ALIASING ERROR OF SAMPLING SERIES IN WAVELET SUBSPACES

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Abstract. Alising error arises whenever a sampling formula, valid for a prescribed space, is applied to a function in a bigger space. In this work we estimate the aliasing error of classical and average sampling expansions in wavelet subspaces of a multiresolution analysis.

Key words : sampling, wavelet subspace, aliasing error

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1. Introduction

The Shannon sampling theorem states that any function f in the classical Paley-Wiener space $PW_\pi := \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \text{supp}\widehat{f} \subseteq [-\pi, \pi]\}$, where $\widehat{f}$ stands for the Fourier transform of f, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\text{sinc}(t-n), \quad t \in \mathbb{R}\,,$$

where $\text{sinc}(t) = \sin \pi t/\pi t$ denotes the cardinal sinc function. The space PW_π can be seen as the subspace V_0 of the Shannon multiresolution analysis, whose scaling function is precisely $\phi(t) = \text{sinc}(t)$. Although Shannon's sampling theory has had an enormous impact, it has a number of drawbacks, as pointed out by Unser in [11]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of finite duration signal; the bandlimiting operation generates Gibbs oscillations, and finally, the sinc function has a very slow decay, which makes computation in the signal domain very inefficient. Moreover, many applied problems impose different a priori constraints on the type of functions. For these reasons, the sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces.

In [12], Walter extended, under appropriate hypotheses, the Shannon sampling theorem to the subspace V_0 of a general multiresolution analysis $\{V_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$: For any $f \in V_0$ the sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n), \quad t \in \mathbb{R}$$

holds, where $\widehat{S}(\xi) := \widehat{\phi}(\xi)/(\sum_{n\in\mathbb{Z}} \phi(n)e^{-in\xi})$ and ϕ denotes the scaling function. Later on, Unser and Aldroubi introduced in [10], under suitable conditions, the average sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} (\mathcal{L}f)(n)\ S_{\mathcal{L}}(t-n)\,, \quad t \in \mathbb{R}\,,$$

which uses the average samples $\{(\mathcal{L}f)(n)\}_{n\in\mathbb{Z}}$ obtained from $f \in V_0$ by means of a linear time-invariant system $\mathcal{L}f := f * \mathsf{h}$ defined on V_0. Notice that, in practice, the measurements of a function f in V_0 are taken not from the function itself but from some filtered version $\mathcal{L}f$.

Whenever these sampling formulas are applied to a function f which does not belong to V_0, the so-called aliasing error arises:

$$E^A f(t) := f(t) - \sum_{n\in\mathbb{Z}} f(n)S(t-n) \quad\text{or}\quad E^A_{\mathcal{L}} f(t) := f(t) - \sum_{n\in\mathbb{Z}} (\mathcal{L}f)(n)S_{\mathcal{L}}(t-n)\,, \quad t \in \mathbb{R}\,.$$

Concerning this error in Shannon's setting, a classic result by Brown [1] states that if $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then

$$\Big| f(t) - \sum_{n\in\mathbb{Z}} f(n)\operatorname{sinc}(t-n)\Big| \le \frac{2}{\sqrt{2\pi}} \int_{|\xi|>\pi} |\widehat{f}(\xi)|d\xi\,, \quad t \in \mathbb{R}\,. \tag{1.1}$$

In addition, the function $f(t) = \operatorname{sinc}(2t-1)$ is an extremal solution for (1.1), i.e., there exists a value of t for which (1.1) becomes an equality. Notice that if $f \in V_1 = PW_{2\pi}$, then (1.1) can be written as

$$|E^A f(t)| \le \frac{2}{\sqrt{2\pi}} \|P_{W_0} f\|_{L^1(\mathbb{R})}\,,$$

where P_{W_0} denotes the orthogonal projection onto W_0, the orthogonal complement of V_0 in V_1. The aliasing error in Shannon's setting has been largely studied: see [7] and references therein. Besides, Walter [12] has proved a similar result for functions in the subspace V_1 of a general multiresolution analysis. Specifically, for any $f \in V_1$, there exists a constant C such that

$$|E^A f(t)| \le C \|P_{W_0} f\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}\,. \tag{1.2}$$

Notice that $\|P_{W_0} f\|_{L^2(\mathbb{R})}$ can be expressed in terms of the wavelet coefficients of the function f.

On the other hand, Janssen generalized in [9] Walter's sampling formula by using shifted samples $\{f(n+\sigma)\}_{n\in\mathbb{Z}}$, where $\sigma \in [0,1)$. As to the corresponding aliasing error $E^A f$, he proved the inequalities

$$K_0 \|P_{W_0} f\|_{L^2(\mathbb{R})} \le \|E^A f\|_{L^2(\mathbb{R})} \le K_\infty \|P_{W_0} f\|_{L^2(\mathbb{R})}, \quad f \in V_1\,.$$

In addition, he found the smallest possible value for the constant K_0 and the largest possible value for K_∞. Later on, in [5] the authors dealt with the aliasing error function $E^A f$ for $f \in V_1$. In so doing, they calculate its Fourier transform, $\widehat{E^A f}$, in terms of the Fourier transform of $P_{W_0} f$. Besides recovering Janssen's inequalities, this technique also allows to derive a precise bound like (1.2), exhibiting the extremal solutions in some cases. Some results concerning the aliasing error for functions $f \in V_2$ are also provided. See also references [10] and [13] for the general wavelet setting.

In the present paper we study the aliasing error arising when the classical sampling formula is applied to a function f in the wavelet subspace V_n, $n \ge 1$, of a multiresolution analysis. Estimations both in L^2 and L^∞ norms are provided. The alising error arising when we apply the average sampling formula for V_0 to a function $f \in V_1$ is also included. In particular, this paper improves the results in [5] in different directions: Apart from to work with a non necessarily orthonormal scaling function ϕ, some of the results in [5] are derived under weaker hypotheses.

The paper is organized as follows: In Section 2 the needed preliminaries are included; in particular, a variance of the Poisson summation formula used with profusion in the sequel. Section 3 is devoted to study the aliasing error in classical sampling for wavelet subspaces in a multiresolution analysis. Finally, in Section 4 the aliasing error in average sampling is carried out.

2. Preliminaries

On $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we take the Fourier transform to be normalized as

$$\mathcal{F}[\phi](\xi) = \hat{\phi}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) e^{-it\xi} dt$$

so that $\mathcal{F}[\cdot]$ becomes a unitary operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. Let

$$Z_\phi(t, \xi) := \sum_{n\in\mathbb{Z}} \phi(t+n) e^{-in\xi}$$

be the Zak transform of $\phi(t)$ in $L^2(\mathbb{R})$ (cf. [9]). We first introduce a variance of the Poisson summation formula.

Lemma 2.1. *Let $\phi \in L^2(\mathbb{R})$ be such that $\hat{\phi} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, for any $t \in \mathbb{R}$, the series $\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}$ converges absolutely in $L^1[0, 2\pi]$ and*

$$\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)} \sim \frac{1}{\sqrt{2\pi}} Z_\phi(t, \xi) \tag{2.1}$$

which means that $\frac{1}{\sqrt{2\pi}} Z_\phi(t, \xi)$ is the Fourier series expansion of $\sum_{n\in\mathbb{Z}} e^{it(\xi+2n\pi)} \hat{\phi}(\xi + 2n\pi)$. If moreover for any fixed t in $\mathbb{R}$, $\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}$ converges in $L^2[0, 2\pi]$ or equivalently $\{\phi(t + n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, then

$$\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)} = \frac{1}{\sqrt{2\pi}} Z_\phi(t, \xi) \text{ in } L^2[0, 2\pi]. \tag{2.2}$$

Proof. Since $\hat{\phi} \in L^1(\mathbb{R})$ and

$$\sum_{n\in\mathbb{Z}} \|\hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}\|_{L^1[0,2\pi]} = \sum_{n\in\mathbb{Z}} \int_0^{2\pi} |\hat{\phi}(\xi + 2n\pi)| d\xi = \int_{-\infty}^{+\infty} |\hat{\phi}(\xi)| d\xi,$$

the series $\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}$ converges absolutely in $L^1[0, 2\pi]$. Hence

$$\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)} \sim \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \langle \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}, e^{-ik\xi} \rangle_{L^2[0,2\pi]} e^{-ik\xi},$$

where

$$\langle \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}, e^{-ik\xi} \rangle_{L^2[0,2\pi]} = \int_0^{2\pi} \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)} e^{ik\xi} d\xi$$

$$= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \hat{\phi}(\xi + 2n\pi) e^{i((t+k)\xi+2n\pi))} d\xi = \int_{-\infty}^{+\infty} \hat{\phi}(\xi) e^{i(t+k)\xi} d\xi = \sqrt{2\pi}\phi(t + k).$$

Hence (2.1) holds from which the second claim follows immediately. □

Lemma 2.1 generalizes Lemma 1 in [5] and a claim in the Appendix in [2]. For any $\phi \in L^2(\mathbb{R})$, let

$$\Phi(t) := \sum_{n\in\mathbb{Z}} |\phi(t-n)|^2 \text{ and } G_\phi(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi+2n\pi)|^2.$$

Then $\Phi(t) = \Phi(t+1) \in L^1[0,1]$, $G_\phi(\xi) = G_\phi(\xi+2\pi) \in L^1[0,2\pi]$ and $\|\phi(t)\|^2_{L^2(\mathbb{R})} = \|\Phi(t)\|_{L^1[0,1]} = \|G_\phi(\xi)\|_{L^1[0,2\pi]}$. For any $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, let

$$\hat{\mathbf{c}}^*(\xi) := \sum_{n\in\mathbb{Z}} c(n)e^{-in\xi}$$

be the discrete Fourier transform of $\mathbf{c}$. Then $\hat{\mathbf{c}}^* \in L^2[0,2\pi]$ and $\int_0^{2\pi} |\hat{\mathbf{c}}^*(\xi)|^2 d\xi = 2\pi\|\mathbf{c}\|^2$ where $\|\mathbf{c}\|^2 := \sum_{n\in\mathbb{Z}} |c(n)|^2$. For any $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ and $\mathbf{d} = \{d(n)\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, the discrete convolution product of $\mathbf{c}$ and $\mathbf{d}$ is defined by

$$\mathbf{c}*\mathbf{d} = \left\{(\mathbf{c}*\mathbf{d})(n) := \sum_{k\in\mathbb{Z}} c(k)d(n-k)\right\}_{n\in\mathbb{Z}}.$$

Then $\hat{\mathbf{c}}^*\,\hat{\mathbf{d}}^*$ belongs to $L^1[0,2\pi]$ and its Fourier series is $\sum_{n\in\mathbb{Z}} (\mathbf{c}*\mathbf{d})(n)e^{-in\xi}$ so that $\mathbf{c}*\mathbf{d} \in \ell^\infty(\mathbb{Z})$ and

$$\int_0^{2\pi} |\hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi)|^2 d\xi = 2\pi\|\mathbf{c}*\mathbf{d}\|^2. \tag{2.3}$$

Lemma 2.2. *If $\phi \in L^2(\mathbb{R})$ is such that $H_\phi(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi+2n\pi)| \in L^2[0,2\pi]$, then $\phi \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $\sup_{\mathbb{R}} \Phi(t) < \infty$. In particular, $Z_\phi(t,\cdot) \in L^2[0,2\pi]$ for each t in $\mathbb{R}$.*

Proof. Since $H_\phi \in L^2[0,2\pi] \subset L^1[0,2\pi]$ and $\|H_\phi(\xi)\|_{L^1[0,2\pi]} = \|\hat{\phi}(\xi)\|_{L^1(\mathbb{R})}$, $\hat{\phi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and the series $\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi+2n\pi)e^{it(\xi+2n\pi)}$ converges in $L^2[0,2\pi]$. Hence we have by Lemma 2.1

$$\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi+2n\pi)e^{it(\xi+2n\pi)} = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \phi(t+n)e^{-in\xi} \text{ in } L^2[0,2\pi]$$

so that

$$\Phi(t) = \sum_{n\in\mathbb{Z}} |\phi(t+n)|^2 = \|\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi+2n\pi)e^{it(\xi+2n\pi)}\|^2_{L^2[0,2\pi]} \le \|H_\phi\|^2_{L^2[0,2\pi]}.$$

Hence, $\sup_{\mathbb{R}} \Phi(t) \le \|H_\phi\|^2_{L^2[0,2\pi]} < \infty$ and so $Z_\phi(t,\cdot) \in L^2[0,2\pi]$, $t \in \mathbb{R}$. □

Finally in this section, let us recall the following sampling theorem (cf. [8, 9, 14]), which extends the classical Shannon sampling theorem in the Paley-Wiener space to the general shift invariant space. For any measurable function f on $\mathbb{R}$, let

$$\|f\|_0 := \sup_{|E|=0} \inf_{\mathbb{R}\setminus E} |f(t)| \text{ and } \|f\|_\infty := \inf_{|E|=0} \sup_{\mathbb{R}\setminus E} |f(t)|$$

be the essential infimum and the essential supremum of $|f|$ respectively where $|E|$ is the Lebesgue measure of E.

Proposition 2.3. *Let V_0 be the closed subspace of $L^2(\mathbb{R})$ of which $\{\phi(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis. Assume $\phi \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $\sup_{\mathbb{R}} \Phi(t) < \infty$. Then*

$$0 < \|Z_\phi(\sigma,\xi)\|_0 \leq \|Z_\phi(\sigma,\xi)\|_\infty < \infty \tag{2.4}$$

holds for some σ in $[0,1)$ if and only if there is a Riesz basis $\{S_\sigma(t-n) : n \in \mathbb{Z}\}$ of V_0 such that for each $f \in V_0$

$$f = \sum_{n\in\mathbb{Z}} f(\sigma+n) S_\sigma(\cdot - n) \quad \text{in } V_0\,. \tag{2.5}$$

Moreover, in this case, we have

$$\hat{S}_\sigma(\xi) = Z_\phi(\sigma,\xi)^{-1}\hat{\phi}(\xi) \ \text{a.e. on } \mathbb{R}. \tag{2.6}$$

Note that in Proposition 2.3, any function in V_0 is of the form

$$f(t) = (\mathbf{c} * \phi)(t) := \sum_{n\in\mathbb{Z}} c(n)\phi(t-n)$$

for some $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$ to a continuous function and V_0 is a reproducing kernel Hilbert space. Let $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ be the Fourier coefficients of $Z_\phi(\sigma,\cdot)^{-1} \in L^\infty[0,2\pi]$ so that $Z_\phi(\sigma,\xi)^{-1} = \sum_{n\in\mathbb{Z}} c(n)e^{-in\xi}$. Then $S_\sigma(t) = (\mathbf{c} * \phi)(t)$ and so we have by (2.3)

$$\sum_{n\in\mathbb{Z}} |S_\sigma(t+n)|^2 = \sum_{n\in\mathbb{Z}} |(\mathbf{c} * \phi)(t+n)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\hat{\mathbf{c}}^*(\xi) Z_\phi(t,\xi)|^2 d\xi.$$

Hence

$$\sum_{n\in\mathbb{Z}} |S_\sigma(t+n)|^2 \leq \|Z_\phi(\sigma,\xi)\|_0^{-2} \sum_{n\in\mathbb{Z}} |\phi(t+n)|^2,\ t \in \mathbb{R} \tag{2.7}$$

so that $\sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |S_\sigma(t+n)|^2 < \infty$. Therefore, the sampling series (2.5) converges not only in $L^2(\mathbb{R})$ but also absolutely and uniformly on $\mathbb{R}$.

3. Aliasing error in classical sampling

From now on let ϕ be a scaling function of a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ (cf. [3, 4]), where the V_j's are closed subspaces of $L^2(\mathbb{R})$ satisfying

- *(i)* $V_j \subset V_{j+1},\ j \in \mathbb{Z}$;
- *(ii)* $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
- *(iii)* $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1},\ j \in \mathbb{Z}$;
- *(iv)* $\{\phi(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_0.

We also assume that $H_\phi \in L^2[0,2\pi]$ and the condition (2.4) holds for some σ in $[0,1)$ so that there is a regular shifted sampling expansion (2.5) on V_0. Then $\phi \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and

$\sup_{\mathbb{R}} \Phi(t) < \infty$ by Lemma 2.2. Moreover for any $f \in V_j$ $(j \in \mathbb{Z})$, $f(2^{-j}t) = (\mathbf{c} * \phi)(t) \in V_0$ for some $\mathbf{c}$ in $\ell^2(\mathbb{Z})$ so that

$$\|\hat{f}(\xi)\|_{L^1(\mathbb{R})} = \|2^j \hat{f}(2^j\xi)\|_{L^1(\mathbb{R})} = \|\hat{\mathbf{c}}^*(\xi)\hat{\phi}(\xi)\|_{L^1(\mathbb{R})} = \|\hat{\mathbf{c}}^*(\xi)H_\phi(\xi)\|_{L^1[0,2\pi]} < \infty\,,$$

by using the Cauchy-Schwarz inequality. Hence $\hat{f} \in L^1(\mathbb{R})$ for any $f \in V_j$ $(j \in \mathbb{Z})$ so that $V_j \subset L^2(\mathbb{R}) \cap C(\mathbb{R}), j \in \mathbb{Z}$. Now for any integer $j \geq 1$, let

$$V_j(\sigma) := \{f \in V_j : \{f(\sigma + n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\}$$

and

$$E^A f(t) := f(t) - \sum_{n\in\mathbb{Z}} f(\sigma + n)S_\sigma(t - n),\ \ f \in V_j(\sigma)$$

be the aliasing error of f, which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$ to a continuous function since $\sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |S_\sigma(t - n)|^2 < \infty$. Then $E^A f \in V_j$ so that the function $\widehat{E^A f}(\xi) = \hat{f}(\xi) - Z_f(\sigma, \xi)Z_\phi(\sigma, \xi)^{-1}\hat{\phi}(\xi)$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Lemma 3.1. *For any fixed integer $j \geq 1$, if $Z_\phi(2^j\sigma, \cdot) \in L^\infty[0, 2\pi]$, then $V_j(\sigma) = V_j$. In particular, if $\sigma = 0$, then $V_j(0) = V_j$ for any $j \geq 1$.*

Proof. Assume $Z_\phi(2^j\sigma, \cdot) \in L^\infty[0, 2\pi]$. Let $f \in V_j$. Then $f(t) = \sum_{k\in\mathbb{Z}} c(k)\phi(2^jt - k)$ for some $\mathbf{c} = \{c(k)\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Hence, we have by (2.3)

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |f(\sigma + n)|^2 &= \sum_{n\in\mathbb{Z}} \Big|\sum_{k\in\mathbb{Z}} c(k)\phi(2^j\sigma + 2^jn - k)\Big|^2 \leq \sum_{n\in\mathbb{Z}} \Big|\sum_{k\in\mathbb{Z}} c(k)\phi(2^j\sigma + n - k)\Big|^2 \\ &= \frac{1}{2\pi}\|\hat{\mathbf{c}}^*(\xi)Z_\phi(2^j\sigma, \xi)\|^2_{L^2[0,2\pi]} \leq \frac{1}{2\pi}\|Z_\phi(2^j\sigma, \xi)\|^2_\infty \|\hat{\mathbf{c}}^*(\xi)\|^2_{L^2[0,2\pi]} < \infty\end{aligned}$$

so that $f \in V_j(\sigma)$. Hence $V_j = V_j(\sigma)$. If $\sigma = 0$, then $Z_\phi(0, \cdot) \in L^\infty[0, 2\pi]$ by (2.4). Hence $V_j(0) = V_j$, $j \geq 1$. □

Corollary 3.2. *If $H_\phi \in L^\infty[0, 2\pi]$, then $Z_\phi(t, \cdot) \in L^\infty[0, 2\pi]$ for any t in $\mathbb{R}$ so that $V_j(\sigma) = V_j$ for any $j \geq 1$.*

Proof. By Lemma 2.1, for any t in $\mathbb{R}$, $Z_\phi(t, \xi) = \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi)e^{it(\xi+2n\pi)}$ in $L^2[0, 2\pi]$. Hence $|Z_\phi(t, \xi)| \leq \sqrt{2\pi}H_\phi(\xi)$ a.e. on $\mathbb{R}$ so that $Z_\phi(t, \cdot) \in L^\infty[0, 2\pi]$ for any t in $\mathbb{R}$. □

We also note that if $\mathbf{c} \in \ell^2(\mathbb{Z})$ is such that $\hat{\mathbf{c}}^* \in L^\infty[0, 2\pi]$, then $f(t) = \sum_{k\in\mathbb{Z}} c(k)\phi(2^jt - k) \in V_j(\sigma)$ for any $j \geq 1$.

Let $\psi \in V_1$ be a wavelet associated to the scaling function ϕ, that is, $\{\psi(t - n) : n \in \mathbb{Z}\}$ is a Riesz basis of W_0, the orthogonal complement of V_0 in V_1. We may express any $f \in V_1$ uniquely as

$$f(t) = \sum_{n\in\mathbb{Z}} c(n)\phi(2t - n) = g(t) + h(t),$$

where $g(t) = \sum_{n\in\mathbb{Z}} a(n)\phi(t-n) \in V_0$, $h(t) = \sum_{n\in\mathbb{Z}} b(n)\psi(t-n) \in W_0$, and $\mathbf{a} = \{a(n)\}_{n\in\mathbb{Z}}$, $\mathbf{b} = \{b(n)\}_{n\in\mathbb{Z}}$, $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ are in $\ell^2(\mathbb{Z})$. Then

$$\hat{f}(\xi) = m_f\Big(\frac{\xi}{2}\Big)\hat{\phi}\Big(\frac{\xi}{2}\Big) = u_f(\xi)\hat{\phi}(\xi) + v_f(\xi)\hat{\psi}(\xi) \tag{3.1}$$

where $m_f(\xi) = \frac{1}{2}\hat{\mathbf{c}}^*(\xi)$, $u_f(\xi) = \hat{\mathbf{a}}^*(\xi)$, and $v_f(\xi) = \hat{\mathbf{b}}^*(\xi)$. In particular, m_ϕ and m_ψ are in $L^\infty[0,2\pi]$ so that ϕ and ψ belong to $V_1(\sigma)$. Note also that for any $f = g + h$ with $g \in V_0$ and $h \in W_0$, $f \in V_1(\sigma)$ if and only if $h \in V_1(\sigma)$.

Lemma 3.3. *For any $f \in V_1(\sigma)$,*

$$Z_f(\sigma,\xi) = m_f\Big(\frac{\xi}{2}\Big)Z_\phi\Big(2\sigma,\frac{\xi}{2}\Big) + m_f\Big(\frac{\xi}{2}+\pi\Big)Z_\phi\Big(2\sigma,\frac{\xi}{2}+\pi\Big) \text{ in } L^2[0,2\pi] \tag{3.2}$$

Proof. See Lemma 2 in [5]. □

In [5], (3.2) was proved under a stronger condition on ϕ, say, $H_\phi \in L^\infty[0,2\pi]$. But we can easily see that (3.2) holds also when $H_\phi \in L^2[0,2\pi]$ by using Lemma 2.1.

Proposition 3.4. *For any $f \in V_1(\sigma)$,*

$$\widehat{E^A f}(\xi) = v_f(\xi)\hat{\phi}\Big(\frac{\xi}{2}\Big)Z_\phi\Big(2\sigma,\frac{\xi}{2}+\pi\Big)Z_\phi(\sigma,\xi)^{-1}M\Big(\frac{\xi}{2}\Big) \tag{3.3}$$

where v_f is the one given in (3.1) and

$$M(\xi) := m_\psi(\xi)m_\phi(\xi+\pi) - m_\phi(\xi)m_\psi(\xi+\pi). \tag{3.4}$$

Proof. For any $f = g + h \in V_1$ with $g \in V_0$ and $h \in W_0$, we have by applying Lemma 2.1 to $Z_h(\sigma,\xi)$ and $Z_\psi(\sigma,\xi)$

$$\begin{aligned}
\widehat{E^A f}(\xi) &= \widehat{E^A h}(\xi) = \hat{h}(\xi) - Z_h(\sigma,\xi)\hat{S}(\xi) \\
&= v_f(\xi)m_\psi\Big(\frac{\xi}{2}\Big)\hat{\phi}\Big(\frac{\xi}{2}\Big) - v_f(\xi)Z_\psi(\sigma,\xi)\hat{S}(\xi) \\
&= v_f(\xi)\hat{\phi}\Big(\frac{\xi}{2}\Big)Z_\phi(\sigma,\xi)^{-1}\Big[m_\psi\Big(\frac{\xi}{2}\Big)Z_\phi(\sigma,\xi) - m_\phi\Big(\frac{\xi}{2}\Big)Z_\psi(\sigma,\xi)\Big].
\end{aligned}$$

Then (3.3) comes immediately by applying (3.2) to ϕ and ψ. □

Proposition 3.4 was proved in [5] (see Theorem 1 in [5]) assuming that ϕ is an orthonormal scaling function satisfying $\phi(t) = O((1+|t|)^{-s})$ with $s > \frac{1}{2}$ and $H_\phi \in L^\infty[0,2\pi]$. In such a case, we may take the wavelet ψ with $m_\psi(\xi) = e^{i\xi}\overline{m_\phi(\xi+\pi)}$ as in [5]. Then

$$M(\xi) = e^{i\xi}(|m_\phi(\xi)|^2 + |m_\phi(\xi+\pi)|^2) = e^{i\xi} \text{ a.e. on } \mathbb{R}$$

so that the equation (3.3) becomes the equation (13) in [5]:

$$\widehat{E^A f}(\xi) = e^{i\frac{\xi}{2}}v_f(\xi)\hat{\phi}\Big(\frac{\xi}{2}\Big)Z_\phi\Big(2\sigma,\frac{\xi}{2}+\pi\Big)Z_\phi(\sigma,\xi)^{-1}.$$

Theorem 3.5. *Assume $Z_\phi(2\sigma,\cdot) \in L^\infty[0,2\pi]$ so that $V_1(\sigma) = V_1$. Then we have for any $f \in V_1$,*

$$\begin{aligned}
|E^A f(t)| &\leq \frac{1}{\pi}\|Z_\phi(2\sigma,\xi+\pi)Z_\phi(\sigma,2\xi)^{-1}Z_\phi(2t,\xi)M(\xi)\|_{L^2[0,2\pi]}\|v_f(\xi)\|_{L^2[0,2\pi]} \\
&\leq \sqrt{\frac{2}{\pi}}\|Z_\phi(2\sigma,\xi+\pi)Z_\phi(\sigma,2\xi)^{-1}H_\phi(\xi)M(\xi)\|_{L^2[0,2\pi]}\|v_f(\xi)\|_{L^2[0,2\pi]},\ t \in \mathbb{R}. \qquad (3.5)
\end{aligned}$$

Moreover, the equality holds in the first inequality of (3.5) at $t = \sigma + k + \frac{1}{2}$ $(k \in \mathbb{Z})$ *for any* $f \in V_1$ *satisfying*

$$\overline{v_f(2\xi)} = \lambda e^{i(2k+1)\xi} Z_\phi(2\sigma, \xi) Z_\phi(2\sigma, \xi+\pi) Z_\phi(\sigma, 2\xi)^{-1} M(\xi) \ (\lambda \in \mathbb{C}). \tag{3.6}$$

Proof. Since $\widehat{E^A f} \in L^1(\mathbb{R})$, we have by (3.3) and the Poisson summation formula

$$\begin{aligned} E^A f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{E^A f}(\xi) e^{it\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_f(\xi) \hat{\phi}\Big(\frac{\xi}{2}\Big) Z_\phi\Big(2\sigma, \frac{\xi}{2} + \pi\Big) Z_\phi(\sigma, \xi)^{-1} M\Big(\frac{\xi}{2}\Big) e^{it\xi} d\xi \\ &= \frac{1}{\pi} \int_0^{2\pi} Z_\phi(2\sigma, \xi+\pi) Z_\phi(\sigma, 2\xi)^{-1} Z_\phi(2t, \xi) M(\xi) v_f(2\xi) d\xi. \end{aligned} \tag{3.7}$$

Then (3.5) follows from (3.7) since $|Z_\phi(2t, \xi)| \leq \sqrt{2\pi} H_\phi(\xi)$ a.e. on $\mathbb{R}$. Now the equality holds in the first inequality of (3.5) if and only if for some constant λ

$$\overline{v_f(2\xi)} = \lambda Z_\phi(2\sigma, \xi+\pi) Z_\phi(\sigma, 2\xi)^{-1} Z_\phi(2t, \xi) M(\xi), \tag{3.8}$$

which requires that the right hand side of (3.8) must be π-periodic. For $t = \sigma + k + \frac{1}{2}$ $(k \in \mathbb{Z})$, $Z_\phi(2\sigma + 2k + 1, \xi) = e^{i(2k+1)\xi} Z_\phi(2\sigma, \xi)$ so that (3.8) becomes (3.6), of which the right hand side is indeed π-periodic since $M(\xi + \pi) = -M(\xi)$ (cf. (3.4)). □

When ϕ is an orthonormal scaling function and $m_\psi(\xi) = e^{i\xi} \overline{m_\phi(\xi + \pi)}$ so that $M(\xi) = e^{i\xi}$ a.e. on $\mathbb{R}$, the first inequality (3.5) and the equation (3.7) become the equations (19) and (21) in [5] respectively. In [5], the extremal solutions of the inequality (3.5) were found (see Corollary 3 in [5]) when $\arg \hat{\phi}(\xi)$ is 2π-periodic. For a L^2-estimate of the aliasing error, we have:

Theorem 3.6. *Assume* $Z_\phi(2\sigma, \cdot) \in L^\infty[0, 2\pi]$ *so that* $V_1(\sigma) = V_1$*. Then we have for any* $f \in V_1$,

$$K_0 \|v_f\|^2_{L^2[0,2\pi]} \leq \|E^A(f)\|^2_{L^2(\mathbb{R})} \leq K_\infty \|v_f\|^2_{L^2[0,2\pi]} \tag{3.9}$$

where $K_0 := \|K(\xi)\|_0$, $K_\infty := \|K(\xi)\|_\infty$, *and*

$$K(\xi) := |M(\xi)|^2 |Z_\phi(\sigma, 2\xi)|^{-2} \Big[G_\phi(\xi) |Z_\phi(2\sigma, \xi+\pi)|^2 + G_\phi(\xi+\pi) |Z_\phi(2\sigma, \xi)|^2 \Big].$$

Moreover, K_0 *and* K_∞ *are the optimal constants for (3.9).*

Proof. By using (3.3) and $M(\xi + \pi) = -M(\xi)$ we have

$$\begin{aligned} &\|E^A(f)\|^2_{L^2(\mathbb{R})} = \|\widehat{E^A(f)}\|^2_{L^2(\mathbb{R})} \\ &= 2 \int_0^{2\pi} |M(\xi)|^2 |Z_\phi(\sigma, 2\xi)|^{-2} G_\phi(\xi) |Z_\phi(2\sigma, \xi+\pi)|^2 |v_f(2\xi)|^2 d\xi \\ &= 2 \int_0^{\pi} K(\xi) |v_f(2\xi)|^2 d\xi \end{aligned}$$

from which (3.9) follows. Note that $K(\xi) = K(\xi + \pi) \in L^\infty[0, \pi]$. If $K_\infty = 0$, i.e., $K(\xi) = 0$ a.e. in $[0, \pi]$, then $K_0 = K_\infty = 0$ are trivially optimal. When $K_\infty > 0$, choose any λ with $0 < \lambda < K_\infty$. Then there is a subset of D of $[0, \pi]$ with $|D| > 0$ and $K(\xi) > \lambda$ a.e. on D.

Let $f \in V_1$ be such that $\hat{f}(\xi) = v_f(\xi)\hat{\psi}(\xi)$, where $v_f(\xi) = \chi_{2D}(\xi)$ the characteristic function of $2D$. Then

$$\|E^A f\|^2_{L^2(\mathbb{R})} = 2\int_0^\pi K(\xi)|v_f(2\xi)|^2 d\xi > 2\lambda \int_0^\pi |v_f(2\xi)|^2 d\xi = \lambda \|v_f\|^2_{L^2[0,2\pi]}$$

so that K_∞ is optimal. Similarly we can see that K_0 is also optimal. □

Aliasing error estimates in Theorem 3.5 and Theorem 3.6 can also be expressed via $\|P_{W_0}f\|_{L^2(\mathbb{R})}$ where $P_{W_0}f(=h)$ is the orthogonal projection of $f \in V_1$ onto W_0. In fact, since $\{\psi(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of W_0, there are positive constants A and B such that $A \le G_\psi(\xi) \le B$ a.e. in $[0, 2\pi]$. Then we have

$$A\|v_f\|^2_{L^2[0,2\pi]} \le \|P_{W_0}f\|^2_{L^2(\mathbb{R})} \le B\|v_f\|^2_{L^2[0,2\pi]}.$$

Moreover, if $\{\psi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal basis of W_0, then $\|P_{W_0}f\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|v_f\|^2_{L^2[0,2\pi]}$.

We now estimate the aliasing error for signals in $V_j(\sigma)$ for arbitrary $j \ge 1$.

Proposition 3.7. *For any integer $j \ge 1$ and any $f \in V_j(\sigma)$*

(3.10)
$$\widehat{E^A f}(\xi) = 2^{-\frac{j-1}{2}}\hat{\phi}\Big(\frac{\xi}{2^j}\Big)\sum_{k=1}^{2^{j-1}}\Big[A_k\Big(\frac{\xi}{2^j}\Big)u_g\Big(\frac{\xi+2(k-1)\pi}{2^{j-1}}\Big) + B_k\Big(\frac{\xi}{2^j}\Big)v_g\Big(\frac{\xi+2(k-1)\pi}{2^{j-1}}\Big)\Big]$$

where $g(t) := 2^{-\frac{j-1}{2}}f(\frac{t}{2^{j-1}}) \in V_1$ so that $\hat{g}(\xi) = u_g(\xi)\hat{\phi}(\xi) + v_g(\xi)\hat{\psi}(\xi)$ (cf. (3.1)) and

$$A_k(\xi) := m_\phi(\xi)\delta_{1,k} - Z_\phi(\sigma, 2^j\xi)^{-1}Z_\phi(2^{j-1}\sigma, 2\xi + 2^{2-j}(k-1)\pi)\prod_{l=0}^{j-1} m_\phi(2^l\xi) \quad (3.11)$$

$$B_k(\xi) := m_\psi(\xi)\delta_{1,k} - Z_\phi(\sigma, 2^j\xi)^{-1}Z_\psi(2^{j-1}\sigma, 2\xi + 2^{2-j}(k-1)\pi)\prod_{l=0}^{j-1} m_\phi(2^l\xi) \quad (3.12)$$

for $1 \le k \le 2^{j-1}$.

Proof. From $g(t) = 2^{-\frac{j-1}{2}}f(\frac{t}{2^{j-1}})$ and $\hat{g}(\xi) = u_g(\xi)\hat{\phi}(\xi) + v_g(\xi)\hat{\psi}(\xi)$, we obtain

$$\begin{aligned} \hat{f}(\xi) &= 2^{-\frac{j-1}{2}}\Big[u_g\Big(\frac{\xi}{2^{j-1}}\Big)\hat{\phi}\Big(\frac{\xi}{2^{j-1}}\Big) + v_g\Big(\frac{\xi}{2^{j-1}}\Big)\hat{\psi}\Big(\frac{\xi}{2^{j-1}}\Big)\Big] \\ &= 2^{-\frac{j-1}{2}}\Big[u_g\Big(\frac{\xi}{2^{j-1}}\Big)m_\phi\Big(\frac{\xi}{2^j}\Big) + v_g\Big(\frac{\xi}{2^{j-1}}\Big)m_\psi\Big(\frac{\xi}{2^j}\Big)\Big]\hat{\phi}\Big(\frac{\xi}{2^j}\Big). \end{aligned} \quad (3.13)$$

We then have by Lemma 2.1

$$
\begin{aligned}
(3.14)\; Z_f(\sigma,\xi) &= \sqrt{2\pi}\sum_{n\in\mathbb{Z}} \hat{f}(\xi+2n\pi)e^{i\sigma(\xi+2n\pi)} \\
&= 2^{-\frac{j-1}{2}}\sqrt{2\pi}\sum_{n\in\mathbb{Z}}\Big[u_g\Big(\frac{\xi+2n\pi}{2^{j-1}}\Big)\hat{\phi}\Big(\frac{\xi+2n\pi}{2^{j-1}}\Big)+v_g\Big(\frac{\xi+2n\pi}{2^{j-1}}\Big)\hat{\psi}\Big(\frac{\xi+2n\pi}{2^{j-1}}\Big)\Big]e^{i\sigma(\xi+2n\pi)} \\
&= 2^{-\frac{j-1}{2}}\sqrt{2\pi}\sum_{k=1}^{2^{j-1}}\Big[u_g\Big(\frac{\xi+2(k-1)\pi}{2^{j-1}}\Big)\sum_{n\in\mathbb{Z}}\hat{\phi}\Big(\frac{\xi+2(k-1)\pi}{2^{j-1}}+2n\pi\Big) \\
&\qquad + v_g\Big(\frac{\xi+2(k-1)\pi}{2^{j-1}}\Big)\sum_{n\in\mathbb{Z}}\hat{\psi}\Big(\frac{\xi+2(k-1)\pi}{2^{j-1}}+2n\pi\Big)\Big]e^{i2^{j-1}\sigma(\frac{\xi+2(k-1)\pi}{2^{j-1}}+2n\pi)} \\
&= 2^{-\frac{j-1}{2}}\sum_{k=1}^{2^{j-1}}\Big[u_g\Big(\frac{\xi+2(k-1)\pi}{2^{j-1}}\Big)Z_\phi\Big(2^{j-1}\sigma,\frac{\xi+2(k-1)\pi}{2^{j-1}}\Big) \\
&\qquad + v_g\Big(\frac{\xi+2(k-1)\pi}{2^{j-1}}\Big)Z_\psi\Big(2^{j-1}\sigma,\frac{\xi+2(k-1)\pi}{2^{j-1}}\Big)\Big].
\end{aligned}
$$

Therefore we obtain (3.10) together with (3.11) and (3.12) by applying (3.13) and (3.14) to

$$
\begin{aligned}
\widehat{E^A f}(\xi) &= \hat{f}(\xi)-Z_f(\sigma,\xi)Z_\phi(\sigma,\xi)^{-1}\hat{\phi}(\xi) \\
&= \hat{f}(\xi)-Z_f(\sigma,\xi)Z_\phi(\sigma,\xi)^{-1}\Big[\prod_{l=1}^{j}m_\phi\Big(\frac{\xi}{2^l}\Big)\Big]\hat{\phi}\Big(\frac{\xi}{2^j}\Big).
\end{aligned}
$$

□

The method of deriving (3.10), expressing the Fourier transform of the aliasing error $E^A f$ is much easier than the one in [5], where authors employed the subspace $M_\sigma := \{f \in V_1 : f(\sigma+n)=0, n\in\mathbb{Z}\}$ to derive the corresponding expressions for $j=1$ and 2 (see Theorems 1 and 4 in [5]).

Note that $A_k(\xi)=A_k(\xi+2\pi)$ and $B_k(\xi)=B_k(\xi+2\pi)$ belong to $L^2[0,2\pi]$ for $1\le k\le 2^{j-1}$. On the other hand, by applying (3.2) to ϕ and ψ we have

$$
\begin{aligned}
A_k(\xi) &= m_\phi(\xi)\delta_{1,k}-Z_\phi(\sigma,2^j\xi)^{-1}[m_\phi(\xi+2^{1-j}(k-1)\pi)Z_\phi(2^j\sigma,\xi+2^{1-j}(k-1)\pi) \\
&\quad + m_\phi(\xi+\pi+2^{1-j}(k-1)\pi)Z_\phi(2^j\sigma,\xi+\pi+2^{1-j}(k-1)\pi)\prod_{l=0}^{j-1}m_\phi(2^l\xi)
\end{aligned}
$$

and

$$
\begin{aligned}
B_k(\xi) &= m_\psi(\xi)\delta_{1,k}-Z_\phi(\sigma,2^j\xi)^{-1}[m_\psi(\xi+2^{1-j}(k-1)\pi)Z_\phi(2^j\sigma,\xi+2^{1-j}(k-1)\pi) \\
&\quad + m_\psi(\xi+\pi+2^{1-j}(k-1)\pi)Z_\phi(2^j\sigma,\xi+\pi+2^{1-j}(k-1)\pi)\prod_{l=0}^{j-1}m_\phi(2^l\xi).
\end{aligned}
$$

Hence A_k and B_k belong to $L^\infty[0,2\pi]$ for $1\le k\le 2^{j-1}$ provided that $Z_\phi(2^j\sigma,\cdot)\in L^\infty[0,2\pi]$.

Theorem 3.8. *Assume* $Z_\phi(2^j\sigma,\cdot)\in L^\infty[0,2\pi]$ *so that* $V_j(\sigma)=V_j$ *for an integer* $j\geq 1$*. Then we have for any* $f\in V_j$ *and for any* t *in* $\mathbb{R}$

$$\begin{aligned}|E^Af(t)| &\leq \frac{1}{\pi}2^{-\frac{j-1}{2}}\sum_{k=1}^{2^{j-1}}[\|A_k(\xi)Z_\phi(2^jt,\xi)\|_{L^2[0,2\pi]}\|u_g(\xi)\|_{L^2[0,2\pi]}\\ &\qquad +\|B_k(\xi)Z_\phi(2^jt,\xi)\|_{L^2[0,2\pi]}\|v_g(\xi)\|_{L^2[0,2\pi]}]\\ &\leq \sqrt{\frac{2^j}{\pi}}\sum_{k=1}^{2^{j-1}}[\|H_\phi(\xi)A_k(\xi)\|_{L^2[0,2\pi]}\|u_g(\xi)\|_{L^2[0,2\pi]}\\ &\qquad +\|H_\phi(\xi)B_k(\xi)\|_{L^2[0,2\pi]}\|v_g(\xi)\|_{L^2[0,2\pi]}]\end{aligned}\tag{3.15}$$

and

$$\|E^Af\|^2_{L^2(\mathbb{R})} \leq 2^{j+1}\|G_\phi(\xi)\|_\infty\sum_{k=1}^{2^{j-1}}[\|A_k(\xi)\|_\infty\|u_g(\xi)\|_{L^2[0,2\pi]} +\|B_k(\xi)\|_\infty\|v_g(\xi)\|_{L^2[0,2\pi]}].\tag{3.16}$$

Proof. Since $\widehat{E^Af}\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, we have by (3.10) and Lemma 2.1

$$\begin{aligned}E^Af(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{E^Af}(\xi)e^{it\xi}d\xi = \frac{2^N}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{E^Af}(2^N\xi)e^{it2^N\xi}d\xi\\ &= \frac{2^{\frac{j+1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{k=1}^{2^{j-1}}[A_k(\xi)u_g(2\xi+2^{2-j}(k-1)\pi)+B_k(\xi)v_g(2\xi+2^{2-j}(k-1)\pi)]\hat{\phi}(\xi)e^{it2^N\xi}d\xi\\ &= \frac{2^{\frac{j+1}{2}}}{\sqrt{2\pi}}\int_{0}^{2\pi}\sum_{k=1}^{2^{j-1}}[A_k(\xi)u_g(2\xi+2^{2-j}(k-1)\pi)+B_k(\xi)v_g(2\xi+2^{2-j}(k-1)\pi)]\\ &\qquad\cdot\sum_{n\in\mathbb{Z}}\hat{\phi}(\xi+2n\pi)e^{it2^N(\xi+2n\pi)}d\xi\\ &= \frac{2^{\frac{j+1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{k=1}^{2^{j-1}}[A_k(\xi)u_g(2\xi+2^{2-j}(k-1)\pi)+B_k(\xi)v_g(2\xi+2^{2-j}(k-1)\pi)]Z_\phi(2^Nt,\xi)d\xi\end{aligned}$$

from which the first inequality of (3.15) follows. Then the second inequality of (3.15) follows since $|Z_\phi(t,\xi)|\leq\sqrt{2\pi}H_\phi(\xi)$ for any t in $\mathbb{R}$. Similarly (3.16) also comes from (3.10). □

In Theorem 3.8, we assume $Z_\phi(2^j\sigma,\cdot)\in L^\infty[0,2\pi]$ only to guarantee the finiteness of the upper bounds in (3.15) and (3.16). For example if we assume $H_\phi\in L^4[0,2\pi]$, then we can easily see that $H_\phi A_k$ and $H_\phi B_k$ belong to $L^2[0,2\pi]$ for $1\leq k\leq 2^{j-1}$ so that the estimate (3.15) remains to hold for any $f\in V_j(\sigma)$ even if we do not assume $Z_\phi(2^j\sigma,\cdot)\in L^\infty[0,2\pi]$.

Note that $Z_\phi(2^j\sigma,\cdot)\in L^\infty[0,2\pi]$ for any $j\geq 1$ if either $\sigma=0$ or $H_\phi\in L^\infty[0,2\pi]$ since $\|Z_\phi(\sigma,\cdot)\|_\infty<\infty$ by (2.4) and $|Z_\phi(t,\xi)|\leq\sqrt{2\pi}H_\phi(\xi)$ for any t in $\mathbb{R}$ by (2.2). It is also worth to note that if $Z_\phi(2^j\sigma,\cdot)\in L^\infty[0,2\pi]$ for some integer $j\geq 1$, then $Z_\phi(2^k\sigma,\cdot)\in L^\infty[0,2\pi]$ for $0\leq k\leq j$. It follows immediately from (3.2) applied to ϕ since $m_\phi\in L^\infty[0,2\pi]$.

4. Aliasing error in average sampling

Suppose that $\mathcal{L}$ is a linear time-invariant system defined on V_0 as

$$\big(\mathcal{L}f\big)(t) := [f * \mathsf{h}](t) = \int_{-\infty}^{\infty} f(x)\mathsf{h}(t-x)dx\,, \quad t \in \mathbb{R}\,,$$

where the impulse response h of $\mathcal{L}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. As a consequence, the sequence $\{\mathcal{L}\phi(n)\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ (see [6, Lemma 1]) and $Z_{\mathcal{L}\phi}(0,\xi)$ belongs to $L^2[0,2\pi]$. Moreover, for each $f \in L^2(\mathbb{R})$ we have that $\widehat{\mathcal{L}f}(\xi) = \widehat{\mathsf{h} * f}(\xi) = \sqrt{2\pi}\,\widehat{\mathsf{h}}(\xi)\widehat{f}(\xi)$ a.e. in $\mathbb{R}$.

Unser and Aldroubi derived in [10] a sampling formula that allows to recover any function f in V_0 from the average samples $\{\big(\mathcal{L}f\big)(n)\}_{n\in\mathbb{Z}}$. The following theorem, which proof can be found in [6], gives a necessary and sufficient condition for this formula to hold.

Theorem 4.1. *Assume that $\phi \in L^2(\mathbb{R}) \cap C(\mathbb{R})$, $\sup_{\mathbb{R}} \Phi(t) < \infty$ and $\|Z_{\mathcal{L}\phi}(0,\cdot)\|_\infty < \infty$. Then, there exists a Riesz basis $\big\{S_{\mathcal{L}}(t-n) : n \in \mathbb{Z}\big\}$ for V_0 such that the sampling formula*

$$f(t) = \sum_{n\in\mathbb{Z}} \big(\mathcal{L}f\big)(n)\; S_{\mathcal{L}}(t-n)\,, \quad t \in \mathbb{R}\,, \tag{4.1}$$

holds for each $f \in V_0$ if and only if $\|Z_{\mathcal{L}\phi}(0,\cdot)\|_0 > 0$. Moreover, in this case $\widehat{S}_{\mathcal{L}}(\xi) = Z^{-1}_{\mathcal{L}\phi}(0,\xi)\widehat{\phi}(\xi)$. The convergence of the series in (4.1) *is in the $L^2(\mathbb{R})$-sense, absolute and uniform on $\mathbb{R}$.*

A similar formulation can be given in terms of the shifted samples $\{\big(\mathcal{L}f\big)(\sigma+n)\}_{n\in\mathbb{Z}}$. Throughout this section we assume the hypotheses in Theorem 4.1 and that the sampling formula (4.1) holds. In the following, we assume $H_\phi \in L^\infty[0,2\pi]$. Then we can easily see that $\sup\limits_{\mathbb{R}} \sum\limits_{n\in\mathbb{Z}} |f(t+n)|^2 < \infty$ so that $\{\mathcal{L}f(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for any $f \in V_1$ (cf. [6, Lemma 1]). For each $f \in V_1$ we define the average aliasing error as

$$E^A_{\mathcal{L}}f(t) := f(t) - \sum_{n\in\mathbb{Z}} \big(\mathcal{L}f\big)(n)\; S_{\mathcal{L}}(t-n)\,, \quad t \in \mathbb{R}\,.$$

Proposition 4.2. *Assume that $H_\phi \in L^\infty[0,2\pi]$. For any $f \in V_1$,*

$$\widehat{E^A_{\mathcal{L}}f}(\xi) = v_f(\xi)N_{\mathcal{L}}(\xi/2)\widehat{\phi}(\xi/2), \tag{4.2}$$

where $v_f(\xi)$ is the function given in (3.1) *and*

$$N_{\mathcal{L}}(\xi) := m_\psi(\xi) - Z_{\mathcal{L}\psi}(0,2\xi)Z^{-1}_{\mathcal{L}\phi}(0,2\xi)m_\phi(\xi).$$

Proof. First, we check that we can apply the Poisson summation formula to $\mathcal{L}f$ for any $f \in V_1$. Indeed,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} |\widehat{\mathcal{L}f}(\xi+2n\pi)| &= \sum_{n\in\mathbb{Z}} |\widehat{\mathsf{h}}(\xi+2n\pi)\widehat{f}(\xi+2n\pi)| \le \|\widehat{\mathsf{h}}\|_{L^\infty(\mathbb{R})} \sum_{n\in\mathbb{Z}} |\widehat{f}(\xi+2n\pi)| \\
&\le \frac{1}{\sqrt{2\pi}} \|\mathsf{h}\|_{L^1(\mathbb{R})} \sum_{n\in\mathbb{Z}} \Big| m_f\Big(\frac{\xi}{2}+n\pi\Big)\widehat{\phi}\Big(\frac{\xi}{2}+n\pi\Big)\Big| \\
&= \frac{1}{\sqrt{2\pi}} \|\mathsf{h}\|_{L^1(\mathbb{R})} \Big[\Big|m_f\Big(\frac{\xi}{2}\Big)\Big| \sum_{n\in\mathbb{Z}} \Big|\widehat{\phi}\Big(\frac{\xi}{2}+2n\pi\Big)\Big| + \Big|m_f\Big(\frac{\xi}{2}+\pi\Big)\Big| \sum_{n\in\mathbb{Z}} \Big|\widehat{\phi}\Big(\frac{\xi}{2}+\pi+2n\pi\Big)\Big|\Big].
\end{aligned}$$

Since we have assumed that $\mathsf{h} \in L^1(\mathbb{R})$ and $H_\phi(\xi) = \sum_{n\in\mathbb{Z}} |\widehat{\phi}(\xi + 2n\pi)| \in L^\infty[0, 2\pi]$ we have that the series $\sum_{n\in\mathbb{Z}} |\widehat{\mathcal{L}f}(\xi + 2n\pi)|$ belongs to $L^2[0, 2\pi]$. Then, using Lemma 2.1 we have that, for each $f \in V_1$,

$$Z_{\mathcal{L}f}(0, \xi) = \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \widehat{\mathcal{L}f}(\xi + 2n\pi), \quad a.e.\ \xi \in [0, 2\pi].$$

Now, for $f \in V_1$, we consider the orthogonal projection h onto W_0, i.e., $f = g + h$ with $g \in V_0$ and $h \in W_0$. We have

$$\widehat{E^A_{\mathcal{L}} f}(\xi) = \widehat{E^A_{\mathcal{L}} h}(\xi) = \widehat{h}(\xi) - \sum_{n\in\mathbb{Z}} (\mathcal{L}h)(n)\ \widehat{S}_{\mathcal{L}}(\xi) e^{-in\xi} = \widehat{h}(\xi) - Z_{\mathcal{L}h}(0, \xi)\widehat{S}_{\mathcal{L}}(\xi)$$

Using that $\widehat{h}(\xi) = v_f(\xi)\widehat{\psi}(\xi)$ and the Poisson summation formula we obtain

$$\begin{aligned}
\widehat{E^A_{\mathcal{L}} f}(\xi) &= \widehat{h}(\xi) - \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \widehat{\mathcal{L}h}(\xi + 2n\pi)\widehat{S}_{\mathcal{L}}(\xi) \\
&= v_f(\xi)\hat{\psi}(\xi) - 2\pi \sum_{n\in\mathbb{Z}} \widehat{\mathsf{h}}(\xi + 2n\pi)\widehat{h}(\xi + 2n\pi)\widehat{S}_{\mathcal{L}}(\xi) \\
&= v_f(\xi)\Big[\widehat{\psi}(\xi) - 2\pi \sum_{n\in\mathbb{Z}} \widehat{\mathsf{h}}(\xi + 2n\pi)\widehat{\psi}(\xi + 2n\pi)\widehat{S}_{\mathcal{L}}(\xi)\Big] \\
&= v_f(\xi)\Big[\widehat{\psi}(\xi) - Z_{\mathcal{L}\psi}(0, \xi)\widehat{S}_{\mathcal{L}}(\xi)\Big] = v_f(\xi)\Big[\widehat{\psi}(\xi) - Z_{\mathcal{L}\psi}(0, \xi) Z^{-1}_{\mathcal{L}\phi}(0, \xi)\widehat{\phi}(\xi)\Big] \\
&= v_f(\xi)\hat{\phi}(\xi/2)\Big[m_\psi(\xi/2) - Z_{\mathcal{L}\psi}(0, \xi) Z^{-1}_{\mathcal{L}\phi}(0, \xi) m_\phi(\xi/2)\Big],
\end{aligned}$$

which concludes the proof. □

Theorem 4.3. *Assume that $H_\phi \in L^\infty[0, 2\pi]$. For any $f \in V_1$,*

$$|E^A_{\mathcal{L}} f(t)| \le \frac{1}{\pi} \|N_{\mathcal{L}}\, Z_\phi(2t, \cdot)\|_{L^2[0,2\pi]} \|v_f\|_{L^2[0,2\pi]}, \quad t \in \mathbb{R}.$$

Moreover, the following uniform bound holds

$$|E^A_{\mathcal{L}} f(t)| \le \sqrt{\frac{2}{\pi}}\, \|N_{\mathcal{L}}\, H_\phi\|_{L^2[0,2\pi]}\, \|v_f\|_{L^2[0,2\pi]}, \quad t \in \mathbb{R}.$$

Proof. We have

$$\begin{aligned}
\|\widehat{E^A_{\mathcal{L}} f}\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{\infty} |\widehat{E^A_{\mathcal{L}} f}(\xi)| d\xi = \int_{-\infty}^{\infty} |v_f(\xi)\widehat{\phi}(\xi/2) N_{\mathcal{L}}(\xi/2)| d\xi \\
&= 2\int_{-\infty}^{\infty} |v_f(2\xi)\widehat{\phi}(\xi) N_{\mathcal{L}}(\xi)| d\xi = 2\int_0^{2\pi} |v_f(2\xi) N_{\mathcal{L}}(\xi)| H_\phi(\xi) d\xi.
\end{aligned}$$

Since we have assumed that $H_\phi \in L^\infty[0, 2\pi]$ and $N_{\mathcal{L}}, v_f \in L^2[0, 2\pi]$, the Fourier transform of $E^A_{\mathcal{L}} f$ belongs to $L^1(\mathbb{R})$. Using the inverse Fourier transform and the Poisson summation formula we obtain

$$\begin{aligned}
E^A_{\mathcal{L}} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{E^A_{\mathcal{L}} f}(\xi) e^{it\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_f(\xi)\widehat{\phi}(\xi/2) N_{\mathcal{L}}(\xi/2) e^{it\xi} d\xi \\
&= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_f(2\xi)\widehat{\phi}(\xi) N_{\mathcal{L}}(\xi) e^{2it\xi} d\xi = \frac{2}{\sqrt{2\pi}} \int_0^{2\pi} v_f(2\xi) N_{\mathcal{L}}(\xi) \sum_{n\in\mathbb{Z}} \widehat{\phi}(\xi + 2n\pi) e^{2it(\xi+2n\pi)} d\xi \\
&= \frac{1}{\pi} \int_0^{2\pi} v_f(2\xi) N_{\mathcal{L}}(\xi) Z_\phi(2t, \xi) d\xi.
\end{aligned}$$

Then

$$
\begin{aligned}
|E_{\mathcal{L}}^A f(t)| &\le \frac{1}{\pi}\|v_f(2\xi)\|_{L^2[0,2\pi]}\|N_{\mathcal{L}}(\xi)Z_\phi(2t,\xi)\|_{L^2[0,2\pi]}\\
&= \frac{1}{\pi}\|N_{\mathcal{L}}(\xi)Z_\phi(2t,\xi)\|_{L^2[0,2\pi]}\|v_f(\xi)\|_{L^2[0,2\pi]}.
\end{aligned}
$$

Having in mind that

$$
|Z_\phi(2t,\xi)| = \sqrt{2\pi}\Big|\sum_{n\in\mathbb{Z}}\widehat{\phi}(\xi+2n\pi)e^{i2t(\xi+2n\pi)}\Big| \le \sqrt{2\pi}\sum_{n\in\mathbb{Z}}|\widehat{\phi}(\xi+2n\pi)| = \sqrt{2\pi}H_\phi(\xi)
$$

the uniform bound follows. □

Theorem 4.4. *Assume that* $H_\phi \in L^\infty[0,2\pi]$. *For any* $f \in V_1$,

$$
K_0\|v_f\|^2_{L^2[0,2\pi]} \le \|E_{\mathcal{L}}^A f\|^2_{L^2(\mathbb{R})} \le K_\infty\|v_f\|^2_{L^2[0,2\pi]},
$$

where

$$
\begin{aligned}
K_0 &:=\| \, |N_{\mathcal{L}}(\xi)|^2G_\phi(\xi) + |N_{\mathcal{L}}(\xi+\pi)|^2G_\phi(\xi+\pi)\|_0\\
K_\infty &:=\| \, |N_{\mathcal{L}}(\xi)|^2G_\phi(\xi) + |N_{\mathcal{L}}(\xi+\pi)|^2G_\phi(\xi+\pi)\|_\infty.
\end{aligned}
$$

The constants K_0 *and* K_∞ *are the optimal constants for these inequalities.*

Proof. We have

$$
\begin{aligned}
\|E_{\mathcal{L}}^A f\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty}|\widehat{E_{\mathcal{L}}^A f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty}|v_f(\xi)\widehat{\phi}(\xi/2)N_{\mathcal{L}}(\xi/2)|^2 d\xi\\
&= 2\int_{-\infty}^{\infty}|v_f(2\xi)\widehat{\phi}(\xi)N_{\mathcal{L}}(\xi)|^2 d\xi = 2\int_0^{2\pi}|v_f(2\xi)N_{\mathcal{L}}(\xi)|^2G_\phi(\xi)d\xi\\
&= 2\int_0^{\pi}|v_f(2\xi)N_{\mathcal{L}}(\xi)|^2G_\phi(\xi)d\xi + 2\int_\pi^{2\pi}|v_f(2\xi)N_{\mathcal{L}}(\xi)|^2G_\phi(\xi)d\xi\\
&= 2\int_0^{\pi}|v_f(2\xi)|^2\Big(|N_{\mathcal{L}}(\xi)|^2G_\phi(\xi) + |N_{\mathcal{L}}(\xi+\pi)|^2G_\phi(\xi+\pi)\Big)d\xi,
\end{aligned}
$$

from which the inequalities are easily obtained. The optimality of the constants K_0 and K_∞ can be proved similarly as in the proof of Theorem 3.6. □

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