Regular multivariate sampling and approximation in L^p shift-invariant spaces

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Abstract

In this paper some results about regular multivariate generalized sampling in the L^p setting ($1 \le p \le \infty$) are proven. Thus, stable multivariate regular sampling formulas are derived for L^p shift-invariant spaces V_Φ^p, i.e., the L^p-closure of the linear span of the shifts of a finite set Φ of generators. These sampling formulas include regular samples of the function, its derivatives and/or some filtered versions of the function itself taken at a lattice of $\mathbb{R}^d$. Approximation schemes using these generalized sampling formulas are also included.

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1 Statement of the problem

The classical Whittaker-Shannon-Kotel'nikov sampling formula has its counterpart in d dimensions. Thus, any function f band-limited to the d-dimensional cube $[-1/2, 1/2]^d$, i.e., $f(t) = \int_{[-1/2,1/2]^d} \widehat{f}(x)e^{2\pi i x^\top t}dx, \quad t \in \mathbb{R}^d$, may be reconstructed from its sequence of samples $\{f(\alpha)\}_{n\in\mathbb{Z}^d}$ as

$$f(t) = \sum_{\alpha\in\mathbb{Z}^d} f(\alpha)\,\mathrm{sinc}(t_1 - \alpha_1)\ldots\mathrm{sinc}(t_d - \alpha_d)\,, \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d\,.$$

Although Shannon's sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in [34, 35]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite duration signal; the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has

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a very slow decay, which makes computation in the signal domain very inefficient. Besides, in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval. In addition, many applied problems impose different a priori constraints on the type of functions. For these reasons, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces. Also to model the decay conditions of real signals, the sampling theory is developed in weighted shift-invariant spaces. See, for instance, [2, 3, 4, 5, 6, 12, 33, 35, 38, 39, 40] and the references therein.

In many practical applications, signals are assumed to belong to some shift-invariant space of the form

$$V_\Phi^2 := \overline{\operatorname{span}}_{L^2(\mathbb{R}^d)}\big\{\phi_j(t-\alpha) \,:\, \alpha \in \mathbb{Z}^d,\, j=1,2,\ldots,r\big\}\,,$$

where $\Phi := \{\phi_j\}_{j=1}^r$ in $L^2(\mathbb{R}^d)$ is the set of generators of V_Φ^2. Assuming that Φ is a stable set of generators, i.e., the sequence $\{\phi_j(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,r}$ is a Riesz basis for V_Φ^2, the shift-invariant space V_Φ^2 can be described as

$$V_\Phi^2 = \Big\{\sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} a_{j,\alpha}\ \phi_j(t-\alpha) \,:\, \{a_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)\,, j=1,2,\ldots,r\Big\} \subset L^2(\mathbb{R}^d)\,. \tag{1}$$

On the other hand, in many common situations the available data are samples of some filtered versions $f * \mathsf{h}_l$ of the signal f itself. This leads to generalized sampling (also called average sampling in some recent papers [8, 32]) in V_Φ^2. Suppose that s convolution systems (linear time-invariant systems or filters in engineering jargon) Υ_l, $l=1,2,\ldots,s$, are defined on the shift-invariant subspace V_Φ^2 of $L^2(\mathbb{R}^d)$. The goal is to recover any function f in V_Φ^2 from the set of samples $\{(\Upsilon_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ l=1,2,\ldots,s}$, taken at the lattice $M\mathbb{Z}^d$ in $\mathbb{R}^d$ (M denotes a matrix of integer entries with positive determinant), by means of a stable sampling formula like

$$f(t) = \sum_{l=1}^s \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_l(t-M\alpha)\,, \quad t\in\mathbb{R}^d\,. \tag{2}$$

By stable sampling we mean that there exist two positive constants $0 < A \le B$ such that

$$A\|f\|_2 \le \sum_{l=1}^s \|\{\Upsilon_l f(M\alpha)\}\|_{\ell^2} \le B\|f\|_2\,, \quad f\in V_\Phi^2\,. \tag{3}$$

The regular sampling L^2-theory, which involves the well-known frame theory, has been well-established by several authors (see, among others, [8, 21, 40]).

The aim of this paper is to prove some regular multivariate stable L^p-sampling results ($1\le p\le\infty$) and to construct approximation schemes, valid in appropriate Sobolev spaces, by means of them. To this end we consider the L^p shift-invariant spaces

$$V_\Phi^p := \overline{\operatorname{span}}_{L^p(\mathbb{R}^d)}\big\{\phi_j(t-\alpha) \,:\, \alpha \in \mathbb{Z}^d,\, j=1,2,\ldots,r\big\} \quad (1\le p\le\infty)\,.$$

Under appropriate hypotheses (see infra Section 2), and assuming that the set of generators $\Phi := \{\phi_j\}_{j=1}^r$ has L^p-stable shifts (see [30]) or, equivalently, it forms a p-Riesz basis for V_Φ^p (see [4, 24]), the space V_Φ^p becomes a Banach space which can be described as

$$V_\Phi^p = \Big\{\sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} a_{j,\alpha}\ \phi_j(t-\alpha) \,:\, \{a_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d} \in \ell^p(\mathbb{Z}^d)\,, j=1,2,\ldots,r\Big\} \subset L^p(\mathbb{R}^d)\,,$$

for $1 \le p < \infty$, and

$$V_\Phi^\infty = \Big\{ \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} a_{j,\alpha}\ \phi_j(t-\alpha) \ :\ \{a_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d} \in \mathsf{c}_0(\mathbb{Z}^d)\,, j=1,2,\ldots,r \Big\} \subset L^\infty(\mathbb{R}^d)\,,$$

where $\mathsf{c}_0(\mathbb{Z}^d)$ denotes the space of sequences on $\mathbb{Z}^d$ vanishing at ∞.

In order to prove a sampling result for V_Φ^p, $1 \le p \le \infty$, like (2), we first prove it for $\text{span}\{\phi_j(t-\alpha) \,:\, \alpha \in \mathbb{Z}^d,\, j=1,2,\ldots,r\}$, and then we extend it to the whole space V_Φ^p, $1 \le p \le \infty$, by means of a density argument. In regarding formula (2), we note that the inequality $r(\det M) \le s$ necessarily holds (see Lemma 1 infra). Also it is worth to mention here that the reconstruction functions S_l in formula (2) are explicitly given. In this L^p context, the stability condition (3) reads

$$A_p\|f\|_p \le \sum_{l=1}^{s} \|\{\Upsilon_l f(M\alpha)\}\|_{\ell^p} \le B_p\|f\|_p\,, \quad f \in V_\Phi^p\,.$$

We also consider the space $V_\Phi(\infty)$ involving $\ell^\infty(\mathbb{Z}^d)$ sequences

$$V_\Phi(\infty) := \Big\{ \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} a_{j,\alpha}\ \phi_j(t-\alpha) \ :\ \{a_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d} \in \ell^\infty(\mathbb{Z}^d)\,, j=1,2,\ldots,r \Big\} \subset L^\infty(\mathbb{R}^d)\,,$$

endowed with the metric topology giving uniform convergence on compact subsets of $\mathbb{R}^d$. The corresponding sampling theorem (2) for $V_\Phi(\infty)$ is also obtained but the corresponding stability condition does not remain true (see infra Section 4.6).

The last part of the paper concerns with the study of the approximation properties of the scaled version of the sampling operator

$$\Gamma f(t) = \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_l(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,.$$

In other words, we want to obtain a good approximation for a smooth function f (in a Sobolev space) by means of the scaled operator Γ^h defined by $\Gamma^h := \sigma_{1/h}\Gamma\sigma_h$, where $\sigma_h f(\cdot) := f(\cdot h)$, $h > 0$. The goal is to obtain an estimation for the L^p-approximation error of the type $\|\Gamma^h f - f\|_p = \mathcal{O}(h^k)$ as $h \to 0^+$, where $k \in \mathbb{N}$ denotes the approximation order which coincides, in general, with the order of the Strang-Fix conditions satisfied by the set of generators Φ.

The possibility of generalized sampling for obtaining approximation schemes in appropriate Sobolev spaces was derived in [20]; here we give some complementary results. For approximation schemes constructed by using shift-invariant spaces see Refs. [10, 11, 26, 28, 29, 30] and the references therein. Compared to the approximation results in Refs. [10, 15], the results here included have been proved by using a different technique which allows samples of derivatives. All these steps will be carried out throughout the remaining sections.

2 The shift-invariant spaces V_Φ^p $(1 \le p \le \infty)$

We start this section by introducing some notations and preliminaries used in the sequel. For $1 \le p \le \infty$, $L^p(\mathbb{R}^d)$ denotes the classical Lebesgue space. We denote by $\ell^p(\mathbb{Z}^d)$

$(1 \le p < \infty)$ the space of pth power summable sequences on $\mathbb{Z}^d$, by $\ell^\infty(\mathbb{Z}^d)$ the bounded sequences, and by $\mathsf{c}_0(\mathbb{Z}^d)$ the space of sequences on $\mathbb{Z}^d$ vanishing at ∞.

Given a Lebesgue measurable function $\phi : \mathbb{R}^d \longrightarrow \mathbb{C}$, set

$$|\phi|_p := \Big(\int_{[0,1)^d} \Big(\sum_{\alpha \in \mathbb{Z}^d} |\phi(t-\alpha)| \Big)^p dt \Big)^{1/p} \quad \text{when } 1 \le p < \infty \,,$$
$$|\phi|_\infty := \operatorname*{ess\,sup}_{t \in [0,1)^d} \sum_{\alpha \in \mathbb{Z}^d} |\phi(t-\alpha)| \quad \text{when } p = \infty \,.$$

For $1 \le p \le \infty$, let $\mathcal{L}^p(\mathbb{R}^d) := \big\{ f : \mathbb{R}^d \longrightarrow \mathbb{C} \text{ measurable} : |f|_p < \infty \big\}$. Equipped with the norm $|\cdot|_p$, $\mathcal{L}^p(\mathbb{R}^d)$ becomes a Banach space. These spaces are profusely used by Jia and Micchelli in [27].

Clearly, $|\phi|_1 = \|\phi\|_1$, $\|\phi\|_p \le |\phi|_p$ and $|\phi|_{p'} \le |\phi|_p$ for $1 \le p' \le p \le \infty$. This shows that $\mathcal{L}^p(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ and $\mathcal{L}^\infty(\mathbb{R}^d) \subset \mathcal{L}^p(\mathbb{R}^d) \subset \mathcal{L}^{p'}(\mathbb{R}^d) \subset \mathcal{L}^1(\mathbb{R}^d) = L^1(\mathbb{R}^d)$ for $1 \le p' \le p \le \infty$. Observe that if there are constants $C > 0$ and $\delta > 0$ such that

$$|\phi(t)| \le \frac{C}{(1+|t|)^{d+\delta}} \,, \quad t \in \mathbb{R}^d \,,$$

then $\phi \in \mathcal{L}^\infty(\mathbb{R}^d)$. Thus, the Kth-order B-spline $N_K := \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ (K times) belongs to $\mathcal{L}^\infty(\mathbb{R}^d)$. The Wiener amalgam space $W(L^\infty, \ell^1)$ defined as

$$W(L^\infty, \ell^1) := \Big\{ f \,:\, \|f\|_W := \sum_{\alpha \in \mathbb{Z}^d} \operatorname*{ess\,sup}_{t \in [0,1)^d} |f(t+\alpha)| < \infty \Big\} \,,$$

becomes a Banach space when considering the norm $\|\cdot\|_W$. Analogously, we can consider the amalgam space $W(C_0\,, \ell^1)$, where $C_0 := C_0(\mathbb{R}^d)$ denote the space of continuous functions on $\mathbb{R}^d$ vanishing at infinity. We have that

$$W(C_0\,, \ell^1) \subset W(L^\infty, \ell^1) \subset \mathcal{L}^\infty(\mathbb{R}^d) \,.$$

Given a function $\phi \in \mathcal{L}^p(\mathbb{R}^d)$ and a sequence $a \in \ell^\infty(\mathbb{Z}^d)$, the semi-discrete convolution product is defined by

$$\phi *' a := \sum_{\alpha \in \mathbb{Z}^d} a(\alpha) \phi(\cdot - \alpha) \,.$$

In [7, 27] we can find the following useful inequalities:

- If $\phi \in \mathcal{L}^p(\mathbb{R}^d)$ $(1 \le p \le \infty)$ then
$$|\phi *' a|_p \le |\phi|_p \, \|a\|_1 \qquad \text{and} \qquad \|\phi *' a\|_p \le |\phi|_p \, \|a\|_p. \tag{4}$$
- If $f \in \mathcal{L}^p(\mathbb{R})$ (respectively $f \in L^p(\mathbb{R}^d)$) and $\mathsf{h} \in \mathcal{L}^q(\mathbb{R}^d)$ $(1 \le p \le \infty,\ 1/p + 1/q = 1)$ then
$$\big\| \{\mathsf{h} * f(\alpha)\}_{\alpha \in \mathbb{Z}^d} \big\|_1 \le |\mathsf{h}|_q \, |f|_p \qquad \text{and} \qquad \big\| \{\mathsf{h} * f(\alpha)\}_{\alpha \in \mathbb{Z}^d} \big\|_p \le |\mathsf{h}|_q \, \|f\|_p \,, \tag{5}$$
where, as usual, the convolution is given by $\mathsf{h} * f := \int_{\mathbb{R}^d} f(x) \mathsf{h}(\cdot - x) dx$.
- If $\phi \in W(L^\infty, \ell^1)$ and $a \in \ell^1(\mathbb{Z}^d)$ then
$$\|\phi *' a\|_W \le \|\phi\|_W \, \|a\|_1 \,. \tag{6}$$

We denote the Fourier transform of f by $\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(t) e^{-2\pi i \xi t} dt$.

First, we state the L^p-stable shifts concept ($1 \le p \le \infty$) as established in [30].

Definition 1 *Let $1 \le p \le \infty$. A finite subset $\Phi = \{\phi_j\}_{j=1}^r$ of $\mathcal{L}^\infty(\mathbb{R}^d)$ is said to have L^p-stable shifts if there exist positive constants $0 < A \le B$ (depending on p and Φ) such that*

$$A \sum_{j=1}^r \|a_j\|_{\ell^p} \le \Big\| \sum_{j=1}^r \phi_j *' a_j \Big\|_{L^p} \le B \sum_{j=1}^r \|a_j\|_{\ell^p} , \tag{7}$$

for any sequence $a_j \in \ell^p(\mathbb{Z}^d)$, $j = 1, 2, \ldots, r$, when $1 \le p < \infty$, and for any sequence $a_j \in \mathsf{c}_0(\mathbb{Z}^d)$, $j = 1, 2, \ldots, r$, when $p = \infty$.

A necessary and sufficient condition for $\Phi = \{\phi_j\}_{j=1}^r$ of $\mathcal{L}^\infty(\mathbb{R}^d)$ to have L^p-stable shifts, regardless p, reads as follows: There are sequences $b_j \in \ell^1(\mathbb{Z}^d)$, $j = 1, 2, \ldots, r$, such that the functions $\widetilde{\phi}_j := \phi_j *' b_j$ are dual to the functions ϕ_j in the sense that

$$\langle \phi_j(\cdot - \nu), \widetilde{\phi}_k(\cdot - \mu) \rangle = \delta_{\nu\mu} \delta_{jk} , \quad j, k = 1, 2, \ldots, r , \ \nu, \mu \in \mathbb{Z}^d ,$$

where δ is the Kronecker symbol (see [27]). Thus we may drop the affiliation L^p from the word stability. Notice that the sum in the middle term of (7) is independent of the order in which the sum is performed.

Let V_Φ^p be the L^p-closure of the linear span of the shifts of $\Phi = \{\phi_j\}_{j=1}^r$. If the integer translates of $\Phi = \{\phi_j\}_{j=1}^r$ in $\mathcal{L}^\infty(\mathbb{R}^d)$ are L^p-stable, then this space can be expressed as

$$V_\Phi^p = \Big\{ \sum_{j=1}^r \phi_j *' a_j \, : \, a_j \in \ell^p(\mathbb{Z}^d) \, , j = 1, 2, \ldots, r \Big\} \quad \text{if } 1 \le p < \infty ,$$

or

$$V_\Phi^\infty = \Big\{ \sum_{j=1}^r \phi_j *' a_j \ : \ a_j \in \mathsf{c}_0(\mathbb{Z}^d) \, , \ j = 1, 2, \ldots, r \Big\} \quad \text{if } p = \infty .$$

(See the proof of Lemma 5.1 in [30]). As a consequence, for $1 \le p' < p \le \infty$ we have the set inclusion $V_\Phi^{p'} \subset V_\Phi^p$. For more details and properties on shift-invariant spaces in $L^p(\mathbb{R}^d)$ see [7].

Saying that the subset $\Phi = \{\phi_j\}_{j=1}^r$ of $\mathcal{L}^\infty(\mathbb{R}^d)$ has L^p-stable shifts is equivalent to that the sequence $\{\phi_j(\cdot - \alpha) \, : \, \alpha \in \mathbb{Z}^d \, , j = 1, 2, \ldots, r\}$ is a p-Riesz basis for V_Φ^p, $1 \le p \le \infty$. Recall that (see [7, 24]):

Definition 2 *Let B be a normed linear space. We say that $\{g_\alpha\}_{\alpha \in \mathbb{Z}^d}$ is a p-Riesz basis in B if there exists a positive constant C such that*

$$C^{-1} \|c\|_{\ell^p} \le \Big\| \sum_{\alpha \in \mathbb{Z}^d} c(\alpha) g_\alpha \Big\|_B \le C \|c\|_{\ell^p} ,$$

for all $c = \{c(\alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^p(\mathbb{Z}^d)$ when $1 \le p < \infty$, and for all $c = \{c(\alpha)\}_{\alpha \in \mathbb{Z}^d} \in \mathsf{c}_0(\mathbb{Z}^d)$ when $p = \infty$.

Thus, the shift-invariant space V_Φ^p, $1 \le p < \infty$, is a Banach space isomorphic to the product space $\ell^p(\mathbb{Z}^d) \times \ldots \times \ell^p(\mathbb{Z}^d)$ (r times), whilst V_Φ^∞ is a Banach space isomorphic to $\mathsf{c}_0(\mathbb{Z}^d) \times \ldots \times \mathsf{c}_0(\mathbb{Z}^d)$ (r times). Obviously, a p-Riesz basis is unconditional.

Observe that a p-Riesz basis $\{\phi_j(\cdot - \alpha) \, : \, \alpha \in \mathbb{Z}^d \, , j = 1, 2, \ldots, r\}$ for the shift-invariant space V_Φ^p can be characterized in terms of the Gramian of Φ (see, for instance, [7]).

3 Generalized sampling in V_Φ^p: preliminaries

Assume that the generators $\Phi = \{\phi_j\}_{j=1}^r$ are continuous in $\mathbb{R}^d$, they satisfy that

$$\sup_{t\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} |\phi_j(t-\alpha)| < \infty\,, \quad j = 1,2,\dots,r$$

(hence, $\phi_j \in \mathcal{L}^\infty(\mathbb{R}^d)$ for each for $j = 1,2,\dots,r$), and they have L^p-stable shifts (i.e., regardless p as said before). As a consequence, $V_\Phi^p \subset C(\mathbb{R}^d)$.

We are mainly interested in obtaining generalized regular sampling formulas like (2) valid for the shift-invariant spaces V_Φ^p covering the full range $1 \le p \le \infty$.

A sampling formula like (2) involves s convolution systems Υ_l, $1 \le l \le s$, and samples taken at the lattice $M\mathbb{Z}^d$ which we should precise:

3.1 The convolution systems Υ_l

First of all, we introduce some notation; for a point $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ and a d-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}^d$ we denote $\alpha^\top x := \sum_{k=1}^d \alpha_k x_k$. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{N}_0^d$, D^β stands for the differential operator $D^\beta := D_1^{\beta_1} D_2^{\beta_2} \dots D_d^{\beta_d}$, and $|\beta| := \sum_{j=1}^d |\beta_j|$ for its order.

Throughout this paper we consider s convolution systems Υ_l, $1 \le l \le s$, of the following types:

(a) Whenever we are working in the space V_Φ^p, the impulse response h_l of the system Υ_l belongs to $\mathcal{L}^q(\mathbb{R}^d)$, where p and q are conjugate exponents, i.e.,

$$\big(\Upsilon_l f\big)(t) := [f * \mathsf{h}_l](t) = \int_{\mathbb{R}^d} f(x)\mathsf{h}_l(t-x)dx\,, \quad t \in \mathbb{R}^d\,,$$

for $\mathsf{h}_l \in \mathcal{L}^q(\mathbb{R}^d)$ and q satisfying $1/p + 1/q = 1$.

(b) The impulse response is a shifted Dirac delta, i.e., $\big(\Upsilon_l f\big)(t) := f(t + c_l), \quad t \in \mathbb{R}^d$.

(c) The impulse response is a linear combination of partial derivatives of shifted deltas, i.e.,

$$\big(\Upsilon_l f\big)(t) := \sum_{|\beta|\le N_l} c_{l,\beta} D^\beta f(t + d_{l,\beta})\,, \quad t \in \mathbb{R}^d\,.$$

If there is a system of this type, we also assume that $D^\beta \phi_j \in C(\mathbb{R}^d)$ and satisfies $\displaystyle\sup_{t\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} |D^\beta \phi_j(t-\alpha)| < \infty$ for $|\beta| \le N_l$, $j = 1,2,\dots,r$.

Any system of type (b) is a particular case of a system of type (c), but for the sake of clarity we treat both cases separately. We denote by m the largest order among the partial derivatives that appear in the systems of type (c) ($m = 0$ if there are only systems of types (a) and/or (b)). From now on, we consider s systems Υ_l, $l = 1,2,\dots,s$, of the types (a), (b), (c) or a linear combination of them.

Let $\mathcal{A}$ be the Wiener algebra of the functions of the form $f(x) = \sum_{\alpha\in\mathbb{Z}^d} a(\alpha) e^{2\pi i\alpha x}$ with $a := \{a(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$. The space $\mathcal{A}$, normed by $\|f\|_\mathcal{A} := \|a\|_1$ and with pointwise multiplication becomes a commutative Banach algebra. If $f \in \mathcal{A}$ and $f(x) \ne 0$ for every $x \in \mathbb{R}^d$, the function $1/f$ is also in $\mathcal{A}$ by Wiener's Lemma (see, for instance, [23]).

The sequence $\{\Upsilon_l \phi_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^1(\mathbb{Z}^d)$ (for systems of type (b) or (c) it is obvious having in mind the assumptions on ϕ_j and $D^\beta\phi_j$; use (5) for systems of type (a)). The Fourier transform of this sequence, which belongs to the Banach algebra $\mathcal{A}$, will play an important role in the sequel. We denote it by

$$g_{l,j}(x) := \sum_{\alpha\in\mathbb{Z}^d} \big(\Upsilon_l \phi_j\big)(\alpha) e^{-2\pi i\alpha x}\,, \quad x\in\mathbb{R}^d\,,$$

and

$$\mathbf{g}_l^\top(x) := \big(g_{l,1}(x), g_{l,2}(x), \dots, g_{l,r}(x)\big)\,, \quad 1\le l\le s\,. \tag{8}$$

3.2 Lattices in $\mathbb{Z}^d$

Given a nonsingular matrix M with integer entries, we consider the lattice in $\mathbb{Z}^d$ generated by M, i.e.,

$$M\mathbb{Z}^d := \{M\alpha : \alpha\in\mathbb{Z}^d\}\subset\mathbb{Z}^d\,.$$

Without loss of generality we can assume that $\det M>0$; otherwise we can consider $M' = ME$ where E is some $d\times d$ integer matrix satisfying $\det E=-1$; trivially, $M\mathbb{Z}^d = M'\mathbb{Z}^d$. We denote by $M^\top$ and $M^{-\top}$ the transpose matrices of M and M^{-1} respectively. The following useful generalized orthogonal relationship holds

$$\sum_{k\in\mathcal{N}(M^\top)} e^{-2\pi i\alpha^\top M^{-T}k} = \begin{cases}\det M, & \alpha\in M\mathbb{Z}^d\\ 0 & \alpha\in\mathbb{Z}^d\setminus M\mathbb{Z}^d\end{cases} \tag{9}$$

where

$$\mathcal{N}(M^\top) := \mathbb{Z}^d\cap\{M^\top x : x\in[0,1)^d\}$$

The set $\mathcal{N}(M^\top)$ has $\det M$ elements (see [37]). One of these elements is zero, say $i_1=0$; we denote the rest of elements by $i_2,\dots,i_{\det M}$ ordered in any form.

Notice that the sets, defined as $Q_k := M^{-\top}i_k + M^{-\top}[0,1)^d$, $k=1,2,\dots,\det M$, satisfy (see [37, p. 110])

$$Q_k\cap Q_{k'}=\emptyset \ \text{ if } \ k\ne k' \quad \text{ and } \quad \operatorname{Vol}\Big(\bigcup_{k=1}^{\det M} Q_k\Big)=1\,.$$

Thus, for any function F integrable in $[0,1)^d$ and $\mathbb{Z}^d$-periodic we have $\int_{[0,1)^d}F(x)dx = \sum_{k=1}^{\det M}\int_{Q_k}F(x)dx$.

In order to recover any function $f\in V_\Phi^p$ from its generalized samples at a lattice $M\mathbb{Z}^d$, i.e., from the sequence of samples $\big\{(\Upsilon_l f)(M\alpha)\big\}_{\alpha\in\mathbb{Z}^d,\ l=1,2,\dots,s}$, a suitable expression for the samples will be useful

3.3 An expression for the samples

First, consider the map

$$\begin{array}{rccl}\mathcal{T}_\Phi : & \mathcal{A}\times\ldots\times\mathcal{A} & \longrightarrow & L^p(\mathbb{R}^d)\\ & \mathbf{F}^\top := (f_1,\dots,f_r) & \longmapsto & \sum_{j=1}^r \phi_j *' a_j\,,\end{array} \tag{10}$$

where $f_j(x)=\sum_{\alpha\in\mathbb{Z}^d}a_j(\alpha)e^{-2\pi i\alpha x}\in\mathcal{A}$, $j=1,2,\dots,r$. Notice that (4) ensures that $\mathcal{T}_\Phi$ is a well-defined bounded operator by considering in $\mathcal{A}\times\ldots\times\mathcal{A}$ the norm $\|\mathbf{F}\| := \sum_{j=1}^r \|a_j\|_1$.

For $f \in \operatorname{span}\{\phi_j(\cdot - \alpha) : \alpha \in \mathbb{Z}^d, j = 1, 2, \ldots, r\}$ let $\mathbf{a} = \{(a_1(\alpha), \ldots, a_r(\alpha))\}$ be the finite sequence such that $f = \sum_{j=1}^r \phi_j *' a_j$ and the corresponding trigonometric polynomial $\mathbf{F}^\top(x) := \Big(\sum_\alpha a_1(\alpha) e^{-2\pi i \alpha x}, \ldots, \sum_\alpha a_r(\alpha) e^{-2\pi i \alpha x}\Big) = \sum_\alpha \mathbf{a}(\alpha) e^{-2\pi i \alpha x}$, so that $\mathcal{T}_\Phi \mathbf{F} = f$. For any $l = 1, 2, \ldots, s$ and $\alpha \in \mathbb{Z}^d$, we have

$$
\begin{aligned}
(\Upsilon_l f)(M\alpha) &= \sum_\mu \sum_{j=1}^r a_j(\mu)(\Upsilon_l \phi_j)(M\alpha - \mu) = \langle \mathbf{F}, \overline{\mathbf{g}}_l e^{-2\pi i \alpha^\top M^\top x} \rangle_{L^2[0,1)^d} \\
&= \int_{[0,1)^d} \mathbf{F}^\top(x) \mathbf{g}_l(x) e^{2\pi i \alpha^\top M^\top x} dx\,.
\end{aligned} \tag{11}
$$

As the sequence $\{e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d}$ is an orthogonal basis for $L^2(M^{-\top}[0,1)^d)$, we can exploit this fact in computing the above integral as follows

$$
\begin{aligned}
(\Upsilon_l f)(M\alpha) &= \sum_{k=1}^{\det M} \int_{Q_k} \mathbf{F}^\top(x) \mathbf{g}_l(x) e^{2\pi i \alpha^\top M^\top x} dx \\
&= \int_{M^{-\top}[0,1)^d} \sum_{k=1}^{\det M} \mathbf{F}^\top(x + M^{-\top} i_k) \mathbf{g}_l(x + M^{-\top} i_k)\; e^{2\pi i \alpha^\top M^\top x} dx\,.
\end{aligned} \tag{12}
$$

This leads us to introduce the $s \times (\det M) r$ matrix of functions $\mathbb{G}(x)$, $x \in [0,1)^d$, which, involving the functions in (8), is given by

$$
\begin{aligned}
\mathbb{G}(x) &:= \begin{bmatrix}
\mathbf{g}_1^\top(x) & \mathbf{g}_1^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_1^\top(x + M^{-\top} i_{\det M}) \\
\mathbf{g}_2^\top(x) & \mathbf{g}_2^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_2^\top(x + M^{-\top} i_{\det M}) \\
\vdots & \vdots & \vdots & \vdots \\
\mathbf{g}_s^\top(x) & \mathbf{g}_s^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_s^\top(x + M^{-\top} i_{\det M})
\end{bmatrix} \\
&= \Big[\mathbf{g}_l^\top\big(x + M^{-\top} i_k\big)\Big]_{\substack{l=1,2,\ldots,s \\ k=1,2,\ldots,\det M}}\,.
\end{aligned} \tag{13}
$$

As we will see in next section, the reconstruction functions S_l, $l = 1, 2, \ldots, s$, appearing in formula (2) rely on the existence of left inverse matrices of $\mathbb{G}(x)$ having entries in the algebra $\mathcal{A}$.

Lemma 1 *There exists an $r \times s$ matrix $\mathbf{d}(x) := \big(\mathbf{d}_1(x), \mathbf{d}_2(x), \ldots, \mathbf{d}_s(x)\big)$ with entries $d_{j,l} \in \mathcal{A}$, $j = 1, 2, \ldots, r$, $l = 1, 2, \ldots, s$ and satisfying*

$$
\mathbf{d}(x)\mathbb{G}(x) = \begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \ldots & 1 & 0 & \ldots & 0
\end{bmatrix} = [\mathbb{I}_r, \mathbb{O}_{r \times (\det M - 1) r}]\,, \quad x \in [0,1)^d\,, \tag{14}
$$

if and only if rank $\mathbb{G}(x) = (\det M) r$ for all $x \in \mathbb{R}^d$.

Proof. Notice that $\operatorname{rank} \mathbb{G}(x) = (\det M) r$ if and only if $\det(\mathbb{G}^*(x)\mathbb{G}(x)) \neq 0$ where $\mathbb{G}^*(x)$ denotes the conjugate transpose of $\mathbb{G}(x)$. If rank $\mathbb{G}(x) = (\det M) r$ then the first r rows of the pseudo inverse of $\mathbb{G}(x)$, $\mathbb{G}^\dagger(x) := \big(\mathbb{G}^*(x)\mathbb{G}(x)\big)^{-1}\mathbb{G}^*(x)$, satisfy (14); moreover, according to Wiener's Lemma the entries of $\mathbb{G}^\dagger$ belong to $\mathcal{A}$.

Conversely, assume that the $r \times s$ matrix $\mathbf{d}(x) = \big(\mathbf{d}_1(x), \mathbf{d}_2(x), \dots, \mathbf{d}_s(x)\big)$ satisfies (14). We consider the periodic extension of $d_{j,l}$, i.e., $d_{j,l}(x+\alpha) = d_{j,l}(x)$, $\alpha \in \mathbb{Z}^d$. For all $x \in [0,1)^d$, the matrix

$$\mathbb{D}^\top(x) := \begin{bmatrix} \mathbf{d}_1(x) & \mathbf{d}_2(x) & \cdots & \mathbf{d}_s(x) \\ \mathbf{d}_1(x+M^{-\top}i_2) & \mathbf{d}_2(x+M^{-\top}i_2) & \cdots & \mathbf{d}_s(x+M^{-\top}i_2) \\ \vdots & \vdots & & \vdots \\ \mathbf{d}_1(x+M^{-\top}i_{\det M}) & \mathbf{d}_2(x+M^{-\top}i_{\det M}) & \cdots & \mathbf{d}_s(x+M^{-\top}i_{\det M}) \end{bmatrix} \tag{15}$$

is a left inverse matrix of $\mathbb{G}(x)$. Therefore, necessarily $\operatorname{rank}\mathbb{G}(x) = (\det M)r$, for all $x \in [0,1)^d$. □

Provided that the condition (14) in Lemma 1 is satisfied, it can be easily checked that all matrices $\mathbf{d}(x)$ with entries in $\mathcal{A}$, and satisfying (14) correspond to the first r rows of the matrices of the form

$$\mathbb{D}^\top(x) = \mathbb{G}^\dagger(x) + \mathbb{U}(x)\big[\mathbb{I}_s - \mathbb{G}(x)\mathbb{G}^\dagger(x)\big]\,, \tag{16}$$

where $\mathbb{U}(x)$ is any $(\det M)r \times s$ matrix with entries in $\mathcal{A}$. Notice that if $s = (\det M)r$ there exists a unique matrix $\mathbf{d}(x)$, given by the first r rows of $\mathbb{G}^{-1}(x)$; if $s > (\det M)r$ there are many solutions according to (16).

Notice that the result in Lemma 1 has also its counterpart for Beurling weighted variants $\mathcal{A}_v := \mathcal{F}^{-1}\ell^1_v(\mathbb{Z}^d)$ of the Wiener's algebra $\mathcal{A}$ (see [23]).

4 Multivariate generalized sampling in V_Φ^p $(1 \le p \le \infty)$

As we have pointed out in the introductory section, our sampling result for V_Φ^p, $1 \le p \le \infty$, rely on its version for the linear span of $\{\phi_j(\cdot - \alpha) : \alpha \in \mathbb{Z}^d, j = 1, 2, \dots, r\}$. In so doing, assume that the set of continuous generators $\Phi = \{\phi_j\}_{j=1}^r$ satisfy, for $j = 1, 2, \cdots, r$ that $\sup_{t\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} |\phi_j(t-\alpha)| < \infty$. Consider also s convolution systems Υ_l, $l = 1, 2, \cdots, s$, satisfying that $|\mathsf{h}_l|_1 < \infty$ whenever Υ_l is a system of type (a), and satisfying that $\sup_{t\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} |D^\beta\phi_j(t-\alpha)| < \infty$ whenever the derivative $D^\beta\phi_j$ appears in a system of type (c). The following lemma holds for the span of the integer shifts of $\Phi = \{\phi_j\}_{j=1}^r$:

Lemma 2 *Let $\mathbf{d}(x) = \big(d_1(x), d_2(x), \dots, d_s(x)\big)$ be an $r \times s$ matrix with entries $d_{j,l} \in \mathcal{A}$, $j = 1, 2, \dots, r$, $l = 1, 2, \dots, s$, and satisfying (14). Then, for any $f \in \operatorname{span}\{\phi_j(\cdot - \alpha) : \alpha \in \mathbb{Z}^d, j = 1, 2, \dots, r\}$ the following sampling expansion holds:*

$$f(t) = \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,, \tag{17}$$

where the reconstruction function $S_{l,\mathbf{d}}$ is given by

$$S_{l,\mathbf{d}}(t) = (\det M) \sum_{\alpha\in\mathbb{Z}^d} \sum_{j=1}^{r} \widehat{d}_{j,l}(\alpha)\phi_j(t-\alpha)\,, \quad t \in \mathbb{R}^d\,, \tag{18}$$

with $\widehat{d}_{j,l}(\alpha) := \int_{[0,1)^d} d_{j,l}(x) e^{2\pi i\alpha x} dx$, $\alpha \in \mathbb{Z}^d$, the Fourier coefficients of the functions $d_{l,j} \in \mathcal{A}$, $j = 1, 2, \dots, r$ and $l = 1, 2, \dots, s$. The convergence of the sampling series is in the L^p-norm sense and uniform on $\mathbb{R}^d$.

Proof. For $f \in \operatorname{span}\{\phi_j(\cdot-\alpha) : \alpha \in \mathbb{Z}^d, j=1,2,\dots,r\}$ let $\mathbf{a}=\{(a_1(\alpha),\dots,a_r(\alpha))\}$ be the finite sequence such that $f=\sum_{j=1}^r \phi_j *' a_j$ and

$$\mathbf{F}^\top(x) := \Big(\sum_\alpha a_1(\alpha)e^{-2\pi i\alpha x},\dots,\sum_\alpha a_r(\alpha)e^{-2\pi i\alpha x}\Big) = \sum_\alpha \mathbf{a}(\alpha)e^{-2\pi i\alpha x}$$

the corresponding trigonometric polynomial such that $\mathcal{T}_\Phi\mathbf{F}=f$ (see (10)).

Having in mind expression (12), the sequence of samples $\{(\Upsilon_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d}$ forms the Fourier coefficients of the continuous function $\sum_{k=1}^{\det M}\mathbf{F}^\top(x+M^{-\top}i_k)\mathbf{g}_l(x+M^{-\top}i_k)$ with respect to the orthogonal basis $\{e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d}$ for $L^2(M^{-\top}[0,1)^d)$.

Since $\{\Upsilon_l\phi_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}\in\ell^1(\mathbb{Z}^d)$ we have that $\{\Upsilon_l f(M\alpha)\}_{\alpha\in\mathbb{Z}^d}\in\ell^1(\mathbb{Z}^d)$ (remind that $(\Upsilon_l f)(M\alpha)$ is a finite sum $\sum\sum_{j=1}^r a_j(\mu)(\Upsilon_l\phi_j)(M\alpha-\mu)$). Therefore, for $l=1,2\dots,s$, we have

$$\sum_{k=1}^{\det M}\mathbf{F}^\top(x+M^{-\top}i_k)\mathbf{g}_l(x+M^{-\top}i_k) = (\det M)\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)e^{-2\pi i\alpha^\top M^\top x}\,,$$

$x\in M^{-\top}[0,1)^d$. By periodicity, the above equality also holds for all $x\in[0,1)^d$. Hence we can write

$$\mathbb{G}(x)\mathbb{F}(x) = (\det M)\Big(\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_1 f)(M\alpha)e^{-2\pi i\alpha^\top M^\top x},\dots,\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_s f)(M\alpha)e^{-2\pi i\alpha^\top M^\top x}\Big)^\top$$

where $\mathbb{G}(x)$ is the $s\times(\det M)r$ matrix, defined in (13) and

$$\mathbb{F}(x) := \big(\mathbf{F}^\top(x),\mathbf{F}^\top(x+M^{-\top}i_2),\cdots,\mathbf{F}^\top(x+M^{-\top}i_{\det M})\big)^\top\,.$$

Multiplying on the left by the matrix $\mathbf{d}(x)$ we obtain $\mathbf{F}(x)$ by means of the generalized samples

$$\mathbf{F}(x) = (\det M)\sum_{l=1}^s\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)\mathbf{d}_l(x)e^{-2\pi i\alpha^\top M^\top x},\quad x\in[0,1)^d\,. \tag{19}$$

Since $\{(\Upsilon_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^1(\mathbb{Z}^d)$ and $d_{l,j}\in\mathcal{A}$, the series in (19) also converges in the norm of $\mathcal{A}\times\ldots\times\mathcal{A}$. Indeed, for $N\in\mathbb{N}$,

$$\begin{aligned}\Big\|\sum_{|\alpha|>N}(\Upsilon_l f)(M\alpha)\mathbf{d}_l(x)e^{-2\pi i\alpha^\top M^\top x}\Big\| &\le \|\mathbf{d}_l\|\Big\|\sum_{|\alpha|>N}(\Upsilon_l f)(M\alpha)e^{-2\pi i\alpha^\top M^\top x}\Big\|_{\mathcal{A}}\\ &= \|\mathbf{d}_l\|\sum_{|\alpha|>N}|(\Upsilon_l f)(M\alpha)|\,.\end{aligned}$$

Applying $\mathcal{T}_\Phi$ to both sides of the equality (19), and using that

$$\big[\mathcal{T}_\Phi\mathbf{d}_l(\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\big](t) = \big[\mathcal{T}_\Phi\mathbf{d}_l\big](t-M\alpha)\,,\quad \alpha\in\mathbb{Z}^d\,,$$

we deduce that

$$f=\sum_{l=1}^s\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)S_{l,\mathbf{d}}(\cdot-M\alpha)\quad\text{in } L^p(\mathbb{R}^d)\,,$$

where $S_{l,\mathbf{d}} = (\det M)\mathcal{T}_\Phi \mathbf{d}_l$, for $l = 1, 2, \ldots, s$.

The reconstruction functions $S_{l,\mathbf{d}}$, $l = 1, 2, \ldots, s$, are determined from the Fourier coefficients of $d_{j,l}$, $\widehat{d}_{j,l}(\alpha) := \int_{[0,1)^d} d_{j,l}(x) e^{2\pi i \alpha x} dx$. More specifically,

$$S_{l,\mathbf{d}}(t) = (\det M) \sum_{\alpha \in \mathbb{Z}^d} \sum_{j=1}^{r} \widehat{d}_{j,l}(\alpha) \phi_j(t - \alpha)\,, \quad t \in \mathbb{R}^d\,.$$

The sequence $\widehat{d}_{j,l} \in \ell^1(\mathbb{Z}^d)$ because the function $d_{j,l}(x) = \sum_{\alpha \in \mathbb{Z}^d} \widehat{d}_{j,l}(\alpha) e^{-2\pi i \alpha x}$ belongs to $\mathcal{A}$. As a consequence, $S_{l,\mathbf{d}} \in V_\Phi^1 \subset V_\Phi^\infty$. Hence, the partial sums of the above sampling series are in V_Φ^∞, i.e., are continuous functions and, as a consequence, they converge uniformly on $\mathbb{R}^d$. $\square$

Some comments about Lemma 2 are in order:

1. We are assuming that rank $\mathbb{G}(x) = (\det M) r$ for all $x \in \mathbb{R}^d$ and, consequently $s \geq r(\det M)$.

2. Since $d_{j,l} \in \mathcal{A}$ and $\phi_j \in \mathcal{L}^\infty(\mathbb{R}^d)$, by using (4) the reconstruction functions (18) satisfy that $S_{l,\mathbf{d}} \in \mathcal{L}^\infty(\mathbb{R}^d)$, $l = 1, 2, \ldots, s$. Similarly, whenever $\phi_j \in W(L^\infty, \ell^1)$, inequality (6) shows that the reconstruction function $S_{l,\mathbf{d}}$ belongs to $W(L^\infty, \ell^1)$ for $l = 1, 2, \ldots, s$.

3. The Fourier transform of $S_{l,\mathbf{d}}$ can be determined from the functions $d_{j,l}$. Indeed, from (18), we obtain that

$$\widehat{S}_{l,\mathbf{d}}(w) = (\det M) \sum_{j=1}^{r} d_{j,l}(w) \widehat{\phi}_j(w)\,, \quad w \in \mathbb{R}^d\,.$$

4. In the case $s = (\det M) r$, there is a unique $r \times s$ matrix $\mathbf{d}(x)$ satisfying (14), which is those formed with the first r rows of the matrix $\mathbb{G}^{-1}(x) = \mathbb{D}^\top(x)$ in the notation of (15). Then, using (12), we obtain that the reconstruction functions $S_{l,\mathbf{d}}$ satisfy in this case an interpolatory property; namely

$$\begin{aligned}
(\Upsilon_{l'} S_{l,\mathbf{d}})(M\alpha) &= (\det M) \int_{M^{-\top}[0,1)^d} \sum_{k=1}^{\det M} \mathbf{d}_l(x + M^{-\top} i_k) \mathbf{g}_{l'}(x + M^{-\top} i_k)\; e^{2\pi i \alpha^\top M^\top x} dx \\
&= \delta_{l',l}\, (\det M) \int_{M^{-\top}[0,1)^d} e^{2\pi i \alpha^\top M^\top x} dx = \begin{cases} 1 & \text{if } l = l' \text{ and } \alpha = 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

4.1 The sampling result in V_Φ^p $(1 \leq p < \infty)$

Assume that the set of generators $\Phi = \{\phi_j\}_{j=1}^r$ has also L^p-stable shifts, and that the hypotheses in Lemma 2 hold with $|\mathsf{h}_l|_q < \infty$ when appearing systems Υ_l of type (a) (which implies that $|\mathsf{h}_l|_1 < \infty$). Then, a density argument allows us to prove that sampling formula (17) in Lemma 2 is also valid for the whole space V_Φ^p:

Theorem 1 *Under the above assumptions, for any $f \in V_\Phi^p$ $(1 \leq p < \infty)$, the sampling formula*

$$f = \sum_{l=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(\cdot - M\alpha)\,, \tag{20}$$

holds in the L^p-sense. The series in (20) *also converges absolutely and uniformly on $\mathbb{R}^d$.*

Proof. We define on V_Φ^p the sampling operator

$$\begin{array}{rccl} \Gamma_{\mathbf{d}}: & V_\Phi^p & \longrightarrow & V_\Phi^p \\ & f & \longmapsto & \Gamma_{\mathbf{d}} f := \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(\cdot - M\alpha)\,. \end{array}$$

It is a well-defined and bounded operator regardless the type of convolution systems Υ_l: Having in mind (4), notice that $\|\{(\Upsilon_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_p \le C_l \|f\|_p$; for systems of type (a) we use (5), whilst for systems of type (b) or (c) we are using the inequality $\|a*b\|_p \le \|a\|_p \|b\|_1$ for sequences, the hypotheses on ϕ_j and $D^\beta \phi_j$, and the left inequality in (7) (since Φ has L^p-stable shifts).

Given $f \in V_\Phi^p$, there exists a sequence $\{f_N\}$ in $\operatorname{span}\{\phi_j(\cdot - \alpha) \,:\, \alpha \in \mathbb{Z}^d\,, j = 1,2,\dots,r\}$ such that $\|f_N - f\|_p \to 0$ as $N \to \infty$. By using Lemma 2 we have,

$$0 \le \|f - \Gamma_{\mathbf{d}} f\|_p = \|f - f_N + \Gamma_{\mathbf{d}} f_N - \Gamma_{\mathbf{d}} f\|_p \le (1 + \|\Gamma_{\mathbf{d}}\|)\|f_N - f\|_p \to 0\,,\ N \to \infty\,,$$

which implies that $\Gamma_{\mathbf{d}} f = f$ in $L^p(\mathbb{R}^d)$, i.e., the validity of the sampling result (17).

The series $\sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)$ converges, absolutely and uniformly on $\mathbb{R}^d$, to the continuous function f. Indeed,

$$\sum_{|\alpha|>N} \big|(\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)\big| \le \sup_{|\alpha|>N} \big|(\Upsilon_l f)(M\alpha)\big| \sup_{t\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} \big|S_{l,\mathbf{d}}(t - M\alpha)\big| \to 0\,,$$

uniformly on $\mathbb{R}^d$ as $N \to \infty$.

$\square$

For average sampling, i.e., where we are only dealing with systems of the type (a), Theorem 1 still remains valid by relaxing the hypotheses to the generators ϕ_j belong to $\mathcal{L}^p(\mathbb{R}^d)$ and $\sup_{t\in[0,1)^d} \big\|\{\phi_j(t-\alpha)\}_{\alpha\in\mathbb{Z}^d}\big\|_q < \infty$, $j = 1,2,\dots,r$ $\ (1/p + 1/q = 1)$.

4.2 Some comments on the case $p = 2$

In the case $p = 2$ we can exploit the hilbertian structure of V_Φ^2 whenever the generators $\phi_j \in L^2(\mathbb{R}^d)$, and $\sup_{t\in[0,1)^d} \big\|\{\phi_j(t-\alpha)\}_{\alpha\in\mathbb{Z}^d}\big\|_2 < \infty$, $j = 1,2,\dots,r$. Based on previous work of the authors [18, 19, 21], we can state that the entries of the function $\mathbf{g}_l$ in (8) belong to $L^2(\mathbb{R}^d)$ whenever $\mathsf{h}_l \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $l = 1,2,\dots,s$ (see [18, Lemma 1]). Associated with the matrix $\mathbb{G}$ defined in (13), consider its related constants

$$A_{\mathbb{G}} := \operatorname*{ess\,inf}_{x\in[0,1)^d} \lambda_{\min}[\mathbb{G}^*(x)\mathbb{G}(x)], \quad B_{\mathbb{G}} := \operatorname*{ess\,sup}_{x\in[0,1)^d} \lambda_{\max}[\mathbb{G}^*(x)\mathbb{G}(x)]\,,$$

where $\mathbb{G}^*(x)$ denotes the transpose conjugate of the matrix $\mathbb{G}(x)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbb{G}^*(x)\mathbb{G}(x)$. Observe that $0 \le A_{\mathbb{G}} \le B_{\mathbb{G}} \le \infty$. Notice that in the definition of the matrix $\mathbb{G}(x)$ we are considering the $\mathbb{Z}^d$-periodic extension of the involved functions $\mathbf{g}_l$, $l = 1,2,\dots,s$. Under these circumstances, the following result holds (see [18, Theorems 1,2], [19, Theorem 1] and [21, Theorem 1]):

Theorem 2 *Assume that the entries of the functions $\mathbf{g}_l$ in (8) belong to $L^\infty[0,1)^d$ for $l = 1,2,\dots,s$. The following statements are equivalent:*

(a) $A_{\mathbb{G}} > 0$

(b) There exists an $r\times s$ matrix $\mathbf{d}(x) := \big(\mathbf{d}_1(x), \mathbf{d}_2(x), \dots, \mathbf{d}_s(x)\big)$ with entries in $L^\infty[0,1)^d$ such that

$$\mathbf{d}(x)\mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{r\times(\det M-1)r}]\,, \qquad a.e.\ in\ [0,1)^d\,. \tag{21}$$

(c) There exists a frame for V_Φ^2 having the form $\{S_{l,\mathbf{d}}(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ l=1,2,\dots,s}$ such that for any $f\in V_\Phi^2$,

$$f = \sum_{\alpha\in\mathbb{Z}^d}\sum_{l=1}^{s}\big(\Upsilon_l f\big)(M\alpha)\ S_{l,\mathbf{d}}(\cdot - M\alpha) \quad in\ L^2(\mathbb{R}^d)\,. \tag{22}$$

In case the equivalent conditions are satisfied, the reconstruction functions $S_{l,\mathbf{d}}$, $l = 1,2,\dots,s$, are given by (18). *The series in* (22) *also converges absolutely and uniformly on $\mathbb{R}^d$.*

The assumption that the entries of the functions $\mathbf{g}_l$ in (8) belong to $L^\infty[0,1)^d$ for $l = 1,2,\dots,s$ (which is equivalent to $B_\mathbb{G} < \infty$) means that $\big\{\mathbf{g}_l(x)e^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d, l=1,2,\dots,s}$ is a Bessel sequence for the product Hilbert space $L^2[0,1)^d\times\cdots\times L^2[0,1)^d$ (r times) (see [19, Lemma 2]).

All the admissible solutions of (21) are given by the first r rows of the matrix (16) where $\mathbb{U}(x)$ denotes now any $(\det M)r\times s$ matrix with entries in $L^\infty[0,1)^d$.

Notice that if $s = (\det M)r$ there exists a unique matrix $\mathbf{d}(x)$, given by the first r rows of $\mathbb{G}^{-1}(x)$; if $s > (\det M)r$ there are many solutions according to (16). Something more can be said in the case where $s = (\det M)r$ (see [19, Theorem 2]):

Theorem 3 *Assume that the entries of the functions $\mathbf{g}_l$ belong to $L^\infty[0,1)^d$ for $l = 1,2,\dots,s$ and $s = (\det M)r$. The following statements are equivalent:*

(i) $A_\mathbb{G} > 0$

(ii) There exists a Riesz basis $\{S_{l,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ l=1,2,\dots,s}$ for V_Φ^2 such that for any $f\in V_\Phi^2$, the expansion

$$f = \sum_{\alpha\in\mathbb{Z}^d}\sum_{l=1}^{s}\big(\Upsilon_l f\big)(M\alpha)\ S_{l,\alpha} \quad in\ L^2(\mathbb{R}^2)\,, \tag{23}$$

holds.

In case the equivalent conditions are satisfied, necessarily $S_{l,\alpha}(t) = S_{l,\mathbf{d}}(t - M\alpha)\,,\ t\in\mathbb{R}^d$ where $S_{l,\mathbf{d}}$, $l = 1,2,\dots,s$, is given by (18) *being $\mathbf{d}(x)$ the $r\times s$ matrix formed with the r first rows of $\mathbb{G}^{-1}$. Moreover, the sampling functions $S_{l,\mathbf{d}}$, $l = 1,2,\dots,s$, satisfy the interpolation property $(\Upsilon_{l'}S_{l,\mathbf{d}})(M\alpha) = \delta_{l,l'}\delta_{\alpha,0}$, where $l,l' = 1,2,\dots,s$ and $\alpha\in\mathbb{Z}^d$.*

4.3 The sampling result in V_Φ^∞

Assume that the set of continuous generators $\Phi = \{\phi_j\}_{j=1}^r$ has L^p-stable shifts, and that the hypotheses in Lemma 2 hold with $|\mathsf{h}_l|_1 < \infty$. Associated with an $r\times s$ matrix $\mathbf{d}(x) = \big(\mathbf{d}_1(x), \mathbf{d}_2(x), \dots, \mathbf{d}_s(x)\big)$ with entries $d_{j,l}\in\mathcal{A}$, $j = 1,2,\dots,r$, $l = 1,2,\dots,s$, and satisfying (14), we consider the sampling operator $\Gamma_\mathbf{d}$, formally defined as

$$\big(\Gamma_\mathbf{d} f\big)(t) := \sum_{l=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)S_{l,\mathbf{d}}(t - M\alpha)\,, \quad t\in\mathbb{R}^d\,. \tag{24}$$

Recall that m stands for the largest order among the partial derivatives appearing in systems of type (c) ($m = 0$ if there are only systems of types (a) and (b)). The space $C_b^m(\mathbb{R}^d)$ consists of all functions f which, together with all their partial derivatives $D^\beta f$ of order $|\beta| \le m$, are continuous and bounded on $\mathbb{R}^d$. The space $C_b^m(\mathbb{R}^d)$ becomes a Banach space with the norm $\|f\|_{C_b^m} := \max_{|\beta|\le m} \sup_{t\in\mathbb{R}^d} |D^\beta f(t)|$.

Lemma 3 *For any $r \times s$ matrix $\mathbf{d}$ with entries in $\mathcal{A}$ and satisfying (14), there exists a constant $K > 0$ such that, for each $f \in C_b^m(\mathbb{R}^d)$,*

$$\big|\big(\Gamma_{\mathbf{d}} f\big)(t)\big| \le K\|f\|_{C_b^m} \quad \textit{for all } t \in \mathbb{R}^d.$$

Proof. If the system Υ_l is of the type (a), then for all $f \in C_b^m(\mathbb{R}^d)$,

$$\big|\Upsilon_l f(\alpha)\big| \le \|h_l\|_1 \, \|f\|_\infty \le \|h_l\|_1 \, \|f\|_{C_b^m}, \quad \alpha \in \mathbb{Z}^d\,.$$

If the system Υ_l is of the type (c) (including in particular the type (b)) then for all $f \in C_b^m(\mathbb{R}^d)$,

$$|\Upsilon_l f(\alpha)| \le \sum_{|\beta|\le N_l} |c_{l,\beta}| \; |D^\beta f(\alpha + d_{l,\beta})| \le M \max_{|\beta|\le N_l} |c_{l,\beta}| \; \|f\|_{C_b^m}, \quad \alpha \in \mathbb{Z}^d\,,$$

for some constant M. Since $S_{l,\mathbf{d}} \in \mathcal{L}^\infty(\mathbb{R}^d)$ then, for any $f \in C_b^m(\mathbb{R}^d)$,

$$\begin{aligned}|\Gamma_{\mathbf{d}} f(t)| &\le \sum_{l=1}^s \Big| \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(\alpha) S_{l,\mathbf{d}}(t-\alpha) \Big| \le \sum_{l=1}^s \big\|\{\Upsilon_l f(\alpha)\}_{n\in\mathbb{Z}^d}\big\|_\infty |S_{l,\mathbf{d}}|_\infty \\ &\le K\|f\|_{C_b^m}\,, \quad t \in \mathbb{R}^d,\end{aligned}$$

where K is a constant independent of f. □

Theorem 4 *Let $\mathbf{d}(x) = \big(\mathbf{d}_1(x), \mathbf{d}_2(x), \dots, \mathbf{d}_s(x)\big)$ be an $r\times s$ matrix with entries $d_{j,l} \in \mathcal{A}$, $j = 1, 2, \dots, r$, $l = 1, 2, \dots, s$, and satisfying (14). Then, for any $f \in V_\Phi^\infty$, the following sampling formula holds:*

$$f(t) = \sum_{l=1}^s \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,, \tag{25}$$

where the reconstruction functions $S_{l,\mathbf{d}}$, $l = 1, 2, \dots, s$, are given by (18). Assuming that the continuous functions $\phi_j, D^\beta\phi_j$, $|\beta| \le m$, $j = 1, 2, \dots, r$, vanish at infinity, then the convergence of the sampling series is also absolute and uniform on $\mathbb{R}^d$.

Proof. Let $f \in V_\Phi^\infty$; then $f(t) = \sum_{\alpha\in\mathbb{Z}^d}\sum_{j=1}^r a_j(\alpha)\phi_j(t-\alpha)$, when $a_j \in \mathsf{c}_0(\mathbb{Z}^d)$, $j = 1, 2, \dots, r$. For $M \in \mathbb{N}$ we define

$$f_M(t) := \sum_{|\alpha|\le M} \sum_{j=1}^r a_j(\alpha)\phi_j(t-\alpha)\,.$$

From the assumptions on the generators ϕ_j we have that $f_M \in C_b^m(\mathbb{R}^d)$. Moreover, let $|\beta| \le m$ and $M > N > 0$, for any $t \in \mathbb{R}^d$ we have

$$|D^\beta(f_M - f_N)(t)| \le \sum_{N<|\alpha|\le M} \sum_{j=1}^r |a_j(\alpha)| \; |D^\beta\phi_j(t-\alpha)| \le \sup_{N<|\alpha|\le M} \sum_{j=1}^r |a_j(\alpha)| \; |D^\beta\phi_j|_\infty\,.$$

Since the sequences $a_j \in \mathsf{c}_0(\mathbb{Z}^d)$, the sequence $\{f_M\}_{M=1}^\infty$ is a Cauchy sequence in the Banach space $C_b^m(\mathbb{R}^d)$, we deduce that f_M converges in the C_b^m-norm to f as $M \to \infty$. In particular $f \in C_b^m(\mathbb{R}^d)$. Using Lemmas 2 and 3 we obtain that, for all $t \in \mathbb{R}^d$,

$$0 \le |f_M(t) - \Gamma_{\mathbf{d}} f(t)| = \big|[\Gamma_{\mathbf{d}}(f_M - f)](t)\big| \le K\|f_M - f\|_{C_b^m} \to 0 \quad \text{as} \quad M \to \infty,$$

and then $\Gamma_{\mathbf{d}} f(t) = f(t)$ for all $t \in \mathbb{R}^d$. This proves that the sampling formula (25) holds pointwise. It remains to prove the absolute and uniform convergence of the series in (25). For $|\beta| \le m$, assuming that $D^\beta \phi_j \in C_0(\mathbb{R}^d)$ we have that $D^\beta f_M \in C_0(\mathbb{R}^d)$. Since $D^\beta f_M$ converges uniformly to $D^\beta f$ on $\mathbb{R}^d$, and $C_0(\mathbb{R}^d)$ is a closed subspace in $L^\infty(\mathbb{R}^d)$, we obtain that $D^\beta f \in C_0(\mathbb{R}^d)$. From this fact and using the Lebesgue dominated convergence theorem (whenever Υ_l is a system of type (a)), we obtain that $\{(\Upsilon_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \mathsf{c}_0(\mathbb{R}^d)$ for each $l = 1, 2, \dots, s$. Hence, by using that $S_{l,\mathbf{d}} \in \mathcal{L}^\infty(\mathbb{R}^d)$ and the inequality

$$\sum_{|\alpha|>N} |(\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)| \le \sup_{|\alpha|>N} |(\Upsilon_l f)(M\alpha)|\, |S_{l,\mathbf{d}}|_\infty\,, \quad t \in \mathbb{R}^d,\ N \in \mathbb{N}\,,$$

we obtain that the series in (25) converges absolutely and uniformly on $\mathbb{R}^d$. □

Under the assumed hypotheses, observe that in the proof of the theorem we have obtained that $V_\Phi^\infty \subset C_b^m(\mathbb{R}^d)$.

In case the continuous functions ϕ_j and $D^\beta\phi_j$, $|\beta| \le m$, $j = 1, 2, \dots, r$, belong to the Wiener space $W(L^\infty, \ell^1)$, then they also belong to $\mathcal{L}^\infty(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$.

4.4 Some illustrative examples

We include here some examples illustrating Theorems 1 and 4 by taking B-splines as generators. They belong to $\mathcal{L}^\infty$ since they are compactly supported, and they have L^p-stable shifts for $1 \le p \le \infty$, fulfilling the required assumptions. Moreover, they certainly are important for practical purposes [34].

4.4.1 The case $r = 1$, $M = \mathbb{I}$ and $s = 1$

Here a sampling formula (20) exists whenever $g(x) = \sum_{\alpha\in\mathbb{Z}^d} \Upsilon\phi(\alpha)e^{-2\pi i\alpha x} \ne 0,\ \forall x \in \mathbb{R}^d$. It is unique and it can be written as: For any $f \in V_\phi^p$,

$$f(t) = \sum_{\alpha\in\mathbb{Z}^d} \Upsilon f(\alpha) S(t - \alpha)\,, \quad t \in \mathbb{R}^d\,, \tag{26}$$

where $S(t) = \sum_{\alpha\in\mathbb{Z}^d} \widehat{d}(\alpha)\phi(t - \alpha)$ and $\widehat{d}(\alpha)$ are the Fourier coefficients of $d(x) = 1/g(x)$. Taking the centered quadratic B-spline $\beta_3 := \chi_{[-1/2,1/2)} * \chi_{[-1/2,1/2)} * \chi_{[-1/2,1/2)}$ as generator, we obtain the space $V_{\beta_3}^p$, $1 \le p \le \infty$, of quadratic B-spline $f \in C^2(\mathbb{R}) \cap L^p(\mathbb{R})$ with knots on $\mathbb{Z} + 1/2$. In $V_{\beta_3}^p$ formula (26) for $\Upsilon f = f$ reads:

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) S(t - n)\,, \quad t \in \mathbb{R}\,, \tag{27}$$

where the reconstruction function is $S(t) = \sqrt{2}\sum_{n\in\mathbb{Z}} (2\sqrt{2} - 3)^{|n|}\beta_3(t - n)$. This formula, as well as that corresponding to the cubic B-spline, is very useful in practice (see [34, 35]).

Formulas (26) corresponding to $\Upsilon f = f * \mathsf{h}$ recover any function $f \in V_\phi^p$ from the local average $f * \mathsf{h}(\alpha) = \int_{\mathbb{R}^d} f(t)\mathsf{h}(\alpha - t)\, dt$, $\alpha \in \mathbb{Z}^d$, where the averaging function h is related to the acquisition device (see [8, 32]).

4.4.2 The case $d = 1$, $r = 1$, $M = 2$ and $s = 2$

Here a sampling formula (20) exists in V_ϕ^p whenever the matrix

$$\mathbb{G}(x) = \begin{bmatrix} g_1(x) & g_1(x+1/2) \\ g_2(x) & g_2(x+1/2) \end{bmatrix}, \quad \text{where} \quad g_l(x) = \sum_{n\in\mathbb{Z}} \Upsilon_l\phi(n)e^{-2\pi inx},\ l = 1, 2\,,$$

is no singular for all $x \in [0, 1)$. The reconstruction functions are unique and the sampling formula reads:

$$f(t) = \sum_{n\in\mathbb{Z}} \big[\Upsilon_1 f(2n)S_1(t-2n) + \Upsilon_2 f(2n)S_2(t-2n)\big]\,, \quad t \in \mathbb{R}\,, \tag{28}$$

where the reconstruction functions are given by $S_l(t) = 2\sum_{n\in\mathbb{Z}} \widehat{d_l}(n)\phi(t-n)$, $l = 1, 2$, being $\{\widehat{d_l}(n)\}$ the Fourier coefficients of the functions

$$d_1(x) = \frac{g_2(x+1/2)}{\det \mathbb{G}(x)}, \quad d_2(x) = \frac{-g_1(x+1/2)}{\det \mathbb{G}(x)}\,.$$

For instance, taking $\Upsilon_1 f = f$ and $\Upsilon_2 f = f'$ we obtain a sampling formula that allows to recover any function of V_ϕ^p from samples of the function and of its first derivative (see [36])

$$f(t) = \sum_{n\in\mathbb{Z}} \big[f(2n)S_1(t-2n) + f'(2n)S_2(t-2n)\big]\,, \quad t \in \mathbb{R}\,. \tag{29}$$

Next we give three examples where $s > r(\det M)$, i.e., in the oversampling setting.

4.4.3 The case $d = 1$, $r = 1$, $M = 1$ and $s = 2$

In the oversampling setting we are using a higher sampling rate but in contrast they are infinitely many reconstruction functions (provided $\mathbb{G}(x)$ has full rank for all $x \in \mathbb{R}$). So, we can choose among different reconstruction functions S_l. This flexibility can be used in order to obtain appropriate sampling formula (see [21, 22]). For example, to recover any function $f \in V_{\beta_3}^p$ from its samples we can use formula (27) which uses sampling rate 1. We can also take $s = 2$ and $M = 1$ with $\Upsilon_1 f(t) = f(t)$ and $\Upsilon_2 f(t) = f(t+1/2)$, obtaining sampling formulas as

$$f(t) = \sum_{n\in\mathbb{Z}} \big[f(n)S_1(t-n) + f(n+1/2)S_2(t-n)\big]\,, \quad t \in \mathbb{R}\,,$$

where the reconstruction functions are

$$S_1(t) = \sum_{n\in\mathbb{Z}} \widehat{d_1}(n)\beta_3(t-n), \quad S_2(t) = \sum_{n\in\mathbb{Z}} \widehat{d_2}(n)\beta_3(t-n)$$

where the functions d_1, d_2 can be chosen among those satisfying

$$d_1(x)(e^{2\pi ix} + 6 + e^{-2\pi ix})/8 + d_2(x)(e^{2\pi ix} + 1)/2 = 1\,.$$

By choosing $d_1(x) = 2$ and $d_2(x) = -(1 + e^{-2\pi ix})/2$, we obtain $S_1(t) = 2\beta_3(t)$ and $S_2(t) = -(1/2)[\beta_3(t) + \beta_3(t-1)]$. These reconstruction function have a small support and are computationally efficient.

4.4.4 The case $d = 1$, $r = 1$, $M = 2$ and $s = 3$

Let $N_3(t) := \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]}(t)$ be the quadratic B-spline and let Υ_j be the systems:

$$\Upsilon_1 f(t) = f(t); \quad \Upsilon_2 f(t) = f(t + \frac{2}{3}) \quad \text{and} \quad \Upsilon_3 f(t) = f(t + \frac{4}{3}).$$

Since the functions $\Upsilon_j \phi$, $j = 1, 2, 3$, have compact support, then the entries of the 3×2 matrix $\mathbb{G}(x)$ are trigonometric polynomials and we can try to search a vector $(d_1(x), d_2(x), d_3(x))$ satisfying (14) with trigonometric polynomials entries also. This implies reconstruction functions S_l, $l = 1, 2, 3$, with compact support. Proceeding as in [22] we obtain that any function $f \in V^p_{N_3}$ can be recovered through the sampling formula:

$$f(t) = \sum_{n \in \mathbb{Z}} \big[f(2n) S_1(t - 2n) + f(2n + 2/3) S_2(t - 2n) + f(2n + 4/3) S_3(t - 2n) \big], \quad t \in \mathbb{R},$$

where the reconstruction functions are given by

$$\begin{aligned}
S_1(t) &= \frac{1}{16} \big(N_3(t+3) - 3N_3(t+2) - 3N_3(t+1) + N_3(t) \big), \\
S_2(t) &= \frac{1}{16} \big(27N_3(t+1) - 9N_3(t) \big), \\
S_3(t) &= \frac{1}{16} \big(-9N_3(t+1) + 27N_3(t) \big), \quad t \in \mathbb{R}.
\end{aligned}$$

4.4.5 The case $d = 1$, $r = 2$, $M = 1$ and $s = 3$

Consider the Hermite cubic splines defined as

$$\varphi_1(t) = \begin{cases} (t+1)^2(1-2t), & t \in [-1, 0] \\ (1-t)^2(1+2t), & t \in [0, 1] \\ 0, & |t| > 1 \end{cases} \quad \text{and} \quad \varphi_2(t) = \begin{cases} (t+1)^2 t, & t \in [-1, 0] \\ (1-t)^2 t, & t \in [0, 1] \\ 0, & |t| > 1 \end{cases}.$$

They are stable generators for the space $V^p_{\varphi_1, \varphi_2}$ (see [16]). Take the sampling period $M = 1$ and the systems defined by

$$\Upsilon_1 f(t) := \Upsilon f(t) := \int_t^{t+1/3} f(u) du, \quad \Upsilon_2 f(t) := \Upsilon f\big(t + \frac{1}{3}\big), \quad \Upsilon_3 f(t) := \Upsilon f\big(t + \frac{2}{3}\big).$$

Searching for reconstruction functions S_l with compact support as in [19] we obtain in $V^p_{\varphi_1, \varphi_2}$ the following sampling formula:

$$f(t) = \sum_{n \in \mathbb{Z}} \big[\Upsilon f(n) S_1(t - n) + \Upsilon f\big(n + \frac{1}{3}\big) S_2(t - n) + \Upsilon f\big(n + \frac{2}{3}\big) S_3(t - n) \big], \quad t \in \mathbb{R},$$

where the sampling functions are:

$$\begin{aligned}
S_1(t) &:= \frac{85}{44} \varphi_1(t) + \frac{1}{11} \varphi_1(t-1) + \frac{85}{4} \varphi_2(t) - \varphi_2(t-1) \\
S_2(t) &:= \frac{-23}{44} \varphi_1(t) - \frac{23}{44} \varphi_1(t-1) - \frac{23}{4} \varphi_2(t) + \frac{23}{4} \varphi_2(t-1) \\
S_3(t) &:= \frac{1}{11} \varphi_1(t) + \frac{85}{44} \varphi_1(t-1) + \varphi_2(t) - \frac{85}{4} \varphi_2(t-1), \quad t \in \mathbb{R}.
\end{aligned}$$

4.5 Sampling formulas as p-frames and p-Riesz bases expansions

The reconstruction method in V_Φ^p ($1 \le p \le \infty$) given by formula (25) is stable in the following way: For $f, g \in V_\Phi^p$ and $l = 1, 2, \dots, s$ the sum $\sum_{l=1}^s \|\{(\Upsilon_l f)(M\alpha) - (\Upsilon_l g)(M\alpha)\}\|_p$ is small if and only if $\|f-g\|_p$ is also small. Indeed, consider $\Delta_{l,\alpha} := (\Upsilon_l f)(M\alpha) - (\Upsilon_l g)(M\alpha)$. We have

$$\|f-g\|_p = \Big\| \sum_{l=1}^s \sum_{\alpha\in\mathbb{Z}^d} \Delta_{l,\alpha} S_{l,\mathbf{d}}(\cdot - M\alpha) \Big\|_p = \Big\| \sum_{l=1}^s \sum_{\alpha\in\mathbb{Z}^d} \widetilde{\Delta}_{l,\alpha} S_{l,\mathbf{d}}(\cdot - \alpha) \Big\|_p ,$$

where

$$\widetilde{\Delta}_{l,\alpha'} := \begin{cases} \Delta_{l,\alpha} & \text{if } \alpha' = M\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, denoting $\widetilde{\Delta}_l = \{\widetilde{\Delta}_{l,\alpha}\}_{\alpha\in\mathbb{Z}^d}$ and $\Delta_l = \{\Delta_{l,\alpha}\}_{\alpha\in\mathbb{Z}^d}$ we obtain

$$\|f-g\|_p = \big\| \sum_{l=1}^s S_{l,\mathbf{d}} *' \widetilde{\Delta}_l \big\|_p \le \sum_{l=1}^s \|\Delta_l\|_p |S_{l,\mathbf{d}}|_p \le \Big(\max_{1\le l\le s} |S_{l,\mathbf{d}}|_p \Big) \sum_{l=1}^s \|\Delta_l\|_p ,$$

where we have used that for each $l = 1, 2, \dots, s$, the reconstruction function $S_{l,\mathbf{d}} \in \mathcal{L}^\infty(\mathbb{R}^d) \subset \mathcal{L}^p(\mathbb{R}^d)$, $1 \le p < \infty$; the corresponding inequality in (4); and also that $\|\widetilde{\Delta}_l\|_p = \|\Delta_l\|_p$ for each $l = 1, 2, \dots, s$. Moreover, as in the proof of Theorems 1 and 4 we have that $\|\Delta_l\|_p \le K_l \|f-g\|_p$, $l = 1, 2, \dots, s$.

In particular, we have proved that, in V_Φ^p, $1 \le p \le \infty$, there exist two positive constants $0 < A_p \le B_p$ such that

$$A_p \|f\|_p \le \sum_{l=1}^s \|\{\Upsilon_l f(M\alpha)\}\|_{\ell^p} \le B_p \|f\|_p , \quad f \in V_\Phi^p .$$

In other words, we have

$$\frac{1}{B_p} \sum_{l=1}^s \|\{\Upsilon_l f(M\alpha)\}\|_{\ell^p} \le \Big\| \sum_{l=1}^s S_{l,\mathbf{d}} *' \{\Upsilon_l f\} \Big\|_p \le \frac{1}{A_p} \sum_{l=1}^s \|\{\Upsilon_l f(M\alpha)\}\|_{\ell^p} .$$

In fact, we have the following result:

Theorem 5 *Assume that $s = (\det M) r$; then the sequence $\big\{S_{l,\mathbf{d}}(\cdot - M\alpha) : \alpha \in \mathbb{Z}^d, l = 1, 2, \dots, s\big\}$ is a p-Riesz basis for V_Φ^p, $1 \le p < \infty$. The result is also true in V_Φ^∞ by assuming that the continuous functions ϕ_j, $D^\beta \phi_j$, $|\beta| \le m$, $j = 1, 2, \dots, r$, belong to $W(L^\infty, \ell^1)$.*

Proof. It is sufficient to prove that the map $f \mapsto \{(\Upsilon_1 f(M\alpha), \dots, \Upsilon_s f(M\alpha))\}_{\alpha\in\mathbb{Z}^d}$ is surjective from $V_\Phi^p \longrightarrow \ell^p(\mathbb{Z}^d) \times \dots \times \ell^p(\mathbb{Z}^d)$ (s times) when $1 \le p < \infty$, or from $V_\Phi^\infty \longrightarrow \mathsf{c}_0(\mathbb{Z}^d) \times \dots \times \mathsf{c}_0(\mathbb{Z}^d)$ (s times) when $p = \infty$. To this end, let $\{b_l(\alpha)\}$ be a sequence in $\ell^p(\mathbb{Z}^d$ for $1 \le p < \infty$ (in $\mathsf{c}_0(\mathbb{Z}^d)$ for $p = \infty$), $l = 1, 2, \dots, s$. Define $g = \sum_{l=1}^s \sum_{\alpha\in\mathbb{Z}^d} b_l(\alpha) S_{l,\mathbf{d}}(\cdot - M\alpha) \in L^p(\mathbb{R}^d)$. The truncated series belong to V_Φ^p; taking the L^p limit we obtain that $g \in V_\Phi^p$. Finally, the interpolatory of $S_{l,\mathbf{d}}$, which holds whenever $s = (\det M) r$, shows that $\Upsilon_l g(M\alpha) = b_l(\alpha)$ for all $\alpha \in \mathbb{Z}^d$ and $l = 1, 2, \dots, s$. $\square$

In the light of the p-frames theory (see Refs. [7, 14, 24]), it can be derived that the sequence $\big\{S_{l,\mathbf{d}}(\cdot - M\alpha) : \alpha \in \mathbb{Z}^d, l = 1, 2, \dots, s\big\}$ is a p-frame for V_Φ^p, $1 \le p < \infty$, i.e., there exists a positive constant C (depending on Φ and p) such that

$$C^{-1} \|f\|_{L^p} \le \sum_{l=1}^s \Big\| \Big\{ \int_{\mathbb{R}^d} f(t) \overline{S_{l,\mathbf{d}}(t - M\alpha)} dt \Big\}_{\alpha\in\mathbb{Z}^d} \Big\|_{\ell^p} \le C \|f\|_{L^p} \quad \text{for all } f \in V_\Phi^p .$$

Indeed, since

$$V_\Phi^p = \Big\{ \sum_{l=1}^{s} S_{l,\mathbf{d}} *' a_l \,:\, a_l \in \ell^p(\mathbb{Z}^d)\,, l = 1, 2, \ldots, s \Big\} \quad \text{if } 1 \le p < \infty\,,$$

and V_Φ^p is closed in $L^p(\mathbb{R}^d)$, Theorem 1 in [7] gives the result whenever $\Phi \subset \mathcal{L}^\infty(\mathbb{R}^d)$ for $1 < p < \infty$, and $\Phi \subset W(L^\infty, \ell^1)$ for $p = 1$.

4.6 The sampling result in $V_\Phi(\infty)$

The aim in this section is to prove a generalized sampling result in the bigger space considered in [7]:

$$V_\Phi(\infty) := \Big\{ \sum_{j=1}^{r} \phi_j *' a_j \;:\; a_j \in \ell^\infty(\mathbb{Z}^d)\,, j = 1, 2, \ldots, r \Big\}\,.$$

To this end we assume that the generators ϕ_j are continuous in $\mathbb{R}^d$ and $\phi_j \in W(L^\infty, \ell^1)$, $j = 1, 2, \ldots, r$. As a consequence, the series $\sum_{\alpha\in\mathbb{Z}^d} |\phi_j(t-\alpha)|$, $j = 1, 2, \ldots, r$, converges uniformly on compact subsets of $\mathbb{R}^d$; thus, the space $V_\Phi(\infty)$ is a subset of $L^\infty(\mathbb{R}^d)$ of continuous functions.

Acting on $V_\Phi(\infty)$ we consider s convolution systems Υ_l, $l = 1, 2, \cdots, s$, satisfying that $|\mathsf{h}_l|_1 < \infty$ whenever Υ_l is of type (a), and satisfying any derivative $D^\beta\phi_j$ appearing in systems of type (c) that $D^\beta\phi_j \in C(\mathbb{R}^d) \cap W(L^\infty, \ell^1)$. Finally, let $\mathbf{d}(x) = \big(d_1(x), d_2(x), \ldots, d_s(x)\big)$ be a $r \times s$ matrix with entries $d_{j,l} \in \mathcal{A}$, $j = 1, 2, \ldots, r$, $l = 1, 2, \ldots, s$, and satisfying (14), and let $S_{l,\mathbf{d}}$ be the associated reconstruction functions given in (18). We obtain the following result:

Theorem 6 *Under the above assumptions, for any function $f \in V_\Phi(\infty)$ the following sampling theorem holds*

$$f(t) = \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

where the series converges absolutely and uniformly on compact subsets of $\mathbb{R}^d$.

Proof. Truncating the series that defines $f \in V_\Phi(\infty)$ yields a sequence $\{f_m\}$ in $\operatorname{span}\{\phi_j(\cdot - \alpha) \,:\, \alpha \in \mathbb{Z}^d\,, j = 1, 2, \ldots, r\}$ such that $f(t) = \lim_{m\to\infty} f_m(t)$, uniformly on compact subsets of $\mathbb{R}^d$, and satisfying that $\sup_m \|f_m\|_\infty < \infty$.

By using Lemma 2, for any $f_m \in \operatorname{span}\{\phi_j(\cdot - \alpha) \,:\, \alpha \in \mathbb{Z}^d\,, j = 1, 2, \ldots, r\}$ we have

$$f_m(t) = \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f_m)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

uniformly on $\mathbb{R}^d$. In order to compute correctly the limit as $m \to \infty$ we can use the Moore-Smith theorem which statement we include for the sake of completeness. Its proof can be found in [9, p. 236]:

Lemma 4 *Let M be a complete metric space with metric ρ, and let $\{x_{n,m}\}$, $n, m \in \mathbb{N}$, be given. Assume there are sequences $\{y_n\}$, $\{z_m\}$ in M such that*

1. $\lim_{n\to\infty} \rho(x_{n,m}, z_m) = 0$ *uniformly in* m*, and*

2. for each $n \in \mathbb{N}$*,* $\lim_{m\to\infty} \rho(x_{n,m}, y_n) = 0$*.*

Then there is $x \in M$ *such that*

$$\lim_{m\to\infty} \lim_{n\to\infty} \rho(x_{n,m}, x) = \lim_{n\to\infty} \lim_{m\to\infty} \rho(x_{n,m}, x) = \lim_{m,n\to\infty} \rho(x_{n,m}, x) = 0\,.$$

Let (M, ρ) be the completion of $\operatorname{span}\{\phi_j(\cdot - \alpha) : \alpha \in \mathbb{Z}^d\,, j = 1, 2, \dots, r\}$, where ρ denotes the metric giving the uniform convergence on compact subsets of $\mathbb{R}^d$. Hence, $V_\Phi(\infty) \subset M$. Consider the double series

$$x_{N,m}(t) := \sum_{l=1}^{s} \sum_{|\alpha|\le N} (\Upsilon_l f_m)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)\,.$$

Now it is easy to check the hypotheses in the Moore-Smith theorem. Indeed, for condition 1 in Lemma 4 we have the inequality

$$\left|x_{N,m}(t) - f_m(t)\right| \le \sum_{l=1}^{s} \sum_{|\alpha|>N} |(\Upsilon_l f_m)(M\alpha)||S_{l,\mathbf{d}}(t - M\alpha)|\,,$$

and $\|\{(\Upsilon_l f_m)(M\alpha)\}\|_\infty \le K$ uniformly in m. For systems of the type (a), we have $\|\{(\Upsilon_l f_m)(M\alpha)\}\|_\infty \le |\mathsf{h}_j|_1 \|f_m\|_\infty \le |\mathsf{h}_j|_1 \sup_m \|f_m\|_\infty < \infty$; for systems of types (b) and (c) it comes from the assumptions. In other words,

$$\left|x_{N,m}(t) - f_m(t)\right| \le K \sum_{l=1}^{s} \sum_{|\alpha|>N} |S_{l,\mathbf{d}}(t - M\alpha)| \to 0 \quad \text{as } N \to \infty$$

uniformly on compact subsets of $\mathbb{R}^d$ regardless m. Notice that $S_{l,\mathbf{d}} = \sum_{j=1}^{r} \phi_j *' d_{j,l}$ where $d_{j,l} \in \ell^1(\mathbb{Z}^d)$. Hence, for $l = 1, 2, \dots, s$, the reconstruction function $S_{l,\mathbf{d}}$ is a continuous function on $\mathbb{R}^d$ belonging to $W(L^\infty, \ell^1)$ (see (6)). Consequently, the series $\sum_{\alpha\in\mathbb{Z}^d} |S_{l,\mathbf{d}}(t - M\alpha)|$ converges uniformly on compact subsets of $\mathbb{R}^d$.

Regarding condition 2, for each $N \in \mathbb{N}$ we have that $\lim_{m\to\infty} \|x_{N,m} - x_N\|_\infty = 0$, where $x_N(t) := \sum_{l=1}^{s} \sum_{|\alpha|\le N} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)$; notice that we have used the Lebesgue dominated convergence theorem for systems of type (a).

Since $\lim_{m\to\infty} \lim_{N\to\infty} \rho(x_{N,m}, f) = \lim_{m\to\infty} \rho(f_m, f) = 0$, the Moore-Smith theorem gives that

$$f(t) = \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

uniformly on compact subsets of $\mathbb{R}^d$. The convergence absolute comes easily from the inequality

$$\sum_{l=1}^{s} \sum_{|\alpha|>N} |(\Upsilon_l f)(M\alpha)||S_{l,\mathbf{d}}(t - M\alpha)| \le C \sum_{l=1}^{s} \sum_{|\alpha|>N} |S_{l,\mathbf{d}}(t - M\alpha)| \to 0$$

as $N \to \infty$. □

5 Approximation order

Assume the hypotheses in Sections 4.1 and 4.3. Consider an $r \times s$ matrix $\mathbf{d}(x) := \big(\mathbf{d}_1(x), \mathbf{d}_2(x), \dots, \mathbf{d}_s(x)\big)$ with entries $d_{j,l} \in \mathcal{A}$, $j = 1, 2, \dots, r$, $l = 1, 2, \dots, s$, and satisfying (14). Let $S_{l,\mathbf{d}}$ be the associated reconstruction functions, $l = 1, 2, \dots, s$. Recall that $\mathcal{N}(M^\top) := \mathbb{Z}^d \cap \{M^\top x : x \in [0,1)^d\} = \{i_j\}_{j=1,2,\dots,\det M}$. Consider also the new points $r_j = M^{-\top} i_j$, $j = 1, 2, \dots, \det M$ (notice that $i_1 = r_1 = 0$).

The aim of this section is to show that if the set of generators Φ satisfies the Strang-Fix conditions, then the sampling operator

$$\Gamma_{\mathbf{d}} f(t) = \sum_{l=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

takes advantage of the good approximation properties of the scaled spaces $\sigma_{1/h} V_\Phi^p$ $(h > 0)$; we are using the notation: $\sigma_A f(t) := f(At)$, where A denotes a number or a matrix.

The set of generators $\Phi = \{\phi_j\}_{j=1}^r$ is said to satisfy the Strang-Fix conditions of order k if there exist finitely supported sequences $b_j : \mathbb{Z}^d \to \mathbb{C}$ such that the function $\phi = \sum_{j=1}^r \phi_j *' b_j$ satisfies the Strang-Fix conditions of order k, i.e.,

$$\widehat{\phi}(0) \neq 0, \quad D^\beta \widehat{\phi}(\alpha) = 0, \quad |\beta| < k, \quad \alpha \in \mathbb{Z}^d \setminus \{0\}\,. \tag{30}$$

We denote by $W_p^k(\mathbb{R}^d) := \{f : \|D^\gamma f\|_p < \infty\,, |\gamma| \le k\}$ the usual Sobolev space (see [1]), and by $|f|_{k,p} := \sum_{|\beta|=k} \|D^\beta f\|_p$ the seminorm of a function $f \in W_p^k(\mathbb{R}^d)$. When $kp > d$ we identify $f \in W_p^k(\mathbb{R}^d)$ with its continuous choice.

It is well known that if Φ satisfies the Strang-Fix conditions of order k, and the generators ϕ_j satisfy a suitable decay condition, the space V_Φ^p provides approximation order k, in the L^p-sense $(1 \le p \le \infty)$, for any function f regular enough. For instance, Lei, Jia and Cheney proved in [30, Theorem 5.2] that if a set $\Phi = \{\phi_j\}_{j=1}^r$ of L^p-stable generators satisfies the Strang-Fix conditions of order k and the decay condition, $\phi_j(t) = O\big([1 + |t|]^{-d-k-\epsilon}\big)$ for $\epsilon > 0$, then, for any $f \in W_p^k(\mathbb{R}^d)$, there exists $f_{approx} \in \sigma_{1/h} V_\Phi^p$ such that

$$\|f - f_{approx}\|_p \le C\, |f|_{k,p}\, h^k\,, \tag{31}$$

where the constant C does not depend on h and f.

5.1 L^∞-approximation order of the sampling operator $\Gamma_{\mathbf{d}}$

The sampling operator $\Gamma_{\mathbf{d}} : C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ is bounded and satisfies $\Gamma_{\mathbf{d}} g = g$ for $g \in V_\Phi(\infty)$, whenever only systems of type (a) and (b) appear (see Lemma 3 and Theorem 4). Thus, by applying Lebesgue's Lemma [17, p. 30] we prove in the following corollary that an approximation $f_{approx} \in \sigma_{1/h} V_\Phi^\infty$ satisfying the estimation (31) for $p = \infty$ can be obtained by means of the operator $\Gamma_{\mathbf{d}}$. Specifically, $f_{approx} = \Gamma_{\mathbf{d}}^h f$ where

$$\Gamma_{\mathbf{d}}^h := \sigma_{1/h} \Gamma_{\mathbf{d}} \sigma_h, \qquad h > 0.$$

Moreover, it proves that the approximation given by $\Gamma_{\mathbf{d}}^h$ is, up to a constant factor, as good as the best approximation from $\sigma_{1/h} V_\Phi^\infty$.

Corollary 1 *Assume that there are only systems of types (a) and (b). Then, the following estimation holds:*

$$\|f - \Gamma_{\mathbf{d}}^h f\|_\infty \leq (1 + \|\Gamma_{\mathbf{d}}\|) \inf_{g \in \sigma_{1/h} V_\Phi^\infty} \|f - g\|_\infty, \quad f \in W_\infty^k(\mathbb{R}^d),$$

where $\|\Gamma_{\mathbf{d}}\|$ *denotes the operator norm. If the set of generators* $\Phi = \{\phi_j\}_{j=1}^r$ *satisfies* $\phi_j(t) = O\big([1 + |t|]^{-d-k-\epsilon}\big)$ *with* $\epsilon > 0$ *and the Strang-Fix conditions of order* k *then,*

$$\|f - \Gamma_{\mathbf{d}}^h f\|_\infty \leq C\, h^k\, |f|_{k,\infty}\ , \quad f \in W_\infty^k(\mathbb{R}^d),$$

where the constant C *does not depend on* h *and* f*.*

Proof. Notice that the scaled operator $\Gamma_{\mathbf{d}}^h : C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ is a bounded operator with the same norm as $\Gamma_{\mathbf{d}}$, that $\sigma_{1/h} V_\Phi^\infty \subset C_b(\mathbb{R}^d)$, and that $\Gamma_{\mathbf{d}}^h g = g$ for $g \in \sigma_{1/h} V_\Phi^\infty$. Then, for $f \in W_\infty^k(\mathbb{R}^d)$ and $g \in \sigma_{1/h} V_\Phi(\infty)$ we have

$$\|f - \Gamma_{\mathbf{d}}^h f\|_\infty \leq \|f - g\|_\infty + \|\Gamma_{\mathbf{d}}^h g - \Gamma_{\mathbf{d}}^h f\|_\infty \leq (1 + \|\Gamma_{\mathbf{d}}\|)\|f - g\|_\infty.$$

As $\sigma_{1/h} V_\Phi^\infty \subset C_b(\mathbb{R})$, the second assertion of the corollary is a consequence of the mentioned result in [30]. $\square$

5.2 L^p-approximation order in case of systems of type (a) $(1 \leq p < \infty)$

For $1 \leq p < \infty$ we prove an analogous result to Corollary 1 whenever only systems of type (a) appear.

Corollary 2 *Assume* $kp > d$ *and that all the systems* Υ_l *satisfy* $\Upsilon_l f = f * \mathsf{h}_l$ *with* $\mathsf{h}_l \in \mathcal{L}^q(\mathbb{R}^d)$*,* $l = 1, \ldots, s$ $(1/p + 1/q = 1)$*. Then*

$$\|f - \Gamma_{\mathbf{d}}^h f\|_p \leq (1 + \|\Gamma_{\mathbf{d}}\|) \min_{g \in \sigma_{1/h} V_\Phi^p} \|f - g\|_p, \quad f \in W_p^k(\mathbb{R}^d),$$

where $\|\Gamma_{\mathbf{d}}\|$ *denotes the norm of the operator* $\Gamma_{\mathbf{d}} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$*. If the set of generators* $\Phi = \{\phi_j\}_{j=1}^r$ *satisfies* $\phi_j(t) = O\big([1 + |t|]^{-d-k-\epsilon}\big)$ *with* $\epsilon > 0$*, and the Strang-Fix conditions of order* k*, then*

$$\|f - \Gamma_{\mathbf{d}}^h f\|_p \leq C\, h^k\, |f|_{k,p}\ , \quad f \in W_p^k(\mathbb{R}^d),$$

where the constant C *does not depend on* h *and* f*.*

Proof. Using inequalities (4) and (5) we easily prove that $\Gamma_{\mathbf{d}} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a bounded operator. Theorem 1 proves that $\Gamma_{\mathbf{d}} g = g$ for $g \in V_\Phi^p$. Now, proceeding as in Corollary 1, Lebesgue's Lemma jointly with (31) yield the result. $\square$

5.3 L^2-approximation order

For systems of type (b) or (c), the operator $\Gamma_{\mathbf{d}}$ is not bounded in the L^p-norm and, consequently, in order to obtain an L^p-approximation result we cannot apply Lebesgue's Lemma. However, returning to the Hilbert space case, the results by Jetter and Zhou in [25] allow us to prove that the sampling operator $\Gamma_{\mathbf{d}}$ gives approximation order k, in case of a unique generator ϕ satisfying the Strang-Fix conditions of order k. The aforementioned work [25] deals with the approximation order by means of linear operators of the type:

$$Qf(t)=\sum_{l=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big[\sum_{|\beta|\le N_l}\int D^\beta f(x+\alpha)d\overline{\mu}_{l,\beta}(x)\Big]\varphi_l(t-\alpha)\,, \tag{32}$$

where $\mu_{l,\beta}$ denotes a Borel finite measure. According to the notation in [25], let $\overline{\Omega}_l$ denote $\overline{\Omega}_l(w):=\sum_{|\beta|\le N_l}(2\pi i w)^\beta\widehat{\overline{\mu}}_{l,\beta}(w)$. In [25] it is proved that if the following conditions:

(C1) $\quad \Big|1-\sum_{l=1}^{s}\widehat{\varphi}_l(w)\overline{\Omega}_l(w)\Big|\le C_1\,|w|^k,$

(C2) $\quad \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}}\Big|\sum_{l=1}^{s}\widehat{\varphi}_l(w+\alpha)\overline{\Omega}_l(w)\Big|^2\le\, C_2\,|w|^{2k},$

(C3) $\quad \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}}|w+\alpha|^{-2k}\Big|\sum_{l=1}^{s}\widehat{\varphi}_l(w)\overline{\Omega}_l(w+\alpha)\Big|^2\le\, C_3$, and

(C4) $\quad \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}}|w+\alpha|^{-2k}\sum_{\beta\in\mathbb{Z}^d\setminus\{0\}}\Big|\sum_{l=1}^{s}\widehat{\varphi}_l(w+\beta)\overline{\Omega}_l(w+\alpha)\Big|^2\le\, C_4,$

are satisfied, a.e. on $[-1/2,1/2)^d$, then the operator Q gives approximation order k. In next theorem we prove that the sampling property $\Gamma_{\mathbf{d}}f=f$ for $f\in V_\phi^2$ and the Strang-Fix conditions of order k for ϕ imply that the operator $Q:=\sigma_M\Gamma_{\mathbf{d}}\sigma_{M^{-1}}$ satisfies the above conditions and, as a consequence, the operator $\Gamma_{\mathbf{d}}$ gives approximation order k. Recall that m denotes the largest order of the partial derivatives that appear in systems of type (c) ($m=0$ if there are only systems of type (a) or (b)).

Theorem 7 *Assume that $k>d/2+m$ and that the generator ϕ of the space V_ϕ^2 satisfies $\widehat{D^\beta\phi}\in W_1^{k+d}(\mathbb{R}^d)$ for $|\beta|\le m$, $|w|^m\widehat{\phi}(w)\in\mathcal{L}^2(\mathbb{R}^d)$, and the Strang-Fix conditions of order k. Then,*

$$\|\Gamma_{\mathbf{d}}^h f-f\|_{L^2(\mathbb{R}^d)}\le C\,|f|_{k,2}\,h^k,\quad f\in W_2^k(\mathbb{R}^d), \tag{33}$$

where the constant C is independent of f and $h>0$.

First of all, notice that if the generator ϕ has compact support and the function $|\widehat{\phi}(w)|(1+|w|)^{d+m+\epsilon}$ is bounded for some $\epsilon>0$, then the hypotheses, $\widehat{D^\beta\phi}\in W_1^{k+d}(\mathbb{R}^d)$ for $|\beta|\le m$, and $|w|^m\widehat{\phi}(w)\in\mathcal{L}^2(\mathbb{R}^d)$, in the above theorem, are satisfied. Indeed, in this case, $|w|^m\widehat{\phi}(w)\in\mathcal{L}^\infty(\mathbb{R}^d)\subset\mathcal{L}^2(\mathbb{R}^d)$ and $\widehat{D^\beta\phi}\in L^1(\mathbb{R}^d)$. As a consequence of Bernstein's inequalities (see [31]), we obtain that $\widehat{D^\beta\phi}\in W_1^n(\mathbb{R}^d)$ for all $n\in\mathbb{N}$.

Before proving the theorem we need two technical lemmas.

Lemma 5 *Assume that $|\beta|\le k$ and $\widehat{D^\beta\phi}\in W_1^{k+d}(\mathbb{R}^d)$. If ϕ satisfies the Strang-Fix conditions of order k (30), then there exists a constant C such that*

$$\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}}|(w+\alpha)^\beta\widehat{\phi}(w+\alpha)|\le C\,|w|^k,\quad w\in[-1/2,1/2)^d\,.$$

Proof. From the Strang-Fix conditions we have that $D^\gamma \widehat{D^\beta \phi}(\alpha) = D^\gamma \left[(2\pi i\cdot)^\beta \widehat{\phi}\right](\alpha) = 0$, for $\alpha \in \mathbb{Z}^d \setminus \{0\}$, $|\gamma| < k$. Using the Taylor formula we obtain that, for all $\alpha \in \mathbb{Z}^d \setminus \{0\}$,

$$|\widehat{D^\beta \phi}(w+\alpha)| \le C\,|w|^k \max_{|\gamma|=k} \|D^\gamma \widehat{D^\beta \phi}\|_{L^\infty(U+\alpha)}, \quad w \in U, \tag{34}$$

where $U := [-1/2, 1/2)^d$ and the constant C does not depend on α. Using the Sobolev imbedding $W_1^{k+d}(U) \hookrightarrow W_\infty^k(U)$ (see [1, p. 85]), we obtain that

$$\max_{|\gamma|=k} \|D^\gamma \widehat{D^\beta \phi}\|_{L^\infty(U+\alpha)} \le \|\widehat{D^\beta \phi}\,(\cdot+\alpha)\|_{W_\infty^k(U)} \le C\,\|\widehat{D^\beta \phi}\,(\cdot+\alpha)\|_{W_1^{k+d}(U)}\,.$$

Hence,

$$|\widehat{D^\beta \phi}(w+\alpha)| \le\; C|w|^k \|\widehat{D^\beta \phi}\,(\cdot+\alpha)\|_{W_1^{k+d}(U)}\,, \quad w \in U\,,$$

where the constant C does not depend on α; as a consequence, the inequality

$$\sum_{\alpha \in \mathbb{Z}^d \setminus \{0\}} |\widehat{D^\beta \phi}(w+\alpha)| \le C|w|^k \|\widehat{D^\beta \phi}\|_{W_1^{k+d}(\mathbb{R}^d)}$$

holds. □

Lemma 6 *Let g be a function such that $\widehat{g} \in \mathcal{L}^2(\mathbb{R}^d)$. Then, the Poisson summation formula $\sum_{\alpha \in \mathbb{Z}^d} \widehat{g}(w+\alpha) = \sum_{\alpha \in \mathbb{Z}^d} g(-\alpha) e^{2\pi i \alpha \cdot w}$ holds in the $L^2([0,1)^d)$ sense.*

Proof. Since $\widehat{g} \in \mathcal{L}^2(\mathbb{R}^d)$ the function $\sum_{\alpha \in \mathbb{Z}^d} \widehat{g}(w+\alpha) \in L^2[0,1)^d$; as $\widehat{g} \in \mathcal{L}^1(\mathbb{R}^d) = L^1(\mathbb{R}^d)$, the dominated convergence theorem gives the Fourier coefficients

$$\begin{aligned} &\int_{[-\frac{1}{2},\frac{1}{2})^d} \sum_{\alpha \in \mathbb{Z}^d} \widehat{g}(w+\alpha) e^{-2\pi i \beta \cdot w} dw = \int_{[-\frac{1}{2},\frac{1}{2})^d} \sum_{\alpha \in \mathbb{Z}^d} \widehat{g}(w+\alpha) e^{-2\pi i \beta \cdot (w+\alpha)} dw \\ &= \sum_{\alpha \in \mathbb{Z}^d} \int_{[-\frac{1}{2},\frac{1}{2})^d} \widehat{g}(w+\alpha) e^{-2\pi i \beta \cdot (w+\alpha)} dw = \int_{\mathbb{R}^d} \widehat{g}(w) e^{-2\pi i \beta \cdot w} dw = g(-\alpha)\,. \end{aligned}$$

□

Proof of Theorem 7. For the sake of simplicity, we prove the theorem for the case of systems of type (c) (any system of type (b) can be considered as a particular case). For systems of type (a) the proof follows the same steps in an easier way.

Throughout the proof C denotes a generic constant, not necessarily the same in any place. An equivalent estimation to (33) is $\|\Gamma_{\mathbf{d}} f - f\|_{L^2(\mathbb{R}^d)} \le C\,|f|_{k,2}$ for $f \in W_2^k(\mathbb{R}^d)$. Since $|\sigma_M f|_{k,2} \le C|f|_{k,2}$ for some constant independent of f, in order to prove this last estimation, it is sufficient to prove that $\|Qf - f\|_{L^2(\mathbb{R}^d)} \le C\,|f|_{k,2}$ for all $f \in W_2^k(\mathbb{R}^d)$, where $Q := \sigma_M \Gamma_{\mathbf{d}} \sigma_{M^{-1}}$. Thus, we can derive our theorem by proving that the operator Q satisfies the conditions of Theorem 2.1 in [25].

First, notice that $Q = \sigma_M \Gamma_{\mathbf{d}} \sigma_{M^{-1}}$ is an operator of the type considered in [25]. Namely, it can be expressed as in (32) where

$$\varphi_l = \sigma_M S_l \quad \text{and} \quad d\overline{\mu}_{l,\beta}(x) = \sum_{|\beta| \le |\gamma| \le N_l} c_{l,\gamma} a_{\gamma,\beta} \delta(x - M^{-1} d_{l,\gamma}) dx\,,$$

being $a_{\beta,\gamma}$ the constants satisfying $D^\beta \sigma_{M^{-1}} = \sum_{|\gamma| \le |\beta|} a_{\beta,\gamma} \sigma_{M^{-1}} D^\gamma$.

The properties of the Fourier transform allow us to check that $a_{\beta,\gamma}$ are the constants satisfying $(2\pi i w)^\beta = \sum_{|\gamma|\le|\beta|} a_{\beta,\gamma}(2\pi i M^\top w)^\gamma$. By using this fact it is easy to obtain that $\overline{\Omega}_l(w) := \sum_{|\beta|\le N_l} c_{l,\beta}(2\pi i M^{-\top}w)^\beta e^{2\pi i d_{l,\beta}\cdot M^{-\top}w}$.

In order to apply [25, Theorem 2.1], along with conditions (C1)–(C4), the equality $\widehat{Qf}(w) = \sum_{l=1}^s \widehat{\sigma_M S_l}(w)\sum_{\alpha\in\mathbb{Z}^d}\widehat{f}(w+\alpha)\overline{\Omega}_l(w+\alpha)$ a.e., must be verified in the Fourier domain for any function $f\in W_2^k(\mathbb{R}^d)$. This representation can be obtained by applying Lemma 7.2.1 in [13], and the Poisson summation formula in Lemma 6 with $g(t) = D^\gamma f(t+M^{-1}d_{l,\beta})$, $|\gamma|\le|\beta|$. To this end, we need $\widehat{g}\in\mathcal{L}^2(\mathbb{R}^d)$. Indeed,

$$\Big[\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} |w+\alpha|^\gamma|\widehat{f}(w+\alpha)|\Big]^2 \le \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} |w+\alpha|^{2\gamma-2k}\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} |w+\alpha|^{2k}|\widehat{f}(w+\alpha)|^2 .$$

Hence, since $2k-2\gamma>d$,

$$\int_{[0,1)^d}\Big[\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} |w+\alpha|^\gamma|\widehat{f}(w+\alpha)|\Big]^2 dw \le \Big\|\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} |w+\alpha|^{2\gamma-2k}\Big\|_{L^\infty([0,1)^d)}|f|_{k,2}^2 < \infty.$$

Next, we obtain an expression for the projection condition $\mathbf{d}(w)\mathbb{G}(w) = [1,0,\ldots,0]$ in Lemma 1, in terms of $\widehat{\phi}$ and Ω_l. Applying the Poisson summation formula, Lemma 6, [13, Lemma 7.2.1], and that $|w|^m\widehat{\phi}(w)\in\mathcal{L}^2(\mathbb{R}^d)$, we have

$$\begin{aligned} g_{l,1}(w) &= \sum_{\alpha\in\mathbb{Z}^d}\Upsilon_l\phi(\alpha)e^{-2\pi i\alpha\cdot w} = \sum_{\alpha\in\mathbb{Z}^d}\sum_{|\beta|\le N_l} c_{l,\beta}D^\beta\phi(\alpha+d_{l,\beta})e^{-2\pi i\alpha\cdot w} \\ &= \sum_{|\beta\le N_l} c_{l,\beta}\sum_{\alpha\in\mathbb{Z}^d}(2\pi i(w+\alpha))^\beta\widehat{\phi}(w+\alpha)e^{2\pi i d_{l,\beta}\cdot(w+\alpha)} = \sum_{\alpha\in\mathbb{Z}^d}\widehat{\phi}(w+\alpha)\overline{\Omega}_l(M^\top[w+\alpha]), \text{ a.e.}\end{aligned}$$

As a consequence, the expression $\sum_{l=1}^s d_l(w+r_j)g_{l,1}(w) = \delta_{j-1}$ can be written as

$$\sum_{l=1}^s d_l(w+r_j)\sum_{\alpha\in\mathbb{Z}^d}\widehat{\phi}(w+\alpha)\overline{\Omega}_l(M^\top[w+\alpha]) = \delta_{j-1}, \quad \textit{a.e.}$$

Having in mind Lemma 5 and that the functions d_l are bounded, we obtain

$$\begin{aligned}\Big|\delta_{j-1} - \sum_{l=1}^s d_l(w+r_j)\widehat{\phi}(w)\overline{\Omega}_l(M^\top w)\Big| &= \Big|\sum_{l=1}^s d_l(w+r_j)\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}}\widehat{\phi}(w+\alpha)\overline{\Omega}_l(M^\top(w+\alpha))\Big| \\ &\le C\,|w|^k, \quad \text{a.e. on } [-1/2,1/2)^d,\end{aligned}$$

which proves that condition (C1) is satisfied. Besides, since $\widehat{\phi}\in C(\mathbb{R}^d)$ and $\widehat{\phi}(0)\neq 0$, we have that $\Big|\sum_{l=1}^s d_l(w+r_j)\overline{\Omega}_l(M^\top w)\Big| \le C\,|w|^k$, a.e. on $[-1/2,1/2)^d$, for $j=2,\ldots,s$. Then, taking into account that $\mathbb{Z}^d = M^\top\mathbb{Z}^d + \{i_1,i_2,\ldots,i_{\det M}\}$, Lemma 5 and the stability condition $\sup_{w\in\mathbb{R}^d}\sum_{\alpha\in\mathbb{Z}^d}|\widehat{\phi}(w+\alpha)|^2<\infty$ a.e., and denoting

$$\Lambda_{j,\alpha}(w) = \Big|\sum_{l=1}^s d_l(w+r_j)\widehat{\phi}(w+\alpha+r_j)\overline{\Omega}_l(M^\top w)\Big|^2,$$

we obtain that

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \Big|\sum_{l=1}^{s} \widehat{\sigma_M S_l}(M^\top w+\alpha)\overline{\Omega}_l(M^\top w)\Big|^2 \\
&= \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \Big|\sum_{l=1}^{s} d_l(w+M^{-\top}\alpha)\widehat{\phi}(w+M^{-\top}\alpha)\overline{\Omega}_l(M^\top w)\Big|^2 \\
&= \sum_{j=2}^{s}\sum_{\alpha\in\mathbb{Z}^d} \Lambda_{j,\alpha}(w) + \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \Lambda_{1,\alpha}(w) \\
&= \sum_{j=2}^{s}\Big|\sum_{l=1}^{s} d_l(w+r_j)\overline{\Omega}_l(M^\top w)\Big|^2 \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \Big|\widehat{\phi}(w+\alpha+r_j)\Big|^2 + \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \Lambda_{1,\alpha}(w) \\
&\le C\,|w|^{2k}, \quad \text{a.e. on } [-1/2,1/2)^d,
\end{aligned}$$

which proves condition (C2). The weaker conditions (C3) and (C4) in [25] are easily checked having in mind that we have assumed that $k > d/2+m$. □

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