Reconstruction of splines from local average samples

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Abstract

We study the reconstruction of cardinal splines $f(t)$, from their average samples $y_n = f * h(n)$, $n \in \mathbb{Z}$, when the average function $h(t)$ has support in $[-1/2, 1/2]$. We investigate the existence and uniqueness of the solution of the following problem: for given dates y_n, find a cardinal spline $f(t)$, of a given degree, satisfying $y_n = f * h(n)$, $n \in \mathbb{Z}$.

1 Spline interpolation and average sampling

First, let us introduce some notation and recall some of the fundamental results of Schoenberg's magnificent cardinal interpolation spline theory (see [4]).

Let β_d be the cardinal central B-spline of degree d,

$$\beta_d := \mathcal{X}_{[-1/2,1/2]} * \ldots * \mathcal{X}_{[-1/2,1/2]} \quad (d + 1 \text{ terms})$$

and let $\mathcal{S}_d$ the space generated by the shift of β_d, i.e. the set of functions $f(t)$ that admit a representation of the form

$$f(t) = \sum_{n\in\mathbb{Z}} a_n \, \beta_d(t - n)$$

with appropriate coefficients a_n. When d is odd, the space $\mathcal{S}_d$ is the set of cardinal splines of degree d with knot sequence $\mathbb{Z}$, i.e. the set of functions $f \in C^{d-1}(\mathbb{R})$ such that for all $k \in \mathbb{Z}$, $f|_{[k,k+1)}$ belongs to Π_d the class of polynomials of degree not exceeding d. When d is even, the knot sequence is $\mathbb{Z} + 1/2$, i.e. $\mathcal{S}_d := \big\{ f \in C^{d-1}(\mathbb{R}) : f|_{[k-1/2,k+1/2)} \in \Pi_d,\, k \in \mathbb{Z} \big\}$.

We consider the cardinal spline interpolation problem: Given a sequence of real numbers $\{y_n\}_{n\in\mathbb{Z}}$, find a spline $f \in \mathcal{S}_d$ such that

$$f(n) = y_n, \quad n \in \mathbb{Z}.$$

For $d = 1$ the problem has a unique solution, the linear spline obtained by linear interpolation between every pair of consecutive data, but for $d > 1$ it has infinitely many solutions forming a linear manifold in $\mathcal{S}_d$ of dimension $d-1$ when d is odd, and of dimension d when d is even. Growth conditions give the uniqueness. Specifically, denoting

$$\mathcal{S}_{d,\gamma} := \big\{ f(t) \in \mathcal{S}_d \,:\, f(t) = (|t|^\gamma) \text{ as } t \mapsto \pm\infty \big\}, \quad D_\gamma := \big\{ \{y_n\}_{n\in\mathbb{Z}} \,:\, y_n = O(|n|^\gamma) \text{ as } n \mapsto \pm\infty \big\},$$

for $\gamma \geq 0$, Shoenberg proved that for a given sequence of real numbers $\{y_n\}_{n\in\mathbb{Z}} \in D_\gamma$, the problem:

$$\textit{Find a spline } f \in \mathcal{S}_{d,\gamma} \textit{ satisfying } f(n) = y_n, \quad n \in \mathbb{Z},$$

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has a unique solution.

In many applications it is more realistic to assume that the available samples, are not the value of the function at the integers n, but local averages near n,

$$f * h\,(n) = \int_{n-1/2}^{n+1/2} f(t)\, h(n-t)\, dt, \quad n \in \mathbb{Z},$$

where the average function $h(t)$, with support in $[-1/2, 1/2]$, reflects the characteristic of the acquisition device. Thus regular average sampling has been studied for spaces of cardinal splines and for general shift invariant spaces (see [1, 2, 3, 5, 6, 8, 7, 9] and references therein).

In this paper we carry some results of the Shoenberg's cardinal interpolation spline theory over the averaging sampling context. If the average function $h(t)$ verifies

$$\begin{aligned} &\bullet\ \operatorname{supp} h \subseteq [-1/2, 1/2], \quad h(t) \geq 0,\ t \in \mathbb{R}, \\ &\bullet\ 0 < \int_{-1/2}^{0} h(t)dt < \infty \quad \text{and} \quad 0 < \int_{0}^{1/2} h(t)dt < \infty, \end{aligned} \tag{1}$$

then, given a sequence of numbers $\{y_n\}_{n\in\mathbb{Z}}$, the problem: Find $f \in \mathcal{S}_d$ such that

$$f * h(n) = y_n, \qquad n \in \mathbb{Z}, \tag{2}$$

even in the linear case, has infinitely many solutions forming a linear manifold in $\mathcal{S}_d$ of dimension $d+1$ when d is odd, and of dimension d when d is even (see Lemma 1). Thus, we require growth conditions to get uniqueness. Specifically, in the next section we prove the following result.

Theorem. *In the linear, quadratic, cubic and quartic cases ($d = 1, 2, 3, 4$), and assuming that h satisfies* (1)*, for a given sequence of real numbers $\{y_n\}_{n\in\mathbb{Z}} \in D_\gamma$, the problem:*

$$\text{Find a spline } f \in \mathcal{S}_{d,\gamma} \text{ satisfying } f * h(n) = y_n, \quad n \in \mathbb{Z}, \tag{3}$$

has a unique solution.

We conjecture that the theorem holds for all $d \in \mathbb{N}$, but we are unable to prove it.

2 Proof

Lemma 1 *Let $d \in \mathbb{N}$ and assume that $h(t)$ verifies the conditions* (1)*. For a given sequence of real numbers $\{y_n\}_{n\in\mathbb{Z}}$ the problem* (2) *has infinitely many solutions in $\mathcal{S}_d$ forming a linear manifold of dimension $d+1$ when d is odd, and of dimension d when d is even.*

Proof. When d is odd, the space $\mathcal{S}_d$ can be described as the functions $f(t)$ that admit a representation of the form

$$\begin{aligned} f(t) = P(t) &+ a_1(t-1)_+^d + a_2(t-2)_+^d + \dots \\ &+ a_0(-t)_+^d + a_{-1}(-t-1)_+^d + \dots \end{aligned} \tag{4}$$

with appropriate coefficients a_n, where $t_+ := \max(0, t)$ and $P(t) \in \Pi_d$ (see [4, p. 33]). The representation (4) is unique, since $P(t) = f(t)$ in $[0, 1]$ and the coefficient a_n is the value of the jump of the step function $f^{(d)}(t)/d!$ at n, for all $n \in \mathbb{Z}$.

Now, for a fixed polynomial $P(t) \in \Pi_d$, we prove that there exist unique coefficients a_n in (4) such that $f(t)$ satisfies (2). By using (1), we obtain

$$h * f(1) = h * P(1) + a_1 \int_{\frac{1}{2}}^{\frac{3}{2}} h(1-t)(t-1)_+^d\, dt \quad \text{and} \quad \int_{\frac{1}{2}}^{\frac{3}{2}} h(1-t)(t-1)_+^d\, dt > 0$$

Hence, there exists a unique a_1 such that $h*f(1)=y_1$. In this way, we can determinate successively and uniquely a_2 such that $h*f(2)=y_2$, a_3 such that $h*f(3)=y_3$, etc. Indeed, assuming that we have determined $a_1,a_2,\dots,a_{n-1}$, and denoting $Q(t)=P(t)+a_1(t-1)_+^d+\dots+a_{n-1}(t-[n-1])_+^d$, we have

$$h*f(n)=h*Q(n)+a_n\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}h(n-t)(t-n)_+^d\,dt \quad \text{and} \quad \int_{n-\frac{1}{2}}^{n+\frac{1}{2}}h(n-t)(t-n)_+^d dt\;>\;0$$

and then, there exists a unique coefficient a_n such that $h*f(n)=y_n$.

Likewise, we can determinate successively and uniquely $a_0,a_{-1},a_{-2},\dots$ by $h*f(n)=y_n$ for $n=0,-1,-2,\dots$.

Therefore, for any $P(t)\in\Pi_d$, there exists a unique $f(t)$ satisfying (2). Since $P(t)$ depends linearly on $d+1$ parameters, we can deduce easily that the lemma holds when n is odd.

When d is even, any $f\in\mathcal{S}_d$ can be uniquely represented in the form

$$\begin{aligned}f(t)=P(t)&+b_1(t-1/2)_+^d+b_2(t-3/2)_+^d+\dots\\&+b_{-1}(-t-1/2)_+^d+b_{-2}(-t-3/2)_+^d+\dots\;,\end{aligned}$$

where $P(t)\in\Pi_d$. Since $h*f(0)=\int_{-1/2}^{1/2}h(-t)f(t)dt=\int_{-1/2}^{1/2}h(-t)P(t)dt$, the set of polynomials $P(t)\in\Pi_d$ such that the corresponding $f(t)$ satisfy $h*f(0)=y_0$ is a linear manifold of dimension d. Now, we can prove the result similarly that in the odd case. □

In the proof of the Theorem, the function

$$G_{h,d}(z):=\sum_{n\in\mathbb{Z}}h*\beta_d\,(n)\;z^{-n}$$

plays a principal role. If h satisfies (1), $G_{h,d}(z)$ is a Laurent polynomial and it can be written as

$$G_{h,d}(z)=\int_{-\frac{1}{2}}^{\frac{1}{2}}h(t)\Upsilon_{z,d}(t)\,dt, \tag{5}$$

where

$$\Upsilon_{z,d}(t):=\sum_{n\in\mathbb{Z}}z^{-n}\,\beta_d(n-t).$$

The functions $\Upsilon_{z,d}$ are splines of $\mathcal{S}_d$ related to the exponential splines considered in [4]. In the following lemma we collect some properties of these splines.

Lemma 2 *For $d\in\mathbb{N},n\in\mathbb{Z}$ and $z\in\mathbb{C}\setminus\{0\}$, we have*

(i) $\Upsilon_{z^{-1},d}(-t)=\Upsilon_{z,d}(t)$

(ii) $\Upsilon_{z,d}(t+n)=z^{-n}\,\Upsilon_{z,d}(t)$

(iii) $\Upsilon'_{z,d+1}(t)=(1-z)\Upsilon_{z,d}(t+1/2)$

(iv) $\Upsilon_{-1,d}(t)$ *is even,* $\Upsilon_{-1,d}(1/2)=0$ *and* $\Upsilon_{-1,d}(t)>0$ *for* $t\in(-1/2,1/2)$

Proof. Since $\sum_{n\in\mathbb{Z}}z^n\beta_d(n+t)=\sum_{n\in\mathbb{Z}}z^n\beta_d(-n-t)=\sum_{n\in\mathbb{Z}}z^{-n}\beta_d(n-t)$ we have (i), and since $\sum_{k\in\mathbb{Z}}z^{-k}\,\beta_d(k-[t+n])=\sum_{k\in\mathbb{Z}}z^{-k-n}\,\beta_d(k-t)$ we have (ii). Using that $\beta'_{d+1}(t)=\beta_d(t+1/2)-\beta_d(t-1/2)$ we deduce (iii). From (i), $\Upsilon_{-1,d}(t)$ is even. We have $\Upsilon_{-1,d}(1/2)=\dots+\beta_d(3/2)-\beta_d(1/2)+\beta_d(-1/2)-\beta_d(-3/2)+\dots=0$. Finally, we prove that $\Upsilon_{-1,d}(t)>0$ for $t\in(-1/2,1/2)$ by induction over d. For $d=1$ it is trivial. Assume that it is true for d. Then $\Upsilon'_{-1,d+1}(t)=2\,\Upsilon_{-1,d}(t+1/2)>0,\quad t\in(-1/2,0)$. Hence, using that $\Upsilon_{-1,d+1}(-1/2)=0$ and that $\Upsilon_{-1,d+1}$ is even, we deduce that $\Upsilon_{-1,d+1}(t)>0$ for $t\in(-1/2,1/2)$. □

Lemma 3 *Let $d \in \mathbb{N}$, $\gamma \geq 0$ and assume that $h(t)$ verifies (1). If all the roots of $G_{h,d}(z)$ are simple and negative different from -1, then for a given sequence $\{y_n\}_{n\in\mathbb{Z}} \in D_\gamma$, the problem (3) has a unique solution. This solution can be expressed as*

$$f(t) = \sum_{n\in\mathbb{Z}} y_n \, L_{h,d}(t-n) \tag{6}$$

where the reconstruction spline $L_{h,d}$ is given by

$$L_{h,d}(t) := \sum_{n\in\mathbb{Z}} c_n \, \beta_d(t-n) \tag{7}$$

and c_n are the coefficients of the expansion $G_{h,d}(z)^{-1} = \sum_{n\in\mathbb{Z}} c_n \, z^{-n}$, $|z| = 1$. The reconstruction spline $L_{h,d}$ has exponential decay, i.e. there exists a constant $\mu_{h,d} \in (0,1)$ such that $L_{h,d}(t) = O(\mu_{h,d}^{|t|})$. The series in (6) converges uniformly and absolutely in every finite interval.

Proof. Since $G_{h,d}(z)$ has no zeros on the unit circle $|z| = 1$, the coefficients c_n of the Laurent expansion $C(z) := G_{h,d}^{-1}(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n}$, $|z| = 1$, have exponential decay, i.e. there exists a constant $\mu_{h,d} \in (0,1)$ such that $c_n = O(\mu_{h,d}^{|n|})$. Hence, since β_d has compact support, $L_{h,d}(t) := \sum_{n\in\mathbb{Z}} c_n \beta_d(t-n) = O(\mu_{h,d}^{|t|})$.

On the other hand, if $|t| > 2$ and $[t]$ is the integer part of t, we have

$$\frac{\sum_{n\in\mathbb{Z}} |n|^\gamma \mu_{h,d}^{|t-n|}}{(|t|+1)^\gamma} \leq \frac{\sum_{n\in\mathbb{Z}} |n|^\gamma \mu_{h,d}^{|[t]-n|-1}}{|[t]|^\gamma} \leq \frac{\sum_{n\in\mathbb{Z}} |[t]-n|^\gamma \mu_{h,d}^{|n|-1}}{|[t]|^\gamma} \leq \sum_{n\in\mathbb{Z}} (1+|n|)^\gamma \mu_{h,d}^{|n|-1} < \infty$$

By using this inequality, that $y_n = O(|n|^\gamma)$ as $n \mapsto \pm\infty$ and that $L_{h,d}(t) = O(\mu_{h,d}^{|t|})$, we deduce

$$f(t) = \sum_{n\in\mathbb{Z}} y_n \, L_{h,d}(t-n) = O(|t|^\gamma), \quad \text{as } t \to \pm\infty.$$

As $|y_n \, L_{h,d}(t-n)| < K((|n|+1)^\gamma \mu_{h,d}^{|n|-|t|})$ for a big enough constant K, the series $\sum_{n\in\mathbb{Z}} y_n \, L_{h,d}(t-n)$ converges uniformly and absolutely in every finite interval. Besides,

$$f(t) = \sum_{n\in\mathbb{Z}} y_n L_{h,d}(t-n) = \sum_{n\in\mathbb{Z}} y_n \sum_{k\in\mathbb{Z}} c_k \, \beta_d(t-n-k) = \sum_{k\in\mathbb{Z}} \Big[\sum_{n\in\mathbb{Z}} y_n c_{k-n} \Big] \beta_d(t-k).$$

Therefore $f \in \mathcal{S}_{d,\gamma}$. We denote $g_n := h * \beta_d(n)$. Then $G_{h,d}(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n}$. Using that $C(z)G_{h,d}(z) = 1$ we obtain

$$h * L_{h,d}(n) = \sum_{k\in\mathbb{Z}} c_k \, [h * \beta_d](n-k) = \sum_{k\in\mathbb{Z}} c_k \, g_{n-k} = \delta_n,$$

where $\delta_n = 0$, $n \neq 0$, and $\delta_0 = 1$. Hence, $f(t) = \sum_{n\in\mathbb{Z}} y_n L_{h,d}(t-n)$ satisfies $h * f(n) = y_n$, $n \in \mathbb{Z}$. Therefore, $f(t)$ is a solution of the problem (3).

In order to prove the uniqueness, we first find a basis of the linear space

$$\Lambda := \{\xi(t) \in \mathcal{S}_d \, : h * \xi(n) = 0, \, n \in \mathbb{Z}\}.$$

Since Lemma 1, the dimension of Λ is

$$\ell := \begin{cases} d+1, & d \text{ is odd} \\ d, & d \text{ is even} \end{cases}$$

We will find ℓ independent functions of Λ, related to the roots of $G_{h,d}(z)$.

First, we see that $G_{h,d}(z)$ has ℓ roots in $\mathbb{C}\setminus\{0\}$. Since $\operatorname{supp}\beta_d=\left[-\frac{d+1}{2},\frac{d+1}{2}\right]$ and (1),

$$p(z):=z^{\ell/2}G_{h,d}(z)=h*\beta_d\Big(\frac{\ell}{2}\Big)+h*\beta_d\Big(\frac{\ell}{2}-1\Big)z+\ldots+h*\beta_d\Big(-\frac{\ell}{2}\Big)z^{\ell}$$

is a polynomial of degree ℓ with coefficients strictly positive. It has the same roots of $G_g(z)$, and they are different from zero. Since we have supposed that they are simple, these roots are ℓ.

We denote by $z_1,\ldots,z_\ell$ the ℓ roots of $G_{d,h}$. Using Lemma 2 and (5) we obtain

$$\begin{aligned}h*\Upsilon_{z_j^{-1},d}(n)&=\int_{-1/2}^{1/2}h(t)\Upsilon_{z_j^{-1},d}(n-t)dt=z_j^n\int_{-1/2}^{1/2}h(t)\Upsilon_{z_j^{-1},d}(-t)dt\\&=z_j^n\int_{-1/2}^{1/2}h(t)\Upsilon_{z_j,d}(t)dt=z_j^n\,G_{d,h}(z_j)=0,\quad n\in\mathbb{Z},j=1,\ldots,\ell.\end{aligned}$$

Then $\Upsilon_{z_1^{-1},d},\Upsilon_{z_2^{-1},d},\ldots,\Upsilon_{z_\ell^{-1},d}$ belong to Λ. Since they are linearly independent, they form a basis of Λ.

Now it is easy to prove the uniqueness. Assume that the splines $f,g\in\mathcal{S}_{d,\gamma}$ satisfy (2). Then $f(t)-g(t)\in\Lambda$, and thus there exist coefficients c_j such that $f(t)-g(t)=\sum_{j=1}^{\ell}c_j\Upsilon_{z_j^{-1},d}(t)$. Since $z_j\in\mathbb{R}^-\setminus\{-1\}$, $j=1,\ldots,\ell$, and $f(t)-g(t)=O(|t|^\gamma)$, the behavior of $\Upsilon_{z_j^{-1},d}(t)$ at $\pm\infty$ (see Lemma 2 (ii)) shows that $c_j=0$ for $j=1,\ldots,\ell$ and then $f=g$. □

Proof of the Theorem. As a consequence of Lemma 3 is is sufficient to prove that for $d=1,2,3,4$, all the roots of $G_{h,d}(z)$ are simple and negative different from -1.

The linear and quadratic cases ($d=1,2$) are easy to deal with, since, in both cases

$$p(z)=z\,G_{h,d}(z)=h*\beta_d(-1)\,z^2+h*\beta_d(0)z+h*\beta_d(-1)$$

is a polynomial of degree 2. Its principal coefficient is positive and it satisfies $p(0)>0$. Besides, using (5) and Lemma 2, we obtain $p(-1)=-\,G_{h,d}(-1)=-\int_{-1/2}^{1/2}h(t)\Upsilon_{-1,d}(t)dt<0$, from which the result follows.

In the cubic and quartic cases ($d=3,4$) we have

$$p(z)=z^2\,G_{h,d}(z)=h*\beta_d(-2)\,z^4+h*\beta_d(-1)z^3+h*\beta_d(0)\,z^2+h*\beta_d(1)z+h*\beta_d(2)$$

is a polinomial of degree 4, whose principal coefficient is positive. It satisfies

$$p(0)>0\quad\text{and}\quad p(-1)=\,G_{h,d}(-1)=\int_{-1/2}^{1/2}h(t)\Upsilon_{-1,d}(t)dt>0.$$

Thus, it is sufficient to find $z_0\in(-1,0)$ such that

$$\Upsilon_{z_0,d}(t)<0\text{ for all }t\in(-1/2,1/2).\tag{8}$$

Indeed for such z_0 we would have (see (5) and Lemma 2)

$$\begin{aligned}&p(z_0)=z_0^2\int_{-1/2}^{1/2}h(t)\Upsilon_{z_0,d}(t)dt<0,\quad z_0\in(-1,0)\\&p(z_0^{-1})=z_0^{-2}\int_{-1/2}^{1/2}h(t)\Upsilon_{z_0^{-1},d}(t)dt=z_0^{-2}\int_{-1/2}^{1/2}h(t)\Upsilon_{z_0,d}(-t)dt<0,\quad z_0^{-1}\in(-\infty,-1).\end{aligned}$$

Since $\Upsilon_{z_0,d}(1/2)=\Upsilon_{z_0,d}(-1/2+1)=z_0^{-1}\Upsilon_{z_0,d}(-1/2)$ (see Lemma 2 (ii)) a necessary condition for (8) is $\Upsilon_{z_0,d}(1/2)=0$. The unique solution of this equation in $(-1,0)$ is $z_0=-11+2\sqrt{30}$ when $d=3$ and it is $z_0=-5+2\sqrt{6}$ when $d=4$. For these values of z_0 it is straightforward to check that (8) is satisfied. □

3 Conclusions

We have proved that if the condition (see Lemma 3):

All the roots of the Laurent polynomial $G_{h,d}(z)$ are simple and negative different from -1

holds, then, for a given sequence $\{y_n\}_{n\in\mathbb{Z}} \in D_\gamma$, the average interpolation problem (3) has a unique solution, given by $f(t) = \sum_{n\in\mathbb{Z}} y_n \, L_{h,d}(t-n)$, where the reconstruction spline $L_{h,d}$ are defined by (7). In particular, if $\{y_n\}_{n\in\mathbb{Z}}$ is a bounded sequence, there is a unique bounded spline $f \in \mathcal{S}_d$ satisfying $h * f(n) = y_n$.

We have proved that this condition holds for the cases with more practical interest, $d = 1, 2, 3, 4$. Based on that and on numerical simulations, we conjecture that this condition holds for all $d \in \mathbb{N}$, but we can not find a proof.

Acknowledgments: This work has been supported by the grant MTM2009–08345 from the D.G.I. of the Spanish *Ministerio de Ciencia y Tecnología*.

References

[1] I. Djokovic and P. P. Vaidyanathan. Generalized sampling theorems in multiresolution subspaces. *IEEE Trans. Signal Process.*, 45(3):583–599, 1997.

[2] A. G. García, M. J. Muñoz-Bouzo and G. Pérez-Villalón. Regular multivariate sampling and approximation in L^p shift-invariant spaces *J. Math. Anal. Appl.*, 380 :607–627, 2011.

[3] S. Kang and K.H. Kwon Generalized average sampling in shift invariant spaces. *J. Math. Anal. Appl.* 377 :70–78, 2011.

[4] I. J. Schoenberg. *Cardinal Spline Interpolation.* SIAM Regional Conference Series in Appl. Math., 1973.

[5] W. Sun and X. Zhou. Average sampling in spline subspaces. *Applied Mathematics Letters*, 15 (12):233–237, 2002.

[6] W. Sun and X. Zhou. Average sampling in shift invariant subspaces with symmetric averaging functions. *J. Math. Anal. Appl.*, 287 :279–295, 2003.

[7] M. Unser and A. Aldroubi. A general sampling theory for nonideal acquisition devices. *IEEE Trans. Signal Process.*, 42 (11):2915–2925, 1994.

[8] M. Unser. Sampling-50 years after Shannon. *Proc. IEEE*, 88 (4):569–587, 2000.

[9] J. Xian and W. Sun. Local sampling and reconstruction in shift-invariant spaces and their applications in spline subspaces. *Numerical Functional Analysis and Optimization.*, 31(3):366–386, 2010.