A note on uniform average sampling in frame-generated weighted shift-invariant spaces

Héctor R. Fernández-Morales* and **Antonio G. García†**

Departamento de Matemáticas, Universidad Carlos III de Madrid,
Avda. de la Universidad 30, 28911 Leganés-Madrid, Spain.

Abstract

Weighted shift-invariant spaces are suitable model spaces for sampling and reconstruction problems. They avoid most of the problems associated with classical Shannon's theory, and they also model the natural decay conditions of real signals. In this paper we consider a frame-generated weighted shift-invariant space $V_\nu^p(\Phi)$, where Φ denotes the finite set of generators. The aim is to obtain regular sampling formulas in $V_\nu^p(\Phi)$ where the available data are samples of some filtered (convolved) versions of the signals in $V_\nu^p(\Phi)$, taken at a suitable lattice of $\mathbb{Z}^d$. The reconstruction functions are obtained by using Wiener's Lemma in the associated weighted Wiener algebra $\mathcal{A}_\nu$.

Keywords: Amalgam Wiener spaces; Weighted shift-invariant spaces; Weighted Wiener algebras; Sampling formulas.
AMS: 42C15; 46E15; 94A20.

1 Introduction

The classical Whittaker-Shannon-Kotel'nikov sampling theorem states that any band-limited function f, say for instance to the d-dimensional cube $[-1/2, 1/2]^d$ or in other words, $f(t) = \int_{[-1/2,1/2]^d} \widehat{f}(x) e^{2\pi i x^\top t} dx$, $t \in \mathbb{R}^d$, may be reconstructed from its sequence of samples $\{f(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ at $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$ by means of the formula

$$f(t) = \sum_{\alpha\in\mathbb{Z}^d} f(\alpha)\,\mathrm{sinc}(t_1 - \alpha_1)\dots\mathrm{sinc}(t_d - \alpha_d)\,, \quad t = (t_1, \dots, t_d) \in \mathbb{R}^d\,,$$

where sinc denotes the cardinal sine function. Here we are used the notation $x^\top t := x_1\,t_1 + \cdots + x_d\,t_d$ identifying elements in $\mathbb{R}^d$ with column vectors.

Although Shannon's sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in [18, 19]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite duration signal;

*E-mail:hfernand@math.uc3m.es
†E-mail:agarcia@math.uc3m.es

the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has a very slow decay which makes computation in the time domain very inefficient. Besides, in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval. In addition, many applied problems impose different a priori constraints on the type of functions. For these reasons, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces. See, for instance, [1, 5, 6, 7, 8, 18, 19, 22] and the references therein.

Thus, in many practical applications, the signals are assumed to belong to some shift-invariant space of the form

$$V^2(\Phi) = \Big\{ \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\, \phi_j(t-\alpha) \,:\, \{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)\,, j=1,2,\dots,r \Big\} \subset L^2(\mathbb{R}^d)\,.$$

where $\Phi := \{\phi_j\}_{j=1}^r$ in $L^2(\mathbb{R}^d)$ is the set of generators of $V^2(\Phi)$. On the other hand, in many common situations the available data are samples of some filtered versions $f * \mathsf{h}_l$ of the signal f itself; this leads to average sampling in $V^2(\Phi)$ (see, for instance, [4, 9, 10, 11]). Suppose that s convolution systems (linear time-invariant systems or filters in engineering jargon) Υ_l, $l = 1, 2, \dots, s$, are defined on the shift-invariant subspace $V^2(\Phi)$ of $L^2(\mathbb{R}^d)$. The goal is to recover any function f in $V^2(\Phi)$ from the set of samples $\{(\Upsilon_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, l=1,2,\dots,s}$, taken at the lattice $M\mathbb{Z}^d$ in $\mathbb{R}^d$ (M denotes a matrix of integer entries with positive determinant), by means of a sampling formula like

$$f(t) = \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_l(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,. \tag{1}$$

Roughly speaking we will need $s \geq r(\det M)$ convolution systems Υ_l. Also, one can consider shift-invariant subspaces $V^p(\Phi)$ of $L^p(\mathbb{R}^d)$ spaces, with $1 \leq p \leq \infty$ (see, for instance, [4, 12, 16]).

Besides, to model decay or growth of real signals one can assume that they belong to a $L^p_\nu(\mathbb{R}^d)$ space with weight function ν. Recall that a function f belongs to $L^p_\nu(\mathbb{R}^d)$ if νf belongs to $L^p(\mathbb{R}^d)$. If the weight function ν grows rapidly as $|t| \to \infty$, then the functions in $L^p_\nu(\mathbb{R}^d)$ decay roughly at a corresponding rate. Conversely, if the weight function ν decays rapidly, then the functions in $L^p_\nu(\mathbb{R}^d)$ may grow as $|t| \to \infty$ (see, for instance, [2, ?, 20]). In this paper we deal with average regular sampling in a weighted shift-invariant space $V^p_\nu(\Phi)$ in $L^p_\nu(\mathbb{R}^d)$, formally defined as

$$V^p_\nu(\Phi) := \Big\{ \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\, \phi_j(t-\alpha) \,:\, \{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^p_\nu(\mathbb{Z}^d),\, j=1,2,\dots,r \Big\}\,.$$

That is, we derive sampling formulas like (1) valid in $V^p_\nu(\Phi)$. The set of generators $\Phi := \{\phi_j\}_{j=1}^r$ is contained in the Wiener amalgam space $W(L^1_\nu)$, i.e., the generators are functions locally in $L^\infty(\mathbb{R}^d)$ and globally in $L^1_\nu(\mathbb{R}^d)$. The sequence $\{\phi_j(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ is assumed to be a p-frame for $V^p_\nu(\Phi)$; thus $V^p_\nu(\Phi)$ is a closed subspace in $L^p_\nu(\mathbb{R}^d)$. See Section 2 below for the precise results.

In order to obtain our appropriate sampling functions S_l, $l = 1, 2, \dots, s$, we use Wiener's Lemma for the weighted Wiener algebra $\mathcal{A}_\nu$; thus, we need a submultiplicative weight ν satisfying also the so called GRS-condition (see [13]). Typical subexponential or Sobolev weights satisfy our requirements. Our main sampling result (see Theorem 6 in Section 3) will be first proved in $\operatorname{span}\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ (see Lemma 5 in Section 3) and then extended to $V_\nu^p(\Phi)$ by means of a density argument. Finally, the recovery of any function $f \in V_\nu^p(\Phi)$ from a sequence of its own samples is also treated in Section 3; in this case we assume in addition that the set of generators $\Phi = \{\phi_j\}_{j=1}^r$ has L_ν^p-stable shifts.

2 Weighted shift-invariant spaces $V_\nu^p(\Phi)$ $(1 \le p \le \infty)$

In this section we introduce the notations and previous results needed throughout the paper.

2.1 Preliminaries on the weighted spaces $V_\nu^p(\Phi)$ $(1 \le p \le \infty)$

Let ν be a weight function which in general means a non-negative function on $\mathbb{R}^d$. Given a sequence $c := \{c(\alpha)\}_{\alpha\in\mathbb{Z}^d}$, for $1 \le p < \infty$ the weighted $\ell_\nu^p(\mathbb{Z}^d)$ space is defined by the norm $\|c\|_{\ell_\nu^p} := \Big(\sum_{\alpha\in\mathbb{Z}^d} |c(\alpha)|^p \nu(\alpha)^p\Big)^{1/p}$, and for $p = \infty$, we have $\|c\|_{\ell_\nu^\infty} := \sup_{\alpha\in\mathbb{Z}^d} |c(\alpha)|\nu(\alpha)$. A function f belongs to $L_\nu^p(\mathbb{R}^d)$ if the function νf belongs to $L^p(\mathbb{R}^d)$. The norm is defined by $\|f\|_{L_\nu^p(\mathbb{R}^d)} := \|\nu f\|_{L^p(\mathbb{R}^d)}$. Equipped with these norms, the spaces $\ell_\nu^p(\mathbb{Z}^d)$ and $L_\nu^p(\mathbb{R}^d)$ are Banach spaces; when $\nu = 1$, we obtain the usual ℓ^p and L^p spaces.

Given a set of functions $\Phi := \{\phi_j\}_{j=1}^r$, the weighted multiply generated shift-invariant space $V_\nu^p(\Phi)$ is formally defined as

$$V_\nu^p(\Phi) := \Big\{ \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\, \phi_j(t-\alpha) \,:\, \{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell_\nu^p(\mathbb{Z}^d),\ j = 1, 2, \dots, r \Big\}.$$

In order to give a complete sense to these spaces as (closed) subspaces of $L_\nu^p(\mathbb{R}^d)$, the convergence properties of the series $\sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\, \phi_j(t-\alpha)$ should be studied. Thus, appropriate hypotheses on the set of generators Φ must be imposed (see Section 2.2 below).

Throughout the paper the weight function ν is always assumed to be continuous, symmetric, i.e., $\nu(x) = \nu(-x)$, positive and submultiplicative, i.e.,

$$0 < \nu(x+y) \le \nu(x)\nu(y), \quad \text{for all } x, y \in \mathbb{R}^d.$$

It is straightforward to deduce that $\nu(x) \ge 1$ for all $x \in \mathbb{R}^d$. Some typical examples of weight functions are the subexponential weight $\nu(x) = \mathrm{e}^{\alpha|x|^\beta}$ with $\alpha \ge 0$, $\beta \in [0, 1]$, and the Sobolev weight $\nu(x) = (1 + |x|)^\alpha$, with $\alpha \ge 0$.

For $1 \le p < \infty$ we consider the amalgam Wiener spaces

$$W(L_\nu^p) := \Big\{ f \text{ measurable} \,:\, \|f\|_{W(L_\nu^p)}^p := \sum_{\alpha\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d} \big\{ |f(x+\alpha)|^p \nu(\alpha)^p \big\} < \infty \Big\},$$

and for $p = \infty$

$$W(L^\infty_\nu) := \Big\{ f \text{ measurable} : \|f\|_{W(L^\infty_\nu)} := \sup_{\alpha\in\mathbb{Z}^d} \big\{ \operatorname*{ess\,sup}_{x\in[0,1]^d} \{|f(x+\alpha)|\nu(\alpha)\}\big\} < \infty \Big\},$$

Endowed with above norms, these spaces become Banach spaces. Furthermore, they are also translation-invariant spaces.

The subspace of continuous functions in $W(L^p_\nu)$, denoted as $W_0(L^p_\nu)$, is a closed subspace of $W(L^p_\nu)$ and thus also a Banach space. The inclusion $W_0(L^p_\nu) \subset W_0(L^q_\nu)$, where $1 \le p \le q \le \infty$, holds (see [2]). We also have the following inclusions $W(L^p_\nu) \subset W(L^q_\nu) \subset L^q_\nu$, where $1 \le p \le q \le \infty$ (see [17]).

Given a function ϕ and a sequence a, the semi-discrete convolution product is formally defined by

$$\phi *' a := \sum_{\alpha\in\mathbb{Z}^d} a(\alpha)\phi(\cdot - \alpha)\,.$$

As a submultiplicative weight ν is, in particular, ν-moderate (with constant $C = 1$) we have the following inequalities (see, for instance, [2, 17]):

Lemma 1. *(a) If $f \in L^p_\nu$, $g \in L^1_\nu$ and $1 \le p \le \infty$, then $\|f * g\|_{L^p_\nu} \le \|f\|_{L^p_\nu}\|g\|_{L^1_\nu}$.*

(b) If $f \in L^p_\nu$, $g \in W(L^1_\nu)$ and $1 \le p \le \infty$, then $\|f * g\|_{W(L^p_\nu)} \le C\|f\|_{L^p_\nu}\|g\|_{W(L^1_\nu)}$, for some positive constant C.

(c) If $a \in \ell^p_\nu$, $b \in \ell^1_\nu$ and $1 \le p \le \infty$, then $\|a * b\|_{\ell^p_\nu} \le \|a\|_{\ell^p_\nu}\|b\|_{\ell^1_\nu}$.

(d) If $f \in W(L^p_\nu)$, $c \in \ell^1_\nu$ and $1 \le p \le \infty$, then $\|f *' c\|_{W(L^p_\nu)} \le \|c\|_{\ell^1_\nu}\|f\|_{W(L^p_\nu)}$.

(e) If $f \in W(L^1_\nu)$, $c \in \ell^p_\nu$ and $1 \le p \le \infty$, then $\|f *' c\|_{W(L^p_\nu)} \le \|c\|_{\ell^p_\nu}\|f\|_{W(L^1_\nu)}$.

Lemma 2. *If $f \in L^p_\nu$ and $g \in W(L^1_\nu)$, then the sequence d defined by*

$$d := \Big\{ \int_{\mathbb{R}^d} f(x)\overline{g(x-\alpha)}dx \Big\}_{\alpha\in\mathbb{Z}^d}$$

belongs to ℓ^p_ν, and we have $\|d\|_{\ell^p_\nu} \le \|f\|_{L^p_\nu}\|g\|_{W(L^1_\nu)}$.

Notice that from Lemma 2, for $f \in L^p_\nu$ and $h \in W(L^1_\nu)$, it is easy to deduce the inequality $\|\{(f * h)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^p_\nu} \le C\|f\|_{L^p_\nu}\|h\|_{W(L^1_\nu)}$ for some positive constant C.

The following result has been taken from [2, Theorem 3.1]):

Lemma 3. *Assume that $\{\phi_j\}_{j=1}^r \subset W_0(L^1_\nu)$ and $1 \le p \le \infty$. Then the following inclusion holds:*

$$V^p_\nu(\Phi) \subset W_0(L^p_\nu).$$

2.2 Preliminaries on the generators of $V_\nu^p(\Phi)$ $(1 \le p \le \infty)$

Let $\Phi = \{\phi_j\}_{j=1}^r$ be the set of generators for $V_\nu^p(\Phi)$; in most of the papers in the mathematical literature it is assumed that the sequence $\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ is a p-Riesz basis for $V_\nu^p(\Phi)$ (see, for instance, Refs. [2, 12, 16]). Here we assume a more general condition: the sequence $\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ is a p-frame for $V_\nu^p(\Phi)$. Following [3] or [17], we introduce the concept of p-frame:

Definition 1. *A collection $\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ is said to be a p-frame for $V_\nu^p(\Phi)$ if there exists a positive constant C (depending on Φ, p and ν) such that*

$$C^{-1}\|f\|_{L^p_\nu} \le \sum_{j=1}^r \Big\| \Big\{ \int_{\mathbb{R}^d} f(x)\overline{\phi_j(x-\alpha)}dx \Big\}_{\alpha\in\mathbb{Z}^d} \Big\|_{\ell^p_\nu} \le C\|f\|_{L^p_\nu}, \qquad f \in V_\nu^p(\Phi). \tag{2}$$

From Theorem 3.11 in [17] (see also [20] and [3] for the non-weighted case) we know that under the inclusion $\Phi = \{\phi_j\}_{j=1}^r \subset W(L^1_\nu)$ the closedness of $V_\nu^p(\Phi)$ in L^p_ν and the p-frame condition on $\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ are equivalent properties. From now on we assume on the generators the inclusion on $W(L^1_\nu)$ as well as the p-frame condition.

Some important comments are in order:

1. If the sequence $\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ is a p_0-frame for $V_\nu^{p_0}(\Phi)$, then it is a p-frame for $V_\nu^p(\Phi)$ for any $1 \le p \le \infty$. This fact is proved in Corollary 3.13 in [17], and in Corollary 1 in [3] for the non-weighted case.
2. Theorem 2.4 in [2] assures us that if $\Phi \subset W(L^1_\nu)$ then the space $V_\nu^p(\Phi)$ is a subspace (not necessarily closed) of L^p_ν and $W(L^p_\nu)$ for any $1 \le p \le \infty$. Hence we have $V_\nu^p(\Phi) \subset \overline{\operatorname{span}}_{L^p_\nu}\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$. On the other hand, if one of the equivalent statements in [17, Theorem 3.11] is satisfied we have the other inclusion; thus
$$V_\nu^p(\Phi) = \overline{\operatorname{span}}_{L^p_\nu}\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}\,.$$
3. Finally it is worth to mention that for $f \in V_\nu^p(\Phi)$ we do not have uniqueness for the coefficients $\{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^p_\nu(\mathbb{Z}^d)$ in the expansion
$$f = \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\, \phi_j(\cdot - \alpha) \quad \text{in } L^p_\nu(\mathbb{R}^d)\,.$$

3 Uniform average sampling in $V_\nu^p(\Phi)$ $(1 \le p \le \infty)$

Concerning uniform sampling in $V_\nu^p(\Phi)$ $(1 \le p \le \infty)$ we should precise the considered convolution systems and the uniform lattices of $\mathbb{Z}^d$ where they will be sampled:

3.1 The convolution systems Υ_l $(1 \le l \le s)$

Here we consider s convolution systems Υ_l, $1 \le l \le s$, of the following type: the impulse response h_l of the system Υ_l belongs to $W(L^1_\nu)$, i.e.,

$$\big(\Upsilon_l f\big)(t) := [f * \mathsf{h}_l](t) = \int_{\mathbb{R}^d} f(x)\mathsf{h}_l(t-x)dx\,, \quad t \in \mathbb{R}^d\,.$$

Whenever $f \in V^p_\nu(\Phi)$, according to Lemma 1(a), the above convolution $f * \mathsf{h}_l$ is well-defined as a function in L^p_ν. Besides, provided that $\phi_j \in W(L^1_\nu)$, $j = 1, 2, \ldots, r$, the sequence $\{\Upsilon_l \phi_j(\alpha)\}_{\alpha \in \mathbb{Z}^d}$ belongs to $\ell^1_\nu(\mathbb{Z}^d)$; this is a consequence of the inclusion $W(L^1_\nu) \subset L^1_\nu$ and Lemma 2.

For the submultiplicative weight ν, let $\mathcal{A}_\nu$ be the weighted Wiener algebra of the functions $f(x) = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)\, \mathrm{e}^{2\pi i \alpha^\top x}$ with $a := \{a(\alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^1_\nu(\mathbb{Z}^d)$; here we are using the notation $\alpha^\top x := \sum_{k=1}^d \alpha_k x_k$ for $\alpha \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$. This space $\mathcal{A}_\nu$, normed by $\|f\|_{\mathcal{A}_\nu} := \|a\|_{\ell^1_\nu}$ and with pointwise multiplication becomes a commutative Banach algebra.

If in addition the weight ν satisfies the so called GRS-condition (Gelfand-Raikov-Shilov), i.e., for each $\alpha \in \mathbb{Z}^d$, $\lim_{n\to\infty} \nu(n\alpha)^{1/n} = 1$, then the Wiener's Lemma holds: If $f \in \mathcal{A}_\nu$ and $f(x) \neq 0$ for every $x \in \mathbb{R}^d$, the function $1/f$ is also in $\mathcal{A}_\nu$ (see, for instance, [13]).

Thus, for $l = 1, 2, \ldots, s$ and $j = 1, 2, \ldots r$, the Fourier transform of the sequence $\{\Upsilon_l \phi_j(\alpha)\}_{\alpha \in \mathbb{Z}^d}$ belongs to the Wiener algebra $\mathcal{A}_\nu$, and it will play an important role in the sequel. We denote it by

$$g_{l,j}(x) := \sum_{\alpha \in \mathbb{Z}^d} \big(\Upsilon_l \phi_j\big)(\alpha)\, \mathrm{e}^{-2\pi i \alpha^\top x}, \quad x \in \mathbb{R}^d,$$

and

$$\mathbf{g}_l^\top(x) := \big(g_{l,1}(x), g_{l,2}(x), \ldots, g_{l,r}(x)\big), \quad 1 \le l \le s. \tag{3}$$

3.2 Lattices in $\mathbb{Z}^d$

Given a nonsingular matrix M with integer entries, we consider the lattice in $\mathbb{Z}^d$ generated by M, i.e.,

$$M\mathbb{Z}^d := \{M\alpha : \alpha \in \mathbb{Z}^d\} \subset \mathbb{Z}^d.$$

Without loss of generality we can assume that $\det M > 0$; otherwise we can consider $M' = ME$ where E is some $d \times d$ integer matrix satisfying $\det E = -1$; trivially, $M\mathbb{Z}^d = M'\mathbb{Z}^d$. We denote by $M^\top$ and $M^{-\top}$ the transpose matrices of M and M^{-1} respectively. The following useful generalized orthogonal relationship holds

$$\sum_{k \in \mathcal{N}(M^\top)} \mathrm{e}^{-2\pi i \alpha^\top M^{-T} k} = \begin{cases} \det M, & \alpha \in M\mathbb{Z}^d \\ 0 & \alpha \in \mathbb{Z}^d \setminus M\mathbb{Z}^d \end{cases} \tag{4}$$

where

$$\mathcal{N}(M^\top) := \mathbb{Z}^d \cap \{M^\top x : x \in [0,1)^d\}$$

The set $\mathcal{N}(M^\top)$ has $\det M$ elements (see [21]). One of these elements is zero, say $i_1 = 0$; we denote the rest of elements by $i_2, \ldots, i_{\det M}$ ordered in any form.

Notice that the sets, defined as $Q_k := M^{-\top} i_k + M^{-\top}[0,1)^d$, $k = 1, 2, \ldots, \det M$, satisfy (see [21, p. 110])

$$Q_k \cap Q_{k'} = \emptyset \ \text{ if } \ k \neq k' \quad \text{and} \quad \mathrm{Vol}\Big(\bigcup_{k=1}^{\det M} Q_k\Big) = 1.$$

Thus, for any function F integrable in $[0,1)^d$ and $\mathbb{Z}^d$-periodic we have $\int_{[0,1)^d} F(x)dx = \sum_{k=1}^{\det M} \int_{Q_k} F(x)dx$.

In order to recover any function $f \in V_\nu^p(\Phi)$ from its generalized samples at a lattice $M\mathbb{Z}^d$, i.e., from the sequence of samples $\big\{(\Upsilon_l f)(M\alpha)\big\}_{\alpha\in\mathbb{Z}^d;\, l=1,2,\ldots,s}$, a suitable expression for the samples will be useful. The sequence $\big\{(\Upsilon_l f)(\alpha)\big\}_{\alpha\in\mathbb{Z}^d;\, l=1,2,\ldots,s}$ belongs to ℓ_ν^p (see Lemma 2); in order that the sequence of samples $\big\{(\Upsilon_l f)(M\alpha)\big\}_{\alpha\in\mathbb{Z}^d;\, l=1,2,\ldots,s}$ belongs also to ℓ_ν^p, we will need to assume the following compatibility condition:

Definition 2. *Given a submultiplicative weight ν and a lattice $M\mathbb{Z}^d$, we say that ν is M-compatible if the ratio $\nu(\alpha)/\nu(M\alpha)$ remains bounded as $|\alpha|$ goes to infinity.*

The compatibility condition in Definition 2 is not always true; for a subexponential weight there exists a nonsingular matrix M with integer entries for which the condition fails. For instance, consider the matrix

$$M = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \quad \text{and } (\beta, -2\beta)^\top \in \mathbb{Z}^2 \text{ with } \beta \in \mathbb{Z}\text{; we have } M(\beta,-2\beta)^\top = (\beta, 0)^\top .$$

For the weight $\nu(x) = \mathrm{e}^{|x|}$, the ratio $\nu\big((\beta,-2\beta)\big)/\nu\big((\beta,0)\big) = \mathrm{e}^{(\sqrt{5}-1)|\beta|}$ remains unbounded as $|\beta| \to \infty$.

However, one can prove that any Sobolev weight is compatible with respect to any lattice $M\mathbb{Z}^d$. Also, subexponential weights are compatible with respect to any diagonal lattice. From now on, the submultiplicative weight ν will be considered M-compatible.

3.3 An expression for the samples

Recall that ν is a submultiplicative weight so that $\mathcal{A}_\nu$ is a Banach algebra. Consider the map

$$\begin{array}{rccl} \mathcal{T}_\Phi : & \mathcal{A}_\nu \times \ldots \times \mathcal{A}_\nu & \longrightarrow & L_\nu^p(\mathbb{R}^d) \\ & \mathbf{F}^\top := (f_1, \ldots, f_r) & \longmapsto & \sum_{j=1}^r \phi_j *' a_j\,, \end{array} \tag{5}$$

where $f_j(x) = \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\, \mathrm{e}^{-2\pi i \alpha^\top x} \in \mathcal{A}_\nu$, $j = 1, 2, \ldots, r$. It is easy to deduce the existence of a positive constant C such that $\|f\|_{L_\nu^p} \le C \|f\|_{W(L_\nu^p)}$. Thus

$$\Big\| \sum_{j=1}^r \phi_j *' a_j \Big\|_{L_\nu^p} \le C \sum_{j=1}^r \big\|\phi_j *' a_j\big\|_{W(L_\nu^p)} \le C \max_{j=1,2,\ldots,r} \big\{\|\phi_j\|_{W(L_\nu^p)}\big\} \sum_{j=1}^r \big\|a_j\big\|_{\ell_\nu^1}\,,$$

where we have used Lemma 1(d). Now, with the inclusion $W(L_\nu^1) \subset W(L_\nu^p)$ we get that $\mathcal{T}_\Phi$ is a well-defined bounded operator by considering in $\mathcal{A}_\nu \times \ldots \times \mathcal{A}_\nu$ the norm $\|\mathbf{F}\| := \sum_{j=1}^r \|a_j\|_{\ell_\nu^1}$.

For each $f \in \operatorname{span}\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,r}$ let $\mathbf{a} := \{(a_1(\alpha), \ldots, a_r(\alpha))\}$ be the finite sequence such that $f = \sum_{j=1}^r \phi_j *' a_j$ and the corresponding trigonometric polynomial $\mathbf{F}^\top(x) := \Big(\sum_\alpha a_1(\alpha)\, \mathrm{e}^{-2\pi i \alpha^\top x}, \ldots, \sum_\alpha a_r(\alpha)\, \mathrm{e}^{-2\pi i \alpha^\top x} \Big) = \sum_\alpha \mathbf{a}(\alpha)\, \mathrm{e}^{-2\pi i \alpha^\top x}$, so that

$\mathcal{T}_\Phi \mathbf{F} = f$. For any $l = 1, 2, \dots, s$ and $\alpha \in \mathbb{Z}^d$, we have

$$\begin{aligned}(\Upsilon_l f)(M\alpha) &= \sum_{\beta\in\mathbb{Z}^d}\sum_{j=1}^{r} a_j(\beta)(\Upsilon_l \phi_j)(M\alpha-\beta) = \langle \mathbf{F}, \overline{\mathbf{g}}_l\, \mathrm{e}^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2[0,1)^d} \\ &= \int_{[0,1)^d} \mathbf{F}^\top(x)\mathbf{g}_l(x)\, \mathrm{e}^{2\pi i \alpha^\top M^\top x} dx\,.\end{aligned}$$

As the sequence $\{\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d}$ is an orthogonal basis for $L^2(M^{-\top}[0,1)^d)$, we can exploit this fact in computing the above integral as follows

$$\begin{aligned}(\Upsilon_l f)(M\alpha) &= \sum_{k=1}^{\det M}\int_{Q_k} \mathbf{F}^\top(x)\mathbf{g}_l(x)\, \mathrm{e}^{2\pi i\alpha^\top M^\top x} dx \\ &= \int_{M^{-\top}[0,1)^d} \sum_{k=1}^{\det M} \mathbf{F}^\top(x+M^{-\top}i_k)\mathbf{g}_l(x+M^{-\top}i_k)\ \mathrm{e}^{2\pi i\alpha^\top M^\top x} dx\,.\end{aligned} \tag{6}$$

Having in mind expression (6), the sequence of samples $\{(\Upsilon_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d}$ forms the Fourier coefficients of the continuous function $\sum_{k=1}^{\det M} \mathbf{F}^\top(x+M^{-\top}i_k)\mathbf{g}_l(x+M^{-\top}i_k)$ with respect to the orthogonal basis $\{\,\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d}$ for $L^2(M^{-\top}[0,1)^d)$.

Since $\{\Upsilon_l\phi_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^1_\nu(\mathbb{Z}^d)$ we have that $\{\Upsilon_l f(M\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^1_\nu(\mathbb{Z}^d)$; remind that $(\Upsilon_l f)(M\alpha)$ is a finite sum $\sum_\beta \sum_{j=1}^r a_j(\beta)(\Upsilon_l\phi_j)(M\alpha-\beta)$ and ν is M-compatible.

Therefore, for $l = 1, 2\dots, s$, we have

$$\sum_{k=1}^{\det M} \mathbf{F}^\top(x+M^{-\top}i_k)\mathbf{g}_l(x+M^{-\top}i_k) = (\det M)\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)\, \mathrm{e}^{-2\pi i\alpha^\top M^\top x}\,,$$

$x \in M^{-\top}[0,1)^d$. By periodicity, the above equality also holds for all $x \in [0,1)^d$. In matrix form, we have

$$\begin{aligned}&\mathbb{G}(x)\mathbb{F}(x) = \\ &(\det M)\Big(\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_1 f)(M\alpha)\, \mathrm{e}^{-2\pi i\alpha^\top M^\top x}, \dots, \sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_s f)(M\alpha)\, \mathrm{e}^{-2\pi i\alpha^\top M^\top x}\Big)^\top\end{aligned} \tag{7}$$

where $\mathbb{G}(x)$ is the $s\times(\det M)r$ matrix of functions, $x \in [0,1)^d$, which, involving the functions in (3), is given by

$$\mathbb{G}(x) := \begin{bmatrix} \mathbf{g}_1^\top(x) & \mathbf{g}_1^\top(x+M^{-\top}i_2) & \cdots & \mathbf{g}_1^\top(x+M^{-\top}i_{\det M}) \\ \mathbf{g}_2^\top(x) & \mathbf{g}_2^\top(x+M^{-\top}i_2) & \cdots & \mathbf{g}_2^\top(x+M^{-\top}i_{\det M}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_s^\top(x) & \mathbf{g}_s^\top(x+M^{-\top}i_2) & \cdots & \mathbf{g}_s^\top(x+M^{-\top}i_{\det M}) \end{bmatrix} \tag{8}$$

$$= \Big[\mathbf{g}_l^\top\big(x+M^{-\top}i_k\big)\Big]_{\substack{l=1,2,\dots,s \\ k=1,2,\dots,\det M}}$$

and

$$\mathbb{F}(x) := \left(\mathbf{F}^\top(x), \mathbf{F}^\top(x + M^{-\top} i_2), \cdots, \mathbf{F}^\top(x + M^{-\top} i_{\det M})\right)^\top .$$

Having in mind expression (7), to obtain $\mathbf{F}(x)$ in terms of the generalized samples we should have left-inverse matrices of $\mathbb{G}(x)$ having entries in the weighted Wiener algebra $\mathcal{A}_\nu$:

Lemma 4. *Let $\mathbb{G}(x)$ be the matrix defined in (8). Then, there exists an $r \times s$ matrix $\mathbf{d}(x) := \big(\mathbf{d}_1(x), \mathbf{d}_2(x), \dots, \mathbf{d}_s(x)\big)$ with entries $d_{j,l} \in \mathcal{A}_\nu$, $j = 1, 2, \dots, r$, $l = 1, 2, \dots, s$ and satisfying*

$$\mathbf{d}(x)\mathbb{G}(x) = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix} = \left[\mathbb{I}_r, \mathbb{O}_{r\times(\det M-1)r}\right], \quad x \in [0,1)^d, \tag{9}$$

if and only if rank $\mathbb{G}(x) = (\det M)r$ for all $x \in \mathbb{R}^d$.

Proof. The proof mimics the one for Lemma 1 in [12]. In this case, according to Wiener's Lemma for the weighted Wiener algebra $\mathcal{A}_\nu$ (see [13]), the entries of the Moore-Penrose pseudo-inverse $\mathbb{G}^\dagger$ also belong to Wiener algebra $\mathcal{A}_\nu$.

□

Provided that condition (9) in Lemma 4 is satisfied, it can be easily checked that all matrices $\mathbf{d}(x)$ with entries in $\mathcal{A}_\nu$, and satisfying (9) correspond to the first r rows of the matrices of the form

$$\mathbb{D}(x) = \mathbb{G}^\dagger(x) + \mathbb{U}(x)\left[\mathbb{I}_s - \mathbb{G}(x)\mathbb{G}^\dagger(x)\right], \tag{10}$$

where $\mathbb{U}(x)$ is any $(\det M)r \times s$ matrix with entries in $\mathcal{A}_\nu$ and $\mathbb{G}^\dagger(x)$ denotes the Moore-Penrose pseudo-inverse of $\mathbb{G}(x)$.

Note that we are assuming that rank $\mathbb{G}(x) = (\det M)r$ for all $x \in \mathbb{R}^d$ and, consequently $s \geq r(\det M)$. If $s = (\det M)r$ there exists a unique matrix $\mathbf{d}(x)$, given by the first r rows of $\mathbb{G}^{-1}(x)$; if $s > (\det M)r$ there are many solutions according to (10).

As it was pointed out in Section 1, in proving our sampling result for $V_\nu^p(\Phi)$, $1 \leq p \leq \infty$, we first prove it for the linear span of $\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$. In so doing, assume that the set of generators $\Phi = \{\phi_j\}_{j=1}^r$ satisfy, for $j = 1, 2, \cdots, r$, that $\phi_j \in W_0(L_\nu^1)$; this condition ensures that functions in $V_\nu^p(\Phi)$ are continuous (see Lemma 3). Consider also s convolution systems Υ_l, $l = 1, 2, \cdots, s$, with $\mathsf{h}_l \in W(L_\nu^1)$. Under these circumstances we have:

Lemma 5. *Let $\mathbf{d}(x) = \big(d_1(x), d_2(x), \dots, d_s(x)\big)$ be an $r \times s$ matrix with entries $d_{j,l} \in \mathcal{A}_\nu$, $j = 1, 2, \dots, r$, $l = 1, 2, \dots, s$, and satisfying condition (9). Then, for any $f \in \operatorname{span}\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ the following sampling expansion holds:*

$$f = \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\Upsilon_l f)(M\alpha) S_{l,\mathbf{d}}(\cdot - M\alpha) \quad \textit{in } L_\nu^p(\mathbb{R}^d), \tag{11}$$

where the reconstruction function $S_{l,\mathbf{d}}$ *is given by*

$$S_{l,\mathbf{d}}(t) = (\det M) \sum_{\alpha\in\mathbb{Z}^d} \sum_{j=1}^{r} \widehat{d}_{j,l}(\alpha)\phi_j(t-\alpha)\,, \quad t\in\mathbb{R}^d\,, \tag{12}$$

with $\widehat{d}_{j,l}(\alpha) := \int_{[0,1)^d} d_{j,l}(x)\, \mathrm{e}^{2\pi i\alpha^\top x}\, dx$, $\alpha\in\mathbb{Z}^d$, *the Fourier coefficients of the functions* $d_{l,j}\in\mathcal{A}_\nu$, $j=1,2,\ldots,r$ *and* $l=1,2,\ldots,s$.

Proof. The proof is identical to those for Lemma 2 in [12]. A left multiplication by the matrix $\mathbf{d}(x)$ in (7) gives

$$\mathbf{F}(x) = (\det M)\sum_{l=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)\mathbf{d}_l(x)\,\mathrm{e}^{-2\pi i\alpha^\top M^\top x}, \quad x\in[0,1)^d\,, \tag{13}$$

with convergence in the norm of $\mathcal{A}_\nu\times\ldots\times\mathcal{A}_\nu$. Finally, applying $\mathcal{T}_\Phi$ to both sides of the (13) and using the shifting property

$$\big[\mathcal{T}_\Phi \mathbf{d}_l(\cdot)\,\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\big](t) = \big[\mathcal{T}_\Phi\mathbf{d}_l\big](t-M\alpha)\,, \quad \alpha\in\mathbb{Z}^d\,,$$

we deduce (11) with convergence in $L^p_\nu(\mathbb{R}^d)$.

□

3.4 The average sampling result in $V^p_\nu(\Phi)$ $(1\le p\le\infty)$

Assume that $\Phi\subset W_0(L^1_\nu)$ and that we have s systems Υ_l with $\mathsf{h}_l\in W(L^1_\nu)$ such that there exists an $r\times s$ matrix $\mathbf{d}(x) = \big(d_1(x), d_2(x),\ldots,d_s(x)\big)$ with entries $d_{j,l}\in\mathcal{A}_\nu$, $j=1,2,\ldots,r$, $l=1,2,\ldots,s$, and satisfying condition (9). Thus, a density argument allows us to prove that sampling formula (11) in Lemma 5 is also valid for the whole space $V^p_\nu(\Phi)$. In fact, the following theorem holds:

Theorem 6. *Under the above assumptions, for any* $f\in V^p_\nu(\Phi)$, $1\le p\le\infty$, *the sampling formula*

$$f = \sum_{l=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)S_{l,\mathbf{d}}(\cdot - M\alpha)\,, \tag{14}$$

holds in the L^p_ν*-sense. The series in* (14) *also converges absolutely and uniformly to* f *on* $\mathbb{R}^d$.

Proof. We define on $V^p_\nu(\Phi)$ the sampling operator

$$\begin{array}{rccl}\Gamma_{\mathbf{d}}: & V^p_\nu(\Phi) & \longrightarrow & V^p_\nu(\Phi)\\ & f & \longmapsto & \Gamma_{\mathbf{d}} f := \sum_{l=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)S_{l,\mathbf{d}}(\cdot - M\alpha)\,.\end{array}$$

It is a well-defined and bounded operator; indeed, having in mind (12) we have

$$\begin{aligned}(\Gamma_{\mathbf{d}} f)(t) &= (\det M)\sum_{l=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)\sum_{\beta\in\mathbb{Z}^d}\sum_{j=1}^{r}\widehat{d}_{j,l}(\beta)\phi_j(t-M\alpha-\beta)\\ &= (\det M)\sum_{j=1}^{r}\sum_{\delta\in\mathbb{Z}^d}\Big(\sum_{l=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)\widehat{d}_{j,l}(\delta-M\alpha)\Big)\phi_j(t-\delta)\\ &= (\det M)\sum_{j=1}^{r}\sum_{l=1}^{s}\Big(a_{jl} *' \phi_j\Big)(t)\,,\end{aligned}$$

where $a_{jl}(\delta) := \sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)\widehat{d}_{j,l}(\delta - M\alpha)$. Notice that,

$$
\begin{aligned}
|a_{jl}(\delta)|\nu(\delta) &= \Big|\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)\,\widehat{d}_{j,l}(\delta - M\alpha)\Big|\nu(\delta) \le \sum_{\alpha\in\mathbb{Z}^d}\Big|(\Upsilon_l f)(M\alpha)\,\widehat{d}_{j,l}(\delta - M\alpha)\Big|\nu(\delta)\\
&\le \sum_{\alpha\in\mathbb{Z}^d}\Big|(\Upsilon_l f)(\alpha)\,\widehat{d}_{j,l}(\delta - \alpha)\Big|\nu(\delta) \le \sum_{\alpha\in\mathbb{Z}^d}\Big|(\Upsilon_l f)(\alpha)\,\widehat{d}_{j,l}(\delta - \alpha)\Big|\nu(\alpha)\nu(\delta-\alpha)\\
&= \Big(\{|(\Upsilon_l f)(\alpha)|\nu(\alpha)\} * \{|\widehat{d}_{j,l}(\alpha)|\nu(\alpha)\}\Big)(\delta)\,.
\end{aligned}
$$

Thus, Lemma 1(c) gives

$$
\begin{aligned}
\|a_{jl}\|_{\ell^p_\nu} &\le \Big\|\{(\Upsilon_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\Big\|_{\ell^p_\nu}\Big\|\{\widehat{d}_{j,l}(\alpha)\}_{\alpha\in\mathbb{Z}^d}\Big\|_{\ell^1_\nu}\\
&\le \|f\|_{L^p_\nu}\|h_l\|_{W(L^1_\nu)}\Big\|\{\widehat{d}_{j,l}(\alpha)\}_{\alpha\in\mathbb{Z}^d}\Big\|_{\ell^1_\nu}\,.
\end{aligned}
\tag{15}
$$

In the last step we have used Lemma 2. Now, taking into account Lemma 1(e), and the fact that the continuous inclusion $W(L^p_\nu)\subset L^p_\nu$ provides a positive constant C such that $\|f\|_{L^p_\nu}\le C\,\|f\|_{W(L^p_\nu)}$, we obtain

$$
\begin{aligned}
\|\Gamma_{\mathbf{d}} f\|_{L^p_\nu} &\le (\det M)\sum_{j=1}^{r}\sum_{l=1}^{s}\Big\|a_{jl} *' \phi_j\Big\|_{L^p_\nu}\\
&\le C\,(\det M)\sum_{j=1}^{r}\sum_{l=1}^{s}\Big\|a_{jl} *' \phi_j\Big\|_{W(L^p_\nu)}\\
&\le C\,(\det M)\sum_{j=1}^{r}\sum_{l=1}^{s}\|a_{jl}\|_{\ell^p_\nu}\|\phi_j\|_{W(L^1_\nu)}\,.
\end{aligned}
\tag{16}
$$

Combining (16) and (15) we deduce the boundedness of the operator $\Gamma_{\mathbf{d}}$.

Now, given $f\in V^p_\nu(\Phi)$, there exists a sequence $\{f_N\}$ in $\operatorname{span}\{\phi_j(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,r}$ such that $\|f_N - f\|_{L^p_\nu}\to 0$ as $N\to\infty$. By using Lemma 5 we have,

$$
0\le \|f-\Gamma_{\mathbf{d}} f\|_{L^p_\nu} = \|f - f_N + \Gamma_{\mathbf{d}} f_N - \Gamma_{\mathbf{d}} f\|_{L^p_\nu} \le \big(1+\|\Gamma_{\mathbf{d}}\|\big)\|f_N - f\|_{L^p_\nu}\to 0\,,\ N\to\infty\,,
$$

which implies that $\Gamma_{\mathbf{d}} f = f$ in $L^p_\nu(\mathbb{R}^d)$, i.e., the validity of expansion (11) in $V^p_\nu(\Phi)$.

The series $\sum_{l=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\Upsilon_l f)(M\alpha)S_{l,\mathbf{d}}(t-M\alpha)$ converges, absolutely and uniformly on $\mathbb{R}^d$, to the continuous function f. Indeed,

$$
\begin{aligned}
\sum_{|\alpha|>N}\big|(\Upsilon_l f)(M\alpha)S_{l,\mathbf{d}}(t-M\alpha)\big| &\le \sup_{|\alpha|>N}\big|(\Upsilon_l f)(M\alpha)\big|\sup_{t\in[0,1)^d}\sum_{\alpha\in\mathbb{Z}^d}\big|S_{l,\mathbf{d}}(t-M\alpha)\big|\\
&\le \sup_{|\alpha|>N}\big|(\Upsilon_l f)(M\alpha)\big|\nu(M\alpha)\sup_{t\in[0,1)^d}\sum_{\alpha\in\mathbb{Z}^d}\big|S_{l,\mathbf{d}}(t-M\alpha)\big|\\
&\le \sup_{|\alpha|>N}\big|(\Upsilon_l f)(M\alpha)\big|\nu(M\alpha)\,\|S_{l,\mathbf{d}}\|_{W(L^1_\nu)}\longrightarrow 0 \quad \text{as } N\to\infty\,,
\end{aligned}
$$

uniformly on $\mathbb{R}^d$. In the last inequality we have used that $S_{l,\mathbf{d}}\in V^1_\nu(\Phi)\subset W_0(L^1_\nu)$, $l=1,2,\dots,s$ (see Lemma 3), and

$$
\|S_{l,\mathbf{d}}\|_{W(L^1_\nu)} = \sum_{\alpha\in\mathbb{Z}^d}\operatorname*{ess\,sup}_{t\in[0,1)^d}\big|S_{l,\mathbf{d}}(t+\alpha)\big|\nu(\alpha) \ge \sup_{t\in[0,1)^d}\sum_{\alpha\in\mathbb{Z}^d}\big|S_{l,\mathbf{d}}(t-M\alpha)\big|\,.
$$

□

3.5 Dirac's sampling case

This subsection is devoted to study another type of linear systems: their impulse response is a translated Dirac delta, i.e., $\big(\Upsilon_l f\big)(t) := f(t+c_l), \quad t \in \mathbb{R}^d$, where c_l is a fixed vector in $\mathbb{R}^d$. Provided that $\phi_j \in W(L^1_\nu)$, $j = 1, 2, \ldots, r$, the sequence $\{\Upsilon_l \phi_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ also belongs to $\ell^1_\nu(\mathbb{Z}^d)$. Indeed, for a fixed $c_l \in \mathbb{R}^d$, with $c_l = d_l + x_l,\ x_l \in [0,1)^d$ and $d_l \in \mathbb{Z}^d$, we have

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}^d} |\phi(\alpha + c_l)|\nu(\alpha) = \sum_{\beta\in\mathbb{Z}^d} |\phi(\beta + x_l)|\nu(\beta - d_l) \le \sum_{\beta\in\mathbb{Z}^d} |\phi(\beta + x_l)|\nu(\beta)\nu(d_l) \\
&\le \nu(d_l) \sum_{\beta\in\mathbb{Z}^d} |\phi(\beta + x_l)|\nu(\beta) \le \nu(d_l) \sum_{\beta\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d} |\phi(\beta + x)|\nu(\beta) = \nu(d_l)\|\phi\|_{W(L^1_\nu)} .
\end{aligned}$$

Thus, for these new systems the functions defined in (3) make sense.

In order to extend Theorem 6 for the case $1 \le p < \infty$ we need to assume stronger hypotheses on the set of generators $\Phi = \{\phi_j\}_{j=1}^r$ in $W_0(L^1_\nu)$. Next, we state the L^p_ν-stable shifts concept as established in [15] for the non-weighted case. Note that the space $W_0(L^1_\nu)$ is included in the corresponding $\mathcal{L}^\infty_\nu(\mathbb{R}^d)$ space, defined in [15] as

$$\mathcal{L}^\infty_\nu(\mathbb{R}^d) := \Big\{ f \text{ measurable} : \|f\|_{\mathcal{L}^\infty_\nu} := \operatorname*{ess\,sup}_{x\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} |f(x+\alpha)|\nu(x+\alpha) < \infty \Big\} .$$

Definition 3. *For $1 \le p < \infty$, a finite subset $\Phi = \{\phi_j\}_{j=1}^r$ of $W_0(L^1_\nu)$ is said to have L^p_ν-stable shifts if there exist positive constants $0 < A \le B$ (depending on p and Φ) such that*

$$A \sum_{j=1}^r \|a_j\|_{\ell^p_\nu} \le \Big\| \sum_{j=1}^r \phi_j *' a_j \Big\|_{L^p_\nu} \le B \sum_{j=1}^r \|a_j\|_{\ell^p_\nu} , \tag{17}$$

for any sequence $a_j \in \ell^p_\nu(\mathbb{Z}^d)$, $j = 1, 2, \ldots, r$, when $1 \le p < \infty$.

Given $f \in V^p_\nu(\Phi)$, i.e., $f(t) = \sum_{j=1}^r \sum_{\beta\in\mathbb{Z}^d} a_j(\beta)\, \phi_j(t - \beta)$ with $\{a_j(\beta)\}_{\beta\in\mathbb{Z}^d} \in \ell^p_\nu$ for $j = 1, 2, \ldots, r$, we have

$$\begin{aligned}
(\Upsilon_l f)(\alpha) = f(\alpha + c_l) &= \sum_{j=1}^r \sum_{\beta\in\mathbb{Z}^d} a_j(\beta)\, \phi_j(\alpha + c_l - \beta) \\
&= \sum_{j=1}^r \Big(\{a_j(\beta)\}_{\beta\in\mathbb{Z}^d} * \{\phi_j(\beta + c_l)\}_{\beta\in\mathbb{Z}^d} \Big)(\alpha) .
\end{aligned}$$

Having in mind the first inequality in (15), in proving Theorem 6 we just need an inequality like $\big\| \{(\Upsilon_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d} \big\|_{\ell^p_\nu} \le K\, \|f\|_{L^p_\nu}$. Since $\{\phi_j(\beta + c_l)\}_{\beta\in\mathbb{Z}^d} \in \ell^1_\nu$, from Lemma 1($c$) there exists a positive constant K_1 such that

$$\|\{(\Upsilon_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^p_\nu} \le K_1 \sum_{j=1}^r \|\{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^p_\nu} .$$

Finally, from the left inequality in (17) we get

$$\|\{(\Upsilon_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^p_\nu} \le K_2 \|f\|_{L^p_\nu} ,$$

where K_2 is a positive constant. Thus, Theorem 6 can be extended to Dirac's systems, whenever $1 \le p < \infty$. Due to the inequality $\|\{(\Upsilon_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^\infty_\nu} \le \|f\|_{L^\infty_\nu}$, the case $p = \infty$ becomes trivial.

Finally, it is worth to mention that Theorem 6 remains true for linear combinations of average and Dirac systems.

References

[1] E. Acosta-Reyes, A. Aldroubi and I. Krishtal. On stability of sampling-reconstruction models. *Adv. Comput. Math.*, 31:5–34, 2009.

[2] A. Aldroubi and K. Gröchenig. Non-uniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.*, 43(4):585–620, 2001.

[3] A. Aldroubi, Q. Sun and W.S. Tang. p-frames and shift-invariant subspaces of L^p. *J. Fourier Anal. Appl.*, 7(1):1–22, 2001.

[4] A. Aldroubi, Q. Sun and W.S. Tang. Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces. *J. Fourier Anal. Appl.*, 11(2):215–244, 2005.

[5] A. Aldroubi and M. Unser. A general sampling theory for nonideal acquisition devices *IEEE Trans. Signal Process.*, 42(11):2915–2925, 1994.

[6] A. Aldroubi and M. Unser. Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory. *Numer. Funct. Anal. Optimiz.*, 15(1):1–21, 1994.

[7] W. Chen, S. Itoh, and J. Shiki. On sampling in shift invariant spaces. *IEEE Trans. Signal Process.*, 48(10):2802–2810, 2002.

[8] S. Ericsson. Generalized sampling in shift invariant spaces with frames. *Acta Math. Sin. (Engl. Ser.)*, 28(9):1823–1844, 2012.

[9] A. G. García and G. Pérez-Villalón. Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces. *Appl. Comput. Harmon. Anal.*, 20(3):422–433, 2006.

[10] A. G. García, M. A. Hernández-Medina and G. Pérez-Villalón. Generalized sampling in shift-invariant spaces with multiple stable generators. *J. Math. Anal. Appl.*, 337:69–84, 2008.

[11] A. G. García and G. Pérez-Villalón. Multivariate generalized sampling in shift-invariant spaces and its approximation properties. *J. Math. Anal. Appl.*, 355:397–413, 2009.

[12] A. G. García, M. J. Muñoz-Bouzo and G. Pérez-Villalón. Regular multivariate sampling and approximation in L^p shift-invariant spaces. *J. Math. Anal. Appl.*, 380:607–627, 2011.

[13] K. H. Gröchenig. Wiener's Lemma: Theme and Variations. An Introduction to Spectral Invariance and Its Applications. In *Four Short Courses on Harmonic Analysis*, B. Forster and P. Massopust (eds.), Birkhäuser, Boston, pp. 175–234, 2010.

[14] R. Q. Jia and C. A. Micchelli. Using the refinement equations for the construction of pre-waveles II: Powers of two. In *Curves and Surfaces*, P. J. Laurent, A. Le Méhauté, L. L. Schumaker (eds.), Academic Press, Boston, pp. 209–246, 1991.

[15] J. J. Lei, R. Q. Jia, and E. W. Cheney. Approximation from shift-invariant spaces by integral operators. *SIAM J. Math. Anal.*, 28(2):481–498, 1997.

[16] M. Z. Nashed, and Q. Sun. Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$. *J. Funct. Anal.*, 258:2422–2452, 2010.

[17] S. Pilipovic and S. Simic. Frames for weighted shift-invariant spaces. *Mediterr. J. Math.*, 9:897–912, 2012.

[18] M. Unser. Splines: A perfect fit for signal and image processing. *IEEE Signal Process. Mag.*, 16(6):22–38, 1999.

[19] M. Unser. Sampling 50 Years After Shannon. *Proc. IEEE*, 88(4):569–587, 2000.

[20] J. Xian and S. Li. Sampling set conditions in weighted multiply generated shift-invariant spaces and their applications. *Appl. Comput. Harmon. Anal.*, 23:171–180, 2007.

[21] P. Wojtaszczyk. *A Mathematical Introduction to Wavelets.* Cambridge University Press, Cambridge, 1997.

[22] X. Zhou and W. Sun. On the sampling theoren for wavelet subspaces. *J. Fourier Anal. Appl.*, 5(4):347–354, 1999.