Oversampling and reconstruction functions with compact support

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Abstract

Assume that a sequence of samples of a filtered version of a function in a shift-invariant space is given. This paper deals with the existence of a sampling formula involving these samples and having reconstruction functions with compact support. This is done in the light of the generalized sampling theory by using the oversampling technique. A necessary and sufficient condition is given in terms of the Smith canonical form of a polynomial matrix. Finally, we prove that the aforesaid oversampled formulas provide nice approximation schemes with respect to the uniform norm.

Keywords: Shift-invariant spaces; Oversampling; Generalized sampling; Smith canonical form.
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1 Statement of the problem

Let V_φ be a shift-invariant space in $L^2(\mathbb{R})$ with stable generator $\varphi \in L^2(\mathbb{R})$, i.e.,

$$V_\varphi := \Big\{ f(t) = \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n) \;:\; \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R})\,.$$

Nowadays, sampling theory in shift-invariant spaces is a very active research topic (see, for instance, [1, 2, 3, 4, 9, 17, 18] and references therein) since an appropriate choice for the generator φ (for instance, a B-spline) eliminates most of the problems associated with the classical Shannon's sampling theory [16].

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Suppose that a linear time-invariant system $\mathcal{L}$ is defined on V_φ. Under suitable conditions, Unser and Aldroubi [3, 15] have found sampling formulas allowing the recovering of any function $f \in V_\varphi$ from the sequence of samples $\{(\mathcal{L}f)(n)\}_{n\in\mathbb{Z}}$. Concretely, they proved that for any $f \in V_\varphi$,

$$f(t) = \sum_{n\in\mathbb{Z}} \mathcal{L}f(n) S(t-n), \quad t \in \mathbb{R}, \tag{1}$$

where the sequence $\{S(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. Even when the generator φ has compact support, rarely the same occurs with the reconstruction function S in formula (1). Recall that a reconstruction function S with compact support in (1) implies low computational complexities and avoids truncation errors. A way to overcome this difficulty is to use the oversampling technique, i.e, to take samples with a sampling period $T < 1$. This is the main goal in this paper: Assuming that the generator φ and the impulse response of the linear system $\mathcal{L}$ have compact support, we derive stable sampling formulas which allow to recover any $f \in V_\varphi$ from the samples $\{(\mathcal{L}f)(T_s n)\}_{n\in\mathbb{Z}}$, where the sampling period is $T_s := (s-1)/s < 1$ for some $s \in \{2, 3, \dots\}$. This is done in Sections 2,3 in the light of the generalized sampling theory obtained in [10] by following an idea of Djokovic and Vaidyanathan in [9].

For the sake of notational ease we have assumed that only samples from one linear time-invariant system $\mathcal{L}$ are avalaible. Analogous results are still valid in the case of several systems. In [7], a different but related question is studied: Roughly speaking, assuming that φ has compact support a system $\mathcal{L}$ with impulse response compactly supported is found in order to recover any function in V_φ by using the generator itself as the reconstruction function.

Besides, shift-invariant spaces are important in a number of areas of analysis. Many spaces, encountered in approximation theory and in finite element analysis, are generated by the integer shifts of a function φ. Shift-invariant spaces also play a key role in the construction of wavelets [13]. In each of these applications, one is interested in how well a general smooth function (in a potencial Sobolev space) can be approximated by the elements of the scaled spaces $\sigma_h V_\varphi := \{f(\cdot/h) : f \in V_\varphi\}$ (see [5] and references therein). A cornerstone in this theory are the Strang-Fix conditions for the generator φ [14].

On the other hand, as pointed out by Lei et al. in [12], there are many ways to construct approximation schemes associated with shift-invariant spaces. Among them, they cite cardinal interpolation, quasi-interpolation, projection and convolution (see references in [12]). They unify these approaches in a systematic way by viewing all as special cases of the approximation scheme induced by an integral operator. Borrowing a result in [12], in Section 4 we prove that the oversampled formulas with compactly supported reconstruction functions obtained in Section 3 give "good" approximation schemes with respect to the sup norm.

2 A sampling formula in the oversampling setting

From now on, the function $\varphi \in L^2(\mathbb{R})$ is a stable generator for the shift-invariant space

$$V_\varphi := \Big\{ f(t) = \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n) \,:\, \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R}),$$

i.e., the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ if and only if

$$0 < \|\Phi\|_0 \le \|\Phi\|_\infty < \infty\,,$$

where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2$ in $(0,1)$, and $\|\Phi\|_\infty$ its essential supremum. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds [6, p. 143].

We assume throughout the paper that the functions in the shift-invariant space V_φ are continuous on $\mathbb{R}$. Equivalently, the generator φ is continuous on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ (see [18]). Thus, any $f \in V_\varphi$ is defined as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n \varphi(t-n)$ on $\mathbb{R}$. Besides, V_φ is a reproducing kernel Hilbert space where convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence which is uniform on $\mathbb{R}$ (see [10]).

The space V_φ is the image of $L^2(0,1)$ by means of the isomorphism $\mathcal{T}_\varphi : L^2(0,1) \to V_\varphi$ which maps the orthonormal basis $\{e^{-2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{\varphi(t - n)\}_{n\in\mathbb{Z}}$ for V_φ. Namely, for each $F \in L^2(0,1)$ the function $\mathcal{T}_\varphi F \in V_\varphi$ is given by

$$(\mathcal{T}_\varphi F)(t) := \sum_{n\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi i n \cdot} \big\rangle_{L^2(0,1)} \varphi(t-n), \quad t \in \mathbb{R}\,. \tag{2}$$

Suppose that $\mathcal{L}$ is a linear time-invariant system defined on V_φ of one of the following types (or a linear combination of both):

(a) The impulse response h of $\mathcal{L}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, for any $f \in V_\varphi$ we have

$$\big(\mathcal{L}f\big)(t) := [f * \mathsf{h}](t) = \int_{-\infty}^{\infty} f(x)\mathsf{h}(t-x)dx\,, \quad t \in \mathbb{R}\,.$$

(b) $\mathcal{L}$ involves samples of the function itself, i.e., $(\mathcal{L}f)(t) = f(t+d)$, $t \in \mathbb{R}$, for some constant $d \in \mathbb{R}$.

For a fixed $s \in \{2,3,\ldots\}$, consider $T_s = (s-1)/s < 1$. The first goal is to recover any function $f \in V_\varphi$ by using a frame expansion involving the samples $\big\{(\mathcal{L}f)(T_s n)\big\}_{n\in\mathbb{Z}}$. This can be done in the light of the generalized sampling theory developed in [10]. Indeed, since the sampling points $T_s n$, $n \in \mathbb{Z}$, can be expressed as

$$\big\{T_s n\big\}_{n\in\mathbb{Z}} = \big\{(s-1)n + (j-1)T_s\big\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}\,,$$

the initial problem is equivalent to the recovery of $f \in V_\varphi$ from the samples

$$\{\mathcal{L}_j f\big((s-1)n\big)\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$$

where the linear time-invariant systems $\mathcal{L}_j$, $j = 1,2,\ldots,s$, are defined by

$$(\mathcal{L}_j f)(t) := (\mathcal{L}f)\big[t + (j-1)T_s\big]\,, \quad t \in \mathbb{R}\,.$$

Following the notation introduced in [10], consider the functions $g_j \in L^2(0,1)$, $j = 1,2,\ldots,s$, defined as

$$g_j(w) := \sum_{n\in\mathbb{Z}} (\mathcal{L}\varphi)\big[n + (j-1)T_s\big] e^{-2\pi i n w}\,, \tag{3}$$

the $s \times (s-1)$ matrix

$$\mathbf{G}_s(w) := \begin{bmatrix} g_1(w) & g_1(w+\frac{1}{s-1}) & \cdots & g_1(w+\frac{s-2}{s-1}) \\ g_2(w) & g_2(w+\frac{1}{s-1}) & \cdots & g_2(w+\frac{s-2}{s-1}) \\ \vdots & \vdots & & \vdots \\ g_s(w) & g_s(w+\frac{1}{s-1}) & \cdots & g_s(w+\frac{s-2}{s-1}) \end{bmatrix} = \Big[g_j\Big(w+\frac{k-1}{s-1}\Big)\Big]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,s-1}},$$

(in the sequel we omit the subscript s) and its related constants

$$\alpha_{\mathbf{G}} := \operatorname*{ess\,inf}_{w\in(0,1/(s-1))} \lambda_{\min}[\mathbf{G}^*(w)\mathbf{G}(w)], \quad \beta_{\mathbf{G}} := \operatorname*{ess\,sup}_{w\in(0,1/(s-1))} \lambda_{\max}[\mathbf{G}^*(w)\mathbf{G}(w)],$$

where $\mathbf{G}^*(w)$ denotes the transpose conjugate of the matrix $\mathbf{G}(w)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbf{G}^*(w)\mathbf{G}(w)$. Notice that in the definition of the matrix $\mathbf{G}(w)$ we are considering the 1-periodic extensions of the involved functions g_j, $j = 1, 2, \dots, s$.

Thus, the generalized sampling theory in [10] (see Theorem 1, Theorem 2 and its proof) gives the following sampling result in V_φ:

Theorem 1 *Assume that the functions g_j defined in* (3) *belong to* $\in L^\infty(0,1)$ *for each* $j = 1, 2, \dots, s$ *(this is equivalent to $\beta_{\mathbf{G}} < \infty$). Then the following statements are equivalent:*

(i) $\alpha_{\mathbf{G}} > 0$

(ii) There exist functions a_j in $L^\infty(0,1)$, $j = 1, 2, \dots, s$, such that

$$\big[a_1(w), \dots, a_s(w)\big]\, \mathbf{G}(w) = [1, 0, \dots, 0] \quad a.e. \ in \ (0,1). \tag{4}$$

(iii) There exists a frame for V_φ having the form $\{S_j(\cdot - (s-1)n)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ such that, for any $f \in V_\varphi$, we have

$$f = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}f)\big[(s-1)n + (j-1)T_s\big]\ S_j\big(\cdot - (s-1)n\big) \quad in \ L^2(\mathbb{R}). \tag{5}$$

In case the equivalent conditions are satisfied, the reconstruction functions in (5) *are given by $S_j = (s-1)\mathcal{T}_\varphi a_j$, $j = 1, 2, \dots, s$, where the functions a_j, $j = 1, 2, \dots, s$, satisfy* (4). *The convergence of the series in* (5) *is also absolute and uniform on* $\mathbb{R}$.

It is worth to mention that whenever the functions g_j, $j = 1, 2, \dots, s$, are continuous on $\mathbb{R}$, the conditions in Theorem 1 are also equivalent to the new condition:

$$(iv) \qquad \operatorname{rank} \mathbf{G}(w) = s-1 \ \text{ for all } w \in \mathbb{R}.$$

3 Searching for compactly supported reconstruction functions

The main aim in this section is to obtain reconstruction functions S_j, $j = 1, 2, \ldots, s$, in formula (5) with compact support. To this end, assume from now on that the generator φ and $\mathcal{L}\varphi$ have compact support. We introduce the $s \times (s-1)$ matrix

$$\mathsf{G}(z) := \begin{bmatrix} \mathsf{g}_1(z) & \mathsf{g}_1(Wz) & \cdots & \mathsf{g}_1(W^{s-2}z) \\ \mathsf{g}_2(z) & \mathsf{g}_2(Wz) & \cdots & \mathsf{g}_2(W^{s-2}z) \\ \vdots & \vdots & & \vdots \\ \mathsf{g}_s(z) & \mathsf{g}_s(Wz) & \cdots & \mathsf{g}_s(W^{s-2}z) \end{bmatrix} \tag{6}$$

where $W := e^{-2\pi i/(s-1)}$ and $\mathsf{g}_j(z) := \sum_{n\in\mathbb{Z}} (\mathcal{L}\varphi)\big[n + (j-1)T_s\big] z^n$, $j = 1, 2 \ldots, s$. Notice that the matrix $\mathsf{G}(z)$ has Laurent polynomials entries, and $\mathbf{G}(w) = \mathsf{G}(e^{-2\pi i w})$. On the other hand, if the functions $\mathsf{a}_j(z)$, $j = 1, 2 \ldots, s$, are Laurent polynomials satisfying

$$[\mathsf{a}_1(z), \ldots, \mathsf{a}_1(z)]\mathsf{G}(z) = [1, 0, \ldots, 0]\,, \tag{7}$$

then, the trigonometric polynomials $a_j(w) = \mathsf{a}_j(e^{-2\pi i w})$, $j = 1, 2, \ldots, s$, satisfy (4). In this case, the corresponding reconstruction functions S_j, $j = 1, 2, \ldots, s$, have compact support. Indeed, in terms of the coefficients $c_{j,n}$ of $\mathsf{a}_j(z)$, that is, $\mathsf{a}_j(z) = \sum_{n\in\mathbb{Z}} c_{j,n} z^n$, $j = 1, 2, \ldots, s$, the reconstruction function S_j, $j = 1, 2, \ldots, s$, can be written as (see (2))

$$S_j(t) = (s-1) \sum_{n\in\mathbb{Z}} c_{j,n} \varphi(t-n)\,, \quad t \in \mathbb{R}\,. \tag{8}$$

In the sequel we refer to a polynomial matrix (respectively a polynomial vector) for a matrix (respectively a vector) having Laurent polynomial entries. We are interested in finding polynomial solutions of (7) having a small number of nonzero coefficients.

3.1 A theoretical answer via the Smith canonical form

The existence of polynomial solutions of (7) is equivalent to the existence of a left inverse of the matrix $\mathsf{G}(z)$ whose entries are polynomials. This problem has been studied by Cvetković and Vetterli in [8] in the filter banks setting. Applying their result, we obtain a characterization for the existence of polynomial solutions of (7) using the Smith canonical form of the matrix $\mathsf{G}(z)$.

Recall that any $m \times n$ ($m \geq n$) polynomial matrix $\mathsf{H}(z)$ with rank $\mathsf{H}(z) = r$ (recall that the rank of a polynomial matrix is the order of its largest minor that is not equal to the zero polynomial) can be written as the product $\mathsf{H}(z) = \mathbf{V}(z)\mathsf{S}(z)\mathbf{W}(z)$ where $\mathbf{V}(z)$ an $\mathbf{W}(z)$ are unimodular matrices (i.e., the determinants of $\mathbf{V}(z)$ and $\mathbf{W}(z)$ are nonzero constants) of dimension $m \times m$ and $n \times n$ respectively and $\mathsf{S}(z)$ is a diagonal $m \times n$ polynomial matrix $\mathsf{S}(z) := \mathrm{diag}[i_1(z), \ldots, i_r(z), 0, \ldots, 0]$. Moreover, the diagonal entries (the so-called invariant polynomials of $\mathsf{H}(z)$) are given by $i_j(z) = d_j(z)/d_{j-1}(z)$, $j = 1, 2, \ldots, r$, where $d_j(z)$ is the greatest common divisor of all minors of $\mathsf{H}(z)$, $j = 1, 2, \ldots, r$ and $d_0(z) \equiv 1$. The matrix $\mathsf{S}(z)$ is called the Smith canonical form of the matrix $\mathsf{H}(z)$. See [11] for the details.

Assume that the $s \times (s-1)$ matrix

$$\mathsf{S}(z) = \begin{bmatrix} i_1(z) & 0 & \cdots & 0 \\ 0 & i_2(z) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & i_{s-1}(z) \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{9}$$

is the Smith canonical form of the matrix $\mathsf{G}(z)$ (notice that it is the case whenever $\alpha_{\mathbf{G}} > 0$) and consider the unimodular matrices $\mathbf{V}(z)$ and $\mathbf{W}(z)$, of dimension $s \times s$ and $(s-1) \times (s-1)$ respectively, such that $\mathsf{G}(z) = \mathbf{V}(z)\mathsf{S}(z)\mathbf{W}(z)$. The following result holds:

Theorem 2 *Assume that the generator φ and $\mathcal{L}\varphi$ have compact support. Then, there exists a polynomial vector $[\mathsf{a}_1(z), \mathsf{a}_2(z), \cdots, \mathsf{a}_s(z)]$ satisfying (7) if and only if the polynomials $i_j(z)$, $j = 1, 2, \dots, s-1$, on the diagonal of the Smith canonical form (9) of the matrix $\mathsf{G}(z)$ are monomials. Moreover, the polynomial solutions of (7) are the first row of the $(s-1) \times s$ polynomial matrices $R(z)$ having the form*

$$R(z) = R_0(z) + U(z)[\mathbf{I}_s - \mathsf{G}(z)R_0(z)]$$

where $U(z)$ is any $(s-1) \times s$ polynomial matrix and

$$R_0(z) := \mathbf{W}^{-1}(z) \begin{bmatrix} i_1^{-1}(z) & 0 & \cdots & 0 & 0 \\ 0 & i_2^{-1}(z) & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & i_{s-1}^{-1}(z) & 0 \end{bmatrix} \mathbf{V}^{-1}(z)$$

Proof: If the diagonal entries $i_j(z)$, $j = 1, 2, \dots, s-1$ are monomials and $U(z)$ is a $(s-1) \times s$ polynomial matrix then the entries of the matrix $R(z) = R_0(z) + U(z)[\mathbf{I}_s - \mathsf{G}(z)R_0(z)]$ are Laurent polynomials. It can be checked that this matrix satisfies $R(z)\mathsf{G}(z) = \mathbf{I}_{s-1}$. Therefore, the first row of $R(z)$ satisfies (7).

Conversely, if the polynomial vector $[\mathsf{a}_1(z), \mathsf{a}_2(z), \cdots, \mathsf{a}_s(z)]$ satisfies (7) then the matrix $R(z) := \left[\mathsf{a}_j(zW^{k-1})\right]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,s-1}}$ is a polynomial matrix and it satisfies $R(z)\mathsf{G}(z) = \mathbf{I}_{s-1}$. The argument given in [8, Appendix E] proves that $i_j(z)$, $j = 1, 2, \dots, s-1$ are monomials. Moreover, $\mathsf{a}(z)$ is the first row of the polynomial matrix $R(z)$ which can be written as $R(z) := R_0(z) + U(z)[\mathbf{I}_s - \mathsf{G}(z)R_0(z)]$ by taking $U(z) = R(z)$. □

There is an equivalent characterization for the existence of polynomial solutions of (7) which involves the rank of the matrix $\mathsf{G}(z)$ for each $z \in \mathbb{C}$. Notice that if $\mathsf{S}(z)$ is the Smith form of the matrix $\mathsf{G}(z)$ then, taking into account that $\mathbf{V}(z)$ and $\mathbf{W}(z)$ are unimodular matrices, we have

$$\operatorname{rank} \mathsf{S}(z) = \operatorname{rank} \mathsf{G}(z) \text{ for all } z \in \mathbb{C}.$$

Therefore, it is straightforward to deduce that, for each $j = 1, 2, \dots, s-1$, the Laurent polynomial $i_j(z)$ is a monomial if and only if $\operatorname{rank} \mathsf{S}(z) = s-1$ for all $z \in \mathbb{C} \setminus \{0\}$. As a consequence we obtain the following result:

Theorem 3 *Assume that the generator φ and the $\mathcal{L}\varphi$ have compact support. Then, exists a matrix $\mathsf{A}(z)$ whose entries are Laurent polynomials and satisfying $\mathsf{A}(z)\mathsf{G}(z) = \mathbf{I}_{s-1}$ if and only if*

$$\operatorname{rank} \mathsf{G}(z) = s-1 \text{ for all } z \in \mathbb{C} \setminus \{0\}.$$

If $\mathsf{a}(z)$ is the first row of $\mathsf{A}(z)$ then, the reconstruction functions S_j, $j = 1, 2, \ldots, s$, obtained from $\mathsf{a}(z)$ through equation (8) have compact support.

From a practical point of view the Smith canonical form method for solving (7) has some important drawbacks. First, it is not an easy task to compute the Smith canonical form of the matrix $\mathsf{G}(z)$; the polynomial solution given by the first row of the matrix $R_0(z)$ has often a high degree which implies that the corresponding reconstruction functions have long supports; and finally, it is by no means straightforward to find a polynomial matrix $U(z)$ to improve the situation.

3.2 Checking the Smith canonical form condition

The aim here is to study when the Smith canonical form of the matrix $\mathsf{G}(z)$ has monomials in its diagonal for some important cases. Instead of computing directly the Smith canonical form of the matrix $\mathsf{G}(z)$, we compute the Smith canonical form of a simpler related matrix to it. This computation is based on the following decomposition of the matrix $\mathsf{G}(z)$. Without loss of generality, we assume that $\operatorname{supp} \mathcal{L}\varphi \subseteq [0, N]$ for some $N \in \mathbb{N}$; otherwise we might consider an appropriated shifted system.

Since $j-1 \leq jT_s \leq j$, for $j = 0, 1, \ldots, s-1$, the functions $\mathsf{g}_j(z)$ have the form:

$$\begin{aligned}
\mathsf{g}_1(z) &= \mathcal{L}\varphi(1)z + \mathcal{L}\varphi(2)z^2 + \cdots + \mathcal{L}\varphi(N-1)z^{N-1} \\
\mathsf{g}_2(z) &= \mathcal{L}\varphi(T_s) + \mathcal{L}\varphi(1+T_s)z + \cdots + \mathcal{L}\varphi(N-1+T_s)z^{N-1} \\
\mathsf{g}_3(z) &= \mathcal{L}\varphi(2T_s-1)z^{-1} + \mathcal{L}\varphi(2T_s) + \cdots + \mathcal{L}\varphi(N-2+2T_s)z^{N-2} \\
&\ \vdots \\
\mathsf{g}_s(z) &= \mathcal{L}\varphi((s-1)T_s-(s-2))z^{-(s-2)} + \cdots + \mathcal{L}\varphi(N-(s-1)+(s-1)T_s)z^{N-(s-1)}\,.
\end{aligned} \tag{10}$$

We factorize the matrix $\mathsf{G}(z)$ as $\mathsf{G}(z) = [\Gamma_1|\Gamma_2]\mathsf{Z}(z)$, where the matrix $\Gamma_1 \in \mathbb{C}^{s\times s}$ is given by

$$\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \mathcal{L}\varphi(1) \\
0 & 0 & \cdots & 0 & \mathcal{L}\varphi(T_s) & \mathcal{L}\varphi(1+T_s) \\
0 & 0 & \cdots & \mathcal{L}\varphi(2T_s-1) & \mathcal{L}\varphi(2T_s) & \mathcal{L}\varphi(1+2T_s) \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
\mathcal{L}\varphi((s-1)T_s-(s-2)) & \cdots & & \mathcal{L}\varphi(N-(s-1)+(s-1)T_s) & 0 & 0
\end{bmatrix},$$

the matrix $\Gamma_2 \in \mathbb{C}^{s\times(N-2)}$ is given by

$$\begin{bmatrix}
\mathcal{L}\varphi(2) & \cdots & \mathcal{L}\varphi(N-2) & \mathcal{L}\varphi(N-1) \\
\mathcal{L}\varphi(2+T_s) & \cdots & \mathcal{L}\varphi(N-2+T_s) & \mathcal{L}\varphi(N-1+T_s) \\
\mathcal{L}\varphi(2+2T_s) & \cdots & \mathcal{L}\varphi(N-2+2T_s) & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0
\end{bmatrix}$$

and the matrix $\mathsf{Z}(z) \in \mathbb{C}^{(N+s-2)\times(s-1)}$ is given by

$$\mathsf{Z}(z) = \begin{bmatrix} z^{-(s-2)} & (zW)^{-(s-2)} & \cdots & (zW^{s-2})^{-(s-2)} \\ \vdots & \vdots & & \vdots \\ z^0 & (zW)^0 & \cdots & (zW^{s-2})^0 \\ \vdots & \vdots & & \vdots \\ z^{N-1} & (zW)^{N-1} & \cdots & (zW^{s-2})^{N-1} \end{bmatrix}$$

Since Γ_1 is a lower triangular matrix it is straightforward to check when it is invertible. Suppose that Γ_1 is invertible; there exists a $s \times (N-2)$ matrix Λ such that $\Gamma_2 = \Gamma_1 \Lambda$. Therefore, splitting the matrix $\mathsf{Z}(z)$ into blocks we have

$$\mathsf{G}(z) = \Gamma \mathsf{Z}(z) = [\Gamma_1|\Gamma_2]\mathsf{Z}(z) = \Gamma_1[\mathbf{I}_s|\Lambda]\begin{bmatrix} \mathsf{Z}_1(z) \\ \mathsf{Z}_2(z) \end{bmatrix} = \Gamma_1\big(\mathsf{Z}_1(z) + \Lambda \mathsf{Z}_2(z)\big)$$

Since Γ_1 is invertible, the Smith canonical forms of the matrices $\mathsf{G}(z)$ and $\mathsf{Z}_1(z)+\Lambda\mathsf{Z}_2(z)$ coincide. Using this argument we deduce when the Smith canonical form of $\mathsf{G}(z)$ has monomials in its diagonal in two important examples:

Case I: $\operatorname{supp}\mathcal{L}\varphi$ **is a subset of** $[0,2]$

Here $\Gamma_1 = \Gamma$ and $\mathsf{G}(z) = \Gamma_1\mathsf{Z}(z)$. Since the Smith canonical form of $\mathsf{Z}(z)$ has monomials in the diagonal, we conclude that the Smith canonical form of $\mathsf{G}(z)$ has monomials in its diagonal.

Case II: $\operatorname{supp}\mathcal{L}\varphi$ **is a subset of** $[0,3]$

In this case $\Lambda = [a_1\, a_2\, \ldots\, a_s]^T$ is a $s \times 1$ matrix and $\mathsf{Z}_1(z) + \Lambda\mathsf{Z}_2(z) = z^{-(s-2)}\Upsilon(z)$, where

$$\Upsilon(z) = \begin{bmatrix} 1 + a_1 z^s & W^{-(s-2)} + a_1 W^2 z^s & \cdots & W^{-(s-2)^2} + a_1 W^{2(s-2)} z^s \\ z + a_2 z^s & W^{-(s-3)} z + a_2 W^2 z^s & \cdots & W^{-(s-2)(s-3)} z + a_2 W^{2(s-2)} z^s \\ \vdots & \vdots & & \vdots \\ z^{s-2} + a_{s-1} z^s & z^{s-2} + a_{s-1} W^2 z^s & \cdots & z^{s-2} + a_{s-1} W^{2(s-2)} z^s \\ z^{s-1} + a_s z^s & W z^{s-1} + a_s W^2 z^s & \cdots & W^{s-2} z^{s-1} + a_s W^{2(s-2)} z^s \end{bmatrix}$$

It is obvious that the Smith canonical form of $\Upsilon(z)$ has monomials in the diagonal if and only if the Smith canonical form of $\mathsf{Z}_1(z) + \Lambda\mathsf{Z}_2(z)$ has monomials in the diagonal. To compute the Smith canonical form of $\Upsilon(z)$ we reduce it by means of *elementary transformations*. An elementary row (column) transformation on a polynomial matrix is one of the following operations: multiply any row (column) by a nonzero $c \in \mathbb{C}$; interchange any two rows (columns); add to any row (column) any other row (column) multiplied by an arbitrary polynomial $p(z)$. Performing an elementary transformation on a matrix does not change its Smith canonical form [11]. After some of these elementary column operations on $\Upsilon(z)$ we obtain the equivalent $s \times (s-1)$ matrix

$$\Delta(z) = \begin{bmatrix} 1 + a_1 z^s & W^{s-2} + a_1 z^s & \cdots & W + a_1 z^s \\ z + a_2 z^s & z + a_2 z^s & \cdots & z + a_2 z^s \\ \vdots & \vdots & & \vdots \\ z^{s-2} + a_{s-1} z^s & W^{s-3} z^{s-2} + a_{s-1} z^s & \cdots & W^{-2(s-2)} z^{s-2} + a_{s-1} z^s \\ z^{s-1} + a_s z^s & W^{s-2} z^{s-1} + a_s z^s & \cdots & W z^{s-1} + a_s z^s \end{bmatrix}$$

All the $s-1$-minors of $\Delta(z)$ containing the second row have as a factor the polynomial $1+a_2z^{s-1}$. We claim that, if the remainder $s-1$ minor does not have the polynomial $1+a_2z^{s-1}$ as a factor, then the polynomials in the diagonal of the Smith canonical form of $\Delta(z)$ are monomials. Indeed, the $s\times(s-1)$ matrix

$$\Theta(z)=\begin{bmatrix} 1+a_1z^s & W^{s-2}+a_1z^s & \cdots & W+a_1z^s \\ z & z & \cdots & z \\ \vdots & \vdots & & \vdots \\ z^{s-2}+a_{s-1}z^s & W^{s-3}z^{s-2}+a_{s-1}z^s & \cdots & W^{-2(s-2)}z^{s-2}+a_{s-1}z^s \\ z^{s-1}+a_sz^s & W^{s-2}z^{s-1}+a_sz^s & \cdots & Wz^{s-1}+a_sz^s \end{bmatrix}$$

is equal to $\Delta(z)$ except in the second row. Moreover, the polynomial matrix $\Theta(z)$ is equivalent to $z^{s-2}\mathsf{Z}(z)$ (recall that $N=3$) which trivially has monomials in the diagonal of its Smith canonical form. Summarizing we have the following result:
Assume that $\operatorname{supp}\mathcal{L}\varphi\subseteq[0,3]$, $\Gamma_1\in\mathbb{C}^{s\times s}$ *invertible. Let* $\Gamma_2=\Gamma_1\Lambda$ *with* $\Lambda=[a_1\,a_2\,\dots\,a_s]^T$. *If the* $s-1$ *minor obtained from* $\Delta(z)$ *by removing the second row does not have as a factor the polynomial* $1+a_2z^{s-1}$, *then the Smith canonical form of the matrix* $\mathsf{G}(z)$ *has monomials in its diagonal.*

The following example illustrates the result. Assume that $s=4$. In this case, the minor of order 3 which appears in the result is

$$\begin{vmatrix} 1+a_1z^4 & W^2+a_1z^4 & W+a_1z^4 \\ z^2+a_3z^4 & Wz^2+a_3z^4 & W^2z^2+a_3z^4 \\ z^3+a_4z^4 & W^2z^3+a_4z^4 & Wz^3+a_4z^4 \end{vmatrix}=3(W-W^2)z^6(a_4-a_1z^3)$$

As a consequence, if $a_4+\dfrac{a_1}{a_2}\neq 0$ then $1+a_2z^3$ is not a factor of the minor (if $a_2=0$ then it is straightforward to prove that the Smith form of $\mathsf{G}(z)$ has monomials in the diagonal).

3.3 An easy illustrative example

Let $\varphi(t):=N_3(t)$ be the quadratic B-spline

$$N_3(t):=\frac{t^2}{2}\mathcal{X}_{[0,1)}(t)+\Big(3t-t^2-\frac{3}{2}\Big)\mathcal{X}_{[1,2)}(t)+\frac{(3-t)^2}{2}\mathcal{X}_{[2,3)}(t)\,,$$

where $\mathcal{X}_{[a,b)}$ denotes the characteristic function of the interval $[a,b]$, and let $\mathcal{L}$ be the identity system. In this case, $\operatorname{supp}\mathcal{L}\varphi\subseteq[0,3]$. Taking $s=3$ (that is, $T_s=2/3$) we have the matrix

$$\mathsf{G}(z)=\begin{bmatrix} \frac{1}{2}\,z+\frac{1}{2}\,z^2 & -\frac{1}{2}\,z+\frac{1}{2}\,z^2 \\ \frac{2}{9}+\frac{13}{18}\,z+\frac{1}{8}\,z^2 & \frac{2}{9}-\frac{13}{18}\,z+\frac{1}{8}\,z^2 \\ \frac{1}{18}z^{-1}+\frac{13}{18}+\frac{2}{9}\,z & -\frac{1}{18}z^{-1}+\frac{13}{18}-\frac{2}{9}\,z^1 \end{bmatrix}$$

A polynomial solution for $[\mathsf{a}_1(z),\mathsf{a}_2(z),\mathsf{a}_3]\mathsf{G}(z)=[1,0]$, of degrees 7, 10 and 9 respectively, is obtained from the Smith canonical form of $\mathsf{G}(z)$ (see Theorem 2) computed using Maple™. This solution gives reconstruction functions S_j, $j=1,2,3$, supported on the intervals $[-7,2]$, $[-10,3]$ and $[-9,3]$ respectively.

3.4 A case-by-case practical solution: solving a linear system

In this section, a new approach to the problem to seek reconstruction functions of compact support is showed. The method is based on constructing and solving a linear system of equations. Let $[\mathsf{a}_1(z), \mathsf{a}_2(z), \cdots, \mathsf{a}_s(z)]$ be a solution of (7). Assume that $\mathsf{a}_j(z) = \sum_{n\in\mathbb{Z}} a_{j,n} z^n$ for $j = 1, 2, \ldots, s$ with just a finite set of nonzero coefficients $a_{j,n}$. Then, the matrix equation (7) leads to a system of linear equations. The key point is to choose a suitable finite set of nonzero coefficients in such a way that we obtain a compatible linear system. This method avoids the computation of the Smith canonical form of $\mathsf{G}(z)$ and, it gives polynomial solutions of (7) with less terms.

Without loss of generality, assume that $\operatorname{supp} \mathcal{L}\varphi \subseteq [0, N]$ for some $N \in \mathbb{N}$. Thus, the functions $\mathsf{g}_j(z)$ can be written as in (10). Let $p := Ns - s - 2N$; we try a solution $[\mathsf{a}_1(z), \mathsf{a}_2(z), \cdots, \mathsf{a}_s(z)]$ of (7) of the form

$$\mathsf{a}_1(z) = \sum_{n=-N-l'-u'}^{l+u} a_{1,n} z^n\,, \quad \mathsf{a}_j(z) = \sum_{n=-N+j-1-l'}^{j-2+l} a_{j,n} z^n\,, \quad j = 2, 3, \ldots, s\,,$$

where $l = l' = 0$ and $u = u' = -1$ if $p = -2$; $l = l' = 0$, $u = 0$ and $u' = -1$ if $p = -1$; $u = u' = 0$ and $l = l' = \frac{p}{2}$ if $p \geq 0$ is even and $u = u' = 0$, $l = \frac{p+1}{2}$, $l' = l - 1$ if $p \geq 0$ is odd (notice that, since $s \geq 2$ and $N \geq 1$ we have $p \geq -2$). This choice leads to a linear system as many equations as unknows which, in most of cases, comes to be compatible. Otherwise increasing by one l when $l = l'$ or l' when $l \neq l'$ leads to a new linear system with $s-1$ more equations and with s more unknows. Thus, whenever (7) has a polynomial solution, the above procedure gives a solution in a finite number of steps.

3.5 The example revisited

Consider again the example in Section 3.3. The above described method gives the following sampling result:
Any function $f \in V_{N_3}$ can be recovered through the sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} \big[f(2n)S_1(t-2n) + f(2n+2/3)S_2(t-2n) + f(2n+4/3)S_3(t-2n)\big]\,, \quad t \in \mathbb{R}\,,$$

where the reconstruction functions are given by

$$\begin{aligned}
S_1(t) &= \frac{1}{16}\big(N_3(t+3) - 3N_3(t+2) - 3N_3(t+1) + N_3(t)\big),\\
S_2(t) &= \frac{1}{16}\big(27N_3(t+1) - 9N_3(t)\big),\\
S_3(t) &= \frac{1}{16}\big(-9N_3(t+1) + 27N_3(t)\big)\,, \quad t \in \mathbb{R}\,.
\end{aligned}$$

In this case, the reconstruction functions S_j, $j = 1, 2, 3$, are supported on the intervals $[-3, 3]$, $[-1, 3]$ and $[-1, 3]$ respectively.

4 Uniform approximation by using oversampled generalized formulas

In this Section we deal with the uniform approximation property for the generalized sampling formulas appearing in Theorem 1. Concretely, associated with the sampling formula (5) we introduce the operator Γ, formally defined as,

$$(\Gamma f)(t) := \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}f)\big[(s-1)n + (j-1)T_s\big]\ S_j\big(t-(s-1)n\big)\,, \quad t \in \mathbb{R}\,. \tag{11}$$

Under appropriate hypotheses we prove that, if the generator φ satisfies the Strang-Fix conditions of order m, then the operator Γ provides approximation order m in the uniform norm for functions f in the Sobolev space $W^m_\infty(\mathbb{R}) = \big\{f : \|f^{(k)}\|_\infty < \infty, k = 0,1,2,\dots,m\big\}$, i.e.,

$$\|\Gamma_h f - f\|_\infty = \mathcal{O}(h^m) \quad \text{as} \;\; h \to 0^+\,,$$

where $\Gamma_h := \sigma_h \Gamma \sigma_{1/h}$ and $\sigma_h f := f(\cdot/h), \quad h > 0$.

At this point we introduce some further notation. Let $\mathcal{C}_b(\mathbb{R})$ be the Banach space of continuous bounded functions on $\mathbb{R}$ taken with the L^∞-norm. For a measurable function $f : \mathbb{R} \to \mathbb{C}$, we denote $|f|_\infty := \sup_{t\in[0,1)} \sum_{n\in\mathbb{Z}} |f(t+n)|$. Notice that $|f|_\infty < \infty$ for any continuous compactly supported function. Provided that $|\varphi|_\infty < \infty$, the L^∞-closure of the linear span of the integer shifts of φ can be also expressed as (see [12]):

$$V^\infty_\varphi = \Big\{ \sum_{n\in\mathbb{Z}} c_n \varphi(t-n) : \{c_n\}_{n\in\mathbb{Z}} \in \mathsf{c}_0(\mathbb{Z}) \Big\}\,,$$

where $\mathsf{c}_0(\mathbb{Z})$ denotes the Banach space of scalar sequences converging to zero taken with the norm $\|\{c_n\}\|_\infty := \sup_{n\in\mathbb{Z}} |c_n|$. Notice that $V^\infty_\varphi \subset \mathcal{C}_b(\mathbb{R})$. By $\widehat{\varphi}$ we denote the Fourier transform of the generator φ, $\widehat{\varphi}(w) := \int_{-\infty}^{\infty} \varphi(t) e^{-iwt} dt$.

Our approximation result is based on the following theorem whose proof can be found in [12, Theorem 5.2]:

Theorem 4 *Assume that* $\operatorname{ess\,sup}_{t\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\varphi(t+n)|(1+|t+n|)^m < \infty$ *for some* $m \in \mathbb{N}$. *If the generator* φ *satisfies the Strang-Fix conditions of order* m, *i.e.,*

$$\widehat{\varphi}(0) \neq 0, \quad \widehat{\varphi}^{(k)}(2\pi n) = 0, \quad k = 0,1,\dots,m-1, \quad n \in \mathbb{Z}\setminus\{0\}\,,$$

then, for each $f \in W^m_\infty(\mathbb{R})$ *and* $h > 0$ *there exists a function* $g \in \sigma_h V^\infty_\varphi$ *such that*

$$\|g - f\|_\infty \leq\ K\,\|f^{(m)}\|_\infty\ h^m\,,$$

where the constant K *is independent of* f *and* h.

Lemma 1 *Assume that the sampling function* S_j *satisfies* $|S_j|_\infty < \infty$ *for each* $j = 1,2,\dots,s$. *Then, the following statements hold:*

(a) The linear map $\Gamma : \mathcal{C}_b(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$ *defines a bounded operator.*

(b) For any $g \in V^\infty_\varphi$ *we have that* $\Gamma g = g$.

Proof: For $f \in \mathcal{C}_b(\mathbb{R})$, consider the sequence $m_{f,j}$ given by

$$\{m_{f,j}[n] := (\mathcal{L}f)\big((s-1)n + (j-1)T\big)\}_{n\in\mathbb{Z}} .$$

For $\mathcal{L}$ a linear system of the type $(\mathcal{L}f)(t) = f(t+d)$, $t \in \mathbb{R}$), trivially one has $\|m_{f,j}\|_{\ell^\infty} \leq \|f\|_\infty$; whenever $\mathcal{L}$ is a linear system of the type $\mathcal{L}f = f * \mathsf{h}$ $(\mathsf{h} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$ one has $\|m_{f,j}\|_{\ell^\infty} \leq \|\mathsf{h}\|_1 \|f\|_\infty$. Since $|S_j|_\infty < \infty$, the function Γf is well defined for all $t \in \mathbb{R}$, it belongs to $L^\infty(\mathbb{R})$ and Γ is a well defined bounded operator. Indeed,

$$\|\Gamma f\|_\infty \leq \sum_{j=1}^{s} \|m_{f,j}\|_{\ell^\infty} |S_j|_\infty \leq K \|f\|_\infty ,$$

for some constant K independent of f.

Proving (b), notice that $\Gamma f = f$ for each f in $\operatorname{span}\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$. For $g \in V_\varphi^\infty$ let $\{g_k\}_{k\in\mathbb{N}}$ be a sequence in $\operatorname{span}\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ such that $\|g_k - g\|_\infty \to 0$ as $k \to \infty$. Since

$$0 \leq \|g_k - \Gamma g\|_\infty = \|\Gamma g_k - \Gamma g\|_\infty \leq \|\Gamma\| \; \|g_k - g\|_\infty \to 0 \;\text{ as }\; k \to \infty ,$$

we obtain that $\Gamma g = g$. □

Theorem 5 *Assume that* $\operatorname{ess\,sup}_{t\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\varphi(t+n)|(1+|t+n|)^m < \infty$ *for some* $m \in \mathbb{N}$ *and the sampling functions* S_j *satisfy* $|S_j|_\infty < \infty$ *for each* $j = 1, 2, \ldots, s$. *If the generator* φ *satisfies the Strang-Fix conditions of orden* m, *then for each* $f \in W_\infty^m(\mathbb{R})$ *and* $h > 0$, *we have*

$$\|\Gamma_h f - f\|_\infty \leq \; C \, \|f^{(m)}\|_\infty \; h^m ,$$

where the constant C *is independent of* f *and* h.

Proof: Notice that, as a consequence of Lemma 1(a), the linear operator $\Gamma_h : \mathcal{C}_b(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded. Moreover, we can easily deduce that $\|\Gamma_h\| = \|\Gamma\|$. From Lemma 1(b) we obtain that $\Gamma_h g = g$ for each $g \in \sigma_h V_\varphi^\infty$. Thus, for each $f \in W_\infty^m(\mathbb{R})$ and $g \in \sigma_h V_\varphi^\infty$ we obtain

$$\|f - \Gamma_h f\|_\infty \leq \|f - g\|_\infty + \|g - \Gamma_h f\|_\infty \leq (1 + \|\Gamma_h\|)\|f - g\|_\infty = (1 + \|\Gamma\|)\|f - g\|_\infty .$$

Now, the result comes out from Theorem 4. □

The hypotheses in Theorem 5 are clearly satisfied for the sampling formulas in Section 3, where the generator φ, $\mathcal{L}\varphi$ and the sampling functions S_j , $j = 1, 2, \ldots, s$, have compact support. For instance, since the B-spline N_3 satisfies the Strang-Fix conditions of order 3, the operator Γ associated with the sampling formula in Section 3.5 has approximation order 3 . That is, for any $f \in W_\infty^3(\mathbb{R})$,

$$\Big| f(t) - \sum_{n\in\mathbb{Z}} f(2nh) S_1\Big(\frac{t}{h} - 2n\Big) + f\Big(2nh + \frac{2}{3}h\Big) S_2\Big(\frac{t}{h} - 2n\Big) + f\Big(2nh + \frac{4}{3}h\Big) S_3\Big(\frac{t}{h} - 2n\Big) \Big| \leq \; C \; \|f'''\|_\infty \; h^3$$

for all $t \in \mathbb{R}$ and $h > 0$, where the constant C is independent of f and h.

Notice that here the sampling period is $(2/3)h$ instead of h, the sampling period for the corresponding interpolatory formula which satisfies the same approximation order (see [12]). As a counterpart, our reconstruction functions S_j, $j = 1, 2, 3$, have compact support.

5 Conclusions

The recovery of a function f in a shift-invariant space V_φ from the sequence of samples $\{(\mathcal{L}f)(n)\}_{n\in\mathbb{Z}}$, where $\mathcal{L}f$ denotes a filtered version of f, through a sampling formula as

$$f(t)=\sum_{n\in\mathbb{Z}}\mathcal{L}f(n)S(t-n),\quad t\in\mathbb{R}\,,$$

is a well established problem. But, in general, the reconstruction function S is not compactly supported. In this paper we deal with the problem of obtaining reconstruction functions having compact support. This is done in the light of the generalized sampling theory by using the oversampling technique. Under appropriate hypotheses, we obtain a necessary and sufficient condition in this direction. It involves the Smith canonical form of a polynomial matrix (the so-called modulation matrix in the filter bank jargon). Besides, the obtained sampling formulas provide approximation schemes for the functions in a Sobolev space $W^m_\infty(\mathbb{R})$ with respect to the uniform norm. All the results in the paper are illustrated with an example in the shift-invariant space generated by the quadratic B-spline.

To end this section we point out two possible generalizations: The first one concerns the use of a larger oversampling rate, considering the systems

$$(\mathcal{L}_1f)(t)=\mathcal{L}f(t),(\mathcal{L}_2f)(t)=\mathcal{L}f\Big(t+\frac{r}{s}\Big),\dots,(\mathcal{L}_sf)(t)=\mathcal{L}f\Big(t+(s-1)\frac{r}{s}\Big)\,,$$

and the samples $\big\{\mathcal{L}_jf(rn)\big\}_{n\in\mathbb{Z},\ j=1,2,\dots,s}$, where r and s are any natural numbers with $s>r$. The corresponding sampling formulas allow the recovering of the functions in V_φ from their samples in the lattice $(r/s)\mathbb{Z}$. The second one concerns getting compactly supported reconstruction functions in a generalized sampling formulas as

$$f(t)=\sum_{n\in\mathbb{Z}}\sum_{j=1}^{r}\big(L_jf\big)(rn)\ S_j(t-rn)\,,\quad t\in\mathbb{R}\,,$$

where the samples of r filtered versions L_jf of f are included.

Finally, it is worth to mention that, in the present paper, we have only dealed with the univariate case. The multivariate case can be also studied, but in this case the Smith canonical form theory should be substituted by the more involved Gröbner base theory.

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