Sampling in shift invariant spaces of functions with polynomial growth

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Abstract

Under suitable conditions, a function $f(t)$ in a principal shift invariant space $V_\varphi^2 := \{ \sum a_n \varphi(t-n) : a_n \in \ell^2 \}$ can be recovered from its uniform samples $\{f(n)\}_{n\in\mathbb{Z}}$ with a simple sampling formula. Provided that the generator φ has compact support, we consider the sampling problem in the bigger space $V_\varphi := \{ \sum a_n \varphi(t-n) : a_n \in \mathbb{C}^{\mathbb{Z}} \}$. In this space, there exist infinite functions with the same samples $y_n = f(n)$. We show that polynomial growth conditions give the uniqueness: If y_n has polynomial growth, there is a unique function of polynomial growth $f \in V_\varphi$, satisfying $f(n) = y_n$, $n \in \mathbb{Z}$. This function is given by the known sampling formula. The same result is proved also, when we consider average samples $y_n = f * h(n)$.

Keywords: Interpolation, shift invariant spaces, average sampling, cardinal splines.

1 Introduction

Let V_φ^2 be a shift-invariant space in $L^2(\mathbb{R})$ with a continuous stable generator $\varphi \in L^2(\mathbb{R})$, i.e.,

$$V_\varphi^2 := \Big\{ f(t) = \sum_{n\in\mathbb{Z}} a_n \, \varphi(t-n) \, : \, a_n \in \ell^2(\mathbb{Z}) \Big\} .$$

If the generator satisfies suitable conditions and verifies that

$$\sum_{n\in\mathbb{Z}} \varphi(n)\, z^{-n} \quad \text{has no zeros on the unit circle, } |z| = 1 \,, \qquad (1)$$

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then for any $f \in V_\varphi^2$, $f(t) = \sum_{n\in\mathbb{Z}} f(n)L(t-n)$, $t \in \mathbb{R}$, where $L(t)$ is the unique function in V_φ^2 satisfying $L(0) = 1$ and $L(n) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$. Moreover, $\{L(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of V_φ^2, i.e. it is the image of an orthonormal basis by means of a bounded invertible operator.

This sampling formula is very useful, since an appropriate choice for the generator φ eliminates most of the problems associated with the classical Shannon's sampling formula [13], and since wavelet subspaces are shift-invariant spaces.

From the interpolation point of view, we can read the mentioned sampling result as follows: *For any sequence of data $(y_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, there exists a unique function $f \in V_\varphi^2$ satisfying $f(n) = y_n$, which is given by $f(t) = \sum_n y_n L(t-n)$.*

The analogous result holds for samples $\{f(n)\}_{n\in\mathbb{Z}}$ in $\ell^p(\mathbb{Z})$, $1 \le p \le \infty$, when the generators are splines (see references [1, 4]).

When the generator φ has compact support, the bigger shift invariant space

$$V_\varphi := \Big\{ \sum_{n\in\mathbb{Z}} a_n \varphi(t-n) : a_n \in \mathbb{C}^{\mathbb{Z}} \Big\}, \tag{2}$$

where $\mathbb{C}^{\mathbb{Z}}$ denotes the set of complex sequences on $\mathbb{Z}$, is well defined pointwise, because for each $t \in \mathbb{R}$ the series in (2) is a finite sum. Notice that if, for instance, φ is a non-negative continuous function supported in $[0,1]$ such that $\|\varphi\|_2 > 0$, one can find a function $f \in V_\varphi$ such that $f \in V_\varphi \setminus L^2(\mathbb{R})$ by taking $a_n = 1$ for $n \in \mathbb{Z}$. However, the interpolation problem does not have a unique solution in V_φ in general. In fact, for any sequence of data $(y_n)_{n\in\mathbb{Z}}$ there are infinitely many functions in V_φ satisfying $f(n) = y_n$, $n \in \mathbb{Z}$ (see Proposition 1).

In this work, we generalize an interpolation result for cardinal splines due to Schoenberg, showing that polynomial growth conditions give the uniqueness. Specifically, assuming that condition (1) holds, and denoting

$$\begin{aligned} V_{\varphi,\gamma} &:= \big\{ f \in V_\varphi : f(t) = O(|t|^\gamma) \text{ as } t \to \pm\infty \big\}, \\ D_\gamma &:= \big\{ y_n : y_n = O(|n|^\gamma) \text{ as } n \to \pm\infty \big\}, \end{aligned} \tag{3}$$

for $\gamma \ge 0$, we prove that for any given sequence $(y_n)_{n\in\mathbb{Z}} \in D_\gamma$, there exists a unique function $f \in V_{\varphi,\gamma}$ satisfying $f(n) = y_n$ for all $n \in \mathbb{Z}$. This function is given by the sampling formula $f(t) = \sum_n y_n L(t-n)$.

In [10] readers can find a review of the magnificent Schoenberg's cardinal interpolation theory, including the proof of the mentioned result in the particular case where the generator φ is the cardinal central B-spline of degree d,

$$\beta_d := \underbrace{\beta_0 * \cdots * \beta_0}_{d+1 \text{ terms}},$$

where β_0 is the characteristic function of the interval $[-1/2, 1/2]$.

On the other hand, in many applications it is more realistic to assume that the available samples are local averages of the type

$$f * h(n) = \int_{-\infty}^{\infty} f(t)h(n-t)\,dt, \quad n \in \mathbb{Z},$$

where the average function $h(t)$ reflects a characteristic of the acquisition device. Similar sampling results hold for these average samples. See [6, 11, 12, 14, 17] for regular average sampling, and [5] for the case of average samples $\{f * h(n)\}$ in $\ell^p(\mathbb{Z})$. In this work, we study the case of average samples $\{f * h(n)\}$ of polynomial growth.

In order to give a more compact presentation, we study directly the case of average samples $\{f * h(n)\}$, but allowing h to be Dirac's delta, and thus including the case of classical samples.

Nowadays, sampling theory in shift-invariant spaces is a very active research topic because of its significant applications in signal processing (see [14, 13]) and since sampling formulas give efficient methods to compute the wavelet coefficients of a function from its point or average samples (see [3, Chapter 6] and [8]).

2 Hypotheses and notation

We assume that the generator φ is a compactly supported continuous function whose integer shifts are globally linearly independent, i.e. $\sum_{n\in\mathbb{Z}} a_n\varphi(t-n) = 0$ for $t \in \mathbb{R}$ implies $a_n = 0$ for $n \in \mathbb{Z}$. A necessary and sufficient condition for the global linear independence property to hold is (see [9]) that $\{\xi \in \mathbb{C} : \widetilde{\varphi}(\xi + n) = 0\,\forall n \in \mathbb{Z}\} = \varnothing$, where $\widehat{\varphi}$, defined as $\widehat{\varphi}(\xi) := \int_{-\infty}^{\infty} \varphi(t)e^{-2\pi it\xi}\,dt$, denotes the Fourier transform of the generator.

Let V_φ be the space defined in (2). Let $\gamma \geq 0$ and let $V_{\varphi,\gamma}$ and D_γ be the spaces defined in (3). We assume that the average function h is Dirac's delta or it is a compactly supported function in $L^1(\mathbb{R})$ such that $\{n \in \mathbb{Z} : h * \varphi(n) \neq 0\} \neq \varnothing$.

3 Interpolation from the shift invariant spaces V_φ

We consider the following interpolation problem: Given a complex sequence $\{y_n\}_{n\in\mathbb{Z}}$, find $f \in V_\varphi$ such that

$$h * f(n) = y_n\,, \quad \text{for all } n \in \mathbb{Z}. \tag{4}$$

Proposition 1 *Let A be the greatest integer such that $h * \varphi(n) = 0$ for all $n < A$, and let N be the smallest non-negative integer such that $h * \varphi(n) = 0$ for all $n > A + N$. Then, for a given sequence $(y_n)_{n\in\mathbb{Z}}$ the solutions of the problem (4) form a linear manifold in V_φ of dimension N.*

Notice that a consequence of this theorem is that problem (4) has infinitely many solutions for $N \geq 1$.

Proof. Obviously, problem (4) has a unique solution for $N = 0$. For $N > 0$, since the integer shifts of φ are globally linearly independent, the linear application

$$\begin{array}{ccl} \mathbb{C}^{\mathbb{Z}} & \longrightarrow & V_\varphi \\ (a_n)_{n\in\mathbb{Z}} & \longmapsto & \sum_{n\in\mathbb{Z}} a_n \varphi(t-n) \end{array}$$

is an isomorphism. In $\mathbb{C}^{\mathbb{Z}}$ the interpolation condition (4) reads as

$$\sum_{k=0}^{N} h * \varphi(A+k) a_{n-A-k} = y_n, \quad n \in \mathbb{Z}. \tag{5}$$

This is a linear difference equation of order N with constant coefficients. Hence the proposition follows. ■

4 Interpolation from the shift invariant spaces $V_{\varphi,\gamma}$

We need the following lemma in the proof of Theorem 1.

Lemma 1 *Let $\xi_1, \dots, \xi_K$ be K distinct real numbers in the interval $[0, 2\pi)$. Then, it is verified that:*

$$\lim_{n\to\infty} \sum_{k=1}^{K} c_k e^{i\xi_k n} = 0 \quad \textit{implies} \quad c_1 = c_2 = \cdots = c_K = 0.$$

Proof. When $K = 1$ it is obvious. Thus, we assume that $K > 1$. Denote $e_k := e^{i\xi_k}$ and $x_n := \sum_{k=1}^{K} c_k e_k^n$. Since $x_{n+m} = \sum_{k=1}^{K} c_k e_k^{n+m}$ we have that

$$\begin{bmatrix} x_n \\ x_{n+1} \\ \vdots \\ x_{n+K-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e_1 & e_2 & \dots & e_K \\ \vdots & \vdots & \ddots & \vdots \\ e_1^{K-1} & e_2^{K-1} & \dots & e_K^{K-1} \end{bmatrix} \begin{bmatrix} c_1 e_1^n \\ c_2 e_2^n \\ \vdots \\ c_K e_K^n \end{bmatrix}$$

If $\lim_{n\to\infty} x_n = 0$ we have that $\lim_{n\to\infty} x_{n+m} = 0$, for $m = 0, 1, \dots, K-1$, and since the Vandermonde matrix $[e_j^i]_{j=1,\dots,K}^{i=0,\dots,K-1}$ is invertible we deduce that $\lim_{n\to\infty} c_k e_k^n = 0$ for $k = 1, \dots, K$ and then $|c_k| = \lim_{n\to\infty} |c_k e_k^n| = 0$ for $k = 1, \dots, K$. ■

Theorem 1 *The following statements are equivalent.*

a) The function $G(z) := \sum_{n\in\mathbb{Z}} \varphi * h(n)\, z^{-n}$ has no zeros on the unit circle $|z| = 1$.

b) For any given sequence $(y_n)_{n\in\mathbb{Z}} \in D_\gamma$ *the problem* (4) *has a unique solution in the space* $V_{\varphi,\gamma}$.

Besides, this solution can be expressed as

$$f(t) = \sum_{n\in\mathbb{Z}} y_n L(t-n)\,, \quad t \in \mathbb{R}, \tag{6}$$

where the reconstruction function L *is given by*

$$L(t) := \sum_{n\in\mathbb{Z}} c_n \varphi(t-n), \quad t \in \mathbb{R}, \tag{7}$$

and c_n *are the coefficients of the expansion* $G(z)^{-1} = \sum_{n\in\mathbb{Z}} c_n z^{-n}$ *for* $|z| = 1$*. The series in* (6) *converges uniformly and absolutely in every finite interval.*

Proof. As $G(z)$ is indeed a Laurent polynomial because $\varphi * h$ is compactly supported, we can write $G(z) = z^N \widetilde{G}(z)$ where $\widetilde{G}$ is a polynomial and $N \leq 0$. Since $G(z) \neq 0$ for $|z| = 1$, then

$$\frac{1}{G(z)} = \frac{z^{-N}}{\widetilde{G}(z)}$$

is a holomorphic function in an annulus containing $\{z \in \mathbb{C} : |z| = 1\}$, and thus it admits a Laurent series development $G(z)^{-1} = \sum_{n\in\mathbb{Z}} c_n z^{-n}$.

Now assume that condition a) holds. Let A and N be the numbers defined in Proposition 1. Then roots of the polynomial

$$z^{A+N} G(z) = \sum_{n=0}^{N} h * \varphi(A+n) z^{N-n}$$

are different from 0 and they are not on the unit circle $|z| = 1$. Denote these roots by $z_1, \dots, z_K$, and their corresponding multiplicities by $\mu_1, \dots, \mu_K$. The sequences $\big(n^r z_k^n\big)_{n\in\mathbb{Z}}$ for $k = 1, \dots, K$ and $r = 0, \dots, \mu_k - 1$, form a basis of solutions for the homogeneous difference equation associated to (5). Hence, if we assume that $f, g \in V_{\varphi,\gamma}$ are two solutions of (4), then there exist coefficients $a_{k,r}$ such that

$$f(t) - g(t) = \sum_{k=1}^{K} \sum_{r=0}^{\mu_k - 1} a_{k,r} \sum_{n\in\mathbb{Z}} n^r z_k^n \varphi(t-n) = \sum_{k=1}^{K} \sum_{r=0}^{\mu_k - 1} a_{k,r} \Upsilon_{z_k,r}(t) \tag{8}$$

where

$$\Upsilon_{z,r}(t) := \sum_{n\in\mathbb{Z}} n^r z^n \varphi(t-n)\,.$$

Using that $\Upsilon_{z,r}(t+n) = \sum_{m\in\mathbb{Z}} m^r z^m \varphi(t+n-m) = \sum_{m\in\mathbb{Z}} (m+n)^r z^{m+n} \varphi(t-m)$ we obtain

$$\Upsilon_{z,r}(t+n) = z^n \sum_{s=0}^{r} \binom{r}{s} n^{r-s} \Upsilon_{z,s}(t), \quad n \in \mathbb{Z}. \tag{9}$$

Next we prove that the coefficients $a_{k,r}$ in (8) are null. This result is expected from the exponential growth shown in (9). However, the proof is technically complicated since the possibility of roots with the same modulus.

Let Λ the set of index k verifying that $|z_k| > 1$ and that there exist some coefficients $a_{k,r} \neq 0$. First we prove that Λ is the empty set. Assume on the contrary that $\Lambda \neq \varnothing$ and define

$$\rho := \max_{k\in\Lambda} |z_k|, \quad \mathcal{K} := \left\{ k \in \Lambda : |z_k| = \rho \right\}.$$

Let τ be the maximum index such that $a_{k,\tau} \neq 0$ for some k in $\mathcal{K}$. Using (9), that $\rho > 1$, and defining $a_{k,r} := 0$ for $r > \mu_k$, we obtain that for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{f(t+n) - g(t+n)}{n^\tau \rho^n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{K} \sum_{r=1}^{\mu_k} a_{k,r} \Upsilon_{z_k,r}(t+n)}{n^\tau \rho^n}$$
$$= \lim_{n\to\infty} \frac{\sum_{k\in\mathcal{K}} a_{k,\tau} \Upsilon_{z_k,\tau}(t+n)}{n^\tau \rho^n} = \lim_{n\to\infty} \frac{\sum_{k\in\mathcal{K}} a_{k,\tau} z_k^n \sum_{s=0}^{\tau} \binom{\tau}{s} n^{\tau-s} \Upsilon_{z_k,s}(t)}{n^\tau \rho^n},$$

which is equal to

$$\lim_{n\to\infty} \frac{\sum_{k\in\mathcal{K}} a_{k,\tau} z_k^n n^\tau \Upsilon_{z_k,0}(t)}{n^\tau \rho^n} = \lim_{n\to\infty} \sum_{k\in\mathcal{K}} a_{k,\tau} e^{in \arg z_k} \Upsilon_{z_k,0}(t).$$

taking into account that $\lim_{n\to\infty} \frac{1}{n^s} \Upsilon_{z_k,s}(t) = 0$ for $s > 0$. On the other hand, since $f(t) - g(t) = O(|t|^\gamma)$ and $\rho > 1$ we have

$$\lim_{n\to\infty} \frac{f(t+n) - g(t+n)}{n^\tau \rho^n} = 0, \quad t \in \mathbb{R}.$$

Therefore

$$\lim_{n\to\infty} \sum_{k\in\mathcal{K}} a_{k,\mu} e^{in \arg z_k} \Upsilon_{z_k,0}(t) = 0.$$

The real numbers $\arg z_k$, $k \in \mathcal{K}$ are distinct since the complex numbers z_k, $k \in \mathcal{K}$ so are and they have the same modulus. Then, from Lemma 1, we have that

$$a_{k,\tau} \Upsilon_{z_k,0}(t) = 0$$

for $k \in \mathcal{K}$ and $t \in \mathbb{R}$. Since the integer shifts φ are globally linearly independent, for every $k \in \mathcal{K}$, there exists $t_k \in \mathbb{R}$ such that $\Upsilon_{z_k,0}(t_k) \neq 0$. Therefore

$$a_{k,\tau} = 0, \quad \text{for all } k \in \mathcal{K}.$$

This contradiction leads us to infer that Λ is the empty set.

We deduce that the set Λ' of indexes k verifying that $|z_k| < 1$ and that there exist some coefficient $a_{k,r} \neq 0$ is the empty set in the same way, but using in the proof by contradiction the limit

$$\lim_{n\to-\infty} \frac{f(t+n)-g(t+n)}{n^{\tau'}(\rho')^n} = 0 \qquad (t \in \mathbb{R}),$$

where $\rho' := \min_{k\in\Lambda'} |z_k|$, and τ' the maximum index such that $a_{k,\tau'} \neq 0$ for some k in $\mathcal{K}' := \{k \in \Lambda' : |z_k| = \rho'\}$.

Since $\Lambda = \Lambda' = \varnothing$ and since we have assumed that there are no roots z_k such that $|z_k| = 1$, we deduce that all the coefficients $a_{k,r}$ in (8) are null. Hence, we have that $f(t) = g(t)$. This proves the uniqueness.

Now we prove the existence by showing that $f(t)$ defined by (6) belongs to $V_{\varphi,\gamma}$ and satisfies the interpolation condition (4). Since the Laurent polynomial $G(z)$ has no zeros on the unit circle $|z| = 1$, the function $G^{-1}(z)$ is analytic in an annulus $\nu_1 < |z| < 1/\nu_1$ for some $\nu_1 \in (0,1)$. Then, the coefficients of the expansion $G^{-1}(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n}$, $|z| = 1$, can be represented by

$$c_n = \frac{1}{2\pi i}\int_{|z|=1/\nu} \frac{G^{-1}(z)}{z^{n+1}}dz \quad \text{or by} \quad c_n = \frac{1}{2\pi i}\int_{|z|=\nu} \frac{G^{-1}(z)}{z^{n+1}}dz$$

where ν is any number in the interval $(\nu_1, 1)$. From these integral representations of c_n we deduce that there exists a constant C_1 such that

$$|c_n| \leq C_1\, \nu^{|n|}, \quad n \in \mathbb{Z}. \tag{10}$$

Let $N \in \mathbb{N}$ such that $\operatorname{supp}\varphi \subseteq (-N,N)$. Then, we have $|L(t)| \leq \sum_{n\in\mathbb{Z}} |c_n\varphi(t-n)| \leq C_1\|\varphi\|_\infty(\nu^{|[t]-N-1|}+\cdots+\nu^{|[t]+N+1|}) \leq C_1\|\varphi\|_\infty(2N+2)\nu^{|t|-N-2}$ where $[t]$ denotes the integer part of t. Hence, there exists a constant C_2 such that

$$|L(t)| \leq C_2\, \nu^{|t|}, \quad t \in \mathbb{R}.$$

On the other hand, for $|t| > 2$, we have

$$\frac{\sum_{n\in\mathbb{Z}} |n|^\gamma \nu^{|t-n|}}{(|t|+1)^\gamma} \leq \frac{\sum_{n\in\mathbb{Z}} |n|^\gamma \nu^{|[t]-n|-1}}{|[t]|^\gamma} = \sum_{n\in\mathbb{Z}} \frac{|[t]-n|^\gamma}{|[t]|^\gamma}\nu^{|n|-1}$$
$$\leq \sum_{n\in\mathbb{Z}} (1+|n|)^\gamma \nu^{|n|-1} < \infty.$$

Hence, there exist $C_3 > 0$ and $T > 0$ such that $\sum_{n\in\mathbb{Z}} |n|^\gamma\nu^{|t-n|} < C_3|t|^\gamma$, $|t| > T$. Let $C_4 > 0$ such that $|y_n| \leq C_4|n|^\gamma$, $n \in \mathbb{Z}$. Then

$$|f(t)| \leq \sum_{n\in\mathbb{Z}} |y_n L(t-n)| \leq C_4C_2 \sum_{n\in\mathbb{Z}} |n|^\gamma \nu^{|t-n|} \leq C_4C_2C_3|t|^\gamma, \quad |t| > T.$$

Thus,

$$f(t) = O(|t|^\gamma), \quad \text{as } t \to \pm\infty.$$

Moreover, since

$$|y_n L(t-n)| < C_4 C_2 |n|^\gamma \nu^{|t-n|} \le C_4 C_2 |n|^\gamma \nu^{|n|-|t|}, \quad t \in \mathbb{R}, n \in \mathbb{Z},$$

the series $\sum_{n\in\mathbb{Z}} y_n L(t-n)$ converges uniformly and absolutely in every finite interval. We have

$$f(t) = \sum_{n\in\mathbb{Z}} y_n L(t-n) = \sum_{n\in\mathbb{Z}} y_n \sum_{k\in\mathbb{Z}} c_k \varphi(t-n-k) = \sum_{k\in\mathbb{Z}} \Big[\sum_{n\in\mathbb{Z}} y_n c_{k-n}\Big] \varphi(t-k)\,.$$

Hence $f \in V_\varphi$, and therefore $f \in V_{\varphi,\gamma}$.

We denote $g_n := \varphi * h(n)$ and then $G(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n}$. Using that $G^{-1}(z)G(z) = 1$, we obtain that

$$\begin{aligned} h * L(n) &= \int_{-\infty}^{\infty} h(t) \sum_{k\in\mathbb{Z}} c_k \varphi(n-t-k) dt = \sum_{k\in\mathbb{Z}} c_k \int_{-\infty}^{\infty} h(t)\varphi(n-k-t)dt \\ &= \sum_{k\in\mathbb{Z}} c_k [\varphi * h](n-k) = \sum_{k\in\mathbb{Z}} c_k g_{n-k} = \begin{cases} 1, & n = 0 \\ 0, & n \ne 0 \end{cases} \end{aligned}$$

where we can permute the series and the integral because both h and φ are compactly supported, so the integral is over a finite interval I and the series reduces to a finite sum whose terms depend on n but not on $t \in I$.

Hence, $f(t) = \sum_{n\in\mathbb{Z}} y_n L(t-n)$ satisfies $h * f(n) = y_n$, $n \in \mathbb{Z}$. This proves the existence.

To prove that condition a) is also necessary for b) to hold, assume that $G(z)$ has a zero z_0 on the unit circle. Then the function $f(t) := \sum z_0^{-n} \varphi(t-n)$, which belongs to $V_{\varphi,\gamma}$, for all $\gamma \ge 0$, verifies

$$h * f(k) = \sum_{n\in\mathbb{Z}} z_0^{-n} \varphi * h(k-n) = \sum_{n\in\mathbb{Z}} z_0^{-n+k} \varphi * h(n) = z_0^k G(z_0) = 0, \quad k \in \mathbb{Z},$$

and then, b) does not hold. ■

From Theorem 1, the following average sampling formula follows: Provided that condition a) in Theorem 1 holds, for any $f \in V_{\varphi,\gamma}$,

$$f(t) = \sum_{n\in\mathbb{Z}} h * f(n) L(t-n), \quad t \in \mathbb{R}.$$

This sampling formula, as well as condition a) in Theorem 1, are well known in the L^2 setting [14, 13], i.e. for functions $f \in V_\varphi^2$.

Assuming that the hypotheses of Theorem 1 hold, and that

$$\widehat{\varphi}(\xi)\widehat{h}(\xi) = O\Big(\frac{1}{1+|\xi|^p}\Big) \tag{11}$$

for some $p > 1$, we can apply Poisson's Formula (see [2]), to obtain

$$\sum_{n\in\mathbb{Z}} \varphi * h(n) e^{-2\pi i n\xi} = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(\xi+n)\widehat{h}(\xi+n), \quad \xi \in [0,1],$$

and then, condition a) in the Theorem 1 can be written as

$$\sum_{n\in\mathbb{Z}} \widehat{\varphi}(\xi+n)\widehat{h}(\xi+n) \neq 0, \quad \text{for every } \xi \in [0,1]. \tag{12}$$

Notice that condition a) is not satisfied when the generator φ is even and the average function h is odd. Indeed, in this case, $\varphi * h$ is odd and so $G(1) = 0$. The same happens when φ is odd and h is even.

The result in Theorem 1 was established in [1] for the particular case of classical samples, $h = \delta$, and bounded functions, $\gamma = 0$.

For many generators and different average functions, it is easy to check if condition a) in Theorem 1 holds, and determine the coefficients c_n of the reconstruction functions in (7). Examples can be found in [6, 12, 14, 15, 16]. However it is difficult to determine if a whole family of generators satisfies condition a). For instance, in the case of classical samples, $h = \delta$, the orthogonal scaling functions of Daubechies appear to satisfy this condition, but no proof seems to be known [15, p.194]. In next section, we consider the spline case.

5 The spline case

In this section, we study the case where the generator is the centered B-spline $\varphi(t) = \beta_d(t)$, which is important in applications, since the corresponding shift invariant spaces V_{β_d} are the spaces of cardinal splines. Specifically, when d is odd, V_{β_d} is the set of splines of degree d whose knot sequence is $\mathbb{Z}$, i.e.

$$V_{\beta_d} = \mathcal{S}_d^{\mathbb{Z}} := \{f(t) \in \mathcal{C}^{d-1}(\mathbb{R}) : f|_{[k,k+1)} \in \Pi_d\},$$

where Π_d denotes the class of polynomials of degree not exceeding d. When d is even, the knot sequence is $\mathbb{Z} + \frac{1}{2}$, i.e.

$$V_{\beta_d} = \mathcal{S}_d^{\mathbb{Z}+1/2} := \left\{ f(t + \tfrac{1}{2}) : f \in \mathcal{S}_d^{\mathbb{Z}} \right\}.$$

Schoenberg [10] proved that condition a) in Theorem 1 holds in the case of classical samples, $h = \delta$, by using a recurrence relation satisfied by the Euler-Frobenius polynomials, which are related, in this case, with the function $G(z)$ defined in Theorem 1.

In [7] we proved that condition a) holds for any non-negative average function h supported in $[-\frac{1}{2}, \frac{1}{2}]$, but only for the linear, quadratic, cubic and quartic cases ($d = 1, 2, 3, 4$). Notice that $[-\frac{1}{2}, \frac{1}{2}]$ is the greatest support without overlapping between the sample intervals. Thus this result does not hold for an interval containing it strictly. In the next Theorem we state that, for any degree $d \geq d_0$, condition a) is satisfied for any h whose support is strictly contained in $[-\alpha_0, \alpha_0]$, where $\alpha_0 < \frac{1}{2}$ depends on d_0.

Theorem 2 *Consider $d_0 \in \mathbb{N}$, $d_0 \geq 2$. Assume that the average function $h \in L^1(\mathbb{R})$ is a non-negative function, which does not vanish almost everywhere, whose support is contained in $[-\alpha, \alpha]$, where*

$$0 < \alpha < \alpha_0 := \frac{1}{\pi} \arccos \left[\sqrt[3]{\frac{\zeta(d_0+1)}{2^{d_0+1}}} \right], \tag{13}$$

and ζ denotes Riemann's zeta function. Then, if $d \geq d_0$, for any given sequence $(y_n)_{n\in\mathbb{Z}} \in D_\gamma$ the problem (4) has a unique solution in the spline space $V_{\beta_d,\gamma}$.

Proof. Since $d_0 \geq 2$, we can suppose without loss of generality that $\alpha \geq \frac{1}{4}$. Consider $d \geq d_0$. It is sufficient to prove that condition a) in Theorem 1 holds when the generator $\varphi = \beta_d$. Since $\widehat{\beta}_d(\xi) = \operatorname{sinc}^{d+1}(\xi)$ and $\|\widehat{h}\|_\infty \leq \|h\|_1$, we have that (11) holds, so we show that (12) is satisfied.

We write h as the sum of an even function and an odd function: $h = h_0 + h_1$, where $h_0(t) := \frac{1}{2}\big[h(t) + h(-t)\big]$ and $h_1(t) := \frac{1}{2}\big[h(t) - h(-t)\big]$. Since h is non-negative, the even function h_0 is also non-negative. Notice that $\widehat{h} = \widehat{h}_0 + \widehat{h}_1$, where $\widehat{h}_0$ is a real even function and that $\widehat{h}_1$ is an imaginary odd function. Then, by using that $\widehat{\beta}_d$ is even, we have that

$$|\widehat{\beta}_d(\xi)\widehat{h}(\xi) + \widehat{\beta}_d(\xi-1)\widehat{h}(\xi-1)| \geq \widehat{\beta}_d(\xi)\widehat{h}_0(\xi) + \widehat{\beta}_d(1-\xi)\widehat{h}_0(1-\xi)\,. \tag{14}$$

Now we shall prove that, for $\xi \in [0,1]$,

$$g(\xi) := \widehat{\beta}_d(\xi)\widehat{h}_0(\xi) + \widehat{\beta}_d(1-\xi)\widehat{h}_0(1-\xi) > 2\|h\|_1 \Big(\frac{2}{\pi}\Big)^{d+1} \frac{\zeta(d_0+1)}{2^{d_0+1}}\,. \tag{15}$$

Since $g(\xi)$ is symmetric with respect to $\frac{1}{2}$, we can assume that $\xi \in [0, \frac{1}{2}]$. Moreover, from our assumption about α and (13) we have that $\frac{1}{4} \leq \alpha < \frac{1}{2}$. Thus, $0 \leq 2\pi\alpha\xi < \frac{\pi}{2}$ and $\frac{\pi}{2} \leq 2\pi\alpha < \pi$. We obtain the following estimation:

$$\begin{aligned}
&\widehat{\beta}_d(\xi)\widehat{h}_0(\xi) + \widehat{\beta}_d(1-\xi)\widehat{h}_0(1-\xi) \\
&\qquad = \widehat{\beta}_d(\xi) \int_{-\alpha}^{\alpha} h_0(t) e^{-2\pi i t\xi}\, dt + \widehat{\beta}_d(1-\xi) \int_{-\alpha}^{\alpha} h_0(t) e^{-2\pi i t(1-\xi)}\, dt \\
&\qquad = 2\widehat{\beta}_d(\xi) \int_0^{\alpha} h_0(t)\cos(2\pi t\xi)\, dt + 2\widehat{\beta}_d(1-\xi) \int_0^{\alpha} h_0(t)\cos[2\pi t(1-\xi)]\, dt \\
&\qquad \geq \|h_0\|_1 \Big[\widehat{\beta}_d(\xi)\cos(2\pi\alpha\xi) + \widehat{\beta}_d(1-\xi)\cos\big[2\pi\alpha(1-\xi)\big]\Big] \\
&\qquad \geq \|h_0\|_1 \widehat{\beta}_d(\xi)\cos(2\pi\alpha\xi)\Big[1 + \Big(\frac{\xi}{1-\xi}\Big)^{d+1}\cos(2\pi\alpha)\Big]\,.
\end{aligned}$$

Since $\|h_0\|_1 = \|h\|_1$ and $\widehat{\beta}_d$ and $1+\cos(2\pi\alpha)\Big(\frac{\xi}{1-\xi}\Big)^{d+1}$ are non-increasing functions in $[0,\frac{1}{2}]$, and having in mind inequality (13), we obtain that

$$\begin{aligned}\|h_0\|_1\widehat{\beta}_d(\xi)\cos(2\pi\alpha\xi)\Big[1+\Big(\frac{\xi}{1-\xi}\Big)^{d+1}\cos(2\pi\alpha)\Big]&\\ \geq \|h\|_1\Big(\frac{2}{\pi}\Big)^{d+1}\cos(\pi\alpha)\big[1+\cos(2\pi\alpha)\big]&\\ \geq 2\|h\|_1\Big(\frac{2}{\pi}\Big)^{d+1}\cos^3(\pi\alpha) > 2\|h\|_1\Big(\frac{2}{\pi}\Big)^{d+1}\frac{\zeta(d_0+1)}{2^{d_0+1}}&\,.\end{aligned}$$

Finally, by using (14) and (15), we obtain that, for $\xi \in [0,1]$,

$$\begin{aligned}&\Big|\sum_{n\in\mathbb{Z}}\widehat{\beta}_d(\xi+n)\widehat{h}(\xi+n)\Big|\\ &\geq \big|\widehat{\beta}_d(\xi)\widehat{h}(\xi)+\widehat{\beta}_d(\xi-1)\widehat{h}(\xi-1)\big| - \sum_{n\in\mathbb{Z}\setminus\{0,-1\}}\big|\widehat{\beta}_d(\xi+n)\widehat{h}(\xi+n)\big|\\ &\geq \widehat{\beta}_d(\xi)\widehat{h}_0(\xi)+\widehat{\beta}_d(1-\xi)\widehat{h}_0(1-\xi)\\ &\qquad -\|\widehat{h}\|_\infty\sum_{n=1}^{\infty}\Big(\big|\widehat{\beta}_d(\xi+n)\big|+\big|\widehat{\beta}_d(1-\xi+n)\big|\Big)\,,\end{aligned}$$

which is strictly greater than

$$\frac{2}{\pi^{d+1}}\|h\|_1\Big[2^{d-d_0}\zeta(d_0+1)-\sum_{n=1}^{\infty}\frac{1}{n^{d+1}}\Big] = \frac{2}{\pi^{d+1}}\|h\|_1\Big[2^{d-d_0}\zeta(d_0+1)-\zeta(d+1)\Big] \geq 0$$

since $\|\widehat{h}\|_\infty \leq \|h\|_1$. ■

Observe that α_0, defined in (13), converges to $\frac{1}{2}$ when d_0 tends to infinity. In particular, for $d \geq d_0 = 5$, we can take $\alpha = 0.419$ and for $d \geq d_0 = 14$, we can take $\alpha = 0.49$.

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