Lambda_t : \mathsf{c}_0(\mathbb{Z}^n) \to \mathbb{C}$ if $p=\infty$). Since, for each $a \in \ell^p(\mathbb{Z}^n)$ ($a \in \mathsf{c}_0(\mathbb{Z}^n)$ if $p=\infty$) , $(\varphi *' a)(t) = \sum_{\alpha\in\mathbb{Z}^n} a(\alpha)\varphi(t-\alpha)$ is a continuous function on $\mathbb{R}^n$, we have that

$$\sup_{t\in[0,1)^n} |\Lambda_t a| = \sup_{t\in[0,1)^n} |(\varphi *' a)(t)| < \infty \,.$$

Therefore, using again the Banach-Steinhaus Theorem we have that

$$\sup_{t\in[0,1)^n} \|\Lambda_t\| = \sup_{t\in[0,1)^n} \big\|\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^n}\big\|_q < \infty,$$

which proves (b).

Assume now that (b) holds and let $a \in \ell^p(\mathbb{Z}^n)$ ($a \in \mathsf{c}_0(\mathbb{Z}^n)$ if $p=\infty$). For each $N \in \mathbb{N}$, we have

$$\Big|\sum_{|\alpha|>N} a(\alpha)\varphi(t-\alpha)\Big| \le \Big(\sum_{|\alpha|>N} |a(\alpha)|^p\Big)^{1/p} \sup_{t\in\mathbb{R}^n} \big\|\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^n}\big\|_q$$

$$\Big(\,\Big|\sum_{|\alpha|>N} a(\alpha)\varphi(t-\alpha)\Big| \le \sup_{|\alpha|>N} |a(\alpha)| \sup_{t\in[0,1)^n} \big\|\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^n}\big\|_1 < \infty \quad \text{if } p=\infty\,\Big).$$

Then $\sum_{\alpha\in\mathbb{Z}^n} a(\alpha)\varphi(t-\alpha)$ converges uniformly on $\mathbb{R}^n$. Therefore, since $\varphi(t-\alpha)$, $\alpha\in\mathbb{Z}^n$, are continuous functions on $\mathbb{R}^n$, the function $(\varphi *' a)(t) = \sum_{\alpha\in\mathbb{Z}^n} a(\alpha)\varphi(t-\alpha)$ is continuous on $\mathbb{R}^n$.

Moreover, for any $f = \varphi *' a \in V_\varphi^p$, we have

$$|f(t)| \le \|a\|_p \big\|\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^n}\big\|_q \le A_p^{-1} \sup_{t\in\mathbb{R}} \big\|\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^n}\big\|_q \, \|f\|_p, \quad t\in\mathbb{R}^n,$$

where we have used the stability condition (4). This inequality proves that convergence in the L^p-sense of a sequence in V_φ^p implies uniform convergence on $\mathbb{R}^n$. □

Notice that when $p=1$ or $p=\infty$ the condition (b) in Proposition 1 can be written as $\varphi \in C_b(\mathbb{R}^n)$ or $\varphi \in \mathcal{L}^\infty(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ respectively.

3 Generalized regular sampling in shift-invariant spaces

Suppose that s linear time-invariant systems Υ_j, $j=1,2,\dots,s$, are defined on the shift-invariant space V_φ^p. The aim in this Section is to derive stable generalized sampling formulas involving the samples $\{\Upsilon_j f(r\alpha)\}_{\alpha\in\mathbb{Z}^n, j=1,2,\dots,s}$, i.e.,

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^n} (\Upsilon_j f)(r\alpha) S_j(t-r\alpha)\,, \quad t\in\mathbb{R}^n\,,$$

where the sampling period $r\in\mathbb{N}$ necessarily satisfies $r^n \le s$, and S_j, $j=1,2,\dots,s$, are the reconstruction functions.

3.1 The reconstruction functions

Let p be a fixed number satisfying $1 \le p \le \infty$ and assume that the generator φ belongs to $\mathcal{L}^p(\mathbb{R}^n)$, its integer translates are ℓ^p-stable, and it satisfies the equivalent conditions in Theorem 1.

First of all, we introduce some notation. For a point in $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and an n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n$ we denote $\alpha\, x := \sum_{k=1}^n \alpha_k x_k$. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}_0^n$, D^β stands for the differential operator $D^\beta := D_1^{\beta_1} D_2^{\beta_2} \dots D_n^{\beta_n}$, and $|\beta| := \sum_{j=1}^n |\beta_j|$ for its order. The space $C_b^m(\mathbb{R}^n)$ consists of all functions f which, together with all their partial derivatives $D^\beta f$ of order $|\beta| \le m$, are continuous and bounded on $\mathbb{R}^n$. The space $C_b^m(\mathbb{R}^n)$ becomes a Banach space with the norm $\|f\|_{C_b^m} := \max_{|\beta| \le m} \sup_{t \in \mathbb{R}^n} |D^\beta f(t)|$.

We consider s linear-time invariant system Υ_j, $j = 1, 2, \dots, s$, of the following types:

(a) The impulse response h_j of Υ_j belongs to $\mathcal{L}^q(\mathbb{R}^n)$, where p and q are conjugate exponents, i.e.,

$$\big(\Upsilon_j f\big)(t) := [f * \mathsf{h}_j](t) = \int_{\mathbb{R}^n} f(x) \mathsf{h}_j(t - x)\, dx\,, \quad t \in \mathbb{R}^n\,,$$

for $\mathsf{h}_j \in \mathcal{L}^q(\mathbb{R}^n)$ and q satisfiying $1/p + 1/q = 1$.

(b) The impulse response h_j is a shifted Dirac delta, i.e.,

$$\big(\Upsilon_j f\big)(t) := f(t + c_j)\,, \quad t \in \mathbb{R}^n\,.$$

If there is a system of this type, we also assume that $\varphi \in \mathcal{L}^\infty(\mathbb{R}^n)$.

(c) The impulse response h_j is a linear combination of partial derivatives of shifted deltas, i.e.,

$$\big(\Upsilon_j f\big)(t) := \sum_{|\beta| \le N_j} c_{j,\beta} D^\beta f(t + d_{j,\beta})\,, \quad t \in \mathbb{R}^n\,.$$

If there is a system of this type, we also assume that $D^\beta \varphi \in \mathcal{L}^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ for $|\beta| \le N_j$.

Any system of type (b) is a particular case of a system of type (c), but for the sake of clarity we treat both cases separately. We denote by m the largest order among the partial derivatives that appear in the systems of type (c) ($m = 0$ if there are only systems of types (a) and/or (b)).

Let $\mathcal{A}$ be the set of the functions of the form $f(x) = \sum_{\alpha \in \mathbb{Z}^n} a(\alpha) e^{2\pi i \alpha x}$ with $a \in \ell^1(\mathbb{Z}^n)$. The space $\mathcal{A}$, normed by $\|f\|_{\mathcal{A}} := \|a\|_1$ and with pointwise multiplication is a commutative Banach algebra. If $f \in \mathcal{A}$ and $f(x) \ne 0$ for every $x \in \mathbb{R}^n$, the function $1/f$ is also in $\mathcal{A}$ by Wiener's Lemma (see e.g. [20]).

For each $j = 1, 2, \dots, s$, the sequence $\{\Upsilon_j \varphi(\alpha)\}_{\alpha \in \mathbb{Z}^n}$ belongs to $\ell^1(\mathbb{Z}^n)$ (see (3) for systems of type (a); for systems of type (b) or (c) it is obvious, since $\varphi \in \mathcal{L}^\infty(\mathbb{R}^n)$ or $D^\beta \varphi \in \mathcal{L}^\infty(\mathbb{R}^n)$). The Fourier transform of this sequence, which belongs to the Banach algebra $\mathcal{A}$, will play an important role in the sequel. We denote it by

$$g_j(x) := \sum_{\alpha \in \mathbb{Z}^n} \big(\Upsilon_j \varphi\big)(\alpha) e^{-2\pi i \alpha x}\,, \quad x \in \mathbb{R}^n\,.$$

Let $\gamma_1, \gamma_2, \ldots, \gamma_{r^n}$ be the points of the grid

$$\Big\{\Big(\frac{m_1}{r}, \frac{m_2}{r}, \ldots, \frac{m_n}{r}\Big) \,:\, m_k \in \{0, 1, \ldots, r-1\}, k = 1, 2, \ldots, n\Big\}$$

ordered in any form such that $\gamma_1 = (0, 0, \ldots, 0)$. For example, they can be ordered according to the sum $0 \le m_1 + m_2 + \ldots + m_n \le n(r-1)$, and whenever two sums coincide, we can use the lexicographic ordering in $(m_1, m_2, \ldots, m_n)$. Let $\mathbf{G}(x)$ be the $s \times r^n$ matrix

$$\mathbf{G}(x) := \begin{bmatrix} g_1(x) & g_1(x+\gamma_2) & \cdots & g_1(x+\gamma_{r^n}) \\ g_2(x) & g_2(x+\gamma_2) & \cdots & g_2(x+\gamma_{r^n}) \\ \vdots & \vdots & & \vdots \\ g_s(x) & g_s(x+\gamma_2) & \cdots & g_s(x+\gamma_{r^n}) \end{bmatrix} \tag{5}$$

The vectors $\mathbf{d}$ in the next lemma give the reconstruction functions that we are looking for.

Lemma 1 *There exists a vector $\mathbf{d}(x) = \big(d_1(x), d_2(x), \ldots, d_s(x)\big)$ with entries $d_j \in \mathcal{A}$, $j = 1, 2, \ldots, s$, satisfying*

$$\mathbf{d}(x)\mathbf{G}(x) = (1, 0, \ldots, 0)\,, \quad x \in [0,1)^n \tag{6}$$

if and only if rank $\mathbf{G}(x) = r^n$ for all $x \in \mathbb{R}^n$.

Proof. Notice that $\operatorname{rank} \mathbf{G}(x) = r^n$ if and only if $\det(\mathbf{G}^*(x)\mathbf{G}(x)) \neq 0$ where $\mathbf{G}^*(x)$ denotes the conjugate transpose of $\mathbf{G}(x)$. If rank $\mathbf{G}(x) = r^n$ then the first row of the pseudo inverse of $\mathbf{G}(x)$,

$$\mathbf{G}^\dagger(x) := (\mathbf{G}^*(x)\mathbf{G}(x))^{-1}\mathbf{G}^*(x)$$

satisfies (6). Moreover, according to Wiener's Lemma the entries of $\mathbf{G}^\dagger$ belong to $\mathcal{A}$.

Conversely, assume that the vector $\mathbf{d}(x) = (d_1(x), d_2(x), \ldots, d_s(x))$ satisfies (6). We consider the periodic extension of d_j, i.e., $d_j(x+\alpha) = d_j(x)$, $\alpha \in \mathbb{Z}^n$. For all $x \in [0,1)^n$, the matrix

$$\mathbf{D}^\top(x) := \begin{bmatrix} d_1(x) & d_2(x) & \cdots & d_s(x) \\ d_1(x+\gamma_2) & d_2(x+\gamma_2) & \cdots & d_s(x+\gamma_2) \\ \vdots & \vdots & & \vdots \\ d_1(x+\gamma_{r^n}) & d_2(x+\gamma_{r^n}) & \cdots & d_s(x+\gamma_{r^n}) \end{bmatrix} \tag{7}$$

is a left inverse of $\mathbf{G}(x)$. Therefore, necessarily $\operatorname{rank} \mathbf{G}(x) = r^n$, for all $x \in [0,1)^n$. $\square$

Provided that the condition of Lemma 1 is satisfied, it can be easily checked that all the vectors $\mathbf{d}(x)$ with entries in $\mathcal{A}$, satisfying (6) are exactly the first row of the matrices of the form

$$\mathbf{D}^\top(x) = \mathbf{G}^\dagger(x) + \mathbf{U}(x)\big[I - \mathbf{G}(x)\mathbf{G}^\dagger(x)\big]\,, \tag{8}$$

where $\mathbf{U}(x)$ is any matrix with entries in $\mathcal{A}$. Notice that if $s = r^n$ there exists a unique vector $\mathbf{d}(x)$, which is the first row of $\mathbf{G}^{-1}(x)$; if $s > r^n$ there are many solutions according to (8).

For $F(x) = \sum_{\alpha \in \mathbb{Z}^n} a(\alpha) e^{-2\pi i \alpha x} \in \mathcal{A}$, we define $\mathcal{T}_\varphi : \mathcal{A} \longrightarrow L^p(\mathbb{R}^n)$ as $\mathcal{T}_\varphi F := \varphi *' a$. Notice that (2) ensures that $\mathcal{T}_\varphi$ is a well defined bounded operator.

Lemma 2 *Let* $\mathbf{d}(x) = \big(d_1(x), d_2(x), \ldots, d_s(x)\big)$ *be a vector with entries* $d_j \in \mathcal{A}$, $j = 1, 2, \ldots, s$, *satisfying* (6). *Then, for any* $f \in \operatorname{span}\{\varphi(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^n}$ *the following sampling expansion holds:*

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^n} (\Upsilon_j f)(r\alpha) S_{j,\mathbf{d}}(t - r\alpha)\,, \quad t \in \mathbb{R}^n\,,$$

where $S_{j,\mathbf{d}} := r^n \mathcal{T}_\varphi d_j\,, j = 1, 2, \ldots, s$. *The convergence of the sampling series is in the* L^p*-norm sense and uniform on* $\mathbb{R}^n$.

Proof. For $f \in \operatorname{span}\{\varphi(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^n}$ let $a = \{a(\alpha)\}$ be the finite sequence such that $f = \varphi *' a$ and the corresponding trigonometric polynomial $F(x) := \sum_{\alpha \in \mathbb{Z}^n} a(\alpha) e^{-2\pi i \alpha x}$. Notice that $\mathcal{T}_\varphi F = f$. For any $j = 1, 2, \ldots, s$ and $\alpha \in \mathbb{Z}^n$, we have

$$\begin{aligned}(\Upsilon_j f)(r\alpha) &= \sum_{\mu} a(\mu)(\Upsilon_j \varphi)(r\alpha - \mu) = \langle F, \overline{g}_j e^{-2\pi i r\alpha x}\rangle_{L^2([0,1)^n)} \\ &= \int_{[0,1)^n} F(x) g_j(x) e^{2\pi i r\alpha x} dx\,.\end{aligned}$$

The sequence $\{e^{-2\pi i r\alpha x}\}_{\alpha \in \mathbb{Z}^n}$ is an orthogonal basis for $L^2([0, 1/r)^n)$; in order to recover F from the samples $(\Upsilon_j f)(r\alpha)$, $\alpha \in \mathbb{Z}^n$, we use the following expression involving an integral on the cube $[0, 1/r)^n$,

$$\begin{aligned}(\Upsilon_j f)(r\alpha) &= \sum_{k=1}^{r^n} \int_{\gamma_k + [0,1/r)^n} F(x) g_j(x) e^{2\pi i r\alpha x} dx \\ &= \int_{[0,1/r)^n} \sum_{k=1}^{r^n} F(x + \gamma_k) g_j(x + \gamma_k)\; e^{2\pi i r\alpha x} dx\,.\end{aligned} \tag{9}$$

Thus, the samples $\{(\Upsilon_j f)(r\alpha)\}_{\alpha \in \mathbb{Z}^n}$, are the Fourier coefficients of the continuous function $\sum_{k=1}^{r^n} F(x + \gamma_k) g_j(x + \gamma_k)$ with respect to the orthogonal basis $\{e^{-2\pi i r\alpha x}\}_{\alpha \in \mathbb{Z}^n}$ for $L^2([0, 1/r)^n)$. Since $\{\Upsilon_j \varphi(\alpha)\}_{\alpha \in \mathbb{Z}^n} \in \ell^1(\mathbb{Z}^n)$ we have that $\{\Upsilon_j f(r\alpha)\}_{\alpha \in \mathbb{Z}^n} \in \ell^1(\mathbb{Z}^n)$ (remind that $(\Upsilon_j f)(r\alpha) = \sum_{\text{finite}} a(\mu)(\Upsilon_j \varphi)(r\alpha - \mu)$). Therefore, for $j = 1, 2 \ldots, s$, we have

$$\sum_{k=1}^{r^n} F(x + \gamma_k) g_j(x + \gamma_k) = r^n \sum_{\alpha \in \mathbb{Z}^n} (\Upsilon_j f)(r\alpha) e^{-2\pi i r\alpha x}\,, \quad x \in [0, 1/r)^n\,.$$

By periodicity, the above equality also holds for all $x \in [0, 1)^n$. Hence we can write

$$\mathbf{G}(x)\mathbf{F}(x) = r^n \Big(\sum_{\alpha \in \mathbb{Z}^n} (\Upsilon_1 f)(r\alpha) e^{-2\pi i r\alpha x}, \ldots, \sum_{\alpha \in \mathbb{Z}^n} (\Upsilon_s f)(r\alpha) e^{-2\pi i r\alpha x} \Big)^\top$$

where $\mathbf{G}(x)$ is the $s \times r^n$ matrix, defined in (5) and

$$\mathbf{F}(x) := \big(F(x + \gamma_1), F(x + \gamma_2), \cdots, F(x + \gamma_{r^n})\big)^\top\,.$$

Multiplying on the left by the vector $\mathbf{d}(x)$, and having in mind that we have taken $\gamma_1 = (0, 0, \dots, 0)$, we obtain $F(x)$ by means of the generalized samples:

$$F(x) = r^n \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^n} (\Upsilon_j f)(r\alpha) d_j(x) e^{-2\pi i r\alpha x}\,, \quad x \in [0,1)^n\,. \tag{10}$$

Since $\{(\Upsilon_j f)(r\alpha)\}_{\alpha\in\mathbb{Z}^n}$ belongs to $\ell^1(\mathbb{Z}^n)$ and $d_j \in \mathcal{A}$, the series in (10) also converges in the norm of $\mathcal{A}$. Indeed, for $N \in \mathbb{N}$,

$$\begin{aligned}\Big\| \sum_{|\alpha|>N} (\Upsilon_j f)(r\alpha) d_j(x) e^{-2\pi i r\alpha x} \Big\|_{\mathcal{A}} &\le \|d_j\|_{\mathcal{A}} \Big\| \sum_{|\alpha|>N} (\Upsilon_j f)(r\alpha) e^{-2\pi i r\alpha x} \Big\|_{\mathcal{A}} \\ &= \|d_j\|_{\mathcal{A}} \sum_{|\alpha|>N} |(\Upsilon_j f)(r\alpha)|\,.\end{aligned}$$

Applying $\mathcal{T}_\varphi$ to both sides of the equality (10), and using that

$$\big[\mathcal{T}_\varphi d_j(\cdot) e^{-2\pi i r\alpha\cdot}\big](t) = \big[\mathcal{T}_\varphi d_j\big](t - r\alpha)\,, \quad \alpha \in \mathbb{Z}^n\,,$$

we deduce that

$$f = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^n} (\Upsilon_j f)(r\alpha) S_{j,\mathbf{d}}(\cdot - r\alpha) \quad \text{in } L^p(\mathbb{R}^n)\,,$$

where $S_{j,\mathbf{d}} = r^n \mathcal{T}_\varphi d_j$, for $j = 1, 2, \dots, s$.

The reconstruction functions $S_{j,\mathbf{d}}$, $j = 1, 2, \dots, s$, are determined from the Fourier coefficients of d_j, $\widehat{d_j}[\alpha] := \int_{[0,1)^n} d_j(x) e^{2\pi i \alpha x} dx$. More specifically,

$$S_{j,\mathbf{d}}(t) = r^n \sum_{\alpha\in\mathbb{Z}^n} \widehat{d_j}[\alpha] \varphi(t - \alpha)\,, \quad t \in \mathbb{R}^n\,. \tag{11}$$

The sequence $\widehat{d_j} \in \ell^1(\mathbb{Z}^n)$ because the function $d_j(x) = \sum_{\alpha\in\mathbb{Z}^n} \widehat{d_j}[\alpha] e^{-2\pi i \alpha x}$ belongs to $\mathcal{A}$. As a consequence, $S_{j,\mathbf{d}} \in V_\varphi^1 \subset V_\varphi^p$. Hence, the partial sums of the above sampling series are in V_φ^p. Then, as a consequence of Theorem 1, these partial sums also converge uniformly on $\mathbb{R}^n$. $\square$

Since $d_j \in \mathcal{A}$ and $\varphi \in \mathcal{L}^p(\mathbb{R}^n)$, by using (2) we obtain that the above reconstruction functions satisfy:

$$S_{j,\mathbf{d}} \in \mathcal{L}^p(\mathbb{R}^n)\,, \quad j = 1, 2, \dots, s\,. \tag{12}$$

The Fourier transform of $S_{j,\mathbf{d}}$ can be determined from d_j. Indeed, from (11),

$$\widehat{S}_{j,\mathbf{d}}(w) = r^n d_j(w) \widehat{\varphi}(w)\,, \quad w \in \mathbb{R}^n\,.$$

In the case $s = r^n$, there is a unique vector $\mathbf{d}(x)$ satisfying (6), which is the first row of the matrix $\mathbf{G}^{-1}(x) = \mathbf{D}^\top(x)$ in the notation of (7). Then, using (9), we obtain that the reconstruction functions $S_{j,\mathbf{d}}$ satisfy in this case an interpolatory property, namely:

$$\begin{aligned}(\Upsilon_{j'} S_{j,\mathbf{d}})(r\alpha) &= r^n \int_{[0,1/r)^n} \sum_{k=1}^{r^n} d_j(x + \gamma_k) g_{j'}(x + \gamma_k)\, e^{2\pi i r\alpha x} dx \\ &= \delta_{j',j}\, r^n \int_{[0,1/r)^n} e^{2\pi i r\alpha x} dx = \begin{cases} 1 & j = j' \text{ and } \alpha = 0 \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

3.2 The generalized sampling formula in V_φ^∞

We assume the hypotheses in Section 3.1 for $p = \infty$. Recall that m stands for the largest order among the partial derivatives appearing in systems of the type (c) ($m = 0$ if there are only systems of the types (a) and (b)). Associated with a vector $\mathbf{d}(x) = \big(d_1(x), d_2(x), \ldots, d_s(x)\big)$ with entries $d_j \in \mathcal{A}$, $j = 1, 2, \ldots, s$, satisfying (6), we introduce the operator $\Gamma_\mathbf{d}$, formally defined as,

$$\big(\Gamma_\mathbf{d} f\big)(t) := \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^n} (\Upsilon_j f)(r\alpha) S_{j,\mathbf{d}}(t - r\alpha)\,. \tag{13}$$

Lemma 3 *For any vector* $\mathbf{d}$ *satisfying* (6) *with entries in* $\mathcal{A}$, *there exists a constant* $K > 0$ *such that, for each* $f \in C_b^m(\mathbb{R}^n)$,

$$\big|\big(\Gamma_\mathbf{d} f\big)(t)\big| \le K\|f\|_{C_b^m} \quad \textit{for all } t \in \mathbb{R}^n.$$

Proof. If the system Υ_j is of the type (a), then for all $f \in C_b^m(\mathbb{R}^n)$,

$$\big|\Upsilon_j f(\alpha)\big| \le \|h_j\|_1\,\|f\|_\infty \le \|h_j\|_1\,\|f\|_{C_b^m}, \quad \alpha \in \mathbb{Z}^n\,.$$

If the system Υ_j is of the type (c) (including in particular the type (b)) then for all $f \in C_b^m(\mathbb{R}^n)$,

$$|\Upsilon_j f(\alpha)| \le \sum_{|\beta|\le N_j} |c_{j,\beta}|\;|D^\beta f(\alpha + d_{j,\beta})| \le M \max_{|\beta|\le N_j} |c_{j,\beta}|\;\|f\|_{C_b^m}, \quad \alpha \in \mathbb{Z}^n\,,$$

for some constant M. Since $S_{j,\mathbf{d}} \in \mathcal{L}^\infty(\mathbb{R}^n)$ (see (12)), then, for any $f \in C_b^m(\mathbb{R}^n)$,

$$\begin{aligned}|\Gamma_\mathbf{d} f(t)| &\le \sum_{j=1}^{s} \Big| \sum_{\alpha\in\mathbb{Z}^n} (\Upsilon_j f)(\alpha) S_{j,\mathbf{d}}(t-\alpha) \Big| \le \sum_{j=1}^{s} \big\|\{\Upsilon_j f(\alpha)\}_{n\in\mathbb{Z}^n}\big\|_\infty |S_{j,\mathbf{d}}|_\infty \\ &\le K\|f\|_{C_b^m}\,, \quad t \in \mathbb{R}^n,\end{aligned}$$

where K is a constant independent of f. □

Theorem 2 *Let* $\mathbf{d}(x) = \big(d_1(x), d_2(x), \ldots, d_s(x)\big)$ *be a vector with entries* $d_j \in \mathcal{A}$, $j = 1, 2, \ldots, s$, *satisfying* (6). *Then, for any* $f \in V_\varphi^\infty$, *the following sampling formula holds pointwise:*

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^n} (\Upsilon_j f)(r\alpha) S_{j,\mathbf{d}}(t - r\alpha)\,, \quad t \in \mathbb{R}^n\,, \tag{14}$$

where $S_{j,\mathbf{d}} = r^n \mathcal{T}_\varphi d_j \in \mathcal{L}^\infty(\mathbb{R}^n)$, *for each* $j = 1, 2, \ldots, s$. *Assuming that* $\varphi, D^\beta\varphi \in C_0(\mathbb{R}^n)$, $|\beta| \le m$, *and that the impulse responses of the linear time-invariant systems* Υ_j *have compact support, the convergence of the sampling series is also absolute and uniform on* $\mathbb{R}^n$.

Proof. Let $f \in V_\varphi^\infty$ and $a \in \mathsf{c}_0(\mathbb{Z}^n)$ such that $f(t) = \sum_{\alpha\in\mathbb{Z}^n} a(\alpha)\varphi(t-\alpha)$. For $M \in \mathbb{N}$ we define

$$f_M(t) := \sum_{|\alpha|\le M} a(\alpha)\varphi(t-\alpha)\,.$$

From the assumptions on φ we have that $f_M \in C_b^m(\mathbb{R}^n)$. Moreover, for $|\beta| \le m$ and $M > N > 0$, we have

$$|D^\beta(f_M - f_N)(t)| \le \sum_{N<|\alpha|\le M} |a(\alpha)|\,|D^\beta\varphi(t-\alpha)| \le \sup_{N<|\alpha|\le M} |a(\alpha)|\,|D^\beta\varphi|_\infty\,, \quad t\in\mathbb{R}^n\,.$$

Since the sequence $a \in \mathsf{c}_0(\mathbb{Z}^n)$, $\{f_M\}_{M=1}^\infty$ is a Cauchy sequence in the Banach space $C_b^m(\mathbb{R}^n)$, we deduce that f_M converges in the C_b^m-norm to f as $M\to\infty$. In particular $f \in C_b^m(\mathbb{R}^n)$. Using Lemmas 2 and 3 we obtain that, for all $t\in\mathbb{R}^n$,

$$0 \le |f_M(t) - \Gamma_{\mathbf{d}} f(t)| = \big|[\Gamma_{\mathbf{d}}(f_M - f)](t)\big| \le K\|f_M - f\|_{C_b^m} \to 0 \ \text{ as } \ M\to\infty,$$

and then $\Gamma_{\mathbf{d}} f(t) = f(t)$ for all $t\in\mathbb{R}^n$. This proves that the sampling formula (14) holds pointwise. It remains to prove the absolute and uniform convergence of the series in (14). Let $|\beta|\le m$. Assuming that $D^\beta\varphi \in C_0(\mathbb{R}^n)$ we have that $D^\beta f_M \in C_0(\mathbb{R}^n)$. Since $D^\beta f_M$ converges uniformly to $D^\beta f$ on $\mathbb{R}^n$, and $C_0(\mathbb{R}^n)$ is a closed subspace in $L^\infty(\mathbb{R}^n)$, we obtain that $D^\beta f \in C_0(\mathbb{R}^n)$. From this fact and using that the impulse responses of Υ_j has compact support (whenever Υ_j is a system of type (a)), we obtain that $\{(\Upsilon_j f)(r\alpha)\}_{\alpha\in\mathbb{Z}^n} \in \mathsf{c}_0(\mathbb{R}^n)$ for each $j = 1,2,\dots,s$. Hence, by using that $S_{j,\mathbf{d}} \in \mathcal{L}^\infty(\mathbb{R}^n)$ and the inequality

$$\sum_{|\alpha|>N} |(\Upsilon_j f)(r\alpha) S_{j,\mathbf{d}}(t-r\alpha)| \le \sup_{|\alpha|>N} |(\Upsilon_j f)(r\alpha)|\,|S_{j,\mathbf{d}}|_\infty\,, \quad t\in\mathbb{R}^n,\ N\in\mathbb{N}\,,$$

we obtain that the series in (14) converges absolutely and uniformly on $\mathbb{R}^n$. □

Observe that, under the assumed hypotheses, in the proof of the theorem we have obtained that :

$$V_\varphi^\infty \subset C_b^m(\mathbb{R}^n)\,. \tag{15}$$

The reconstruction method in V_φ^p $(1\le p\le\infty)$ given by formula (14) is stable in the following way: If $\|\{(\Upsilon_j f)(r\alpha) - (\Upsilon_j g)(r\alpha)\}\|_p$ is small for $f,g\in V_\varphi^p$ and $j=1,2,\dots,s$, then $\|f-g\|_p$ is also small. More precisely, consider $\Delta_{j,\alpha} := (\Upsilon_j f)(r\alpha) - (\Upsilon_j g)(r\alpha)$. We have

$$\|f-g\|_p = \Big\|\sum_{j=1}^s \sum_{\alpha\in\mathbb{Z}^n} \Delta_{j,\alpha} S_{j,\mathbf{d}}(\cdot - r\alpha)\Big\|_p = \Big\|\sum_{j=1}^s \sum_{\alpha\in\mathbb{Z}^n} \widetilde{\Delta}_{j,\alpha} S_{j,\mathbf{d}}(\cdot - \alpha)\Big\|_p\,,$$

where

$$\widetilde{\Delta}_{j,\alpha'} := \begin{cases} \Delta_{j,\alpha} & \text{is } \alpha' = r\alpha, \\ 0 & \text{otherwise.}\end{cases}$$

Hence, denoting $\widetilde{\Delta}_j = \{\widetilde{\Delta}_{j,\alpha}\}_{\alpha\in\mathbb{Z}^n}$ and $\Delta_j = \{\Delta_{j,\alpha}\}_{\alpha\in\mathbb{Z}^n}$ we obtain

$$\|f-g\|_p = \Big\|\sum_{j=1}^s S_{j,\mathbf{d}} *' \widetilde{\Delta}_j\Big\|_p \le \sum_{j=1}^s \|\Delta_j\|_p |S_{j,\mathbf{d}}|_p \le \Big(\max_{1\le j\le s} |S_{j,\mathbf{d}}|_p\Big) \sum_{j=1}^s \|\Delta_j\|_p\,,$$

where we have used that $S_{j,\mathbf{d}} \in \mathcal{L}^\infty(\mathbb{R}^n) \subset \mathcal{L}^p(\mathbb{R}^n)$, $1\le p<\infty$, for each $j=1,2,\dots,s$, the corresponding inequality in (2) and also that $\|\widetilde{\Delta}_j\|_p = \|\Delta_j\|_p$ for each $j=1,2,\dots,s$.

4 Approximation by means of generalized sampling formulas

We denote by $W_p^k(\mathbb{R}^n) := \{f \,:\, \|D^\gamma f\|_p < \infty\,,\, |\gamma| \le k\}$ the usual Sobolev space, and by $|f|_{j,p} := \sum_{|\beta|=j} \|D^\beta f\|_p$ the seminorm of a function $f \in W_p^k(\mathbb{R}^n)$, $0 \le j \le k$.

Our approximation results are based on known results about approximation from shift-invariant spaces. For $m = 0$ (there are only systems of the types (a) and (b)) we use the following theorem whose proof can be found in [19, Theorem 5.2]:

Theorem 3 *Assume that* $\operatorname{ess\,sup}_{t\in\mathbb{R}^n} \sum_{\alpha\in\mathbb{Z}^n} |\varphi(t+\alpha)|(1+|t+\alpha|)^k < \infty$ *for some* $k \in \mathbb{N}$. *If the generator* φ *satisfies the Strang-Fix conditions of order* k*, i.e.,*

$$\widehat{\varphi}(0) \neq 0, \quad D^\beta \widehat{\varphi}(\alpha) = 0, \quad |\beta| < k, \quad \alpha \in \mathbb{Z}^n \setminus \{0\}\,,$$

then, for each $f \in W_p^k(\mathbb{R}^n)$, $1 \le p \le \infty$, *and* $h > 0$ *there exists a function* $\xi \in \sigma_h V_\varphi^p$ *such that*

$$\|\xi - f\|_p \le \, K\, |f|_{k,p}\; h^k\,,$$

where K *is a constant independent of* f, p *and* h.

See [19, Theorem 5.3] for a converse result. For $m > 0$, we also assume that the generator φ has compact support, and we use the following result by Jia [16]:

Theorem 4 *Let* $\phi \in W_p^j(\mathbb{R}^n)$, *where* $1 \le p \le \infty$ *and* $0 \le j < k$, *and* $\widetilde{\phi} \in L_q(\mathbb{R}^n)$ *(*$1/p+1/q = 1$*) be compactly supported functions and let* Q *be the quasi-projection operator given by*

$$(Qf)(t) := \sum_{\alpha\in\mathbb{Z}^n} \langle f, \widetilde{\phi}(\cdot - \alpha)\rangle_{L^2(\mathbb{R}^n)} \phi(t-\alpha). \tag{16}$$

If $Q\pi = \pi$ *for all* π *polynomial of degree at most* $k-1$, *then*

$$|f - Q_h f|_{j,p} \le K\, h^{k-j} |f|_{k,p}\,, \quad f \in W_p^k(\mathbb{R}^n)\,,$$

where $Q_h := \sigma_h Q \sigma_{1/h}$, $\sigma_h f(\cdot) := f(\cdot/h)$, *and* K *is a constant independent of* $h > 0$, f *and* p.

4.1 L^∞-approximation properties

In this section we assume the hypotheses in Section 3.1 for $p = \infty$. In Theorem 2 we have proved that any vector $\mathbf{d}$ with entries $d_j \in \mathcal{A}$, satisfying (6), provides a generalized sampling formula for the shift-invariant space V_φ^∞. In this section we prove that a scaled version of the sampling formula (14) gives a good approximation scheme for functions f sufficiently smooth. More precisely, we prove that, under appropriate conditions, if the generator φ satisfies the Strang-Fix conditions of order k the operator $\Gamma_{\mathbf{d}}$ provides approximation order $k - m$ in the uniform norm, i.e.,

$$\|\Gamma_{\mathbf{d}}^h f - f\|_\infty = \mathcal{O}(h^{k-m}) \quad \text{as} \;\; h \to 0^+\,,$$

where $\Gamma_{\mathbf{d}}^h := \sigma_h \Gamma_{\mathbf{d}} \sigma_{1/h}$. Recall that $m = 0$ whenever the involved systems are all of types (a) and/or (b), and m denotes the largest order among the partial derivatives appearing in systems of type (c).

Theorem 5 *For any $f \in C_b^m(\mathbb{R}^n)$ and $0 < h \leq 1$, we have*

$$\|f - \Gamma_{\mathbf{d}}^h f\|_\infty \leq K \inf_{\xi \in \sigma_h V_\varphi^\infty} \|\xi - f\|_{C_b^m}\,,$$

where K is a constant independent of f and h.

Proof. From Theorem 2 we have that $\Gamma_{\mathbf{d}}^h \xi = \xi$, for any $\xi \in \sigma_h V_\varphi^\infty$ and $h > 0$. From (15), $\xi \in C_b^m(\mathbb{R}^n)$, and from Lemma 3, there exists a constant $M > 0$ such that $\|\Gamma_{\mathbf{d}} l\|_\infty \leq M\|l\|_{C_b^m}$ for all $l \in C_b^m(\mathbb{R}^n)$. Hence, for any $f \in C_b^m(\mathbb{R}^n)$ and $0 < h \leq 1$, we obtain that

$$\begin{aligned}\|f - \Gamma_{\mathbf{d}}^h f\|_\infty &\leq \|f - \xi\|_\infty + \|\xi - \Gamma_{\mathbf{d}}^h f\|_\infty = \|f - \xi\|_\infty + \|\Gamma_{\mathbf{d}}^h(\xi - f)\|_\infty \\ &= \|f - \xi\|_\infty + \|\Gamma_{\mathbf{d}} \sigma_{1/h}(\xi - f)\|_\infty \leq \|f - \xi\|_\infty + M\ \|\sigma_{1/h}(\xi - f)\|_{C_b^m} \\ &\leq \|f - \xi\|_\infty + M\ \|\xi - f\|_{C_b^m} \leq (1 + M)\|\xi - f\|_{C_b^m},\end{aligned}$$

where we have used that $\sigma_{1/h}(\xi - f) \in C_b^m(\mathbb{R}^n)$. □

Corollary 1 *Assume that the systems Υ_j, $j = 1, 2, \ldots, s$, are of the types (a) and (b), and that* $\operatorname{ess\,sup}_{t\in\mathbb{R}^n} \sum_{\alpha\in\mathbb{Z}^n} |\varphi(t+\alpha)|(1+|t+\alpha|)^k < \infty$ *for some $k \in \mathbb{N}$. If the generator φ satisfies the Strang-Fix conditions of order k then, for each $f \in W_\infty^k(\mathbb{R}^n)$ and $0 < h \leq 1$, we have*

$$\|f - \Gamma_{\mathbf{d}}^h f\|_\infty \leq\ K\, |f|_{k,\infty}\ h^k\,,$$

where the constant K is independent of f and h.

Proof. Here $m = 0$, and in this case Theorem 5 reads:

$$\|f - \Gamma_{\mathbf{d}}^h f\|_\infty \leq K \inf_{\xi \in \sigma_h V_\varphi^\infty} \|\xi - f\|_\infty$$

for all $f \in C_b(\mathbb{R}^n)$ and $0 < h \leq 1$. The result follows from Theorem 3. □

Corollary 2 *Assume that φ is a compactly supported generator in $W_\infty^k(\mathbb{R}^n)$ and $k > m$. If the generator φ satisfies the Strang-Fix conditions of order k then, for each $f \in C_0(\mathbb{R}^n) \cap W_\infty^k(\mathbb{R}^n)$ and $0 < h \leq 1$, the following inequality holds:*

$$\|f - \Gamma_{\mathbf{d}}^h f\|_\infty \leq\ K\, |f|_{k,\infty}\ h^{k-m}\,,$$

where the constant K is independent of f and h.

Proof. If φ satisfies the Strang-Fix conditions of order k, there exists a compactly supported function $\widetilde{\varphi} \in L^1(\mathbb{R}^n)$ such that $Q\pi = \pi$ for all π polynomial of degree at most $k - 1$, where $(Qf)(t) := \sum_{\alpha\in\mathbb{Z}^n} \langle f, \widetilde{\varphi}(\cdot - \alpha)\rangle_{L^2(\mathbb{R}^n)} \varphi(t - \alpha)$. An example of such a function $\widetilde{\varphi}$ can be found in [19, Theorem 6.1]. Notice that for $f \in C_0(\mathbb{R}^n)$ we have that $\{\langle f, \widetilde{\varphi}(\cdot - \alpha)\rangle_{L^2(\mathbb{R}^n)}\}_{\alpha\in\mathbb{Z}^n} \in \mathsf{c}_0(\mathbb{Z}^n)$ and hence $Qf \in V_\varphi^\infty$. Moreover, from Theorem 4, for $j = 0, 1, \ldots, m$, we have that

$$|f - Q_h f|_{j,\infty} \leq K_j\, h^{k-j} |f|_{k,\infty}, \quad f \in W_\infty^k(\mathbb{R}^n),$$

where the constants K_j, $j = 1, 2, \dots, s$, are independent of f and h. By using Theorem 5 and Theorem 4 , for any $f \in C_0(\mathbb{R}^n) \cap W^k_\infty(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
\|f - \Gamma^h_{\mathbf{d}} f\|_\infty &\le C \inf_{\xi \in \sigma_h V^\infty_\varphi} \|\xi - f\|_{C^m_b} \le C\ \|Q_h f - f\|_{C^m_b} \\
&= C \max_{|\beta| \le m} \|D^\beta Q_h f - D^\beta f\|_\infty \le C \sum_{j=1}^m |Q_h f - f|_{j,\infty} \\
&\le C |f|_{k,\infty} \sum_{j=1}^m K_j h^{k-j} \le C \Big(\sum_{j=1}^m K_j \Big)\ |f|_{k,\infty}\, h^{k-m}\,,
\end{aligned}$$

where the constant C is independent of f and h. □

4.2 L^p-approximation properties $(1 \le p < \infty)$

We assume here that all the systems Υ_j, $j = 1, 2, \dots, s$, are of type (a), i.e., $\Upsilon_j f = \mathsf{h}_j * f$ with $\mathsf{h}_j \in \mathcal{L}^q(\mathbb{R}^n)$ $(1/p + 1/q = 1)$, $j = 1, 2, \dots, s$. Besides, we assume the hypotheses in Section 3.1, and that φ satisfies the Strang-Fix conditions of order k. In this Section we prove that $\|\Gamma^h_{\mathbf{d}} f - f\|_p = \mathcal{O}(h^k)$ as $h \to 0^+$ for any function f in the Sobolev space $W^k_p(\mathbb{R}^n)$. First, we prove that $\Gamma_{\mathbf{d}}$ defines a projector onto V^p_φ.

Theorem 6 *Let* $\mathbf{d}(x) = \big(d_1(x), d_2(x), \dots, d_s(x)\big)$ *be a vector with entries* $d_j \in \mathcal{A}$, $j = 1, 2, \dots, s$, *satisfying* (6). *We have that*

$$\Gamma_{\mathbf{d}} : (L^p, \|\cdot\|_p) \longrightarrow (V^p_\varphi, \|\cdot\|_p)\,, \qquad (\Gamma_{\mathbf{d}} f)(t) := \sum_{j=1}^s \sum_{\alpha \in \mathbb{Z}^n} (\Upsilon_j f)(r\alpha) S_{j,\mathbf{d}}(t - r\alpha)$$

is a well defined bounded projector onto V^p_φ. *The series defining* $\Gamma_{\mathbf{d}}$ *converges in the* L^p*-norm and uniformly on* $\mathbb{R}^n$.

Proof. From inequality (3), for any $f \in L^p(\mathbb{R}^n)$ and any $j = 1, 2, \dots, s$, we have

$$\big\|\{\Upsilon_j f(\alpha)\}_{\alpha \in \mathbb{Z}^n}\big\|_p = \big\|\{\mathsf{h}_j * f(\alpha)\}_{\alpha \in \mathbb{Z}^n}\big\|_p \le |\mathsf{h}_j|_q \|f\|_p\,.$$

Besides, we have that $S_{j,\mathbf{d}}(\cdot - \alpha) \in V^1_\varphi \subset V^p_\varphi$, for all $\alpha \in \mathbb{Z}^n$ and $S_{j,\mathbf{d}} \in \mathcal{L}^p(\mathbb{R}^n)$ (see (12)). Hence, from (2),

$$\Big\| \sum_{|\alpha| > N} (\Upsilon_j f)(r\alpha) S_{j,\mathbf{d}}(\cdot - r\alpha) \Big\|_p \le \Big(\sum_{|\alpha| > N} |(\Upsilon_j f)(r\alpha)|^p \Big)^{1/p} |S_{j,\mathbf{d}}|_p\,, \quad N \in \mathbb{N}\,.$$

Therefore, the series defining $\Gamma_{\mathbf{d}} f$ converges in $L^p(\mathbb{R}^n)$ and, as a consequence, $\Gamma_{\mathbf{d}} f \in V^p_\varphi$. By Theorem 1, the series defining $\Gamma_{\mathbf{d}} f$ converges also uniformly on $\mathbb{R}^n$. Moreover, for any $f \in L^p(\mathbb{R}^n)$,

$$\|\Gamma_{\mathbf{d}} f\|_p \le \sum_{j=1}^s \big\|\{\Upsilon_j f(r\alpha)\}_{\alpha \in \mathbb{Z}^n}\big\|_p |S_{j,\mathbf{d}}|_p \le \Big(\sum_{j=1}^s |h_j|_q |S_{j,\mathbf{d}}|_p \Big)\ \|f\|_p\,,$$

and $\Gamma_{\mathbf{d}}$ is a bounded operator.

From Lemma 2, $\Gamma_{\mathbf{d}} f = f$ for each f in $\operatorname{span}\{\varphi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^n}$. Given $f \in V_\varphi^p$, let $\{f_M\}_{M\in\mathbb{N}}$ be a sequence in $\operatorname{span}\{\varphi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^n}$ such that $\|f_M - f\|_p \to 0$ as $M \to \infty$. Since

$$0 \le \|f_M - \Gamma_{\mathbf{d}} f\|_p = \|\Gamma_{\mathbf{d}} f_M - \Gamma_{\mathbf{d}} f\|_p \le \|\Gamma_{\mathbf{d}}\| \|f_M - f\|_p \to 0 \quad \text{as} \quad M \to \infty,$$

we obtain that $\Gamma_{\mathbf{d}} f = f$. □

An application of Theorem 3 and Theorem 6 gives the following Corollary:

Corollary 3 *Assume that* $\operatorname{ess\,sup}_{t\in\mathbb{R}^n} \sum_{\alpha\in\mathbb{Z}^n} |\varphi(t+\alpha)|(1+|t+\alpha|)^k < \infty$ *for some* $k \in \mathbb{N}$. *If the generator* φ *satisfies the Strang-Fix conditions of order* k*, then, for each* $f \in W_p^k(\mathbb{R}^n)$ *and* $h > 0$

$$\|f - \Gamma_{\mathbf{d}}^h f\|_p \le K\, |f|_{k,p}\, h^k\,,$$

where the constant K *is independent of* f*,* p *and* h*.*

Proof. Using that $\Gamma_{\mathbf{d}}^h \xi = \xi$ for each $\xi \in \sigma_h V_\varphi^p$, then, for each $f \in L^p(\mathbb{R}^n)$ and $\xi \in \sigma_h V_\varphi^p$ we obtain that

$$\|f - \Gamma_{\mathbf{d}}^h f\|_p \le \|f - \xi\|_p + \|\xi - \Gamma_{\mathbf{d}}^h f\|_p \le (1 + \|\Gamma_{\mathbf{d}}^h\|)\|f - \xi\|_p\,.$$

It is easy to check that $\|\Gamma_{\mathbf{d}}^h\| = \|\Gamma_{\mathbf{d}}\|$. Therefore, for each $f \in L^p(\mathbb{R}^n)$,

$$\|f - \Gamma_{\mathbf{d}}^h f\|_p \le (1 + \|\Gamma_{\mathbf{d}}\|) \min_{\xi\in\sigma_h V_\varphi^p} \|f - \xi\|_p\,.$$

Finally, Theorem 3 gives the desired result. □

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