MULTI-CHANNEL SAMPLING ON SHIFT-INvariant SPACES WITH FRAME GENERATORS

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Abstract. Let $\varphi$ be a continuous function in $L^2(\mathbb{R})$ such that the sequence $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$ is a frame sequence in $L^2(\mathbb{R})$ and assume that the shift-invariant space $V(\varphi)$ generated by $\varphi$ has a multi-banded spectrum $\sigma(V)$. The main aim in this paper is to derive a multi-channel sampling theory for the shift-invariant space $V(\varphi)$. By using a type of Fourier duality between the spaces $V(\varphi)$ and $L^2[0, 2\pi]$ we find necessary and sufficient conditions allowing us to obtain stable multi-channel sampling expansions in $V(\varphi)$.

Key words: shift-invariant spaces, multi-channel sampling, frames
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1. Introduction

As a natural extension of the classical Shannon sampling theorem, Papoulis introduced in [17] generalized sampling for arbitrary multi-channel sampling in Paley-Wiener spaces $PW_{\pi\sigma}$ of band-limited signals: In many common situations the available data are samples of some filtered versions of the signal itself. Following [17], there have been many generalizations and applications of the multi-channel sampling. See, for example, [6, 7, 16, 19, 20] and references therein.

Although Shannon’s sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in [18]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite duration signal; the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has a very slow decay, which makes computation in the signal domain very inefficient.

Moreover, many applied problems impose different a priori constraints on the type of functions. For this reason, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces. Indeed, in many practical applications, signals are assumed to belong to some shift-invariant space of the form: $V(\varphi) := \text{span}_{L^2(\mathbb{R})} \{\varphi(t - n) : n \in \mathbb{Z}\}$ where the function $\varphi$ in $L^2(\mathbb{R})$ is called the generator of $V(\varphi)$. In most of cases in the mathematical literature, it is supposed that the sequence $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$ forms a Riesz basis for $V(\varphi)$. See, for instance, [1, 2, 3, 4, 12, 15, 18, 21, 22] and the references therein. Throughout this paper we assume that the sequence $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$ is a frame for $V(\varphi)$ and that the spectrum of $V(\varphi)$ is multi-banded in $[0, 2\pi]$ (see Section 3 infra).

On the other hand, suppose that $N$ linear time-invariant systems (filters) $L_j$, $j = 1, 2, \ldots, N$, are defined on the shift-invariant subspace $V(\varphi)$ of $L^2(\mathbb{R})$. In mathematical terms we are dealing with continuous operators which commute with shifts. The recovery of any function $f \in V(\varphi)$ from samples of the functions $L_j f$, $j = 1, 2, \ldots, N$, leads to a generalized sampling in $V(\varphi)$.

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Our challenge problem is the following: Given \( r, N \) positive integers and \( N \) real numbers \( 0 \leq a_j < r \) for \( 1 \leq j \leq N \), find multi-channel sampling expansions like

\[
f(t) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} (\mathcal{L}_j f)(a_j + rn) S_{j,n}(t), \quad t \in \mathbb{R},\]

valid for any \( f \in V(\varphi) \), where the sequence of sampling functions \( \{ S_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z} \} \) forms a frame or a Riesz basis for \( V(\varphi) \).

Recently, García et al. (\[8, 9, 10\]) introduced a novel idea for developing a sampling theory on a shift-invariant space \( V(\varphi) \) by using an analogous of the Fourier duality between the spaces \( V(\varphi) \) and \( L^2[0, 2\pi] \). In particular, García and Pérez-Villalón [9] (see also [14]) developed a multi-channel sampling procedure on a shift-invariant space \( V(\varphi) \), where \( \varphi \) is a continuous Riesz generator. Unlike the author’s claim (see section 4.1 in [9]), the arguments used in [9] for the case of Riesz generator cannot be directly extended to the case of a frame generator.

In the present paper, by assuming that the sequence \( \{ \varphi(t - n) \}_{n \in \mathbb{Z}} \) is a frame for \( V(\varphi) \) and that the spectrum of \( V(\varphi) \) is multi-banded in \([0, 2\pi]\), and allowing more general filters than those used in [9], we obtain necessary and sufficient conditions under which there exists a stable multi-channel sampling expansion on \( V(\varphi) \) like that in (1.1). We also provide some illustrating examples. All these tasks will be carried out throughout the remaining sections.

2. Shift-invariant spaces and Fourier duality type

We start this section by introducing some notation and preliminaries used in the sequel. Let \( \{ \varphi_n \}_{n \in \mathbb{Z}} \) be a sequence of elements in a separable Hilbert space \( \mathcal{H} \). We say that

- the sequence \( \{ \varphi_n \}_{n \in \mathbb{Z}} \) is a Bessel sequence (with Bessel bound \( B \)) in \( \mathcal{H} \) if there exists a constant \( B > 0 \) such that
  \[
  \sum_{n \in \mathbb{Z}} |\langle \varphi, \varphi_n \rangle|^2 \leq B \| \varphi \|^2 \quad \text{for all } \varphi \in \mathcal{H};
  \]

- the sequence \( \{ \varphi_n \}_{n \in \mathbb{Z}} \) is a frame for \( \mathcal{H} \) (with frame bounds \( A \) and \( B \)) if there exist constants \( 0 < A \leq B \) such that
  \[
  A \| \varphi \|^2 \leq \sum_{n \in \mathbb{Z}} |\langle \varphi, \varphi_n \rangle|^2 \leq B \| \varphi \|^2 \quad \text{for all } \varphi \in \mathcal{H};
  \]

- the sequence \( \{ \varphi_n \}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \mathcal{H} \) (with Riesz bounds \( A \) and \( B \)) if it is a complete set in \( \mathcal{H} \) and there exist constants \( 0 < A \leq B \) such that
  \[
  A \| \mathbf{c} \|^2 \leq \sum_{n \in \mathbb{Z}} |c(n) \varphi_n| \leq B \| \mathbf{c} \|^2 \quad \text{for all } \mathbf{c} = \{c(n)\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}),
  \]

where \( \| \mathbf{c} \|^2 := \sum_{n \in \mathbb{Z}} |c(n)|^2 \).

For \( \varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) we take its Fourier transform to be normalized as

\[
\mathcal{F}[\varphi](\xi) = \hat{\varphi}(\xi) := \int_{-\infty}^{\infty} \varphi(t) e^{-it\xi} dt,
\]
so that \( \frac{1}{\sqrt{2\pi}} F : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) becomes a unitary operator. For any \( \varphi \in L^2(\mathbb{R}) \) consider its related functions

\[
C_\varphi(t) := \sum_{n \in \mathbb{Z}} |\varphi(t + n)|^2 \quad \text{and} \quad G_\varphi(\xi) := \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\xi + 2\pi n)|^2.
\]

It is known that the 1-periodic function \( C_\varphi \) belongs to \( L^1[0, 1] \) and the \( 2\pi \)-periodic function \( G_\varphi \) belongs to \( L^1[0, 2\pi] \); moreover,

\[
\|\varphi\|_{L^2(\mathbb{R})}^2 = \|C_\varphi\|_{L^1[0, 1]} = \frac{1}{2\pi} \|G_\varphi\|_{L^1[0, 2\pi]}.
\]

Let \( V(\varphi) := \overline{\sigma(2\pi)} \{ \varphi(t - n) : n \in \mathbb{Z} \} \) be the shift-invariant space generated by \( \varphi \), that is, the closed subspace of \( L^2(\mathbb{R}) \) spanned by \( \{ \varphi(t - n) \}_{n \in \mathbb{Z}} \) and supp \( G_\varphi \) the support of the locally integrable function \( G_\varphi \) as a distribution on \( \mathbb{R} \). Let \( \sigma(V) := \text{supp } G_\varphi \cap [0, 2\pi] \) be the spectrum of \( V(\varphi) \) and \( \tau(V) := [0, 2\pi] \setminus \sigma(V) \). For any \( c = \{c(n)\}_{n \in \mathbb{Z}} \) in \( \ell^2(\mathbb{Z}) \), let

\[
\hat{c}(\xi) := \sum_{n \in \mathbb{Z}} c(n)e^{-in\xi}
\]

be the discrete Fourier transform of the sequence \( c \). In [5] we find the following result:

**Proposition 2.1.** Let \( \varphi \in L^2(\mathbb{R}) \) and \( 0 < A \leq B \). The following statements hold:

(a) The sequence \( \{\varphi(t - n)\}_{n \in \mathbb{Z}} \) is a Bessel sequence with a Bessel bound \( B \) for \( V(\varphi) \) if and only if \( G_\varphi(\xi) \leq B \) a.e. on \( [0, 2\pi] \).

(b) The sequence \( \{\varphi(t - n)\}_{n \in \mathbb{Z}} \) is a frame for \( V(\varphi) \) with frame bounds \( A, B \) if and only if \( A \leq G_\varphi(\xi) \leq B \) a.e. on \( \sigma(V) \).

(c) The sequence \( \{\varphi(t - n)\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( V(\varphi) \) with Riesz bounds \( A, B \) if and only if \( A \leq G_\varphi(\xi) \leq B \) a.e. on \( [0, 2\pi] \).

For any \( \varphi \in L^2(\mathbb{R}) \) and \( c = \{c(k)\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \), let \( T(c) := (c * \varphi)(t) = \sum_{k \in \mathbb{Z}} c(k)\varphi(t - k) \) be the pre-frame operator of \( \{\varphi(t - n)\}_{n \in \mathbb{Z}} \). Proposition 2.1 can be restated as (cf. [5]):

- The sequence \( \{\varphi(t - n)\}_{n \in \mathbb{Z}} \) is a Bessel sequence with a Bessel bound \( B \) if and only if \( T \) is a bounded linear operator from \( \ell^2(\mathbb{Z}) \) into \( V(\varphi) \) with \( \|T\| \leq \sqrt{B} \).
- The sequence \( \{\varphi(t - n)\}_{n \in \mathbb{Z}} \) is a frame for \( V(\varphi) \) with frame bounds \( A, B \) if and only if \( T \) is a bounded linear operator from \( \ell^2(\mathbb{Z}) \) onto \( V(\varphi) \) and

\[
A\|c\|^2 \leq \|T(c)\|_{L^2(\mathbb{R})}^2 \leq B\|c\|^2, \quad c \in N(T)^{\perp},
\]

where \( N(T) := \{c \in \ell^2(\mathbb{Z}) : T(c) = 0\} \) and \( N(T)^{\perp} \) is the orthogonal complement of \( N(T) \) in \( \ell^2(\mathbb{Z}) \).
- The sequence \( \{\varphi(t - n)\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( V(\varphi) \) with Riesz bounds \( A, B \) if and only if \( T \) is an isomorphism from \( \ell^2(\mathbb{Z}) \) onto \( V(\varphi) \) and

\[
A\|c\|^2 \leq \|T(c)\|_{L^2(\mathbb{R})}^2 \leq B\|c\|^2, \quad c \in \ell^2(\mathbb{Z}) .
\]
Lemma 2.2. Let $\varphi \in L^2(\mathbb{R})$ such that the sequence $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$. Then for any $c = \{c(n)\}_{n \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z})$,

$$T(c)(\xi) = \hat{c}(\xi)\hat{\varphi}(\xi),$$

so that

$$\|T(c)\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{c}(\xi)|^2 |\hat{\varphi}(\xi)|^2 d\xi = \frac{1}{2\pi} \int_{0}^{2\pi} |\hat{c}(\xi)|^2 G_{\varphi}(\xi) d\xi. \tag{2.1}$$


In what follows, we always assume that the function $\varphi \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ is a continuous frame generator (i.e., the sequence $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$ is a frame for $V(\varphi)$), and satisfying the condition: $\sup_{\mathbb{R}} C_{\varphi}(t) < \infty$. Thus $V(\varphi) = \{(c * \varphi)(t) : c \in \ell^2(\mathbb{Z})\}$ is a reproducing kernel Hilbert space (RKHS in short) and any $f(t) = (c * \varphi)(t)$ in $V(\varphi)$ converges both in the $L^2(\mathbb{R})$ sense, and absolutely and uniformly on $\mathbb{R}$ to a continuous function on $\mathbb{R}$ (see [15, 22]).

By using (2.1), we have that $N(T) = \{c \in \ell^2(\mathbb{Z}) : \hat{c}(\xi) = 0$ a.e. on $\sigma(V)\}$ and consequently

$$N(T) \perp = \{c \in \ell^2(\mathbb{Z}) : \hat{c}(\xi) = 0$ a.e. on $\tau(V)\}. \tag{2.2}$$

Now, we introduce a Fourier duality for $V(\varphi)$ useful for sampling purposes as we will see in the next section.

Let $T_{\varphi} : L^2[0, 2\pi] \to V(\varphi)$ be the linear operator defined by

$$(T_{\varphi} F)(t) := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \langle F(\xi), e^{-ik\xi} \rangle_{L^2[0, 2\pi]} \varphi(t - k) = \langle F(\xi), \frac{1}{2\pi} Z_{\varphi}(t, \xi) \rangle_{L^2[0, 2\pi]},$$

where $Z_{\varphi}$ denotes the Zak transform of $\varphi$ given as $Z_{\varphi}(t, \xi) := \sum_{k \in \mathbb{Z}} \varphi(t + k)e^{-ik\xi}$ (see [12]).

Notice that $\{\varphi(t - n)\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ for each $t \in \mathbb{R}$. By using (2.2), $T_{\varphi}$ is a bounded linear operator from $L^2[0, 2\pi]$ onto $V(\varphi)$ with kernel

$$N(T_{\varphi}) = \{F(\xi) \in L^2[0, 2\pi] : F(\xi) = 0$ a.e. on $\sigma(V)\}.$$ 

Thus, the operator $T_{\varphi} : L^2[\sigma(V)] \to V(\varphi)$ becomes an isomorphism. We also note the following useful properties of $T_{\varphi}$:

- $T_{\varphi} F(\xi) = F(\xi)\hat{\varphi}(\xi)$;
- $T_{\varphi}[F(\xi)e^{-in\xi}](t) = (T_{\varphi} F)(t - n), n \in \mathbb{Z}$.

3. Multi-channel sampling theory

For $1 \leq j \leq N$, let $L_j$ be an LTI (linear time-invariant) system with impulse response $h_j$, that is,

$$L_j[f](t) := (f * h_j)(t) = \int_{\mathbb{R}} f(s)h_j(t - s)ds.$$

Here, we assume that each system $L_j$ belongs to one of the following three types:

- (i) Its impulse response $h_j(t) = \delta(t + a_j)$, $a_j \in \mathbb{R}$, or
- (ii) $h_j \in L^2(\mathbb{R})$, or
- (iii) $\hat{h}_j \in L^\infty(\mathbb{R})$ whenever $H_{\varphi}(\xi) := \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| \in L^2[0, 2\pi]$. 

For type (i), \( \mathcal{L}[f](t) = f(t + a) \) for any \( f \in L^2(\mathbb{R}) \), so that \( \mathcal{L} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) becomes a unitary operator. In particular, consider \( \psi(t) := \mathcal{L}[\varphi](t) = \varphi(t + a) \); for any \( f(t) = (c \ast \varphi)(t) \in V(\varphi) \) we have that \( \mathcal{L}[f](t) = (c \ast \psi)(t) \) converges absolutely and uniformly on \( \mathbb{R} \) since \( \sup_\mathbb{R} C_\psi(t) = \sum_{n \in \mathbb{Z}} |\psi(t + n)|^2 < \infty \). For types (ii) and (iii) the following result holds:

**Lemma 3.1.** Let \( \mathcal{L} \) be an LTI system with impulse response \( h \) of type (ii) or (iii) as above and consider the function \( \psi(t) := \mathcal{L} [\varphi](t) = (\varphi \ast h)(t) \). Then we have:

(a) The function \( \psi \) belongs to the space \( C^\infty(\mathbb{R}) := \{ u(t) \in C(\mathbb{R}) : \lim_{|t| \rightarrow \infty} u(t) = 0 \} \).
(b) \( \sup_\mathbb{R} C_\psi(t) < \infty \).
(c) For any \( f(t) = (c \ast \varphi)(t) \in V(\varphi) \) with \( c \in \ell^2(\mathbb{Z}) \), \( \mathcal{L}[f](t) = (c \ast \psi)(t) \) converges absolutely and uniformly on \( \mathbb{R} \).
(d) For each fixed \( t \in \mathbb{R} \), \( \text{supp } Z_\psi(t, \cdot) \cap [0, 2\pi] \subset \sigma(V) \).

**Proof.** First assume \( h \in L^2(\mathbb{R}) \). Since \( \hat{\psi}(\xi) = \hat{\varphi}(\xi) \hat{h}(\xi) \in L^1(\mathbb{R}) \), the function \( \psi \in C^\infty(\mathbb{R}) \) by using the Riemann-Lebesgue Lemma. The Poisson summation formula (cf. Lemma 5.1 in [15]) gives:

\[
C_\psi(t) = \sum_{n \in \mathbb{Z}} |\psi(t + n)|^2 = \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \psi(t + n)e^{-int} \right\|^2_{L^2(0, 2\pi)} \\
= \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi)e^{it(\xi + 2n\pi)} \right\|^2_{L^2(0, 2\pi)} \\
\leq \frac{1}{2\pi} \|G^\frac{1}{2}\|L^2(0, 2\pi)\|G_\varphi\|_{L^\infty(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}.
\]

Hence \( \sup_\mathbb{R} C_\psi(t) < \infty \). Since \( f(t) = (c \ast \varphi)(t) \) converges in \( L^2(\mathbb{R}) \) for any \( c \in \ell^2(\mathbb{Z}) \) and the operator \( \mathcal{L} : L^2(\mathbb{R}) \rightarrow L^\infty(\mathbb{R}) \) is bounded by using Young’s inequality on the convolution product, we have that \( \mathcal{L}[f](t) = \sum_{k \in \mathbb{Z}} c(k)\mathcal{L}[\varphi(t - k)] = \sum_{k \in \mathbb{Z}} c(k)\psi(t - k) = (c \ast \psi)(t) \) converges absolutely and uniformly on \( \mathbb{R} \) by using (b).

Now assume that \( H_\varphi \in L^2[0, 2\pi] \) and let \( \hat{h} \in L^\infty(\mathbb{R}) \). Since \( \hat{\varphi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), we obtain that \( \hat{\psi} = \hat{\varphi} \hat{h} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and consequently, \( \psi \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \). Since \( \sum_{n \in \mathbb{Z}} |\hat{\psi}(\xi + 2n\pi)|^2 \leq \|\hat{h}\|^2_{L^\infty(\mathbb{R})}\|H_\varphi\|^2_{L^2[0, 2\pi]} \), we have that \( C_\psi(t) \leq \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \psi(t + n)e^{it(\xi + 2n\pi)} \right\|^2_{L^2(0, 2\pi)} \leq \frac{1}{2\pi} \|\hat{h}\|^2_{L^\infty(\mathbb{R})}\|H_\varphi\|^2_{L^2[0, 2\pi]} \), so that \( \sup_\mathbb{R} C_\psi(t) < \infty \). For any \( f \in L^2(\mathbb{R}) \),

\[
\|\mathcal{L}[f]\|_{L^2(\mathbb{R})} = \|f \ast h\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \left\| \hat{f}(\xi)\hat{h}(\xi) \right\|_{L^2(\mathbb{R})} \leq \|\hat{h}\|_{L^\infty(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}.
\]

Hence, \( \mathcal{L} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) is a bounded linear operator so that, for any \( f(t) = (c \ast \varphi)(t) \in V(\varphi) \), \( \mathcal{L}[f](t) = (c \ast \psi)(t) \) converges in \( L^2(\mathbb{R}) \). Condition (b) implies that \( (c \ast \psi)(t) \) also converges absolutely and uniformly on \( \mathbb{R} \) which proves (c).

Finally to prove (d), consider any \( F \in L^2[0, 2\pi] \) with \( \text{supp } F \subseteq \tau(V) \) and let

\[
F(\xi) = \sum_{k \in \mathbb{Z}} c(k)e^{-ik\xi} \quad \text{where} \quad c(k) = \frac{1}{2\pi} \langle F(\xi), e^{-ik\xi} \rangle_{L^2[0, 2\pi]}, \; k \in \mathbb{Z}.
\]

The sequence \( c \in N(T) \) so that \( T(c) = (c \ast \varphi)(t) = 0 \). Since

\[
\langle F(\xi), Z_\psi(t, \xi) \rangle_{L^2[0, 2\pi]} = 2\pi \langle c \ast \psi)(t) \rangle = 2\pi \mathcal{L}[c \ast \varphi](t) = 0,
\]
we finally obtain that $\text{supp } Z_\psi(t, \cdot) \cap [0, 2\pi] \subset \sigma(V)$. \hfill \Box

In particular, given an LTI system $\mathcal{L}$ of type (i), (ii) or (iii), for any $f = (T_\varphi F) \in V(\varphi)$, where $F \in L^2[\sigma(V)]$, we have

$$
(3.1) \quad \mathcal{L}[f](t) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \langle F(\xi) \chi_{\sigma(V)}(\xi), e^{i k \xi} \rangle_{L^2[0,2\pi]} \psi(t - k) = \langle F(\xi), \frac{1}{2\pi} Z_\varphi(t,\xi) \rangle_{L^2[\sigma(V)]}.
$$

Here $\chi_E(\xi)$ denotes the characteristic function of a measurable set $E$ in $\mathbb{R}$.

As it was said before, in this work we are involved in the following problem: Given $r, N$ positive integers and $N$ real numbers $0 \leq a_j < r$ for $1 \leq j \leq N$, find multi-channel sampling formulas in $V(\varphi)$ such that, for any $f \in V(\varphi)$,

$$
(3.2) \quad f(t) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} (L_j f)(a_j + rn) S_{j,n}(t), \quad t \in \mathbb{R},
$$

where the sequence of sampling functions $\{S_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$ forms a frame or a Riesz basis for $V(\varphi)$.

First of all, notice that convergence in the $L^2(\mathbb{R})$-sense in the sampling series (3.2) implies pointwise convergence since $V(\varphi)$ is a RKHS, which is absolute and uniform on $\mathbb{R}$. Indeed, let $\{\tilde{\varphi}(t - n)\}_{n \in \mathbb{Z}}$ be the canonical dual frame of $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$. Then the reproducing kernel of $V(\varphi)$ is

$$
q(s, t) := \sum_{n \in \mathbb{Z}} \tilde{\varphi}(s - n) \overline{\varphi(t - n)}.
$$

Since $\sup_{\mathbb{R}} C_{\varphi}(t) < \infty$ the function $q(t, t)$ is uniformly bounded on $\mathbb{R}$. Hence, the convergence in the $L^2(\mathbb{R})$-sense implies uniform convergence on $\mathbb{R}$. The pointwise convergence is also absolute due to the unconditional convergence of a frame or Riesz basis expansion.

In this work we solve this problem for the case where $V(\varphi)$ is a shift-invariant space having a continuous frame generator $\varphi$ and the spectrum $\sigma(V)$ of $V(\varphi)$ is a multi-banded region such that

$$
\sigma(V) = \bigcup_{k=1}^{M} [\alpha_k, \beta_k], \quad \text{where } 0 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_M < \beta_M \leq 2\pi.
$$

Notice that through (3.1) and the isomorphism $T_\varphi : L^2[\sigma(V)] \rightarrow V(\varphi)$, the sampling expansion (3.2) on $V(\varphi)$ is equivalent to the expansion in $L^2[\sigma(V)]$:

$$
F(\xi) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} (F(\xi), \frac{1}{2\pi} Z_\psi(a_j, \xi) e^{-i n \xi})_{L^2[\sigma(V)]} s_{j,n}(\xi), \quad F \in L^2[\sigma(V)],
$$

where $\{s_{j,n}(\xi) : 1 \leq j \leq N, n \in \mathbb{Z}\}$ is a frame or a Riesz basis for $L^2[\sigma(V)]$.

From now on we assume that $\sigma(V) = \bigcup_{k=1}^{M} [\alpha_k, \beta_k]$ and we set

$$
s_k := \alpha_k - \left[ \frac{\alpha_k}{2\pi} \right] \frac{2\pi}{r}, \quad \text{and} \quad r_k := \beta_k - \left[ \frac{\beta_k}{2\pi} \right] \frac{2\pi}{r},
$$

so that $0 \leq s_k, r_k < \frac{2\pi}{r}, \ 1 \leq k \leq M$ ($[x]$ denotes the integer part of $x \geq 0$).

Next consider the set of points $\{t_k\}_{k=0}^{m}$ such that $0 = t_0 < t_1 < \cdots < t_m = \frac{2\pi}{r}$ where
\[ \{ t_k \}_{k=1}^{m-1} = \{ s_k, r_k : 1 \leq k \leq M \} \setminus \{ 0 \}. \]

Then,
\[ I := [0, \frac{2\pi}{r}] = \bigcup_{k=1}^{m} \mathcal{B}_k, \quad B_k = (t_{k-1}, t_k). \]

**Lemma 3.2.** For each \( 1 \leq k \leq m \) and each \( 1 \leq n \leq r \), we have that
\[ \text{either } \left( B_k + (n - 1) \frac{2\pi}{r} \right) \cap \sigma(V) = \emptyset \quad \text{or} \quad \left( B_k + (n - 1) \frac{2\pi}{r} \right) \subset \sigma(V). \]

**Proof.** See Lemma 1 in [20]. \(\square\)

For each \( 1 \leq k \leq m \) we consider \( L(k) \), the subset of \( \{1, 2, \ldots, r\} \) defined by
\[ L(k) := \{ 1 \leq n \leq r : B_k + (n - 1) \frac{2\pi}{r} \subset \sigma(V) \} , \]
and \( l(k) := \# L(k) \), i.e., its number of elements. Let \( \mathcal{P} := \{ 1 \leq k \leq m : l(k) > 0 \} \); for each \( k \in \mathcal{P} \), there are \( l(k) \) positive integers \( \{ n_{k,j} \}_{j=1}^{l(k)} \) such that \( 1 \leq n_{k,1} < n_{k,2} < \cdots < n_{k,l(k)} \leq r \) and
\[ B_k + (n_{k,j} - 1) \frac{2\pi}{r} \subset \sigma(V) , \quad 1 \leq j \leq l(k) . \]

For \( k \in \mathcal{P} \), let \( \tilde{B}_k := \bigcup_{j=1}^{l(k)} (B_k + (n_{k,j} - 1) \frac{2\pi}{r}) \). These sets \( \tilde{B}_k \) are disjoint and \( \sigma(V) = \bigcup_{k \in \mathcal{P}} \tilde{B}_k \); hence, \( |\sigma(V)| = \sum_{k \in \mathcal{P}} l(k) |B_k| \), where \( |E| \) denotes the Lebesgue measure of \( E \).

For each \( k \in \mathcal{P} \), consider the unitary operator \( D_k : L^2(\tilde{B}_k) \to L^2_{l(k)}(B_k) \) defined by
\[ D_k(F)(\xi) := \left[ F\left( \xi + \left( n_{k,1} - 1 \right) \frac{2\pi}{r} \right) , \ldots , F\left( \xi + \left( n_{k,l(k)} - 1 \right) \frac{2\pi}{r} \right) \right]^T, \quad F \in L^2{\tilde{B}_k} , \]
where \( L^2_{l(k)}(B_k) \) denotes the Hilbert product space \( L^2(B_k) \times \cdots \times L^2(B_k) \) \( (l(k) \text{ times}) \).

Now, for each \( k \in \mathcal{P} \) we consider the \( N \times l(k) \) matrix with entries in \( L^2(B_k) \)
\[ G_k(\xi) := [D_k(g_1)(\xi), \ldots , D_k(g_N)(\xi)]^T = \left[ g_i(\xi + (n_{k,j} - 1) \frac{2\pi}{r}) \right]_{1 \leq i \leq N, 1 \leq j \leq l(k)} , \]
and the \( l(k) \times l(k) \) matrix with entries in \( L^4(B_k) \)
\[ H_k(\xi) := G_k^*(\xi)G_k(\xi) , \]
where \( G_k^*(\xi) \) denotes the adjoint of the matrix \( G_k(\xi) \), being
\[ g_i(\xi) := \frac{1}{2\pi} Z_{\psi_i}(a_i, \xi) \in L^2[\sigma(V)] , \quad 1 \leq i \leq N . \]

Let \( \lambda_{\min,k}(\xi) \) (respectively \( \lambda_{\max,k}(\xi) \)) be the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix \( H_k(\xi) \) and the constants
\[ \alpha_G := \min_{k \in \mathcal{P}} \| \lambda_{\min,k} \|_{L^0(B_k)} \quad \text{and} \quad \beta_G := \max_{k \in \mathcal{P}} \| \lambda_{\max,k} \|_{L^\infty(B_k)} . \]

Here \( \|u\|_{L^0(E)} \) and \( \|u\|_{L^\infty(E)} \) denote the essential infimum and the essential supremum of a measurable function \( u \) on \( E \). We are now ready to state and prove our main sampling results.

**Theorem 3.3.** Assume that the function \( Z_{\psi_j}(a_j, \xi) \in L^\infty[\sigma(V)] \) for \( 1 \leq j \leq N \). Then the following statements are equivalent:
(iii) \( \alpha_G > 0 \).

**Proof.** Condition (i) implies condition (ii) trivially. Assume condition (ii): applying the isomorphism \( T_\varphi^{-1} : V(\varphi) \rightarrow L^2[\sigma(V)] \) to (3.5) gives:

\[
(3.6) \quad F(\xi) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \langle F(\xi), g_j(\xi)e^{-irn\xi} \rangle_{L^2[\sigma(V)]} s_j,n(\xi), \quad F \in L^2[\sigma(V)],
\]

where \( \{s_j,n(\xi) : 1 \leq j \leq N, n \in \mathbb{Z}\} \) is a frame for \( L^2[\sigma(V)] \). By using Lemma 3.5 (i) below, the sequence \( \{g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z}\} \) is a Bessel sequence in \( L^2[\sigma(V)] \). The expansion (3.6) on \( L^2[\sigma(V)] \) implies that the sequence \( \{g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z}\} \) must be a frame for \( L^2[\sigma(V)] \) (see Lemma 5.6.2 in [5]). Hence, condition (iii) holds by using Lemma 3.5 (ii) below.

Finally assume condition (iii): the sequence \( \{g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z}\} \) with \( g_j(\xi) = \frac{1}{2\pi}Z_\psi(a_j, \xi) \) is a frame for \( L^2[\sigma(V)] \) by Lemma 3.5 (ii) below. Let \( \{s_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z}\} \) be a dual frame of \( \{g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z}\} \) (cf. Lemma 3.6 below). Thus we have the following frame expansion in \( L^2[\sigma(V)]\):

\[
(3.7) \quad F(\xi) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \langle F(\xi), g_j(\xi)e^{-irn\xi} \rangle_{L^2[\sigma(V)]} s_j(\xi)e^{-irn\xi}, \quad F \in L^2[\sigma(V)].
\]

Applying the isomorphism \( T_\varphi : L^2[\sigma(V)] \rightarrow V(\varphi) \) to (3.7) gives (3.4) with \( S_j = T_\varphi(s_j) \), \( 1 \leq j \leq N \), which proves condition (i). \( \square \)

For later use, notice that \( \alpha_G > 0 \) implies \( l(k) \leq N \) for all \( k \in \mathcal{P} \). For \( N = r = 1 \) in Theorem 3.3, we obtain:

**Corollary 3.4.** Let \( \mathcal{L} \) be an LTI system of type (i), (ii) or (iii). There is a frame \( \{S(t-n) : n \in \mathbb{Z}\} \) for \( V(\varphi) \) such that for each \( f \in V(\varphi) \)

\[
(3.8) \quad f(t) = \sum_{n \in \mathbb{Z}} (\mathcal{L}f)(a+n)S(t-n), \quad t \in \mathbb{R}
\]

if and only if

\[
(3.9) \quad 0 < \|Z_\psi(a, \xi)\|_{L^p[\sigma(V)]} \leq \|Z_\psi(a, \xi)\|_{L^\infty[\sigma(V)]} < \infty.
\]
Moreover, in this case,

\begin{equation}
\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\psi(a, \xi)} \chi_{\text{supp} G_\varphi(\xi)}.
\end{equation}

**Proof.** Whenever \( r = 1 \), \( L(k) = \{1\} \) and \( \tilde{B}_k = B_k \) for all \( k \in \mathcal{P} \); thus \( D_k \) becomes the identity operator. Therefore, \( G_k(\xi) = g(\xi) = \frac{1}{2\pi} Z_\psi(a, \xi) \) and \( H_k(\xi) = \frac{1}{(2\pi)^2} |Z_\psi(a, \xi)|^2 \) for \( k \in \mathcal{P} \) and \( \xi \in B_k \). Hence \( 0 < \alpha_G \leq \beta_G < \infty \) if and only if condition (3.9) holds. As a consequence, (3.9) implies (3.8) by Theorem 3.3. Conversely, assume that (3.8) holds. Then \( \varphi(t) = \sum_{n \in \mathbb{Z}} \psi(a + n) S(t - n) \) so that \( \hat{\varphi}(\xi) = Z_\psi(a, \xi) \hat{S}(\xi) \) and \( G_\varphi(\xi) = |Z_\psi(a, \xi)|^2 G_S(\xi) \) from which (3.9) and (3.10) follow. \( \square \)

When the impulse response \( h \) is the Dirac delta distribution \( \delta(t) \), the system \( \mathcal{L} \) is the identity operator, and Corollary 3.4 reduces to a regular shifted sampling in \( V(\varphi) \) (see Theorem 1 in [22] and Theorem 3.4 in [15]). The next technical lemma used in the proof of Theorem 3.3 enlarges the results of Lemma 3 in [9]:

**Lemma 3.5.** Let \( g_j \) be in \( L^2[\sigma(V)] \) for \( 1 \leq j \leq N \) and let \( \alpha_G, \beta_G \) be the constants given by (3.3). Then we have:

(i) The sequence \( \{g_j(\xi)e^{-\imath n \xi} : 1 \leq j \leq N, n \in \mathbb{Z}\} \) is a Bessel sequence in \( L^2[\sigma(V)] \) if and only if \( \beta_G < \infty \), that is, \( g_j(\xi) \in L^\infty[\sigma(V)] \) for each \( 1 \leq j \leq N \).

(ii) The sequence \( \{g_j(\xi)e^{-\imath n \xi} : 1 \leq j \leq N, n \in \mathbb{Z}\} \) is a frame for \( L^2[\sigma(V)] \) if and only if

\begin{equation}
0 < \alpha_G \leq \beta_G < \infty.
\end{equation}

**Proof.** First note that for any \( F \in L^2[\sigma(V)] \) we have

\[
(F(\xi), \overline{g_j(\xi)e^{-\imath n \xi}})_{L^2[\sigma(V)]} = \int_{\sigma(V)} F(\xi) g_j(\xi) e^{\imath n \xi} d\xi
\]

\[
= \sum_{k \in \mathcal{P}} \int_{B_k} [D_k(g_j)]^T D_k(F_k)(\xi)e^{\imath n \xi} d\xi
\]

\[
= \sum_{k \in \mathcal{P}} [D_k(g_j)]^T D_k(F_k) \chi_{B_k}(e^{\imath n \xi})_{L^2(I)},
\]

where \( F_k(\xi) := F(\xi) \chi_{B_k}(\xi) \). Since \( \{\sqrt{2\pi} e^{-\imath n \xi}\}_{n \in \mathbb{Z}} \) is an orthonormal basis for \( L^2(I) \) and the sets \( B_k \) are disjoint, we have

\[
\sum_{n \in \mathbb{Z}} |(F(\xi), \overline{g_j(\xi)e^{-\imath n \xi}})_{L^2[\sigma(V)]}|^2 = \frac{2\pi}{r} \left\| \sum_{k \in \mathcal{P}} D_k(g_j)^T D_k(F_k) \chi_{B_k}(\xi) \right\|^2_{L^2(I)}
\]

\[
= \frac{2\pi}{r} \sum_{k \in \mathcal{P}} \left\| D_k(g_j)^T D_k(F_k) \right\|^2_{L^2(B_k)}
\]

\[
= \frac{2\pi}{r} \sum_{k \in \mathcal{P}} \langle D_k(g_j)^T D_k(F_k), D_k(F_k) \rangle_{L^2(I)}.
\]
satisfies

\[ \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} |\langle F(\xi), g_{j}(\xi)e^{-irn\xi} \rangle|^2 \]

Moreover, if \( \beta \in \mathbb{B} \) can be constructed as in [13, Lemma 2.4]. Extend Lemma 3.6.

For (i), assume that \( \beta_G < \infty \). By using (3.12), for any \( F \in L^2[\sigma(V)] \) we have

\[
\sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} |\langle F(\xi), g_{j}(\xi)e^{-irn\xi} \rangle| L^2[\sigma(V)]^2 \leq \frac{2\pi}{r} \beta_G \sum_{k \in \mathcal{P}} \langle D_k(F_k), D_k(F_k) \rangle L^2_{l(\mathcal{B})}(B_k)
\]

so that \( \left\{ g_{j}(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \right\} \) is a Bessel sequence with bound \( \frac{2\pi}{r} \beta_G \).

On the other hand, for any constant \( K \) with \( 0 \leq K < \beta_G \) we have that \( K < \|\lambda_{\max,k}(\xi)\|_\infty \) for some \( k \in \mathcal{P} \). Then there is a measurable set \( E \subset B_k \) of positive measure such that \( \lambda_{\max,k}(\xi) \geq K \) on \( E \). Choose a measurable vector-valued function \( F_k(\xi) := \{ F_{k,j}(\xi) \} \) on \( E \) such that \( \sum_{j=1}^{l(k)} \| F_{k,j}(\xi) \|^2 = 1 \) on \( E \) and \( H_k(\xi)F_k(\xi) = \lambda_{\max,k}(\xi)F_k(\xi) \) on \( E \). This function can be constructed as in [13, Lemma 2.4]. Extend \( F_k(\xi) \) over \( B_k \) by setting \( F_k(\xi) = 0 \) on \( B_k \setminus E \). Thus \( F_k \in L^\infty_{l(\mathcal{B})(B_k)} \) and \( H_k(\xi)F_k(\xi) = \lambda_{\max,k}(\xi)F_k(\xi) \) on \( B_k \). Let \( F \) be such that \( F = D_k^{-1}(F_k) \) on \( \tilde{B}_k \) and \( F(\xi) = 0 \) on \( \sigma(V) \setminus \tilde{B}_k \). This function \( F \) belongs to \( L^\infty[\sigma(V)] \) and satisfies

\[
\sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} |\langle F(\xi), g_{j}(\xi)e^{-irn\xi} \rangle| L^2[\sigma(V)]^2 \leq \frac{2\pi}{r} \beta_G \sum_{k \in \mathcal{P}} \langle H_k(\xi)F_k(\xi), F_k(\xi) \rangle L^2_{l(\mathcal{B})}(B_k)
\]

As a consequence, \( \frac{2\pi}{r} \beta_G \) is the optimal Bessel bound for \( \left\{ g_{j}(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \right\} \). Moreover, if \( \beta_G = \infty \), the sequence \( \left\{ g_{j}(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \right\} \) cannot be a Bessel sequence. Finally, note that the spectral norm of a matrix is equivalent to its Frobenius norm. Hence \( \beta_G < \infty \) if and only if all entries of \( H_k(\xi) \) for \( k \in \mathcal{P} \) are essentially bounded which is also equivalent to \( g_j \in L^\infty[\sigma(V)] \) for \( 1 \leq j \leq N \).

For (ii), assume that \( 0 < \alpha_G \leq \beta_G < \infty \). A similar reasoning as the one in (i) gives that \( \left\{ g_{j}(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \right\} \) is a frame for \( L^2[\sigma(V)] \), where \( \frac{2\pi}{r} \beta_G \geq \frac{2\pi}{r} \alpha_G \) are the optimal upper and lower bounds. In particular, if either \( \alpha_G = 0 \) or \( \beta_G = \infty \), then \( \left\{ g_{j}(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \right\} \) cannot be a frame for \( L^2[\sigma(V)] \).

It is useful to note that condition (3.11) is equivalent to \( g_j \in L^\infty[\sigma(V)] \) for \( 1 \leq j \leq N \) and \( \min_{k \in \mathcal{P}} \| \det H_k(\xi) \|_{L^0(\mathcal{B}_k)} > 0 \).

**Lemma 3.6.** Let \( g_j \) be in \( L^2[\sigma(V)] \) for \( 1 \leq j \leq N \) such that \( \left\{ g_{j}(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \right\} \) is a frame for \( L^2[\sigma(V)] \). Then any dual frame of \( \left\{ g_{j}(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \right\} \) having
the form \( \{ s_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \) is obtained from the equation
\[
(3.13) \quad \frac{2\pi}{r}S_k(\xi)^\top = G_k(\xi)^\dagger + E_k(\xi)(I_N - G_k(\xi)G_k(\xi)^\dagger), \quad k \in \mathcal{P},
\]
where \( I_N \) is the \( N \times N \) identity matrix, \( E_k(\xi) \) is any arbitrary \( l(k) \times N \) matrix with entries in \( L^\infty(B_k) \), \( G_k(\xi)^\dagger := [G_k(\xi)^*G_k(\xi)]^{-1}G_k(\xi) \) is the pseudo-inverse matrix of \( G_k(\xi) \),
\[
(3.14) \quad S_k(\xi) := [D_k(s_{1,k})(\xi), \ldots, D_k(s_{N,k})(\xi)]^\top
\]
and \( s_{j,k}(\xi) = s_j(\xi)\chi_{\mathbb{R}_k}(\xi) \) for \( 1 \leq j \leq N \).

**Proof.** Assume that the sequence \( \{ s_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \) is a dual frame of the sequence \( \{ g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \). Then \( s_j \in L^\infty[\sigma(V)] \) for \( 1 \leq j \leq N \). For any \( F_1 \) and \( F_2 \) in \( L^2[\sigma(V)] \) we also have (cf. Lemma 5.6.2 in [5]):
\[
\langle F_1, F_2 \rangle_{L^2[\sigma(V)]} = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \langle F_1, s_j e^{-irn\xi} \rangle_{L^2[\sigma(V)]} \langle g_j e^{-irn\xi}, F_2 \rangle_{L^2[\sigma(V)]}
\]
\[
(3.15) \quad = \frac{2\pi}{r} \sum_{k \in \mathcal{P}} \langle D_k(F_1,k), S_k(\xi)G_k(\xi)D_k(F_2,k) \rangle_{L^2[\sigma(V)]}
\]
with \( S_k(\xi) \) as in (3.14). Since
\[
\langle F_1, F_2 \rangle_{L^2[\sigma(V)]} = \sum_{k \in \mathcal{P}} \langle D_k(F_1,k), D_k(F_2,k) \rangle_{L^2[\sigma(V)]},
\]
(3.15) implies that \( \frac{2\pi}{r}S_k(\xi) \) must be a left inverse of the matrix \( G_k(\xi) \). Finally, the right hand side of (3.13) is a left inverse of \( G_k(\xi) \) and any left inverse \( \frac{2\pi}{r}S_k(\xi)^\top \) of \( G_k(\xi) \) is obtained from (3.13) by choosing \( E_k(\xi) = \frac{2\pi}{r}S_k(\xi)^\top \). \( \square \)

One can easily check that the canonical dual frame of \( \{ g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \) is obtained from (3.13) by choosing \( E_k(\xi) = 0 \) for each \( k \in \mathcal{P} \).

Next we give the Riesz basis counterpart to Theorem 3.3:

**Theorem 3.7.** There exists a Riesz basis \( \{ S_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z} \} \) for \( V(\varphi) \) for which the sampling expansion (3.5) holds on \( V(\varphi) \) if and only if
\[
(3.16) \quad 0 < \alpha_G \leq \beta_G < \infty \quad \text{and} \quad l(k) = N \quad \text{for all} \quad 1 \leq k \leq m.
\]
Moreover, in this case,
\[
(3.17) \quad S_{j,n}(t) = S_j(t-rn), \quad 1 \leq j \leq N \quad \text{and} \quad n \in \mathbb{Z};
\]
\[
(3.18) \quad (L_jS_k)(at+rn) = \delta_{j,k}\delta_{n,0}, \quad 1 \leq j,k \leq N \quad \text{and} \quad n \in \mathbb{Z};
\]
\[
(3.19) \quad |\sigma(V)| = 2\pi \frac{N}{r} \quad \text{(which implies} \quad N \leq r \text{).}
\]

**Proof.** Assuming (3.16), Lemma 3.8 below proves that the sequence \( \{ g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \) is a Riesz basis for \( L^2[\sigma(V)] \). Thus we have the Riesz basis expansion (3.7), where \( \{ s_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \) is the dual Riesz basis of \( \{ g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \). The isomorphism \( T_\varphi \) gives the sampling expansion (3.5), where \( S_{j,n}(t) = S_j(t-rn) \) and \( S_j(t) = T_\varphi(s_j(\xi))(t) \). Conversely assume that the Riesz basis expansion (3.5) holds on \( V(\varphi) \). Applying the isomorphism \( T_\varphi^{-1} \) to (3.5) gives the Riesz basis expansion (3.6) on \( L^2[\sigma(V)] \).
Then \( \{g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \) must be the dual Riesz basis of \( \{s_{j,n}(\xi) : 1 \leq j \leq N, n \in \mathbb{Z} \} \) so that (3.16) holds by Lemma 3.8 below. Since \( \{s_{j,n}(\xi) : 1 \leq j \leq N, n \in \mathbb{Z} \} \) is the dual Riesz basis of \( \{g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \), \( s_{j,n}(\xi) = s_j(\xi)e^{-irn\xi} \) where \( s_j \in L^\infty[\sigma(V)] \) (cf. Lemma 3.6). Therefore, \( S_{j,n}(t) = T_{\varphi}(s_j(\xi)e^{-irn\xi})(t) = S_j(t - r n) \), where \( S_j = T_{\varphi}(s_j) \), \( 1 \leq j \leq N \), so that (3.17) holds. Applying the sampling formula (3.4) to \( S_k \) gives

\[
S_k(t) = \sum_{j=1}^{N} \left( \mathcal{L}_{j}S_k \right) (a_j + r n) S_j(t - r n), \quad t \in \mathbb{R},
\]

from which (3.18) follows. Finally (3.19) follows immediately from (3.16) having in mind that \( |\sigma(V)| = \sum_{k \in \mathcal{P}} l(k)|B_k| \).

Choose \( \sigma(V) = [0, 2\pi] \), \( \varphi \) becomes a Riesz generator for \( V(\varphi) \). As a consequence, Theorems 3.3 and 3.7 are the extended frame versions of Theorem 2 and Corollary 1 in [9]; there \( \varphi \) is a Riesz generator and the LTI system \( \mathcal{L}_j \) has impulse response \( h_j \) in \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) for \( 1 \leq j \leq N \).

**Lemma 3.8.** Let \( g_j \) be a function in \( L^2[\sigma(V)] \) for \( 1 \leq j \leq N \) and let \( \alpha_G, \beta_G \) be the constants given by (3.3). Then, the sequence \( \{g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \) is a Riesz basis for \( L^2[\sigma(V)] \) if and only if

\[
0 < \alpha_G \leq \beta_G < \infty \quad \text{and} \quad l(k) = N \quad \text{for all} \quad 1 \leq k \leq m.
\]

**Proof.** Note that \( \{g_j(\xi)e^{-irn\xi} : 1 \leq j \leq N, n \in \mathbb{Z} \} \) is a Riesz basis for \( L^2[\sigma(V)] \) if and only if it is complete set in \( L^2[\sigma(V)] \) and there are constants \( 0 < A \leq B \) such that

\[
A\|\mathbf{c}\|^2 \leq \left\| \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} c_j(n)g_j(\xi)e^{-irn\xi} \right\|^2_{L^2[\sigma(V)]} \leq B\|\mathbf{c}\|^2,
\]

where \( \mathbf{c} = (c_1, \ldots, c_N) \in \ell^2_N(\mathbb{Z}) \) and \( \|\mathbf{c}\|^2 := \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} |c_j(n)|^2 \). For the middle term in (3.21) we have

\[
\left\| \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} c_j(n)g_j(\xi)e^{-irn\xi} \right\|^2_{L^2[\sigma(V)]} = \int_{\sigma(V)} \left\| \sum_{j=1}^{N} g_j(\xi)\tilde{c}_j(r\xi) \right\|^2 \, d\xi
\]

\[
= \sum_{k \in \mathcal{P}} \sum_{j=1}^{l(k)} \int_{B_k} |g^*(\xi + (n_{k,j} - 1)\frac{2\pi}{r})\tilde{c}(r\xi)|^2 \, d\xi
\]

\[
= \sum_{k \in \mathcal{P}} \left( \sum_{j=1}^{l(k)} g(\xi + (n_{k,j} - 1)\frac{2\pi}{r})g^*(\xi + (n_{k,j} - 1)\frac{2\pi}{r})\tilde{c}(r\xi)\tilde{c}(r\xi) \right)_{L^2(B_k)}
\]

\[
= \sum_{k \in \mathcal{P}} \langle \tilde{H}_k(\xi)\tilde{c}(r\xi), \tilde{c}(r\xi) \rangle_{L^2(B_k)}
\]

where \( g(\xi) := [g_1(\xi), \ldots, g_N(\xi)]^\top, \tilde{c}(\xi) := [\tilde{c}_1(\xi), \ldots, \tilde{c}_N(\xi)]^\top \) and \( \tilde{H}_k(\xi) := G_k(\xi)G_k^*(\xi) \).

On the other hand,

\[
\|\mathbf{c}\|^2 = \frac{r}{2\pi} \|\tilde{c}(r\xi)\|^2_{L^2(\mathbb{R})} = \frac{r}{2\pi} \sum_{k=1}^{m} \|\tilde{c}(r\xi)\|^2_{L^2(B_k)}
\]
Hence, condition (3.21) is equivalent to
\begin{equation}
A_r \sum_{k=1}^{m} \|\tilde{c}(r\xi)\|_{L^{2}_{B_k}(B_k)}^2 \leq \sum_{k \in \mathcal{P}} \langle \tilde{H}_k(\xi)\tilde{c}(r\xi), \tilde{c}(r\xi) \rangle_{L^{2}_{B_k}(B_k)} \leq B_r \sum_{k=1}^{m} \|\tilde{c}(r\xi)\|_{L^{2}_{B_k}(B_k)}^2,
\end{equation}
which holds if and only if $\mathcal{P} = \{1, 2, \ldots, m\}$ and $0 < \tilde{\alpha}_G \leq \tilde{\beta}_G < \infty$, where $\tilde{\alpha}_G := \min_{1 \leq k \leq m} \lambda_{\min,k} \|0\|$, $\tilde{\beta}_G := \max_{1 \leq k \leq m} \lambda_{\max,k} \|0\|$, and $\lambda_{\min,k}$ (respectively $\lambda_{\max,k}$) is the smallest (respectively the largest) eigenvalue of the matrix $\tilde{H}_k(\xi)$.

Now assume that $\{\tilde{g}_j(\xi)e^{-i\pi n\xi} : 1 \leq j \leq N, n \in \mathbb{Z}\}$ is a Riesz basis for $L^2[\sigma(V)]$. Since (3.11) holds, we deduce that $l(k) \leq N$ for any $k \in \mathcal{P}$; but we also have (3.22) so that $N \leq l(k)$ for any $1 \leq k \leq m$. Hence, $l(k) = N$ for all $1 \leq k \leq m$. Conversely, assume that (3.20) holds. Thus, $\{\tilde{g}_j(\xi)e^{-i\pi n\xi} : 1 \leq j \leq N, n \in \mathbb{Z}\}$ is a complete set in $L^2[\sigma(V)]$ since it is a frame for $L^2[\sigma(V)]$. For each $1 \leq k \leq m$, since $\alpha_G I_N \leq \tilde{H}_k(\xi) \leq \beta_G I_N$, for any $\mathbf{F}_k \in L^2_N(B_k)$ we have
\begin{equation}
\alpha_G \|\mathbf{F}_k\|_{L^2_N(B_k)}^2 \leq \|G_k(\xi)\mathbf{F}_k(\xi)\|_{L^2_N(B_k)}^2 \leq \beta_G \|\mathbf{F}_k\|_{L^2_N(B_k)}^2,
\end{equation}
and there exists the inverse matrix $G_k(\xi)^{-1}$ a.e. with entries essentially bounded. Then $G_k(\xi)$ and $G_k^*(\xi)$ are isomorphisms from $L^2_N(B_k)$ onto $L^2_N(B_k)$. Hence, for any $k = 1, 2, \ldots, m$ we have
\begin{equation}
\alpha_G \|\mathbf{F}_k\|_{L^2_N(B_k)}^2 \leq \|G_k^*(\xi)\mathbf{F}_k(\xi)\|_{L^2_N(B_k)}^2 = (\tilde{H}_k(\xi)\mathbf{F}_k(\xi), \mathbf{F}_k(\xi))_{L^2_N(B_k)} \leq \beta_G \|\mathbf{F}_k\|_{L^2_N(B_k)}^2,
\end{equation}
for any $\mathbf{F}_k \in L^2_N(B_k)$. Thus (3.22) or, equivalently, (3.21) holds, from which we deduce that the sequence $\{\tilde{g}_j(\xi)e^{-i\pi n\xi} : 1 \leq j \leq N, n \in \mathbb{Z}\}$ is a Riesz basis for $L^2[\sigma(V)]$.

For the particular case $N = 1$, Theorem 3.7 reads:

**Corollary 3.9.** Let $\mathcal{L}$ be an LTI system of type (i), (ii) or (iii). Then, there exists a Riesz basis $\{S_n(t) : n \in \mathbb{Z}\}$ for $V(\varphi)$ such that, for any $f \in V(\varphi)$, the sampling formula
\begin{equation}
f(t) = \sum_{n \in \mathbb{Z}} (\mathcal{L}f)(a + rn) S_n(t), \quad t \in \mathbb{R},
\end{equation}
holds if and only if
\begin{equation}
0 < \|Z_0(a, \xi)\|_{L^0[\sigma(V)]} \leq \|Z_0(a, \xi)\|_{L^\infty[\sigma(V)]} < \infty \quad \text{and} \quad l(k) = 1 \text{ for all } 1 \leq k \leq m.
\end{equation}
Moreover, in this case:
\begin{itemize}
  \item $S_n(t) = \delta(t - rn), ~ n \in \mathbb{Z}$;
  \item $(\mathcal{L}S)(a + rn) = \delta_{0,0}, ~ n \in \mathbb{Z}$;
  \item $|\sigma(V)| = \frac{2\pi}{r}$.
\end{itemize}

**Proof.** Assume $l(k) = 1$ for all $1 \leq k \leq m$; for each $k = 1, 2, \ldots, m$, there is a unique integer $n_k$ with $1 \leq n_k \leq r$ such that $\tilde{B}_k = B_k + (n_k - 1)\frac{2\pi}{r} \subseteq \sigma(V)$. Thus, $G_k(\xi) = D_k(g)(\xi) = \frac{1}{2\pi}Z_0(a, \xi + (n_k - 1)\frac{2\pi}{r})$ and $H_k(\xi) = \frac{1}{(2\pi)^2}Z_0(a, \xi + (n_k - 1)\frac{2\pi}{r})$ for $\xi \in B_k$. Hence, $0 < \alpha_G \leq \beta_G < \infty$ if and only if $0 < \|Z_0(a, \xi)\|_{L^0[\sigma(V)]} \leq \|Z_0(a, \xi)\|_{L^\infty[\sigma(V)]} < \infty$ and, as a consequence, Corollary 3.9 follows from Theorem 3.7.

Furthermore, if $r = 1$ in Corollary 3.9, then $\varphi$ must be a Riesz generator since $\sigma(V) = [0, 2\pi]$ and $\tilde{S}(\xi) = \tilde{\varphi}(\xi)/[Z_0(a, \xi)]$.
Finally, it is worth to notice that in sampling formula (3.2) we may allow a rational sampling period \( r = \frac{p}{q} \), where \( p \) and \( q \) are coprime positive integers, since

\[
\{(L_j f)(a_j + rn) : n \in \mathbb{Z}\} = \{(L_j f)(a_j + r(k - 1) + pm) : 1 \leq k \leq q \text{ and } n \in \mathbb{Z}\}.
\]

4. An illustrative example

Let \( \varphi(t) = \frac{1}{2} \sin\left(\frac{t}{2}\right) = \frac{\sin\pi t}{\pi t} \) so that \( \hat{\varphi}(\xi) = \chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(\xi) \). On \([0, 2\pi]\) we have,

\[
G_\varphi(\xi) = \begin{cases} 
1 & \text{on } [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \\
0 & \text{on } (\frac{\pi}{2}, \frac{3\pi}{2})
\end{cases}
\]

so that \( \varphi \) is a continuous frame generator of \( V(\varphi) \) and \( \sigma(V) = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \). By the Poisson summation formula, we also have

\[
C_\varphi(t) = \sum_{n \in \mathbb{Z}} |\varphi(t + n)|^2 = \frac{1}{2\pi} \left\| Z_{\varphi}(t, \cdot) \right\|_{L^2[0, 2\pi]}^2 \leq \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \hat{\varphi}(\cdot + 2n\pi) e^{it(\cdot + 2n\pi)} \right\|_{L^2[0, 2\pi]}^2 = \frac{1}{2}, \quad t \in \mathbb{R}.
\]

(a) First take \( N = 2 \), \( \hat{h}_j(\xi) = (i\xi)^{-1} \chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(\xi) \) for \( j = 1, 2 \), \( r = 4 \) and \( a_1 = a_2 = 0 \). For any \( f \in V(\varphi) \)

\[
L_j[f](t) = f^{(j-1)}(t) \text{ for } j = 1, 2.
\]

For \( \psi_j(t) = L_j[\varphi](t) \), the Poisson summation formula gives

\[
Z_{\psi_j}(0, \xi) = \sum_{n \in \mathbb{Z}} \psi_j(n) e^{-in\xi} = \sum_{n \in \mathbb{Z}} \hat{\psi}_j(\xi + 2n\pi), \quad j = 1, 2,
\]

so that

\[
Z_{\psi_1}(0, \xi) = \begin{cases} 
1 & \text{on } [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \\
0 & \text{on } (\frac{\pi}{2}, \frac{3\pi}{2})
\end{cases}
\]

and

\[
Z_{\psi_2}(0, \xi) = \begin{cases} 
i\xi & \text{on } [0, \frac{\pi}{2}] \\
0 & \text{on } (\frac{\pi}{2}, \frac{3\pi}{2}) \\
i(\xi - 2\pi) & \text{on } [\frac{3\pi}{2}, 2\pi]
\end{cases}
\]

Hence, \( Z_{\psi_j}(0, \xi) \in L^\infty[0, 2\pi] \) for \( j = 1, 2 \).

On the other hand, since \( \sigma(V) = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \) and \( I = [0, \frac{\pi}{2}] \), \( m = 1 \) and \( L(1) = \{1, 4\} \) so that \( l(1) = 2 \). Hence,

\[
2\pi G_1(\xi) = \begin{bmatrix} 1 & 1 \\ i\xi & i(\xi - \frac{\pi}{2}) \end{bmatrix}, \quad 0 \leq \xi \leq \frac{\pi}{2}
\]

and consequently,

\[
(2\pi)^2 H_1(\xi) = \begin{bmatrix} 1 & \xi^2 \\ 1 + \xi(\xi - \frac{\pi}{2}) & 1 + (\xi - \frac{\pi}{2})^2 \end{bmatrix}, \quad 0 \leq \xi \leq \frac{\pi}{2}.
\]

Hence, \( \det H_1(\xi) = \det G_1(\xi)^2 = 1/(64\pi^2) \) and we deduce that \( \alpha_G > 0 \). Therefore, by using Theorem 3.7, there exists a Riesz basis \( \{S_j(t - 4n) : j = 1, 2 \text{ and } n \in \mathbb{Z}\} \) for \( V(\varphi) \) such that, for any \( f \in V(\varphi) \)

\[
f(t) = \sum_{n \in \mathbb{Z}} \{f(4n)S_1(t - 4n) + f'(4n)S_2(t - 4n)\}, \quad t \in \mathbb{R}.
\]
(b) We now take \( N = 3, \hat{h}_j(\xi) = (i\xi)^{j-1}\chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(\xi) \) for \( j = 1, 2, 3, r = 5 \) and \( a_1 = a_2 = a_3 = 0 \). For \( f \in V(\varphi) \),
\[
\mathcal{L}_j[f](t) = f^{(j-1)}(t) \quad \text{for} \quad j = 1, 2, 3,
\]
and
\[
Z_{\psi_j}(0, \xi) = \begin{cases} 
-\xi^2 & \text{on } [0, \frac{\pi}{2}] \\
0 & \text{on } (\frac{\pi}{2}, \frac{3\pi}{2}) \\
-(\xi - 2\pi)^2 & \text{on } [\frac{3\pi}{2}, 2\pi].
\end{cases}
\]
so that \( Z_{\psi_j}(0, \xi) \in L^\infty[0, 2\pi] \) for \( j = 1, 2, 3 \).

Since \( \sigma(V) = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \) and \( I = [0, \frac{3\pi}{2}] \), \( m = 3 \) and \( \{t_j\}_{j=0}^3 = \{0, \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{5}\} \), so that \( L(1) = \{1, 2, 5\}, L(2) = \{1, 5\}, L(3) = \{1, 4, 5\} \). We then have
\[
2\pi G_1(\xi) = \begin{bmatrix} 1 & i\xi & 1 \\
-\xi^2 & -(\xi + \frac{2\pi}{5})^2 & -(\xi - \frac{2\pi}{5})^2 
\end{bmatrix}, \quad \xi \in B_1 = (0, \frac{\pi}{10});
\]
\[
2\pi G_2(\xi) = \begin{bmatrix} 1 & i\xi \xi(\frac{2\pi}{5}) \\
-\xi^2 & -(\xi + \frac{2\pi}{5})^2 & -(\xi - \frac{2\pi}{5})^2 
\end{bmatrix}, \quad \xi \in B_2 = (\frac{\pi}{10}, \frac{3\pi}{10});
\]
\[
2\pi G_3(\xi) = \begin{bmatrix} 1 & i\xi & 1 \\
-\xi^2 & -(\xi + \frac{2\pi}{5})^2 & -(\xi - \frac{2\pi}{5})^2 
\end{bmatrix}, \quad \xi \in B_3 = (\frac{3\pi}{10}, \frac{2\pi}{5}).
\]
Thus, for \( H_j(\xi) = G_j^*(\xi)G_j(\xi), j = 1, 2, 3 \), we have \( \det H_1(\xi) = \det H_3(\xi) = (2/125)^2 \) and
\[
(2\pi)^4 \det H_2(\xi) = (x^2 - \xi^2)^2 + (x^2 - \xi^2)^2 + (x - \xi)^2 \geq (x - \xi)^2 = \frac{4\pi^2}{25},
\]
where \( x = \xi - \frac{2\pi}{5} \); hence \( \alpha_G > 0 \). Therefore, by Theorem 3.3, there exists a frame \( \{S_j(t-5n) : j = 1, 2, 3 \text{ and } n \in \mathbb{Z}\} \) for \( V(\varphi) \) such that, for each \( f \in V(\varphi) \) we have
\[
f(t) = \sum_{n \in \mathbb{Z}} \{f(5n)S_1(t - 5n) + f'(5n)S_2(t - 5n) + f''(5n)S_3(t - 5n)\}, \quad t \in \mathbb{R}.
\]

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