Generalized sampling in shift-invariant spaces with multiple stable generators

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Abstract

The aim of this paper is to derive stable generalized sampling in a shift-invariant space with ℓ stable generators. This is done in the light of the theory of frames in the product Hilbert space $L^2_\ell(0,1) := L^2(0,1) \times \ldots \times L^2(0,1)$ (ℓ times). The generalized samples are expressed as the frame coefficients of an appropriate function in $L^2_\ell(0,1)$ with respect to some particular frame in $L^2_\ell(0,1)$. Since any multiply stable generated shift-invariant space is the image of $L^2_\ell(0,1)$ by means of a bounded invertible operator, the generalized sampling is obtained from some dual frame expansions in $L^2_\ell(0,1)$. An example in the setting of the Hermite cubic splines is exhibited.

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1 Introduction

The Whittaker-Shannon-Kotel'nikov sampling theorem states that any function f in the classical Paley-Wiener space PW_π

$$PW_\pi := \left\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \,:\, \operatorname{supp} \widehat{f} \subseteq [-1/2, 1/2] \right\},$$

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i.e., band-limited to $[-1/2, 1/2]$, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ on the integers as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}(t-n)\,, \quad t \in \mathbb{R}\,,$$

where sinc denotes the cardinal sine function, $\mathrm{sinc}(t) = \sin \pi t/\pi t$. Thus, the Paley-Wiener space of band-limited functions to $[-1/2, 1/2]$ is generated by the integer shifts of the sinc function.

Although Shannon's sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in [22]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of finite duration signal; the bandlimiting operation generates Gibbs oscillations, and finally, the sinc function has a very slow decay, which makes computation in the signal domain very inefficient. Moreover, many applied problems impose different a priori constraints on the type of functions. For this reason, the sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces. See, for instance, [3, 4, 5, 7, 8, 12, 23, 24] and references therein.

In many practical applications, signals are assumed to belong to some shift-invariant space of the form: $V_\Phi := \overline{\mathrm{span}}_{L^2}\{\varphi_k(t-n) \,:\, k = 1, 2, \dots, \ell \text{ and } n \in \mathbb{Z}\}$ where the functions $\Phi = \{\varphi_1, \dots, \varphi_\ell\}$ in $L^2(\mathbb{R})$ are called a set of generators for V_Φ. Assuming stable generators, the shift-invariant space V_Φ can be described as

$$V_\Phi := \Big\{ \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} d_k(n)\ \varphi_k(t-n) \,:\, d_k \in \ell^2(\mathbb{Z}), k = 1, 2 \dots, \ell \Big\}\,, \tag{1}$$

where the sequence $\{\varphi_k(t-n)\}_{n\in\mathbb{Z}, k=1,2\dots,\ell}$ is a Riesz basis for V_Φ. See Refs. [9, 10, 18] for the general theory of shift-invariant spaces and their applications. These spaces and the scaling functions $\Phi = \{\varphi_1, \dots, \varphi_\ell\}$ appear in the multiwavelet setting. Multiwavelets lead to multiresolution analyses and fast algorithms just as scalar wavelets, but they have some advantages: they can have short support coupled with high smoothness and high approximation order, and they can be both symmetric and orthogonal (see, for instance, Ref. [17]). Classical sampling in multiwavelet subspaces has been studied in Refs. [19, 21].

On the other hand, in many common situations the avalaible data are samples of some filtered versions of the signal itself. This leads to generalized sampling in V_Φ: Suppose that s linear-time invariant systems (filters) $\mathcal{L}_j$, $j = 1, 2, \dots, s$, are defined on the shift-invariant subspace V_Φ of $L^2(\mathbb{R})$. The goal is to recover any function f in V_Φ from the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$, by means of a sampling formula which is a frame expansion in V_Φ [6, 15].

In this paper a new approach for generalized sampling in a shift-invariant space V_Φ with stable generators $\Phi = \{\varphi_1, \dots, \varphi_\ell\}$ in $L^2(\mathbb{R})$ is proposed. It involves the theory of frames in a separable Hilbert space $\mathcal{H}$. Recall that a sequence $\{f_k\}$ is a frame for $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|f\|^2 \leq \sum_k |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}\,.$$

Given a frame $\{f_k\}$ for $\mathcal{H}$ the representation property of any vector $f \in \mathcal{H}$ as a series $f = \sum_k c_k f_k$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for overcomplete frames) is sacrificed. Suitable frame coefficients c_k which depend continuously and linearly on f are obtained by using the dual frames $\{g_k\}$ of $\{f_k\}$, i.e., $\{g_k\}$ is another frame for $\mathcal{H}$ such that $f = \sum_k \langle f, g_k\rangle f_k = \sum_k \langle f, f_k\rangle g_k$ for each $f \in \mathcal{H}$. For more details on frame theory see the superb monograph [13] and references therein.

The shift-invariant space V_Φ is the image of the product Hilbert space $L^2_\ell(0,1) := L^2(0,1) \times \ldots \times L^2(0,1)$ (ℓ times) by means of the isomorphism $\mathcal{T}_\Phi : L^2_\ell(0,1) \longrightarrow V_\Phi$, which maps the standard orthonormal basis for $L^2_\ell(0,1)$ onto the Riesz basis $\{\varphi_k(t-n)\}_{n\in\mathbb{Z},k=1,2\ldots,\ell}$ for V_Φ. The starting point is to write the generalized samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}\,,j=1,2,\ldots,s}$, where the sampling period $r \in \mathbb{N}$ necessarily satisfies $r\ell \leq s$, as the frame coefficients of the function $\mathbf{F} = \mathcal{T}_\Phi^{-1} f \in L^2_\ell(0,1)$ with respect to a particular frame in $L^2_\ell(0,1)$. Searching for its dual frames we obtain those expansions for F in $L^2_\ell(0,1)$ having the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}\,,j=1,2,\ldots,s}$ as frame coefficients. Thus, applying the isomorphism $\mathcal{T}_\Phi$ to the above frame expansions of $\mathbf{F}$ we will obtain sampling expansions for $f = \mathcal{T}_\Phi \mathbf{F}$ in V_Φ involving the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}\,,j=1,2,\ldots,s}$. The case where the sequence $\{\varphi_k(t-n)\}_{n\in\mathbb{Z},k=1,2\ldots,\ell}$ is only a Bessel sequence in $L^2(\mathbb{R})$ is also briefly discussed in Section 5 in the light of the Aldroubi et al. results in [6].

The use of several different dual frames allow us to obtain a variety of reconstruction functions. Thus, following an idea in [12], we can try to find some reconstruction functions with compact support. This idea is illustrated in the case of Hermite cubic splines. All these steps will be carried out throughout the remaining sections.

2 Preliminaries

Let $\Phi := (\varphi_1, \varphi_2, \ldots, \varphi_\ell)^\top$, where $\varphi_k \in L^2(\mathbb{R})$ $k = 1, 2, \ldots, \ell$, such that the sequence $\big\{\varphi_k(t-n)\big\}_{n\in\mathbb{Z},k=1,2\ldots,\ell}$ is a Riesz basis (stable generators) for the shift-invariant space

$$V_\Phi := \Big\{ \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} d_k(n)\; \varphi_k(t-n) \;:\; d_k \in \ell^2(\mathbb{Z}), k = 1, 2\ldots, \ell \Big\} \subset L^2(\mathbb{R})\,.$$

Recall that a Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. There exists a necessary and sufficient condition involving the Gramian matrix-function

$$G_{\mathbf{\Phi}}(w) := \sum_{n\in\mathbb{Z}} \widehat{\Phi}(w+n)\overline{\widehat{\Phi}(w+n)}^\top$$

which assures that the sequence $\{\varphi_k(\cdot - n)\}_{n\in\mathbb{Z},k=1,2\ldots,\ell}$ is a Riesz basis for V_Φ. Namely: there exist two positive constants m and M such that $m\mathbf{I}_\ell \le G_{\boldsymbol{\Phi}}(w) \le M\mathbf{I}_\ell$ a.e. in $w \in (0,1)$ (see, for instance, [1, 2, 6]).

We assume throughout the paper that the functions in the shift-invariant space V_Φ are continuous on $\mathbb{R}$. This is equivalent to the generators Φ being continuous on $\mathbb{R}$ with $\sum_{n\in\mathbb{Z}} |\Phi(t-n)|^2$ uniformly bounded on $\mathbb{R}$. A proof of this equivalence can be found in [20] for the case of a unique generator ($\ell = 1$). The general case can be proved similarly. Thus, any $f \in V_\Phi$ is defined on $\mathbb{R}$ as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} d_k(n)\, \varphi_k(t-n),\quad t \in \mathbb{R}$.

Besides, V_Φ is a reproducing kernel Hilbert space (RKHS) since the evaluation functionals are bounded in V_Φ. Indeed, for each fixed $t \in \mathbb{R}$ we have

$$|f(t)|^2 \le \frac{\|f\|^2}{C} \sum_{n\in\mathbb{Z}} |\Phi(t-n)|^2, \quad f \in V_\Phi\,, \tag{2}$$

where we have used Cauchy-Schwartz's inequality in $f(t) = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} d_k(n)\, \varphi_k(t-n)$, and the inequality $C \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} |d_k(n)|^2 \le \|f\|^2$, satisfied for any lower Riesz bound C of the Riesz basis $\{\varphi_k(\cdot - n)\}_{n\in\mathbb{Z},k=1,2\ldots,\ell}$ for V_Φ. Inequality (2) shows that convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence which is uniform on $\mathbb{R}$.

The product space $L^2_\ell(0,1) := \big\{\mathbf{F} = (F_1, F_2, \ldots, F_\ell)^\top : F_k \in L^2(0,1), k = 1,2,\ldots,\ell\big\}$ with its usual inner product $\langle \mathbf{F}, \mathbf{H}\rangle_{L^2_\ell(0,1)} := \sum_{k=1}^{\ell} \langle F_k, H_k\rangle_{L^2(0,1)} = \int_0^1 \mathbf{H}^*(w)\mathbf{F}(w)dw$ becomes a Hilbert space. Similarly we denote $L^\infty_\ell(0,1) := L^\infty(0,1) \times \ldots \times L^\infty(0,1)$ (ℓ times). The system $\{e^{-2\pi i n w}\mathbf{e}_k\}_{n\in\mathbb{Z},k=1,2,\ldots,\ell}$, where $\mathbf{e}_k$ denotes the vector of $\mathbb{R}^\ell$ with all the components null except the k-th component which is equal to one, is a orthonormal basis for $L^2_\ell(0,1)$.

The space V_Φ is the image of $L^2_\ell(0,1)$ by means of the isomorphism $\mathcal{T}_\Phi : L^2_\ell(0,1) \to V_\Phi$, which maps the orthonormal basis $\{e^{-2\pi i n w}\mathbf{e}_k\}_{n\in\mathbb{Z},k=1,2,\ldots,\ell}$ for $L^2_\ell(0,1)$ onto the Riesz basis $\{\varphi_k(t-n)\}_{n\in\mathbb{Z},k=1,2,\ldots,\ell}$ for V_Φ, i.e.,

$$\mathcal{T}_\Phi \mathbf{F}(t) := \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} \big\langle F_k, e^{-2\pi i n \cdot}\big\rangle_{L^2(0,1)} \varphi_k(t-n), \qquad \mathbf{F} = (F_1, \ldots, F_\ell)^\top \in L^2_\ell(0,1). \tag{3}$$

The isomorphism $\mathcal{T}_\Phi$ can be expressed also by

$$f(t) = \mathcal{T}_\Phi \mathbf{F}(t) = \langle \mathbf{F}, \mathbf{K}_t\rangle_{L^2_\ell(0,1)}\,, \quad t \in \mathbb{R}\,,$$

where the kernel transform $\mathbb{R} \ni t \mapsto \mathbf{K}_t \in L^2_\ell(0,1)$ is defined as $\mathbf{K}_t(x) := \overline{\mathbf{Z}\Phi}(t,x)$, and $\mathbf{Z}\Phi$ denotes the Zak transform of Φ, i.e.,

$$(\mathbf{Z}\Phi)(t,w) := \sum_{n\in\mathbb{Z}} \Phi(t+n) e^{-2\pi i n w}$$

Notice that $(\mathbf{Z}\Phi) = (Z\varphi_1, \ldots, Z\varphi_\ell)^\top$ where Z denotes the usual Zak transform. See [16] for properties and uses of the Zak transform.

The following shifting property of $\mathcal{T}_\Phi$ will be used later: For $\mathbf{F} \in L^2_\ell(0,1)$ and $n \in \mathbb{Z}$ we have

$$\mathcal{T}_\Phi\big[\mathbf{F}(\cdot)e^{-2\pi i n \cdot}\big](t) = \mathcal{T}_\Phi\mathbf{F}(t-n)\,, \quad t \in \mathbb{R}\,. \tag{4}$$

3 An expression for the samples

Throughout the paper we distinguish two types of linear time-invariant system $\mathcal{L}$:

(a) The impulse response h of $\mathcal{L}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, for any $f \in V_\Phi$ we have

$$\big(\mathcal{L}f\big)(t) := [f * \mathsf{h}](t) = \int_{-\infty}^{\infty} f(x)\mathsf{h}(t-x)dx = \langle f, \psi(\cdot - t)\rangle_{L^2(\mathbb{R})}\,, \quad t \in \mathbb{R}\,,$$

where $\psi(t) := \overline{\mathsf{h}(-t)}$. Notice that $\mathcal{L}f$ is a continuous and bounded function in $L^2(\mathbb{R})$.

(b) The impulse response h has the form $\sum_{m=0}^{N} c_m \delta^{(m)}(t+d_m)$ where $\delta^{(m)}$ denotes the m-th derivative of the Dirac delta and c_m, d_m are constants for $m = 0, 1, \ldots, N$. For each $f \in V_\Phi$ we have

$$\big(\mathcal{L}f\big)(t) := \sum_{m=0}^{N} c_m f^{(m)}(t+d_m)\,, \quad t \in \mathbb{R}\,.$$

In this case we also assume that $\Phi^{(N)}$ exists on $\mathbb{R}$, and $\sum_{n\in\mathbb{Z}} |\Phi^{(m)}(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ for each $m = 0, 1, \ldots, N$.

Given a linear time-invariant system $\mathcal{L}$ of the type (a) or (b), for each fixed $t \in \mathbb{R}$ and $k = 1, 2, \ldots, \ell$, we have that the sequence $\{(\mathcal{L}\varphi_k)(t+n)\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$. This statement is proved in [15, Lemma 1]. Thus, for $\mathcal{L}\Phi := (\mathcal{L}\varphi_1, \ldots, \mathcal{L}\varphi_\ell)^\top$ we have

$$(\mathbf{Z}\mathcal{L}\Phi)(t,w) := \sum_{n\in\mathbb{Z}} \mathcal{L}\Phi(t+n)e^{-2\pi i n w} \in L^2_\ell(0,1)\,.$$

Notice that $\mathbf{Z}\mathcal{L}\Phi = \big(Z\mathcal{L}\varphi_1, Z\mathcal{L}\varphi_2, \ldots, Z\mathcal{L}\varphi_\ell\big)^\top$.

Lemma 1 *Let $\mathcal{L}$ be a linear time-invariant system of the type (a) or (b) above. Then, for each $f \in V_\Phi$, we have*

$$\big(\mathcal{L}f\big)(t) = \big\langle \mathbf{F}, \overline{(\mathbf{Z}\mathcal{L}\Phi)}(t,\cdot)\big\rangle_{L^2_\ell(0,1)}\,, \quad t \in \mathbb{R}\,,$$

where $\mathbf{F} = \mathcal{T}_\Phi^{-1} f$.

Proof: Assume that $\mathcal{L}$ is a system of the type (a). For each $t \in \mathbb{R}$ we have

$$
\begin{aligned}
(\mathcal{L}f)(t) &= \langle f, \psi(\cdot - t)\rangle_{L^2(\mathbb{R})} = \Big\langle \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} \langle F_k, e^{-2\pi i n\cdot}\rangle_{L^2(0,1)} \varphi_k(\cdot - n), \psi(\cdot - t)\Big\rangle_{L^2(\mathbb{R})} \\
&= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} \langle F_k, e^{-2\pi i n\cdot}\rangle_{L^2(0,1)} \Big\langle \varphi_k, \psi(\cdot - t + n)\Big\rangle_{L^2(\mathbb{R})} \\
&= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} \langle F_k, e^{-2\pi i n\cdot}\rangle_{L^2(0,1)} \mathcal{L}\varphi_k(t - n)\,.
\end{aligned}
$$

Since $\{(\mathcal{L}\varphi_k)(t+n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, Parseval's equality gives

$$
(\mathcal{L}f)(t) = \sum_{k=1}^{\ell} \langle F_k, \sum_{n\in\mathbb{Z}} \overline{\mathcal{L}\varphi_k}(t-n)\, e^{-2\pi i n\cdot}\rangle_{L^2(0,1)} = \langle \mathbf{F}, \overline{(\mathbf{Z}\mathcal{L}\Phi)}(t,\cdot)\rangle_{L^2_\ell(0,1)}\,.
$$

Assume now that $\mathcal{L}$ is a system of the type (b). Under our hypotheses on $\mathcal{L}$, for $m = 0, 1, 2, \dots, N$, we have that

$$
f^{(m)}(t) = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} \langle F_k, e^{-2\pi i n\cdot}\rangle_{L^2(0,1)} \varphi_k^{(m)}(t-n)\,.
$$

Having in mind that we have assumed that $\sum_{n\in\mathbb{Z}} |\Phi^{(m)}(t-n)|^2$ is uniformly bounded on $\mathbb{R}$, we obtain that

$$
\begin{aligned}
(\mathcal{L}f)(t) &= \sum_{m=0}^{N} c_m f^{(m)}(t + d_m) = \sum_{m=0}^{N} c_m \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} \langle F_k, e^{-2\pi i n\cdot}\rangle_{L^2(0,1)} \varphi_k^{(m)}(t + d_m - n) \\
&= \sum_{k=1}^{\ell} \langle F_k, \sum_{m=0}^{N} \overline{c}_m \sum_{n\in\mathbb{Z}} \overline{\varphi_k}^{(m)}(t + d_m - n) e^{-2\pi i n\cdot}\rangle_{L^2(0,1)} \\
&= \sum_{k=1}^{\ell} \langle F_k, \sum_{n\in\mathbb{Z}} \overline{\mathcal{L}\varphi_k}(t-n) e^{-2\pi i n\cdot}\rangle_{L^2(0,1)} = \sum_{k=1}^{\ell} \langle F_k, \overline{(Z\mathcal{L}\varphi_k)}(t,\cdot)\rangle_{L^2(0,1)}
\end{aligned}
$$

which ends the proof. □

Now, consider s linear time-invariant systems $\mathcal{L}_j$, $j = 1, 2, \dots, s$, of the type (a), (b), or both. The main aim in this work is to recover any function f in the shift-invariant space V_Φ from the sequence of regular samples $\big\{\big(\mathcal{L}_j f\big)(rn)\big\}_{n\in\mathbb{Z},\ j=1,2,\dots,s}$ by means of a stable sampling formula, i.e., the sampling formula will be an expansion with respect to a frame for V_Φ. Notice that the apparently more general set of samples $\{\mathcal{L}_j f(rn + c_j)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$, where $c_j \in \mathbb{R}$ for $j = 1, 2, \dots, s$, is reduced to the case considered here by taking the appropriate shifted systems.

For $j = 1, 2, \dots, s$, the function $\mathbf{g}_j$ in $L^2_\ell(0,1)$ defined by

$$\mathbf{g}_j(w) := (\mathbf{Z}\mathcal{L}_j\Phi)(0,w)\,, \tag{5}$$

plays an important role throughout this paper. Namely, Lemma 1 gives an expression for the regular samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},\ j=1,2,\dots,s}$ which involves the functions $\mathbf{g}_j$, $j = 1, 2, \dots, s$ and the function $\mathbf{F} = \mathcal{T}_\Phi^{-1} f$ in $L^2_\ell(0,1)$:

$$\big(\mathcal{L}_j f\big)(rn) = \big\langle \mathbf{F}, \overline{\mathbf{g}}_j(\cdot) e^{-2\pi i r n \cdot} \big\rangle_{L^2_\ell(0,1)}\,, \quad n \in \mathbb{Z}\,,\ j = 1, 2, \dots, s\,, \tag{6}$$

where we have used that $(\mathbf{F}\mathcal{L}\Phi)(t+rn, w) = e^{2\pi i r n w}(\mathbf{Z}\mathcal{L}\Phi)(t,w)$. This lead us to study when the sequence $\{\overline{\mathbf{g}}_j(w)e^{-2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2_\ell(0,1)$. Having in mind that $\mathcal{T}_\Phi$ is an isomorphism, the above sequence is a frame for $L^2_\ell(0,1)$ if and only if there exist two constants $0 < A \le B$ such that

$$A\|f\|^2 \le \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} |\mathcal{L}_j f(rn)|^2 \le B\|f\|^2 \quad \text{for all } \ f \in V_\Phi\,.$$

Next, we carry out the study of the completeness, Bessel, frame, or Riesz basis properties of a general sequence $\{\overline{\mathbf{b}}_j(w)e^{2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ in $L^2_\ell(0,1)$ with the same structure. To this end, associated with the s vectors $\mathbf{b}_1(w), \mathbf{b}_2(w), \dots, \mathbf{b}_s(w)$ we consider the $s \times r\ell$ matrix

$$\mathbb{B}(w) := \begin{bmatrix} \mathbf{b}_1^\top(w) & \mathbf{b}_1^\top(w+\frac{1}{r}) & \cdots & \mathbf{b}_1^\top(w+\frac{r-1}{r}) \\ \mathbf{b}_2^\top(w) & \mathbf{b}_2^\top(w+\frac{1}{r}) & \cdots & \mathbf{b}_2^\top(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ \mathbf{b}_s^\top(w) & \mathbf{b}_s^\top(w+\frac{1}{r}) & \cdots & \mathbf{b}_s^\top(w+\frac{r-1}{r}) \end{bmatrix} \tag{7}$$

and its related constants

$$\alpha_{\mathbb{B}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{B}^*(w)\mathbb{B}(w)], \quad \beta_{\mathbb{B}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{B}^*(w)\mathbb{B}(w)]\,.$$

Notice that in the definition of the matrix $\mathbb{B}(w)$ we are considering the 1-periodic extensions of the involved functions $\mathbf{b}_j$, $j = 1, 2, \dots, s$.

Lemma 2 *Let $\mathbf{b}_j$ be in $L^2_\ell(0,1)$ for $j = 1, 2, \dots, s$ and let $\mathbb{B}(w)$ be its associated matrix. Then, the following results hold:*

(a) The sequence $\{\overline{\mathbf{b}}_j(w)e^{2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a complete system for $L^2_\ell(0,1)$ if and only if the rank of the matrix $\mathbb{B}(w)$ is $r\ell$ a.e. in $(0,1/r)$.

(b) The sequence $\{\overline{\mathbf{b}}_j(w)e^{2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a Bessel sequence for $L^2_\ell(0,1)$ if and only if $\mathbf{b}_j \in L^\infty_\ell(0,1)$ (or equivalently $\beta_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $\beta_{\mathbb{B}}/r$.

(c) *The sequence* $\{\overline{\mathbf{b}}_j(w)e^{2\pi irnw}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ *is a frame for* $L^2_\ell(0,1)$ *if and only if* $0<\alpha_\mathbb{B}\le\beta_\mathbb{B}<\infty$. *In this case, the optimal frame bounds are* $\alpha_\mathbb{B}/r$ *and* $\beta_\mathbb{B}/r$.

(d) *The sequence* $\{\overline{\mathbf{b}}_j(w)e^{2\pi irnw}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ *is a Riesz basis for* $L^2_\ell(0,1)$ *if and only if is a frame and* $s=r\ell$.

Proof: First, notice that for any $\mathbf{F}\in L^2_\ell(0,1)$ and $j=1,2,\dots,s$ we have

$$\langle \mathbf{F},\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\rangle_{L^2_\ell(0,1)}=\int_0^{1/r}\sum_{k=1}^{r}\mathbf{b}_j^\top\Big(w+\frac{k-1}{r}\Big)\mathbf{F}\Big(w+\frac{k-1}{r}\Big)e^{-2\pi irnw}dw\,. \tag{8}$$

As a consequence, $\langle \mathbf{F},\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\rangle_{L^2_\ell(0,1)}$ are the Fourier coefficients of the j-th component of $\mathbb{B}(w)\mathbb{F}(w)$ with respect to the orthogonal basis $\{e^{2\pi irnw}\}_{n\in\mathbb{Z}}$ for $L^2(0,1/r)$, where

$$\mathbb{F}(w):=\Big[\mathbf{F}^\top(w),\mathbf{F}^\top\Big(w+\frac{1}{r}\Big),\cdots,\mathbf{F}^\top\Big(w+\frac{r-1}{r}\Big)\Big]^\top.$$

To prove (a), assume that $\{\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is not a complete system for $L^2_\ell(0,1)$. Then there exists $\mathbf{F}\neq 0$ in $L^2_\ell(0,1)$ such that $\big\langle \mathbf{F},\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\big\rangle_{L^2_\ell(0,1)}=0$ for all $n\in\mathbb{Z}$ and $j=1,2,\dots,s$. By using (8) one obtains that the components of $\mathbb{B}(w)\mathbb{F}(w)$ (belonging to $L^1(0,1/r)$) have zero Fourier coefficients. Hence, $\mathbb{B}(w)\mathbb{F}(w)=0$ a.e. in $(0,1/r)$. Therefore, $\operatorname{rank}\mathbb{B}(w)<r\ell$ in a subset of $(0,1/r)$ with positive (Lebesgue) measure where $\mathbb{F}(w)\neq 0$.

Conversely, assume that there exists a set $\Omega\subseteq(0,1/r)$ with positive measure such that $\operatorname{rank}\mathbb{B}(w)<\ell r$, $w\in\Omega$. Then, for $w\in\Omega$ there exists a unitary vector $v(w)$ such that $\mathbb{B}(w)v(w)=0$. Define $\mathbf{F}\in L^2_\ell(0,1)$ such that $\mathbb{F}(w)=v(w)$, if $w\in\Omega$, and $\mathbb{F}(w)=\mathbf{0}$, if $w\in(0,1/r)\setminus\Omega$. From (8) one gets $\big\langle \mathbf{F},\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\big\rangle_{L^2_\ell(0,1)}=0$ $n\in\mathbb{Z}$ and $j=1,2,\dots,s$, and consequently $\{\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is not a complete system for $L^2_\ell(0,1)$. This concludes the proof of (a).

By using (8), the proofs of the parts (b) and (c) in the Lemma are completely analogous to those in [15, Lemma 3].

The proof of (d) is based in the following result ([13, Theorem 6.1.1]): A frame is a Riesz basis if and only if it has a biorthogonal sequence. Assume that the sequence $\{\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Riesz basis for $L^2_\ell(0,1)$. Let $\{\mathbf{c}_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ be its biorthogonal sequence, i.e., $\langle \mathbf{c}_{j',n'}(\cdot),\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\rangle_{L^2_\ell(0,1)}=\delta_{j,j'}\delta_{n,n'}$. Then

$$\begin{aligned}&\int_0^{1/r}\sum_{k=0}^{r-1}\mathbf{b}_j^\top\Big(w+\frac{k}{r}\Big)\mathbf{c}_{j',0}\Big(w+\frac{k}{r}\Big)e^{-2\pi irnw}\,dw=\int_0^1\mathbf{b}_j^\top(w)\mathbf{c}_{j',0}(w)e^{-2\pi irnw}\,dw\\&=\big\langle \mathbf{c}_{j',0}(\cdot),\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\big\rangle_{L^2_\ell(0,1)}=\delta_{j,j'}\delta_{n,0}\,,\quad n\in\mathbb{Z}\,.\end{aligned}$$

Therefore, for $j,j'=1,2,\dots,s$, we obtain

$$\sum_{k=0}^{r-1}\mathbf{b}_j^\top\Big(w+\frac{k}{r}\Big)\mathbf{c}_{j',0}\Big(w+\frac{k}{r}\Big)=r\delta_{j,j'}\,,\qquad\text{a.e. in }(0,1)\,.$$

Thus the matrix $\mathbb{B}(w)$ has a right inverse a.e. in $(0,1)$, and in particular, $s \le \ell r$.

On the other hand, $\alpha_{\mathbb{B}} > 0$ implies that $\det[\mathbb{B}^*(w)\mathbb{B}(w)] > 0$, a.e. in $(0,1)$, and there exists the matrix $[\mathbb{B}^*(w)\mathbb{B}(w)]^{-1}\mathbb{B}^*(w)$ a.e. in $(0,1)$. This matrix is a left inverse of the matrix $\mathbb{B}(w)$ which implies $r\ell \le s$. Thus, we obtain that $r\ell = s$.

Conversely, assume that $\{\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ is a frame for $L^2_\ell(0,1)$ and $r\ell = s$. In this case $\mathbb{B}(w)$ is a square matrix and $\det[\mathbb{B}^*(w)\mathbb{B}(w)] > 0$ a.e. in $(0,1)$ implies that $\det\mathbb{B}(w) \ne 0$ a.e. in $(0,1)$. Having in mind the structure of $\mathbb{B}$ its inverse must be

$$\mathbb{B}^{-1}(w) = \begin{pmatrix} \mathbf{c}_1(w) & \ldots & \mathbf{c}_s(w) \\ \mathbf{c}_1(w+1/r) & \ldots & \mathbf{c}_s(w+1/r) \\ \vdots & & \vdots \\ \mathbf{c}_1(w+(r-1)/r) & \ldots & \mathbf{c}_s(w+(r-1)/r) \end{pmatrix}$$

where $\mathbf{c}_j \in L^2_\ell(0,1)$, $j = 1,2,\ldots,s$. Since

$$\big\langle r\mathbf{c}_{j'}(\cdot)e^{2\pi irn'\cdot}, \overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\big\rangle_{L^2_\ell(0,1)} = r\int_0^1 \mathbf{b}_j^\top(w)\mathbf{c}_{j'}(w)e^{2\pi ir(n'-n)w}\,dw$$

$$r\int_0^{1/r}\sum_{k=0}^{r-1}\mathbf{b}_j^\top\Big(w+\frac{k}{s}\Big)\mathbf{c}_{j'}\Big(w+\frac{k}{s}\Big)e^{2\pi ir(n'-n)w}\,dw = \delta_{n,n'}\delta_{j,l}\,,$$

for $n,n' \in \mathbb{Z}$ and $j,j' = 1,2,\ldots,s$, the frame $\{\overline{\mathbf{b}}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ has a biorthogonal sequence and therefore it is a Riesz basis for $L^2_\ell(0,1)$. □

4 Generalized regular sampling

A first question which raises is when the function f in the space V_Φ is uniquely determined by the generalized samples $\{(\mathcal{L}_jf)(rn)\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$. Let $\mathbb{G}(w)$ be the $s \times r\ell$ matrix (7) associated to the functions $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s$ defined in (5). Since $\big(\mathcal{L}_jf\big)(rn) = \big\langle \mathbf{F}, \overline{\mathbf{g}}_j(\cdot)e^{-2\pi irn\cdot}\big\rangle_{L^2_\ell(0,1)}$, $j = 1,2,\ldots,s$, and $n \in \mathbb{Z}$, for each $f = \mathcal{T}_\Phi\mathbf{F} \in V_\Phi$ and $n \in \mathbb{Z}$, the samples $\{(\mathcal{L}_jf)(rn)\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ determine uniquely the function f if and only if the sequence $\big\{\overline{\mathbf{g}}_j(\cdot)e^{-2\pi irn\cdot}\big\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ is a complete system for $L^2_\ell(0,1)$. According with Lemma 2 this is equivalent to

$$\operatorname{rank}\mathbb{G}(w) = r\ell\,, \ \text{ a.e. in } (0,1/r)\,.$$

As a consequence, when the number of systems is smaller than the product of the sampling period and the number of generators, i.e., $\ell r > s$, there exist many functions of V_Φ with the same samples. This makes non-viable the recovery of the functions in V_Φ from their generalized samples.

The main aim in this section is to recover any function $f \in V_\Phi$ from the samples $\{(\mathcal{L}_jf)(rn)\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ by means of a stable sampling formula, i.e., the sampling formula will be an expansion with respect to an appropriate frame for V_Φ.

From now on we assume that the function $\mathbf{g}_j \in L^\infty_\ell(0,1)$, for each $j = 1, 2, \ldots, s$ which is equivalent to $\{\overline{\mathbf{g}}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ being a Bessel sequence for $L^\infty_\ell(0,1)$. As $\{e^{-2\pi irnw}\}_{n\in\mathbb{Z}}$ is an orthogonal basis for $L^2(0,1/r)$, one gets the following expression for the samples

$$(\mathcal{L}_j f)(rn) = \int_0^1 \mathbf{g}_j^\top(w)\mathbf{F}(w)e^{2\pi irnw}dw = \int_0^{1/r} \sum_{k=0}^{r-1} \mathbf{g}_j^\top\Big(w+\frac{k}{r}\Big)\mathbf{F}\Big(w+\frac{k}{r}\Big)e^{2\pi irnw}dw$$

Since we have assumed that the function $\mathbf{g}_j \in L^\infty_\ell(0,1)$, $j = 1, 2, \ldots, s$, we obtain that

$$r\sum_{n\in\mathbb{Z}} \big(\mathcal{L}_j f\big)(rn)e^{-2\pi inrw} = \sum_{k=0}^{r-1} \mathbf{g}_j^\top\Big(w+\frac{k}{r}\Big)\mathbf{F}\Big(w+\frac{k}{r}\Big) \quad \text{in } L^2(0,1/r)\,.$$

The above expansions also hold in $L^2(0,1)$ by considering the 1-periodic extensions of $\mathbf{F}$ and $\mathbf{g}_j$, $j = 1, 2, \ldots, s$. Thus we have the matrix expression

$$\mathbb{G}(w)\mathbb{F}(w) = r\Big[\sum_{n\in\mathbb{Z}} \big(\mathcal{L}_1 f\big)(rn)e^{-2\pi inrw}, \ldots, \sum_{n\in\mathbb{Z}} \big(\mathcal{L}_s f\big)(rn)e^{-2\pi inrw}\Big]^\top \quad \text{in } L^2(0,1)\,, \tag{9}$$

where

$$\mathbb{F}(w) := \Big[\mathbf{F}^\top(w), \mathbf{F}^\top\Big(w+\frac{1}{r}\Big), \cdots, \mathbf{F}^\top\Big(w+\frac{r-1}{r})\Big]^\top.$$

In order to recover $\mathbf{F}(w)$, assume that there exist functions $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s \in L^\infty_\ell(0,1)$ such that the $\ell \times s$ matrix $[\mathbf{a}_1(w), \mathbf{a}_2(w), \ldots, \mathbf{a}_s(w)]$ satisfies

$$[\mathbf{a}_1(w), \mathbf{a}_2(w), \ldots, \mathbf{a}_s(w)]\ \mathbb{G}(w) = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \end{bmatrix} = [\mathbf{I}_\ell, \mathbf{O}_{\ell\times(r-1)\ell}]$$

a.e. in $(0,1)$ (i.e., the matrix $[\mathbf{a}_1(w), \mathbf{a}_2(w), \ldots, \mathbf{a}_s(w)]$ has the first ℓ rows of a left-inverse matrix of $\mathbb{G}(w)$ with $L^\infty(0,1)$ entries). As will be proved later, a necessary and sufficient condition for the existence of such a matrix (not necessarily unique) is that $\alpha_{\mathbb{G}} > 0$. If we left multiply (9) by this matrix, we get

$$\begin{aligned}\mathbf{F}(w) &= r\big[\mathbf{a}_1(w), \ldots, \mathbf{a}_s(w)\big]\Big[\sum_{n\in\mathbb{Z}} \big(\mathcal{L}_1 f\big)(rn)e^{-2\pi inrw}, \ldots, \sum_{n\in\mathbb{Z}} \big(\mathcal{L}_s f\big)(rn)e^{-2\pi inrw}\Big]^\top \\ &= r\sum_{j=1}^{s} \mathbf{a}_j(w) \sum_{n\in\mathbb{Z}} (\mathcal{L}_j f\big)(rn)e^{-2\pi inrw} = r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s} (\mathcal{L}_j f\big)(rn)\mathbf{a}_j(w)e^{-2\pi inrw}\,,\end{aligned} \tag{10}$$

in the $L^2_\ell(0,1)$-sense. Using both the isomorphism $\mathcal{T}_\Phi$ and property (4) gives the following sampling formula in V_Φ: For any $f \in V_\Phi$

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \big(\mathcal{L}_j f\big)(rn)\ S_j(t-rn)\,, \quad t \in \mathbb{R}\,,$$

where the convergence is in the $L^2(\mathbb{R})$-sense, and the sampling functions are given by $S_j = r\mathcal{T}_\Phi(\mathbf{a}_j)$, $j = 1, 2, \dots, s$. As a consequence of (2), the convergence of the series is also uniform on $\mathbb{R}$. Moreover, the sequence $\{S_j(t-rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for V_Φ. In fact, the following result holds:

Theorem 1 *Assume that the functions* $\mathbf{g}_j$ *belong to* $L^\infty_\ell(0,1)$ *for* $j = 1, 2, \dots, s$. *The following statements are equivalent:*

(a) $\alpha_{\mathbb{G}} > 0$

(b) There exist $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s \in L^\infty_\ell(0,1)$ *such that*

$$[\mathbf{a}_1(w), \mathbf{a}_2(w), \dots, \mathbf{a}_s(w)]\mathbb{G}(w) = [\mathbf{I}_\ell, \mathbf{O}_{\ell\times(r-1)\ell}]\,, \qquad a.e.\ in\ (0,1)\,. \tag{11}$$

(c) There exists a frame for V_Φ *having the form* $\{S_j(\cdot - rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ *such that for any* $f \in V_\Phi$,

$$f = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \big(\mathcal{L}_j f\big)(rn)\ S_j(\cdot - rn) \quad in\ \ L^2(\mathbb{R})\,. \tag{12}$$

In case the equivalent conditions are satisfied we have that $S_j = r\mathcal{T}_\Phi(\mathbf{a}_j)$, $j = 1, 2, \dots, s$, *where* $[\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s]$ *is any solution of* (11). *The series in* (12) *also converges absolutely and uniformly on* $\mathbb{R}$.

Proof: First, we prove that (a) implies (b). As the determinant of the semipositive definite matrix $\mathbb{G}^*(w)\mathbb{G}(w)$ is equal to the product of its eigenvalues, (a) implies that $\operatorname{ess\,inf}_{w\in\mathbb{R}} \det[\mathbb{G}^*(w)\mathbb{G}(w)] > 0$. Hence, there exists the left pseudo-inverse matrix $\mathbb{G}^\dagger(w) := [\mathbb{G}^*(w)\mathbb{G}(w)]^{-1}\mathbb{G}^*(w)$, a.e. in $(0,1)$, and it satisfies $\mathbb{G}^\dagger(w)\mathbb{G}(w) = \mathbf{I}_{r\ell}$. The first ℓ rows of $\mathbb{G}^\dagger(w)$ form a $\ell \times s$ matrix $[\mathbf{a}_1(w), \dots, \mathbf{a}_s(w)]$ which satisfies (11). Moreover, the functions $\mathbf{a}_j(w)$, $j = 1, 2, \dots, s$, are essentially bounded since $\operatorname{ess\,inf}_{w\in\mathbb{R}} \det[\mathbb{G}^*(w)\mathbb{G}(w)] > 0$.

Next, we prove that (b) implies (c). Let $\mathbf{a}_j(w)$, $j = 1, 2, \dots, s$, be functions in $L^\infty_\ell(0,1)$ satisfying $[\mathbf{a}_1(w), \dots, \mathbf{a}_s(w)]\mathbb{G}(w) = [\mathbf{I}_\ell, \mathbf{O}_{(r-1)\ell\times\ell}]$. In (10) we have proved that, for each $\mathbf{F} = \mathcal{T}_\Phi^{-1}(f) \in L^2_\ell(0,1)$,

$$\mathbf{F}(w) = r \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \big\langle \mathbf{F}, \overline{\mathbf{g}}_j(\cdot)\, e^{-2\pi i r n \cdot} \big\rangle_{L^2_\ell(0,1)} \mathbf{a}_j(w) e^{-2\pi i n r w} \ \text{ in } L^2_\ell(0,1)\,. \tag{13}$$

Thus, the sequences $\{\overline{\mathbf{g}}_j(\cdot)\, e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ and $\{r\mathbf{a}_j(\cdot)\, e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ are Bessel sequences for $L^2_\ell(0,1)$ (see Lemma 2) satisfying the representation property (13). As a consequence, they are a pair of dual frames for $L^2_\ell(0,1)$ (see [13, Lemma 5.6.2]). In particular, the sequence $\{\mathbf{a}_j(\cdot)\, e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2_\ell(0,1)$. Applying the isomorphism $\mathcal{T}_\Phi$ to (13) one gets the sampling expansion (12) in V_Φ, where $S_j = r\mathcal{T}_\Phi(\mathbf{a}_j)$, $j=1,2,\dots,s$, and the sequence $\{S_j(\cdot - nr)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for V_Φ.

Finally, to prove that (c) implies (a), assume that $\{S_j(\cdot - nr)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for V_Φ such that the expansion (12) holds. Applying the isomorphism $\mathcal{T}_\Phi^{-1}$ we obtain that $\{r\mathbf{a}_j(\cdot)\, e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2_\ell(0,1)$, where $\mathbf{a}_j = (1/r)\mathcal{T}_\Phi^{-1}(S_j)$, $j = 1,2,\dots,s$, for which the expansion (13) holds. Then, reasoning as above we deduce that the sequences $\{\overline{\mathbf{g}}_j(\cdot)\, e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ and $\{r\mathbf{a}_j(\cdot)\, e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ are a pair of dual frames for $L^2_\ell(0,1)$. In particular, the sequence $\{\overline{\mathbf{g}}_j(\cdot)\, e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2_\ell(0,1)$, and Lemma 2 gives $\alpha_{\mathbb{G}} > 0$. Notice that the pointwise convergence in the sampling series is absolute due to the unconditional convergence of a frame expansion. □

If the generalized samples belongs to $\ell^1_s(\mathbb{Z}) := \ell^1(\mathbb{Z}) \times \ldots \times \ell^1(\mathbb{Z})$ (s times), the functions $\mathbf{g}_j$, $j=1,2,\dots,s$, are continuous on $\mathbb{R}$ and the following Corollary holds:

Corollary 1 *Assume that the functions $\mathbf{g}_j$, $j=1,2,\dots,s$, are continuous on $\mathbb{R}$. Then, the following assertions are equivalents:*

(a) rank $\mathbb{G}(w) = r\ell$ for all $w \in \mathbb{R}$.

(b) There exists a frame $\{S_j(\cdot - rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ for V_Φ satisfying the sampling formula (12).

Proof: Whenever the functions $\mathbf{g}_j$, $j=1,2,\dots,s$, are continuous on $\mathbb{R}$, the condition $\alpha_{\mathbb{G}} > 0$ is equivalent to $\det\big[\mathbb{G}^*(w)\mathbb{G}(w)\big] \neq 0$ for all $w\in\mathbb{R}$. Indeed, if $\det \mathbb{G}^*(w)\mathbb{G}(w) > 0$ then the ℓ first rows of the matrix $\mathbb{G}^\dagger(w) := [\mathbb{G}^*(w)\mathbb{G}(w)]^{-1}\mathbb{G}^*(w)$, gives a $\ell \times s$ matrix $[\mathbf{a}_1, \dots, \mathbf{a}_s]$ satisfying the statement (b) in Theorem 1, and therefore $\alpha_{\mathbb{G}} > 0$. The reciprocal follows from the fact that $\det\big[\mathbb{G}^*(w)\mathbb{G}(w)\big] \geq \alpha_{\mathbb{G}}^{r\ell}$ for all $w\in\mathbb{R}$. Since, $\det\big[\mathbb{G}^*(w)\mathbb{G}(w)\big] \neq 0$ is equivalent to rank $\mathbb{G}(w) = r\ell$ for all $w \in \mathbb{R}$, the result is a consequence of Theorem 1. □

The reconstruction functions S_j, $j=1,2,\dots,s$, can be determined from the Fourier coefficients of the components of $\mathbf{a}_j$. Specifically, if we denote by $\mathsf{a}_{j,k}$ the k component of $\mathbf{a}_j$ and by $a_{j,k,n}$ its Fourier coefficients, i.e., $\mathsf{a}_{j,k}(w) = \sum_{n\in\mathbb{Z}} a_{j,k,n} e^{-2\pi i n w}$, then (see (3))

$$S_j(t) = r \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\ell} a_{j,k,n}\varphi_k(t-n)\,, \quad t\in\mathbb{R}\,. \tag{14}$$

The reconstruction function S_j can be also determined from its Fourier transform $\widehat{S}_j$. In fact, taking the Fourier transform in (14), we obtain $\widehat{S}_j(w) = r\sum_{k=1}^{\ell} \mathsf{a}_{j,k}(w)\widehat{\varphi}_k(w)$.

Notice that if $[\mathbf{a}_1(w), \mathbf{a}_2(w), \dots, \mathbf{a}_s(w)]\mathbb{G}(w) = [\mathbf{I}_\ell, \mathbf{O}_{\ell\times(r-1)\ell}]$ then, by periodicity, the transpose of the associated matrix (see (7)) $\mathbb{A}^\top$ is a left inverse of $\mathbb{G}$, i.e., $\mathbb{A}^\top(w)\mathbb{G}(w) = \mathbf{I}_{r\ell}$. Hence, the functions $\mathbf{a}_1(w), \dots, \mathbf{a}_s(w) \in L^\infty_\ell(0,1)$ satisfying (11) are the column vectors from the first ℓ rows of the left inverse matrices of $\mathbb{G}$ with entries in $L^\infty(0,1)$. One of such matrices is precisely the left pseudo-inverse

$$\mathbb{G}^\dagger(w) := [\mathbb{G}^*(w)\mathbb{G}(w)]^{-1}\mathbb{G}^*(w).$$

As a by-product, it can be proved that the $\ell \times s$ matrix $[\mathbf{a}_1^\dagger(w), \mathbf{a}_2^\dagger(w), \dots, \mathbf{a}_s^\dagger(w)]$ obtained by taking the ℓ first rows of $\mathbb{G}^\dagger(w)$ gives precisely the canonical dual frame $\{r\mathbf{a}_j^\dagger(\cdot)e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ of the frame $\{\overline{\mathbf{g}}_j(\cdot)e^{-2\pi i r n\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$. Indeed, the frame operator $\mathcal{S}$ associated to $\{\overline{\mathbf{g}}_j(\cdot)e^{-2\pi i r n\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is given by

$$\mathcal{S}\mathbf{F}(w) = \frac{1}{r}\big[\overline{\mathbf{g}}_1(w), \overline{\mathbf{g}}_2(w), \dots, \overline{\mathbf{g}}_s(w)\big]\mathbb{G}(w)\mathbb{F}(w)\,, \quad \mathbf{F} \in L^2_\ell(0,1)\,,$$

from which one gets

$$\mathcal{S}\big[r\mathbf{a}_j^\dagger(\cdot)e^{-2\pi i r n\cdot}\big](w) = \overline{\mathbf{g}}_j(w)e^{-2\pi i r n w}\,, \quad j = 1, 2, \dots, s\,.$$

Moreover, it is easy to check that any left inverse of the matrix $\mathbb{G}$ with entries in $L^\infty(0,1)$ is given by $\mathbb{G}^\dagger(w) + \mathbf{U}(w)[\mathbf{I}_s - \mathbb{G}(w)\mathbb{G}^\dagger(w)]$, where $\mathbf{U}(w)$ is a $r\ell \times s$ matrix function with entries in $L^\infty(0,1)$.

Something more can be said in the case where $s = r\ell$:

Theorem 2 *Assume that the functions $\mathbf{g}_j$ belong to $L^\infty_\ell(0,1)$ for $j = 1, 2, \dots, s$ and $s = r\ell$. The following statements are equivalent:*

(a) $\alpha_{\mathbb{G}} > 0$

(b) There exists a Riesz basis $\{S_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ for V_Φ such that for any $f \in V_\Phi$, the expansion

$$f = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\big(\mathcal{L}_j f\big)(rn)\ S_{j,n} \quad \text{in} \ \ L^2(\mathbb{R})\,, \tag{15}$$

holds.

In case the equivalent conditions are satisfied, necessarily $S_{j,n}(t) = S_j(t - rn)\,,\ t \in \mathbb{R}$ where $S_j = r\mathcal{T}_\Phi(\mathbf{a}_j)$, $j = 1, 2, \dots, s$, and $[\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s]$ is the $\ell \times s$ matrix formed with the ℓ first rows of $\mathbb{G}^{-1}$. The sampling functions S_j, $j = 1, 2, \dots, s$, satisfy the interpolation property $(\mathcal{L}_{j'}S_j)(rn) = \delta_{j,j'}\delta_{n,0}$, where $j, j' = 1, 2, \dots, s$ and $n \in \mathbb{Z}$.

Proof: Assume that $\alpha_{\mathbb{G}} > 0$; since $\mathbb{G}(w)$ is a square matrix, this implies that $\operatorname{ess\,inf}_{w\in\mathbb{R}} |\det \mathbb{G}(w)| > 0$. Therefore, the ℓ first rows of $\mathbb{G}^{-1}(w)$ gives a solution of the equation $[\mathbf{a}_1(w), \mathbf{a}_2(w), \dots, \mathbf{a}_s(w)]\mathbb{G}(w) = [\mathbf{I}_\ell, \mathbf{O}_{\ell\times(r-1)\ell}]$ with $\mathbf{a}_j \in L^\infty_\ell(0,1)$ for $j = 1, 2, \dots, s$. According to Theorem 1, the sequence $\{S_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s} := \{S_j(t-sn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$, where $S_j = r\mathcal{T}_\Phi(\mathbf{a}_j)$, satisfies the sampling formula (15). Moreover $\{r\mathbf{a}_j(w)e^{-2\pi insw}\}_{n\in\mathbb{Z}, j=1,2,\dots,s} = \{\mathcal{T}_\Phi^{-1} S_j(\cdot - ns)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2_\ell(0,1)$. Since $r\ell = s$, according to Lemma 2 it implies that it is a Riesz basis. Hence, $\{S_j(t-nr)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a Riesz basis for V_Φ and (b) is proved.

Conversely, assume now that $\{S_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a Riesz basis for V_Φ satisfying (15). From the uniqueness of the coefficients in a Riesz basis, we get that the interpolatory condition $(\mathcal{L}_{j'}S_{j,n})(rn') = \delta_{j,j'}\delta_{n,n'}$ holds for $j, j' = 1, 2, \dots, s$ and $n, n' \in \mathbb{Z}$. Since $\mathcal{T}_\Phi^{-1}$ is an isomorphism, $\{\mathcal{T}_\Phi^{-1} S_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a Riesz basis for $L^2_\ell(0,1)$. Expanding the function $\overline{\mathbf{g}}_{j'}(w)e^{-2\pi n'sw}$ with respect to the dual basis of $\{\mathcal{T}_\Phi^{-1} S_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$, denoted by $\{G_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$, we obtain

$$\begin{aligned}\overline{\mathbf{g}}_{j'}(w)e^{-2\pi n'sw} &= \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s} \langle \overline{\mathbf{g}}_{j'}(\cdot)e^{-2\pi n's\cdot}, \mathcal{T}_\Phi^{-1} S_{j,n}\rangle_{L^2(0,1)} G_{j,n}(w) \\ &= \sum_{n\in\mathbb{Z}} \overline{\mathcal{L}_{j'}S_{j,n}}(sn') G_{j,n}(w) = G_{j',n'}(w)\,.\end{aligned}$$

Therefore, the sequence $\{\overline{\mathbf{g}}_j(w)e^{-2\pi nsw}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is the dual basis of the Riesz basis $\{\mathcal{T}_\Phi^{-1} S_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$. In particular it is a Riesz basis for $L^2_\ell(0,1)$, which implies, according to Lemma 2, that $\alpha_{\mathbb{G}} > 0$. This proves (a). Moreover, the sequence $\{\mathcal{T}_\Phi^{-1} S_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is necessarily the unique dual basis of the Riesz basis $\{\overline{\mathbf{g}}_j(w)e^{-2\pi nsw}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$. Therefore, this proves the uniqueness of the Riesz basis $\{S_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ for V_Φ satisfying (15). □

5 The case where the Gramian $G_{\mathbf{\Phi}}$ is only bounded

The sampling and reconstruction problem in the shift-invariant space V_Φ has been dealt in [6] for $r = 1$ under the general assumption that the Gramian $G_{\mathbf{\Phi}}$ must be bounded, i.e., $G_{\mathbf{\Phi}} \le M\mathbf{I}_\ell$ a.e. in $(0,1)$. An equivalent condition on the Gramian is that its components belong to $L^\infty(0,1)$. The results in [6] are given in terms of the Gramian $G_{\mathbf{\Phi}}$ and the $\ell\times\ell$ matrix-function A_Φ^{h} defined by

$$A_\Phi^{\mathsf{h}}(w) := \sum_{j=1}^{s} \left(\sum_{n\in\mathbb{Z}} \widehat{\Phi}(w+n)\widehat{\mathsf{h}}_j(w+n)\right)\overline{\left(\sum_{n\in\mathbb{Z}} \widehat{\Phi}(w+n)\widehat{\mathsf{h}}_j(w+n)\right)}^\top,$$

assuming that the $s\times s$ Gramian matrix-function G_{h} has also bounded entries. Here, h denotes the convolutor vector $\mathsf{h} := (\mathsf{h}_1, \mathsf{h}_2, \dots, \mathsf{h}_s)^\top$.

First of all, it would be worth to express the results in Section 4 of the present paper in the light of those in [6]. To this end, we identify the matrix $A_\Phi^{\mathsf{h}}(w)$ in our context. Indeed, for the sampling period $r = 1$, we have the $s \times \ell$ matrix $\mathbb{G}(w) = [\mathbf{g}_1(w), \mathbf{g}_2(w), \dots, \mathbf{g}_s(w)]^\top$ and, as a consequence, the $\ell \times \ell$ matrix

$$\mathbb{G}^*(w)\mathbb{G}(w) = [\overline{\mathbf{g}_1(w)}, \overline{\mathbf{g}_2(w)}, \dots, \overline{\mathbf{g}_s(w)}][\mathbf{g}_1(w), \mathbf{g}_2(w), \dots, \mathbf{g}_s(w)]^\top = \sum_{j=1}^{s} \overline{\mathbf{g}_j(w)}\mathbf{g}_j(w)^\top .$$

Under our assumptions it is easy to check, by using the Poisson summation formula, that

$$\mathbf{g}_j(w) = \sum_{n\in\mathbb{Z}} \mathcal{L}_j\Phi(n)e^{-2\pi inw} = \sum_{n\in\mathbb{Z}} \widehat{\mathcal{L}_j\Phi}(w+n) = \sum_{n\in\mathbb{Z}} \widehat{\Phi}(w+n)\widehat{\mathsf{h}}_j(w+n) .$$

Hence, we finally obtain that

$$A_\Phi^{\mathsf{h}}(w) = \sum_{j=1}^{s} \mathbf{g}_j(w)\overline{\mathbf{g}_j(w)}^\top = \overline{\mathbb{G}^*(w)}\,\overline{\mathbb{G}(w)} ,$$

i.e., $A_\Phi^{\mathsf{h}}(w)$ is the conjugate matrix of $\mathbb{G}^*(w)\mathbb{G}(w)$.

Having in mind that, in our approach, the operator $\mathcal{T}_\Phi : L_\ell^2(0,1) \to V_\Phi$ is an isomorphism, we have the following:

1. The convolutor vector $\mathsf{h} = (\mathsf{h}_1, \mathsf{h}_2, \dots, \mathsf{h}_s)^\top$ is a *stable uniform averaging sampler* for V_Φ, i.e., there exist two positive constants A, B such that

   $$A\|f\|^2 \le \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} |\mathcal{L}_j f(n)|^2 \le B\|f\|^2 \quad \text{for all } f \in V_\Phi ,$$

   if and only if the sequence $\{\overline{\mathbf{g}}_j(w)e^{-2\pi inw}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L_\ell^2(0,1)$. This is equivalent to the condition $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$ which, in terms of the matrix A_Φ^{h}, reads

   $$\alpha_{\mathbb{G}}\, \mathbf{I}_\ell \le A_\Phi^{\mathsf{h}}(w) \le \beta_{\mathbb{G}}\, \mathbf{I}_\ell \quad \text{a.e. } w \in (0,1) .$$

2. The convolutor vector $\mathsf{h} = (\mathsf{h}_1, \mathsf{h}_2, \dots, \mathsf{h}_s)^\top$ is a *determining uniform averaging sampler* for V_Φ, i.e., the functions in V_Φ are uniquely determined by their generalized samples, if and only if

   $$\text{rank } \mathbb{G}(w) = \text{rank } A_\Phi^{\mathsf{h}}(w) = \ell \quad \text{a.e. } w \in (0,1) .$$

3. Finally, in case that the functions $\mathbf{g}_j$, $j = 1, 2, \dots, s$, are continuous on $\mathbb{R}$, the convolutor vector $\mathsf{h} = (\mathsf{h}_1, \mathsf{h}_2, \dots, \mathsf{h}_s)^\top$ is a stable uniform averaging sampler for V_Φ if and only if

   $$\text{rank } \mathbb{G}(w) = \text{rank } A_\Phi^{\mathsf{h}}(w) = \ell \quad \text{for all } w \in \mathbb{R} .$$

A more interesting question is to obtain stable reconstruction formulas in V_Φ (with sampling period $r = 1$) whenever the Gramian $G_{\mathbf{\Phi}}$ is only bounded as in [6]. A simple calculation shows that there exists a positive constant K (depending on the bounds of the entries in $G_{\mathbf{\Phi}}$) such that

$$\Big\| \sum_{k=1}^{\ell} \sum_{n\in\mathbb{Z}} d_k(n)\ \varphi_k(t-n) \Big\|_2^2 \le K \Big(\sum_{k=1}^{\ell} \|d_k\|_2^2 \Big) .$$

Therefore, the shift-invariant space V_Φ defined as (4) is a subspace (not necessarily closed) of $L^2(\mathbb{R})$, and $\mathcal{T}_\Phi : L^2_\ell(0,1) \to V_\Phi$ defines, in this new setting, a bounded surjective operator. Consider s systems $\mathcal{L}_j$ of the type (a) with impulse responses $\mathsf{h}_j \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $j = 1, 2, \dots, s$.

Assuming that $\sum_{n\in\mathbb{Z}} |\Phi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$, as in Section 3 one can prove that, given $f \in V_\Phi$ there exists a function $\mathbf{F} \in L^2_\ell(0,1)$ such that $\mathcal{T}_\Phi \mathbf{F} = f$ and

$$\big(\mathcal{L}_j f\big)(t) = \big\langle \mathbf{F}, \overline{(\mathbf{Z}\mathcal{L}_j\Phi)}(t,\cdot)\big\rangle_{L^2_\ell(0,1)} , \quad t \in \mathbb{R} ,$$

for each $j = 1, 2, \dots, s$. In particular, we have $(\mathcal{L}_j f)(n) = \big\langle \mathbf{F}, \overline{\mathbf{g}}_j(w) e^{-2\pi i n w} \big\rangle_{L^2_\ell(0,1)}$, $n \in \mathbb{Z}$, where $\mathbf{g}_j(w) = (\mathbf{Z}\mathcal{L}_j\Phi)(0,w)$, $j = 1, 2, \dots, s$.

Assuming that the sequence $\{\overline{\mathbf{g}}_j(w) e^{-2\pi i n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2_\ell(0,1)$, i.e., $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$ where $\mathbb{G}$ is the $s \times \ell$ matrix $\mathbb{G}(w) = [\mathbf{g}_1(w), \mathbf{g}_2(w), \dots, \mathbf{g}_s(w)]^\top$. According to Theorem 1 we can find a sequence $\{\mathbf{a}_j(w) e^{-2\pi i n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ in $L^2_\ell(0,1)$, where

$$[\mathbf{a}_1(w), \mathbf{a}_2(w), \dots, \mathbf{a}_s(w)]\mathbb{G}(w) = \mathbf{I}_\ell \quad \text{a.e. } w \in (0,1) ,$$

such that

$$\mathbf{F} = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \big(\mathcal{L}_j f\big)(n) \mathbf{a}_j(w) e^{-2\pi i n w} \quad \text{in } L^2_\ell(0,1) ,$$

i.e., a dual frame for $\{\overline{\mathbf{g}}_j(w) e^{-2\pi i n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$. Hence, the operator $\mathcal{T}_\Phi$ gives the reconstruction formula

$$f = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \big(\mathcal{L}_j f\big)(n) S_j(\cdot - n) \quad \text{in } V_\Phi , \tag{16}$$

where $S_j = \mathcal{T}_\Phi \mathbf{a}_j$, $j = 1, 2, \dots, s$.

Concerning the stability of the reconstruction formula (16) it is easy to check that $\dfrac{\alpha_{\mathbb{G}}}{M} G_\Phi(w) \le A^{\mathsf{h}}_\Phi(w)$ a.e. $w \in (0,1)$. Assuming that the Gramian G_{h} is also bounded, Theorem 4 in [6] asserts that the convolutor vector $\mathsf{h} = (\mathsf{h}_1, \mathsf{h}_2, \dots, \mathsf{h}_s)^\top$ is a stable uniform averaging sampler for V_Φ.

6 An application: Oversampling and reconstruction functions with compact support

In the oversampling setting, i.e., $s > r\ell$, Theorem 1 allows us different choices for the functions $\mathbf{a}_1(w), \dots, \mathbf{a}_s(w)$ and consequently, different reconstruction functions S_j. One may use this flexibility in order to obtain appropriate sampling functions S_j. For instance, if the generators $\varphi_1, \dots, \varphi_\ell$ and the impulse responses of the systems $\mathcal{L}_j$ have compact support, in general, we can choose $\mathbf{a}_1(w), \dots, \mathbf{a}_s(w)$ in order to obtain sampling functions S_j with compact support. We illustrate this assertion with the following example.

We consider the recovery of a cubic spline f in $C^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with knots at the integers from its averages

$$\mathcal{L}f(n/3) = \int_{n/3}^{n/3+1/3} f(u)du\,, \quad n \in \mathbb{Z}\,.$$

To this end, we take as generators the Hermite cubic splines

$$\varphi_1(t) = \begin{cases} (t+1)^2(1-2t), & t \in [-1,0] \\ (1-t)^2(1+2t), & t \in [0,1] \\ 0, & |t| > 1 \end{cases} \quad \text{and} \quad \varphi_2(t) = \begin{cases} (t+1)^2 t, & t \in [-1,0] \\ (1-t)^2 t, & t \in [0,1] \\ 0, & |t| > 1 \end{cases}$$

which are stable generators for the space V_{φ_1,φ_2} (see [14]), the sampling period $r = 1$ and the systems defined by

$$\mathcal{L}_1 f(t) := \mathcal{L}f(t) := \int_t^{t+1/3} f(u)du\,, \quad \mathcal{L}_2 f(t) := \mathcal{L}f\Big(t + \frac{1}{3}\Big), \quad \mathcal{L}_3 f(t) := \mathcal{L}f\Big(t + \frac{2}{3}\Big)$$

Denoting $z := e^{-2\pi i w}$ we get the matrix

$$\mathsf{G}(z) := \mathbb{G}(w) = \begin{pmatrix} \frac{5}{162}z^{-1} + \frac{49}{162} & \frac{-1}{108}z^{-1} + \frac{11}{324} \\ \frac{1}{6}z^{-1} + \frac{1}{6} & \frac{-13}{324}z^{-1} + \frac{13}{324} \\ \frac{49}{162}z^{-1} + \frac{5}{162} & \frac{-11}{324}z^{-1} + \frac{1}{108} \end{pmatrix}$$

If we find vectors $\mathbf{a}_1(w)$, $\mathbf{a}_2(w)$ and $\mathbf{a}_3(w)$ whose components $\mathsf{a}_{j,k}$ are trigonometric polynomials, i.e., Laurent polynomials in z satisfying $[\mathbf{a}_1(w), \mathbf{a}_2(w), \mathbf{a}_3(w)]\mathbb{G}(w) = \mathbf{I}_2$, the corresponding solution yields reconstruction functions of compact support (see (14)). Trying a solution where the component of $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ are polynomials in z of degree one we arrive to a linear system of 12 equations with 12 unknowns. Solving this system we get the following sampling functions:

$$
\begin{aligned}
S_1(t) &:= \frac{85}{44}\varphi_1(t) + \frac{1}{11}\varphi_1(t-1) + \frac{85}{4}\varphi_2(t) - \varphi_2(t-1) \\
S_2(t) &:= \frac{-23}{44}\varphi_1(t) - \frac{23}{44}\varphi_1(t-1) - \frac{23}{4}\varphi_2(t) + \frac{23}{4}\varphi_2(t-1) \\
S_3(t) &:= \frac{1}{11}\varphi_1(t) + \frac{85}{44}\varphi_1(t-1) + \varphi_2(t) - \frac{85}{4}\varphi_2(t-1)\,, \quad t \in \mathbb{R}\,.
\end{aligned}
$$

The associated sampling formula for $f \in V_{\varphi_1,\varphi_2}$ reads:

$$
f(t) = \sum_{n\in\mathbb{Z}} \Big[\mathcal{L}f(n)S_1(t-n) + \mathcal{L}f\Big(n+\frac{1}{3}\Big)S_2(t-n) + \mathcal{L}f\Big(n+\frac{2}{3}\Big)S_3(t-n)\Big]\,, \quad t \in \mathbb{R}\,,
$$

absolutely and uniformly on $\mathbb{R}$.

The method exhibited in the above example generally applies when $s > r\ell$ and the generators have compact support. Specifically, this method can be applied, provided that the generators have compact support and that the matrix $\mathsf{G}(z)$ is a Laurent polynomial matrix whose canonical Smith form (see [11]) has only monomials in its diagonal. Indeed, Cvetković and Vetterli [11] proved that a Laurent polynomial matrix G has a Laurent polynomial left inverse if and only if the Smith form of G has only monomials in its diagonal.

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