On Lagrange-type interpolation series and analytic Kramer kernels

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Abstract

The classical Kramer sampling theorem provides a method for obtaining orthogonal sampling formulas. A challenging problem is to characterize the situations when these sampling formulas can be written as Lagrange-type interpolation series. This article gives a necessary and sufficient condition to ensure that when the sampling formula is associated with an analytic Kramer kernel, then it can be expressed as a quasi Lagrange-type interpolation series; this latter form is a minor but significant modification of a Lagrange-type interpolation series. Finally, a link with the theory of de Branges spaces is established.

Keywords: Analytic Kramer kernels; Sampling formulas; Lagrange-type interpolation series.
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1 Statement of the problem

The classical Kramer sampling theorem provides a method for obtaining orthogonal sampling theorems [7, 12, 15, 21]. This theorem has played a very significant role in sampling theory, interpolation theory, signal analysis and, generally, in mathematics; see the survey articles [5, 6]. The statement of this general result is as follows: Let

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$K(\omega, \lambda)$ be a function, defined for all λ in an open subset D of $\mathbb{R}$ (or $\mathbb{C}$) such that, as a function of ω, $K(\cdot, \lambda) \in L^2(I)$ for every number $\lambda \in D$, where I is an interval of the real line. Assume that there exists a sequence of distinct real numbers $\{\lambda_n\} \subset D$, with n belonging to an indexing set $\mathbb{I}$ contained in $\mathbb{Z}$, such that $\{K(\omega, \lambda_n)\}$ is a complete orthogonal sequence of functions for $L^2(I)$. Then for any F of the form

$$F(\lambda) = \int_I f(\omega) K(\omega, \lambda)\, d\omega\,, \quad \lambda \in D\,, \tag{1}$$

where $f \in L^2(I)$, we have

$$F(\lambda) = \lim_{N\to\infty} \sum_{\substack{|n|\le N \\ n\in\mathbb{I}}} F(\lambda_n) S_n(\lambda)\,, \tag{2}$$

with

$$S_n(\lambda) = \frac{\int_I K(\omega, \lambda)\overline{K(\omega, \lambda_n)}\, d\omega}{\int_I |K(\omega, \lambda_n)|^2\, d\omega}\,. \tag{3}$$

The series in (2) converges absolutely and uniformly on subsets of D where $\|K(\cdot, \lambda)\|_{L^2(I)}$ is bounded.

The Kramer sampling theorem has been the cornerstone for a significant mathematical literature on the topic of sampling theorems associated with differential or difference problems which has flourished for the past few years. As a small but significant sample of examples see, for instance, [7, 21] and references therein.

In [8] an extension of the Kramer sampling theorem has been obtained to the case when the kernel is analytic in the sampling parameter λ. Namely: Assume that the Kramer kernel K is an entire function for any fixed $\omega \in I$, and that the function $h(\lambda) = \int_I |K(\omega, \lambda)|^2 d\omega$ is locally bounded on D. Then any function F defined by (1) is an entire function, as are all the sampling functions (3). A kernel K satisfying the above additional conditions is called a Kramer analytic kernel.

The discrete version of Kramer sampling theorem has been proved in [1, 2, 9], and its analytic counterpart in [10].

In many cases the sampling functions S_n can be expressed as Lagrange-type interpolation functions, i.e.,

$$S_n(\lambda) = \frac{G(\lambda)}{(\lambda - \lambda_n) G'(\lambda_n)}\,,$$

where G is an entire function having simple zeros at all the points $\{\lambda_n \in D\}$. In other important examples the sampling functions S_n have the form

$$S_n(\lambda) = \frac{A(\lambda)}{A(\lambda_n)} \frac{G(\lambda)}{(\lambda - \lambda_n) G'(\lambda_n)}\,,$$

where the additional function A is an entire function without zeros (see [3] or the Shannon-type interpolation formulae in [7]). In this case we say that the sampling functions can be written as quasi Lagrange-type interpolation functions. Notice that a quasi Lagrange-type interpolation series reduces to a Lagrange-type interpolation series when the function F to be sampled is replaced by F/A.

It is worth mentioning that the problem of whether a sampling theorem involving infinitely many sampling points can be derived as limiting cases of finite Lagrange interpolation has been considered in [14].

The starting point in this paper is an abstract analytic version of the Kramer sampling theorem. To this end, we work in the reproducing kernel Hilbert space (written shortly as RKHS) of entire functions introduced by Saitoh as follows (see his superb monograph [18]): Let $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ be a complex, separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Suppose K is a $\mathbb{H}$-valued function defined on $\mathbb{C}$. For each $x \in \mathbb{H}$, define $f_x(z) = \langle K(z), x\rangle_{\mathbb{H}}$ and let $\mathcal{H}_K$ denote the collection of all such functions f_x. Furthermore, each element in $\mathcal{H}_K$ is an entire function if and only if K is analytic on $\mathbb{C}$ or, equivalently, if and only if $\langle K(z), e_n\rangle$ is entire for each $n \in \mathbb{N}$ and $\|K(\cdot)\|_{\mathbb{H}}$ is bounded on all compact subsets of $\mathbb{C}$. In this setting, an abstract version of the analytic Kramer theorem is obtained assuming the existence of two sequences, $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$, and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C} \setminus \{0\}$, such that $K(z_n) = a_n e_n$ for each $n \in \mathbb{N}$. This is a slight modification of a sampling result derived by Higgins in [13] which also includes the analytic version.

A challenging problem is to give a necessary and sufficient condition to ensure that the corresponding sampling formula can be written as a quasi Lagrange-type interpolation series. This is not always true as a counterexample shows, see Section 3. Roughly speaking, the aforesaid necessary and sufficient condition concerns the stability of the functions belonging in the space $\mathcal{H}_K$, on removing a finite number of zeros. The classical Paley-Wiener spaces satisfy this property. Finally, we prove under suitable hypotheses that the RKHS $\mathcal{H}_K$, whose sampling formula can be written as quasi Lagrange-type interpolation formula, is a de Branges space of entire functions. It is important to mention that the paper [16], by Nashed and Walter, is the first reference where sampling in connection with de Branges spaces is introduced.

2 Analytic Kramer sampling theory

For the sake of completeness we sketch the underlying sampling theory used through this article (see [11] or [18] for details). Let $\mathbb{H}$ be a complex, separable Hilbert space and let K be a $\mathbb{H}$-valued function defined on $\mathbb{C}$, i.e., $K : \mathbb{C} \ni z \mapsto K(z) \in \mathbb{H}$. Using this function K we define a mapping between $\mathbb{H}$ and the set $\mathbb{C}^{\mathbb{C}}$ of all functions between $\mathbb{C}$ and $\mathbb{C}$ as follows:

$$T : \mathbb{H} \ni x \mapsto T(x) = f \text{ such that } f(z) := \langle K(z), x\rangle_{\mathbb{H}} \text{ for } z \in \mathbb{C}\,. \tag{4}$$

The application T is anti-linear, i.e., $T(\alpha x + \beta y) = \overline{\alpha}\, T(x) + \overline{\beta}\, T(y)$ for $x, y \in \mathbb{H}$ and $\alpha, \beta \in \mathbb{C}$.

We denote by $\mathcal{H}_K$ the range of T, i.e., $\mathcal{H}_K = T(\mathbb{H})$. From now on, we refer to the function K as the kernel of the anti-linear application T.

The space $\mathcal{H}_K$ with the norm $\|f\|_{\mathcal{H}_K} := \inf\{\|x\|_{\mathbb{H}} : f = T(x)\}$ becomes a reproducing kernel Hilbert space, RKHS hereafter, i.e., for each $z \in \mathbb{C}$, the evaluation functional $E_z(f) = f(z)$, $f \in \mathcal{H}_K$, is bounded. As a consequence, convergence in the norm $\|\cdot\|_{\mathcal{H}_K}$ implies pointwise convergence which is uniform on those subsets of $\mathbb{C}$ where $\|K(\cdot)\|_{\mathbb{H}}$ is bounded. The reproducing kernel of $\mathcal{H}_K$ is given by $k(z,\omega) = \langle K(z), K(\omega)\rangle_{\mathbb{H}}$, $z,\omega \in \mathbb{C}$.

We note that the anti-linear mapping $T : \mathbb{H} \longrightarrow \mathcal{H}_K$ is injective if and only if it is an isometry, or equivalently, if and only if the set $\{K(z)\}_{z\in\mathbb{C}}$ is complete in $\mathbb{H}$ [18]. In particular, if there exists a sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ such that $\{K(z_n)\}_{n=1}^\infty$ is an orthogonal complete set in $\mathbb{H}$, then T is an anti-linear isometry from $\mathbb{H}$ onto $\mathcal{H}_K$.

On the other hand, to decide whether $\mathcal{H}_K$ is a RKHS of entire functions the following result holds: $\mathcal{H}_K$ is a RKHS of entire functions if and only if the kernel K is analytic in $\mathbb{C}$ ([19, p. 266]). Another characterization of the analyticity of the functions in $\mathcal{H}_K$ is the following: Suppose that an orthonormal basis $\{e_n\}_{n=1}^\infty$ for $\mathbb{H}$ is given; expanding $K(z)$, where $z \in \mathbb{C}$ is fixed, in this basis we obtain

$$K(z) = \sum_{n=1}^{\infty} \langle K(z), e_n\rangle_{\mathbb{H}}\, e_n\,,$$

where the coefficients $S_n(z) := \langle K(z), e_n\rangle_{\mathbb{H}}$ define functions in $\mathcal{H}_K$. Then, the functions in $\mathcal{H}_K$ are entire if and only if the functions $\{S_n\}_{n=1}^\infty$ are entire and $\|K(\cdot)\|_{\mathbb{H}}$ is bounded on compact sets of $\mathbb{C}$ ([11, Theorem 2.2]).

For sampling purposes we suppose the existence of a sequence of points $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ is given such that $K(z_n) = a_n\, e_n$, $n \in \mathbb{N}$, for some nonzero constants $\{a_n\}_{n=1}^\infty$, where $\{e_n\}_{n=1}^\infty$ denotes an orthonormal basis for $\mathbb{H}$. This is equivalent to saying that the sequence of functions $\{S_n\}_{n=1}^\infty$, where $S_n(z) := \langle K(z), e_n\rangle_{\mathbb{H}}$, satisfies, for the sequence $\{z_n\}_{n=1}^\infty$, the interpolatory property:

$$S_n(z_m) = a_n\, \delta_{n,m}\,. \tag{5}$$

In this case, the following sampling result holds:

Theorem 1 *Let $K : \mathbb{C} \longrightarrow \mathbb{H}$ be an analytic kernel. Assume that the interpolation property* (5) *holds for some sequences $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^\infty$ in $\mathbb{C}\setminus\{0\}$. Let $\mathcal{H}_K$ be the corresponding RKHS of entire functions. Then any $f \in \mathcal{H}_K$ can be recovered from its samples $\{f(z_n)\}_{n=1}^\infty$ by means of the sampling series*

$$f(z) = \sum_{n=1}^{\infty} f(z_n)\, \frac{S_n(z)}{a_n}\,, \quad z \in \mathbb{C}\,. \tag{6}$$

This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Proof: First notice that $\lim_{n\to\infty}|z_n| = +\infty$; otherwise the sequence $\{z_n\}_{n=1}^{\infty}$ contains a bounded subsequence and hence, the entire function $S_n \equiv 0$ for all $n \in \mathbb{N}$ which contradicts (5). The anti-linear application T is a bijective isometry between $\mathbb{H}$ and $\mathcal{H}_k$. As a consequence, the functions $\{S_n = T(e_n)\}_{n=1}^{\infty}$ form an orthonormal basis for $\mathcal{H}_K$. Expanding any $f \in \mathcal{H}_K$ in this basis we obtain

$$f(z) = \sum_{n=1}^{\infty} \langle f, S_n\rangle_{\mathcal{H}_K}\ S_n(z)\,.$$

Moreover,

$$\langle f, S_n\rangle_{\mathcal{H}_K} = \overline{\langle x, e_n\rangle_{\mathbb{H}}} = \langle \frac{K(z_n)}{a_n}, x\rangle_{\mathbb{H}} = \frac{f(z_n)}{a_n}\,. \tag{7}$$

Since an orthonormal basis is an unconditional basis, the sampling series will be pointwise unconditionally convergent and hence, absolutely convergent. The uniform convergence is a standard result in the setting of the RKHS theory since $\|K(\cdot)\|_{\mathbb{H}}$ is bounded on compact subsets of $\mathbb{C}$. ■

Theorem 1 is an abstract version of classical Kramer sampling theorem [15]. This leads us to give the following definition:

Definition 1 *An analytic kernel $K : \mathbb{C} \longrightarrow \mathbb{H}$ is said to be an analytic Kramer kernel if there exists a sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ which satisfies $K(z_n) = a_n e_n$, $n \in \mathbb{N}$, for some orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathbb{H}$, and $\{a_n\}_{n=1}^{\infty} \in \mathbb{C} \setminus \{0\}$.*

At this point, a question naturally arises. It concerns the existence of an analytic Kramer kernel associated with an arbitrary sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}|z_n| = +\infty$. The answer to this question is affirmative. Given the sequence $\{z_n\}_{n=1}^{\infty}$, consider a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C} \setminus \{0\}$ such that $\sum_{z_n\neq 0} |a_n/z_n|^2 < \infty$ (set $a_k = 1$ in the case $z_k = 0$). Now, let P be an entire function having only simple zeros at $\{z_n\}_{n=1}^{\infty}$; this is allowed by the theorem of Weierstrass [20, p. 54]. In a separable Hilbert space $\mathbb{H}$ with orthonormal basis $\{e_n\}_{n=1}^{\infty}$, invoking the Riesz-Fisher Theorem, define the mapping $K : \mathbb{C} \longrightarrow \mathbb{H}$ as:

$$K(z) := \sum_{n=1}^{\infty} \frac{a_n P(z)}{z - z_n}\ e_n\,,$$

where the convergence of the series is in the norm of $\mathbb{H}$, and consider the corresponding RKHS $\mathcal{H}_K$. The entire functions $S_n(z) = \dfrac{a_n P(z)}{z - z_n}$, $n \in \mathbb{N}$, satisfy the interpolation condition $S_n(z_m) = a_m P'(z_m)\delta_{n,m}$. In addition, the function

$$\|K(z)\|^2_{\mathbb{H}} = \sum_{n=1}^{\infty} \left| \frac{a_n P(z)}{z - z_n} \right|^2, \quad z \in \mathbb{C}\,,$$

is uniformly bounded on compact subsets. Indeed, given A a compact in $\mathbb{C}$ there exists a closed disk Δ_R centered at the origin with radius $R > 0$ such that $A \subseteq \Delta_R$. Apart from a possible finite number of points $\{z_k\}$, $k \in \mathbb{I}_R$, we have the result that $|z - z_n| \geq \big||z| - |z_n|\big| \geq |z_n| - R$ for all $z \in A$ and $n \in \mathbb{N} \setminus \mathbb{I}_R$. Thus,

$$\sum_{n=1}^{\infty} \left|\frac{a_n P(z)}{z - z_n}\right|^2 \leq \sum_{n \in \mathbb{I}_R} \left|\frac{a_n P(z)}{z - z_n}\right|^2 + \sum_{n \in \mathbb{N} \setminus \mathbb{I}_R} \frac{|a_n|^2 |P(z)|^2}{(|z_n| - R)^2},$$

and both summands are bounded on the compact A. Hence, K is an entire $\mathbb{H}$-valued function [11, Theorem 2.2] satisfying the requirements in Definition 1, i.e., K is an analytic Kramer kernel. As a consequence, Theorem 1 assures that any function $f \in \mathcal{H}_K$ can be recovered from its samples $\{f(z_n)\}_{n=1}^{\infty}$ by means of the Lagrange-type interpolation formula

$$f(z) = \sum_{n=1}^{\infty} f(z_n)\, \frac{a_n P(z)/(z - z_n)}{a_n\ P'(z_n)} = \sum_{n=1}^{\infty} f(z_n)\, \frac{P(z)}{(z - z_n) P'(z_n)}, \quad z \in \mathbb{C}. \tag{8}$$

A more difficult question concerns whether the sampling expansion (6) can be written, in general, as a Lagrange-type interpolation series as in (8). The next Section deals with this problem.

3 Quasi Lagrange-type interpolation

First, we introduce quasi Lagrange-type interpolation series

Definition 2 *The sampling formula* (6) *in a RKHS $\mathcal{H}_K$ associated with an analytic Kramer kernel K is a quasi Lagrange-type interpolation series if it can be written as*

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{(z - z_n) P'(z_n)}, \quad z \in \mathbb{C}, \tag{9}$$

where P is an entire function having only simple zeros at $\{z_n\}_{n=1}^{\infty}$, and A is an entire function without zeros.

In this case, defining a new analytic Kramer kernel as $K_A(z) := K(z)/A(z)$, $z \in \mathbb{C}$, then any function h in the new RKHS $\mathcal{H}_{K_A}$ can be recovered from its samples at $\{z_n\}_{n=1}^{\infty}$ by means of the Lagrange-type interpolation series

$$h(z) = \sum_{n=1}^{\infty} h(z_n)\, \frac{P(z)}{(z - z_n) P'(z_n)}, \quad z \in \mathbb{C}.$$

As we see in the next Theorem, the existence of a quasi Lagrange-type interpolation series in $\mathcal{H}_K$ is intimately related to a stability property in this space with respect to removing a finite number of zeros to functions in $\mathcal{H}_K$.

Definition 3 *A space $\mathcal{H}$ of entire functions has the zero-removing property (ZR property hereafter) if for any $g \in \mathcal{H}$ and any zero w of g the function $g(z)/(z-w)$ belongs to $\mathcal{H}$.*

Next, we give an example taken from [11] of a RKHS $\mathcal{H}_K$ associated with an analytic Kramer kernel where the ZR property fails. Namely: consider $\mathbb{H}$ as the Sobolev Hilbert space $H^1(-\pi,\pi)$ with its usual inner product

$$\langle f, g\rangle_1 = \int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx + \int_{-\pi}^{\pi} f'(x)\,\overline{g'(x)}\,dx\,, \quad f, g \in H^1(-\pi,\pi)\,.$$

The sequence $\{e^{inx}\}_{n\in\mathbb{Z}} \cup \{\sinh x\}$ forms an orthogonal basis for $H^1(-\pi,\pi)$: It is straightforward to prove that the orthogonal complement of $\{e^{inx}\}_{n\in\mathbb{Z}}$ in $H^1(-\pi,\pi)$ is one-dimensional for which $\sinh x$ is a basis. For a fixed $a \in \mathbb{C}\setminus\mathbb{Z}$ we define a kernel

$$\begin{aligned} K_a:\ \mathbb{C} &\longrightarrow \ H^1(-\pi,\pi) \\ z &\longrightarrow \ K_a(z)\,, \end{aligned}$$

by setting

$$[K_a(z)](x) = (z-a)\,e^{izx} + \sin\pi z\,\sinh x\,, \quad \text{for } x \in (-\pi,\pi)\,.$$

Expanding $K_a(z) \in H^1(-\pi,\pi)$ in the former orthogonal basis we obtain

$$K_a(z) = [1 - i(z-a)]\,\sin\pi z\,\sinh x + (z-a)\sum_{n=-\infty}^{\infty} \frac{1+zn}{1+n^2}\,\operatorname{sinc}(z-n)e^{inx}\,.$$

As a consequence, Theorem 1 gives the following sampling result in $\mathcal{H}_{K_a}$: Any function $f \in \mathcal{H}_{K_a}$ can be recovered from its samples $\{f(n)\}_{n\in\mathbb{Z}} \cup \{f(a)\}$ by means of the sampling formula

$$f(z) = [1 - i(z-a)]\,\frac{\sin\pi z}{\sin\pi a}\,f(a) + \sum_{n=-\infty}^{\infty} f(n)\,\frac{z-a}{n-a}\,\frac{1+zn}{1+n^2}\,\operatorname{sinc}(z-n)\,.$$

The function $(z-a)\operatorname{sinc} z$ belongs to $\mathcal{H}_{K_a}$ since $(z-a)\operatorname{sinc} z = \langle K_a(z), 1/2\pi\rangle_1$ for all $z \in \mathbb{C}$. However, by using the sampling formula for $\mathcal{H}_{K_a}$ it is straightforward to check that the function $\operatorname{sinc} z$ does not belong to $\mathcal{H}_{K_a}$.

Theorem 2 *Let $\mathcal{H}_K$ be a RKHS of entire functions obtained from an analytic Kramer kernel K with respect to the sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$. Then, the sampling formula* (6) *for $\mathcal{H}_K$ can be written as a quasi Lagrange-type interpolation series* (9) *if and only if the space $\mathcal{H}_K$ satisfies the ZR property.*

Proof: For the sufficient condition we have to prove that sampling formula (6) can be written as a quasi Lagrange-type interpolation series (9), for some entire functions P and A. First, we prove that the only zeros of the sampling function S_n are given by $\{z_r\}_{r\neq n}$. Suppose that $S_n(w)=0$, then by hypothesis the function $S_n(z)/(z-w)$ is in $\mathcal{H}_K$. Hence, the function

$$\frac{z-z_n}{z-w}S_n(z)=S_n(z)+\frac{w-z_n}{z-w}S_n(z)$$

also belongs to $\mathcal{H}_K$. If $w\notin\{z_r\}_{r\neq n}$, the function $\frac{z-z_n}{z-w}S_n(z)$ in $\mathcal{H}_K$ vanishes at the sequence $\{z_r\}_{r=1}^{\infty}$ which implies that $S_n\equiv 0$, to give a contradiction. In addition, the zeros of S_n are simple; indeed, suppose that z_m is a multiple zero of S_n. Proceeding as above, the function $\frac{z-z_n}{z-z_m}S_n(z)$ belongs to $\mathcal{H}_K$ and vanishes at $\{z_r\}_{r=1}^{\infty}$ which again implies that $S_n\equiv 0$.

Consequently, choose an entire function P having only simple zeros at $\{z_n\}_{n=1}^{\infty}$, then for each $n\in\mathbb{N}$ there exists an entire function without zeros A_n such that $(z-z_n)S_n(z)=P(z)A_n(z)$, $z\in\mathbb{C}$. Next, we prove that there exists an entire function without zeros A and a sequence $\{\sigma_n\}_{n=1}^{\infty}$ in $\mathbb{C}\setminus\{0\}$ such that $A_n(z)=\sigma_nA(z)$ for all $z\in\mathbb{C}$. For $m\neq n$ the function $\frac{z-z_n}{z-z_m}S_n(z)$ in $\mathcal{H}_K$ has its zeros at $\{z_r\}_{r\neq m}$. Thus the sampling formula (6) gives

$$\frac{z-z_n}{z-z_m}S_n(z)=[(z_m-z_n)S_n'(z_m)]S_m(z)\,,\quad z\in\mathbb{C}\,.$$

Fixing $m=1$, we conclude that $A_n(z)=\sigma_nA(z)$ where $A=A_1$ and $\sigma_n=(z_1-z_n)S_n'(z_1)\neq 0$ for $n\in\mathbb{N}\setminus\{1\}$ and $\sigma_1=1$. Hence, $S_n(z)=\frac{\sigma_nP(z)A(z)}{z-z_n}$ for $z\neq z_n$ and $S_n(z_n)=a_n=\sigma_nP'(z_n)A(z_n)$. Substituting in (6) we obtain the quasi Lagrange-type interpolation series (9).

For the necessary condition, assume that the sampling formula in $\mathcal{H}_K$ takes the form of a quasi Lagrange-type interpolation series

$$f(z)=\sum_{n=1}^{\infty}f(z_n)\frac{A(z)}{A(z_n)}\frac{P(z)}{(z-z_n)P'(z_n)}\,,\quad z\in\mathbb{C}\,.$$

Given $g\in\mathcal{H}_K$, there exists $x\in\mathbb{H}$ such that $g(z)=\langle K(z),x\rangle$, $z\in\mathbb{C}$. Assuming that $g(w)=0$, we have to prove that the function $g(z)/(z-w)$ belongs to $\mathcal{H}_K$. The sampling expansion for g at w gives

$$\sum_{n=1}^{\infty}g(z_n)\frac{A(w)}{A(z_n)}\frac{P(w)}{(w-z_n)P'(z_n)}=0\,. \qquad (10)$$

We now distinguish two cases:
(i) $w\in\mathbb{C}\setminus\{z_n\}_{n=1}^{\infty}$. As $P(w)\neq 0$, from (10) we obtain

$$\sum_{n=1}^{\infty}g(z_n)\frac{1}{(w-z_n)A(z_n)P'(z_n)}=0\,.$$

Thus,

$$
\begin{aligned}
g(z) &= \sum_{n=1}^{\infty} g(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{(z-z_n)P'(z_n)} - \sum_{n=1}^{\infty} g(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{(w-z_n)P'(z_n)} \\
&= (w-z) \sum_{n=1}^{\infty} g(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{P'(z_n)} \frac{1}{(z-z_n)(w-z_n)} .
\end{aligned}
$$

Therefore, the entire function $G(z) := g(z)/(w-z)$ can be recovered from its samples at $\{z_n\}_{n=1}^{\infty}$ through the formula

$$
G(z) = \sum_{n=1}^{\infty} G(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{(z-z_n)P'(z_n)} , \quad z \in \mathbb{C} . \tag{11}
$$

Moreover, the function G is in $\mathcal{H}_K$ because $G(z) = \langle K(z), y \rangle_{\mathbb{H}}$, where $y \in \mathbb{H}$ has Fourier coefficients

$$
\left\{ \langle y, e_n \rangle := \frac{1}{\overline{w} - \overline{z}_n} \langle x, e_n \rangle \right\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) .
$$

Indeed, sampling formula (9) for S_n gives $S_n(z) = a_n \frac{A(z)P(z)}{A(z_n)(z-z_n)P'(z_n)}$. Hence, using Parseval's equality we obtain

$$
\langle K(z), y \rangle = \sum_{n=1}^{\infty} \frac{S_n(z)\overline{\langle x, e_n \rangle}}{w - z_n} = G(z) , \quad z \in \mathbb{C} ,
$$

where we have used (11), and the result that $\overline{\langle x, e_n \rangle} = g(z_n)/a_n$, $n \in \mathbb{N}$.

(ii) $w = z_m$ for some $m \in \mathbb{N}$. As $g(z_m) = 0$, the sampling expansion for g reads

$$
g(z) = \sum_{\substack{n=1 \\ n \neq m}}^{\infty} g(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{(z-z_n)P'(z_n)} , \quad z \in \mathbb{C} .
$$

Setting $P(z) = (z - z_m)Q_m(z)$ we have $P'(z) = Q_m(z) + (z - z_m)Q'_m(z)$ and hence

$$
P'(z_k) = \begin{cases} (z_k - z_m)Q'_m(z_k) & \text{if } k \neq m \\ Q_m(z_m) & \text{if } k = m \end{cases}
$$

Hence,

$$
\frac{g(z)}{z - z_m} = \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{g(z_n)}{z_n - z_m} \frac{A(z)}{A(z_n)} \frac{Q_m(z)}{(z-z_n)Q'_m(z_n)} , \quad z \in \mathbb{C} . \tag{12}
$$

Using the uniform convergence of the series in (12) we deduce that this series defines a continuous function. Hence, taking the limit as $z \to z_m$ we obtain

$$g'(z_m) = \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{g(z_n)}{z_n - z_m} \frac{A(z_m)}{A(z_n)} \frac{Q_m(z_m)}{(z_m - z_n)Q'_m(z_n)} \tag{13}$$

Now we prove that

$$\frac{g(z)}{z - z_m} = \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{g(z_n)}{z_n - z_m} \frac{A(z)}{A(z_n)} \frac{P(z)}{(z - z_n)P'(z_n)} + g'(z_m) \frac{A(z)}{A(z_m)} \frac{P(z)}{(z - z_m)P'(z_m)} \,. \tag{14}$$

Indeed, substituting (13) into (14) we obtain

$$\begin{aligned}
&\sum_{\substack{n=1 \\ n \neq m}}^{\infty} \Big[\frac{g(z_n)}{z_n - z_m} \frac{A(z)}{A(z_n)} \frac{P(z)}{(z - z_n)P'(z_n)} + \frac{g(z_n)}{z_n - z_m} \frac{A(z)}{A(z_n)} \frac{Q_m(z)}{(z_m - z_n)Q'_m(z_n)} \Big] \\
&= \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{g(z_n)}{z_n - z_m} \frac{A(z)}{A(z_n)} \frac{Q_m(z)}{Q'_m(z_n)} \Big[\frac{z - z_m}{(z_n - z_m)(z - z_n)} - \frac{1}{z_n - z_m} \Big] \\
&= \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{g(z_n)}{z_n - z_m} \frac{A(z)}{A(z_n)} \frac{Q_m(z)}{(z - z_n)Q'_m(z_n)} \\
&= \frac{g(z)}{z - z_m} \,.
\end{aligned}$$

Thus, defining $y \in \mathbb{H}$ by its Fourier coefficients $\{\langle y, e_n\rangle\}_{n=1}^{\infty}$ in $\ell^2(\mathbb{N})$ as

$$\langle y, e_n\rangle := \begin{cases} \frac{\langle x, e_n\rangle}{\overline{z}_n - \overline{z}_m} & \text{if } n \neq m \\ \\ \overline{g'(z_m)} & \text{if } n = m \end{cases}$$

and proceeding as in case (i), it may be shown that

$$\frac{g(z)}{z - z_m} = \langle K(z), y\rangle \quad z \in \mathbb{C} \,,$$

which proves that $g(z)/(z - z_m)$ belongs to $\mathcal{H}_K$. This concludes the proof of the theorem. ■

It is worth noticing that in the proof of Theorem 2 we have found the relationship between the entire functions A and P appearing in the quasi Lagrange-type interpolation formula; P is an entire function having simple zeros at $\{z_n\}_{n=1}^{\infty}$ and A is an entire function without zeros satisfying

$$(z - z_n)S_n(z) = \sigma_n A(z)P(z) \,,\ z \in \mathbb{C} \,, \quad \text{for all } n \in \mathbb{N} \,,$$

for some sequence $\{\sigma_n\}_{n=1}^{\infty}$ in $\mathbb{C} \setminus \{0\}$.

4 Quasi Lagrange-type interpolation and de Branges spaces

The classical Paley-Wiener space $PW_{\pi\sigma} := \big\{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}),\ \text{supp}\ \widehat{f} \subseteq [-\pi\sigma, \pi\sigma]\big\}$, where $\widehat{f}$ stands for the Fourier transform of f, clearly satisfies the ZR property. Indeed, the Whittaker-Shannon-Kotel'nikov sampling formula for $f \in PW_{\pi\sigma}$ ([12, 20, 21])

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\sigma}\right) \frac{\sin \pi(\sigma t - n)}{\pi(\sigma t - n)}\,, \quad t \in \mathbb{R}\,,$$

can be written as a Lagrange-type interpolation series by taking $P(t) = \sin \pi\sigma t/\pi$. Without using Theorem 2, the ZR property follows from the characterization of $PW_{\pi\sigma}$ which uses the classical Paley–Wiener Theorem [20, p.101], i.e.,

$$PW_{\pi\sigma} = \{f \in \mathcal{H}(\mathbb{C}) \ : \ |f(z)| \leq Ae^{\pi\sigma|z|}, \quad f|_{\mathbb{R}} \in L^2(\mathbb{R})\}\,.$$

As pointed out in [17, p. 234], Paley-Wiener spaces can be seen as special cases of a more general theory of Hilbert spaces of entire functions due to de Branges [4, p. 50]:

Definition 4 *Let E be an entire function verifying $|E(x - iy)| < |E(x + iy)|$ for all $y > 0$. The de Branges space $\mathcal{H}(E)$ is the set of all entire functions F such that*

$$\|F\|_E^2 := \int_{-\infty}^{\infty} \left|\frac{F(t)}{E(t)}\right|^2 dt < \infty,$$

and such that both ratios F/E and F^*/E, where F^* denotes the function $F^*(z) := \overline{F(\overline{z})}$, are of bounded type and of nonpositive mean type in the upper half-plane.

A de Branges space $\mathcal{H}(E)$ is a reproducing kernel Hilbert space of entire functions. In particular, the Paley-Wiener space $PW_{\pi\sigma}$ corresponds to de Branges space $\mathcal{H}(E_\sigma)$ where $E_\sigma(z) = \exp(-i\pi\sigma z)$. The following characterization of a de Branges space can be found in [4, p. 57]:

Theorem 3 *A Hilbert space $\mathcal{H}$ of entire functions is equal isometrically to some de Brange space $\mathcal{H}(E)$ if and only if the following conditions hold:*

B1. Whenever $f \in \mathcal{H}$ and ω is a nonreal zero of f, the function

$$g(z) := \frac{z - \overline{\omega}}{z - \omega}\ f(z)$$

belongs to $\mathcal{H}$ and $\|g\| = \|f\|$.

B2. For each $\omega \notin \mathbb{R}$ the linear mapping $\mathcal{H} \ni f \to f(\omega)$ is continuous.

B3. The function f^* belongs to the space, and $\|f^*\| = \|f\|$.

Whenever the RKHS $\mathcal{H}_K$ associated with a Kramer kernel K is (equal isometrically to) a de Branges space $\mathcal{H}(E)$ such that the entire function E has no real zeros, then the space $\mathcal{H}_K$ satisfies the ZR property [4, p. 52]. As a consequence, the sampling formula (6) in $\mathcal{H}_K$ can be written as a quasi Lagrange-type interpolation formula. There exists a form of converse result in the case when the sequence $\{z_n\}_{n=1}^\infty$ is real, and the functions A and P are real for real z:

Theorem 4 *If the sampling formula in $\mathcal{H}_K$ can be written as a quasi Lagrange-type interpolation formula where $A^* = A$, $P^* = P$ and the sampling points $\{z_n\}_{n=1}^\infty$ are real, then $\mathcal{H}_K$ is a de Branges space.*

Proof: In our case, property B2 holds because $\mathcal{H}_K$ is a RKHS. For B3, consider $f \in \mathcal{H}_K$ such that $f(z) = \langle K(z), x\rangle$ for some $x \in \mathbb{H}$. Then, $f^*(z) = \overline{\langle K(\overline{z}), x\rangle} = \sum_{n=1}^\infty S_n^*(z)\langle x, e_n\rangle$. From (7) we find that $\langle x, e_n\rangle = f^*(z_n)/\overline{a}_n$, $n \in \mathbb{N}$. Since

$$S_n(z) = a_n \frac{A(z)}{A(z_n)} \frac{P(z)}{(z - z_n)P'(z_n)}\,, \quad n \in \mathbb{N}\,,$$

$A^* = A$, $P^* = P$ and $\{z_n\}_{n=1}^\infty \subset \mathbb{R}$ we obtain

$$S_n^*(z) = \overline{a}_n \frac{A(z)}{A(z_n)} \frac{P(z)}{(z - z_n)P'(z_n)}\,, \quad n \in \mathbb{N}\,.$$

Thus we get

$$f^*(z) = \sum_{n=1}^\infty f^*(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{(z - z_n)P'(z_n)}\,. \tag{15}$$

Let y be in $\mathbb{H}$ such that its Fourier coefficients with respect to the orthonormal basis $\{e_n\}_{n=1}^\infty$ are

$$\langle y, e_n\rangle = \frac{a_n}{\overline{a}_n} \overline{\langle x, e_n\rangle}\,, \quad n \in \mathbb{N}\,.$$

The function $g(z) := \langle K(z), y\rangle$ in $\mathcal{H}_K$ satisfies $g(z_n) = a_n \overline{\langle y, e_n\rangle} = \overline{a}_n\langle x, e_n\rangle = f^*(z_n)$ for each $n \in \mathbb{N}$. Taking into account (15) we conclude that $f^* = g$ and, as a consequence, $f^* \in \mathcal{H}_K$. Moreover,

$$\|f^*\|^2 = \|y\|_{\mathbb{H}}^2 = \sum_{n=1}^\infty |\langle y, e_n\rangle|^2 = \sum_{n=1}^\infty |\langle x, e_n\rangle|^2 = \|x\|_{\mathbb{H}}^2 = \|f\|^2$$

Finally we prove property B1. To this end, consider $f \in \mathcal{H}_K$ given by $f(z) = \langle K(z), x\rangle$ for some $x \in \mathbb{H}$, and such that $f(w) = 0$ where $w \in \mathbb{C}\setminus\mathbb{R}$. Since $A(w)P(w) \neq 0$, the quasi Lagrange-type interpolation formula for f gives

$$\sum_{n=1}^\infty \frac{f(z_n)}{A(z_n)(w - z_n)P'(z_n)} = 0\,.$$

Therefore,

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} f(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{(z-z_n)P'(z_n)} - \sum_{n=1}^{\infty} f(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{(w-z_n)P'(z_n)} \\ &= (w-z) \sum_{n=1}^{\infty} f(z_n) \frac{A(z)}{A(z_n)} \frac{P(z)}{P'(z_n)} \frac{1}{(z-z_n)(w-z_n)} \,. \end{aligned}$$

As a consequence, we obtain

$$\frac{f(z)}{z-w} = \sum_{n=1}^{\infty} \frac{f(z_n)}{(z_n-w)} \frac{A(z)}{A(z_n)} \frac{P(z)}{(z-z_n)P'(z_n)} \,. \tag{16}$$

Since

$$\frac{z-\overline{w}}{z-w} f(z) = f(z) + (w-\overline{w}) \frac{f(z)}{z-w} \,,$$

the function $[(z-\overline{w})/(z-w)]f(z)$ belongs to $\mathcal{H}_K$ if and only if the function $f(z)/(z-w)$ belongs to $\mathcal{H}_K$ which follows from Theorem 2. As in the proof of Theorem 2, the function $g \in \mathcal{H}_K$ defined by $g(z) := \langle K(z), y \rangle$, where $y \in \mathbb{H}$ has Fourier coefficients with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$

$$\langle y, e_n \rangle = \frac{1}{z_n - \overline{w}} \langle x, e_n \rangle \,, \quad n \in \mathbb{N} \,,$$

coincides with the entire function $f(z)/(z-w)$. Moreover,

$$\begin{aligned} \left\| \frac{z-\overline{w}}{z-w} f(z) \right\|^2 &= \|f + (w-\overline{w})g\|^2 = \|x + (\overline{w}-w)y\|_{\mathbb{H}}^2 = \\ &= \sum_{n=1}^{\infty} |\langle x + (\overline{w}-w)y, e_n \rangle|^2 = \sum_{n=1}^{\infty} \left| \frac{z_n - w}{z_n - \overline{w}} \right|^2 |\langle x, e_n \rangle|^2 = \|x\|_{\mathbb{H}}^2 = \|f\|^2 \,, \end{aligned}$$

which concludes the proof.

■

In closing the paper, it is worth pointing out that, taking advantage of the general theory of de Branges spaces, we could obtain a characterization of RKHS $\mathcal{H}_K$ independently of the anti-linear transform T (4) (see the characterization of a de Branges space in [4, p. 53]).

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