Computation of wavelet coefficients from average samples

G. Pérez-Villalón* and A. Portal †

Abstract

There exist efficient methods to compute the wavelet coefficients of a function $f(t)$ from its point samples $f\big(T[n+\tau]\big)$, $n \in \mathbb{N}$. However, in many applications the available samples are average samples of the type $\int_{-\infty}^{\infty} f\big(T[t+n+\tau]\big)u(t)dt$, where the averaging function $u(t)$ reflects the characteristic of the acquisition device. In this work, methods to compute the coefficients in a biorthogonal wavelet system from average samples are studied. Error estimations are obtained and using them, the optimal values for the parameters in the proposed approximation rules are calculated. The obtained error estimations can also be applied to the rules that compute the coefficients from point samples, and thus, these estimations can be used to compare and to choose between the different methods proposed in the literature. The methods proposed here also allow to compute the biorthogonal wavelet coefficients from the coefficients in another biorthogonal wavelet system.

Keywords: Wavelet coefficients, Quadrature formulas, Sampling theory, Average samples.

1 Introduction

In this paper we study the computation of the wavelet coefficients of a function in a biorthogonal wavelet system. We denote by φ and $\widetilde{\varphi}$ the dual scaling functions, by ψ and $\widetilde{\psi}$ the corresponding dual wavelets, and by N the order of the wavelet system. For any $f \in L^2(\mathbb{R})$, we denote by $f_{j,k}$ the function $f_{j,k}(t) = 2^{j/2} f(2^j t - k)$.

The wavelet coefficients $\beta_{j,k} = \langle f, \widetilde{\psi}_{j,k} \rangle$, $j < J$, of a function $f \in L^2(\mathbb{R})$ can be calculated using the fast wavelet transform from the coefficients $c_{J,k} = \langle f, \widetilde{\varphi}_{J,k} \rangle$ at a fine scale J. In practice, however, the coefficients $c_{J,k}$ cannot be calculated exactly. Thus, rules to approximate efficiently them with a certain accuracy are necessary.

*Corresponding author. Department of Mathematics, EUITT, Universidad Politécnica, C. de Valencia, Km 7, 28030, Madrid, Spain, E-mail: gperez@euitt.upm.es

†Department of Mathematics, ETSIT, Universidad Politécnica, Av. Complutense 30, 28040, Madrid, Spain

Many methods to approximate the coefficients $c_{J,k}$ from equispaced samples of the function,

$$f\big(T[n+\tau]\big), \qquad n \in \mathbb{Z}, \quad T = 2^{-J},$$

have been studied. See [3, 10, 11, 12, 13, 16, 19, 21, 22, 26, 27, 28] and references therein. The translation parameter τ can be generally chosen. The proposed approximation schemes are, most of the time, of the type

$$c_{J,k} \;\approx\; c_{J,k}^{\text{approx}} \;=\; \sqrt{T}\sum_n \alpha_n\, f\big(T[k+Bn+\tau]\big) \tag{1}$$

for suitable weights α_n and translation parameter $\tau \in \mathbb{R}$. The parameter $B \in \mathbb{N}$ is generally fixed as $B = 1$, but other natural values for B can be suitable depending on the support of $\widetilde{\varphi}$.

When the sequence of weights $\{\alpha_n\}$ is finite, the approximation scheme (1) corresponds to using the quadrature formula

$$\int_{-\infty}^{\infty} g(t)\widetilde{\varphi}(t)dt \;\approx\; \sum_n \alpha_n\, g(Bn+\tau) \tag{2}$$

in the computation of (we assume that $\widetilde{\varphi}$ is real)

$$c_{J,k} = \frac{1}{\sqrt{T}}\int_{-\infty}^{\infty} f(t)\widetilde{\varphi}\Big(\frac{t}{T}-k\Big)dt = \sqrt{T}\int_{-\infty}^{\infty} f(T[t+k])\widetilde{\varphi}(t)dt.$$

In order not to ruin the convergence order of the wavelet system, it is required that the quadrature formula (2) gives order L, i.e. it is exact for polynomials of degree at most $L-1$, for some L greater than or equal to N, the order of the wavelet system. For a fixed translation parameter τ, there exist L-point quadrature formulas of order L. Besides, the translation parameter τ can be fixed, in such a way that there exists an $(L-1)$-point quadrature formula of order L [16, 21]. Gaussian quadratures have been also considered [2, 17]. They give more precision, but they cannot be applied when the samples are equidistant, and they are less efficient since the evaluation of neighboring coefficients does not share common points.

Other important classes of approximation schemes of type (1) are sampling or interpolating schemes, i.e, those satisfying

$$f = \sum_k c_{J,k}^{\text{approx}}\varphi_{J,k}, \quad \text{for all } f \in V_J = \Big\{\sum_k a_k\varphi_{J,k} \,:\, \{a_k\} \in \ell^2(\mathbb{Z})\Big\}. \tag{3}$$

They generalize Shannon's sampling formula, providing a method to recover the functions of the wavelet subspace V_J from their samples. For a fixed translation parameter τ, for $B = 1$, and under suitable hypotheses, a unique weight sequence $\{\alpha_n\}$ satisfying (3) exists, but it is generally not finite [1, 25]. However, in many cases (1) can be computed efficiently [5, 6, 23]. These schemes are interpolating, i.e, for continuous functions f, they satisfy $\sum_k c_{J,k}^{\text{approx}}\varphi_{J,k}\big(T[n+\tau]\big) = f\big(T[n+\tau]\big), \quad n \in \mathbb{Z}$.

In many applications it is more realistic to assume that the available samples are not point samples, but local averages of the type

$$S_n = \sqrt{T} \int_{-\infty}^{\infty} f\big(T[t+n+\tau]\big)u(t)dt, \quad n \in \mathbb{Z}, \tag{4}$$

where the averaging function $u(t)$ reflects the characteristic of the acquisition device.

In particular, the available average samples could be

$$S_n = \frac{1}{a\sqrt{T}} \int_{(n+\tau-\frac{a}{2})T}^{(n+\tau+\frac{a}{2})T} f(t)dt\,,$$

which correspond to the boxcar function $u(t) = a^{-1}\chi_{[-a/2,a/2)}(t)$. The most important case is $u(t) = \chi_{[-1/2,1/2)}(t)$, since the sample intervals cover the whole real line without overlapping.

Notice that the average samples (4) can be represented by the scalar product

$$S_n = \big\langle\, f\,,\, u^{\tau}_{J,n} \,\big\rangle \tag{5}$$

where $u^{\tau}(t) = u(t-\tau)$ (we assume that $u(t)$ is real).

The study of the computation of the wavelet coefficients from samples of the type (4) can be applied to compute the wavelet coefficients in a wavelet system from the coefficients in another wavelet system. Indeed, if we know the coefficients in some wavelet system with dual scaling function $\widetilde{\phi}$, by applying the synthesis relations we can compute the coefficients at a fine scale J, $\langle f, \widetilde{\phi}_{J,n}\rangle$, which can be seen as average samples of the type (5) with average function $u = \widetilde{\phi}$ and $\tau = 0$. If we do not want to lose the flexibility of fixing τ, we can see the known coefficients as $\langle f, \widetilde{\phi}_{J,n}\rangle = \langle f(\cdot - T\tau), \widetilde{\phi}^{\tau}_{J,n}\rangle$, allowing us to compute the wavelet coefficients of the translated version $f(t-\tau T)$.

In order to compute the wavelet coefficients $c_{J,k} = \langle f, \widetilde{\varphi}_{J,k}\rangle$ from the average samples S_n given in (4), it is natural (see (1)) to consider approximation schemes of the type

$$c_{J,k} \;\approx\; c^{\text{approx}}_{J,k} \;=\; \sum_n \alpha_n S_{k+Bn} = \sqrt{T}\sum_n \alpha_n \int_{-\infty}^{\infty} f\big(T[t+k+Bn+\tau]\big)u(t)dt. \tag{6}$$

In the case of point samples, i.e. when $u(t)$ is the Dirac delta, (6) reduces to (1).

An approximation scheme of the type (6) satisfying the sampling condition (3) was proposed by A. Aldroubi and M. Unser in [24] (see also [15, 18, 20]). As in the case of point samples, for a fixed τ and for $B = 1$, a unique weight sequence $\{\alpha_n\}$ satisfying the sampling condition (3) exists, but it is generally not finite.

When the sequence $\{\alpha_n\}$ is finite, the approximation scheme (6) corresponds to the use of the approximation formula

$$\int_{-\infty}^{\infty} g(t)\widetilde{\varphi}(t)dt \;\approx\; \sum_n \alpha_n \int_{-\infty}^{\infty} g(t)u(t-Bn-\tau)dt \tag{7}$$

in the evaluation of $c_{J,k} = T^{-1/2}\int f(t)\widetilde{\varphi}(t/T-k)dt$. In the case of point samples, i.e. when $u(t) = \delta(t)$, (7) becomes the quadrature formula (2). Using the nomenclature of quadrature formulas, we call the formula (7) an n-point approximation formula of order L if the sum has n terms and it is exact for polynomials of degree at most $L-1$. A formula of this type with the same order than the wavelet system ($L = N$) for the case $u(t) = \chi_{[-1/2,1/2)}(t)$, was proposed by B. Delyon and A. Juditsky in [10].

In Section 2 we obtain a simple formula in the frequency domain for the approximation error

$$\mathcal{E}(f) = \left\| \{c_{J,k} - c_{J,k}^{\text{approx}}\}_{k\in\mathbb{Z}} \right\|_{\ell^2(\mathbb{Z})}$$

where the approximations $c_{J,k}^{\text{approx}}$ are computed by (6). From this formula suitable error estimations are deduced. The obtained error estimations are also valid for the point sample case, i.e., when $u = \delta$. Besides, they are valid not only for approximation schemes (6) with a finite sequence of weights $\{\alpha_n\}$ but also for $\{\alpha_n\} \in \ell^1$. Thus, the obtained estimations allow to analyze and to compare many of the methods proposed in the literature to compute wavelet coefficients.

We study the design of approximation formulas of the type (7) (Sections 3 and 4): For fixed parameters τ and B, solving a linear system, it is possible to find L-point formulas which have order L. We give a method to calculate a translation parameter τ for which it is possible to find $(L-1)$-point formulas giving order L. If there exist several of such τ's, the given estimations for the error allow us to choose the most convenient one among them. Generally, the supports of $\widetilde{\varphi}$ and u show the most suitable value of the parameter $B \in \mathbb{N}$. In the case of doubt, the error estimations also allow us to choose the most convenient one. Finally, we illustrate the theory with an example where we design, analyze and compare formulas to compute the wavelet coefficients from several types of samples (Section 5).

2 Estimations of the error

Throughout this work, we assume that φ and $\widetilde{\varphi}$ are a pair of real, continuous and compactly supported dual scaling functions of two multiresolution analysis [6, 7, 8]. We assume that the average function u is a real bounded compactly supported function or it is the Dirac delta. We also assume the normalization

$$\int_{-\infty}^{\infty} \varphi(t)dt = \int_{-\infty}^{\infty} \widetilde{\varphi}(t)dt = \int_{-\infty}^{\infty} u(t)dt = 1.$$

Let

$$\mathcal{E}(f) := \left\|\{c_{J,k} - c_{J,k}^{\text{approx}}\}_{k\in\mathbb{Z}}\right\|_{\ell^2(\mathbb{Z})} = \Big[\sum_k (c_{J,k} - c_{J,k}^{\text{approx}})^2\Big]^{1/2}, \qquad (8)$$

where $c_{J,k} := \langle f, \widetilde{\varphi}_{J,k}\rangle$, $c_{J,k}^{\text{approx}}$ are the approximations defined by (6) with $\alpha_n, \tau \in \mathbb{R}$, $B \in \mathbb{N}$, and f belongs to the Sobolev space $W_2^r := \{f \in L^2(\mathbb{R}) : (1+|w|^2)^{r/2}\widehat{f}(w) \in L^2(\mathbb{R})\}$, for some $r \in \mathbb{N}$. We denote by $\|\cdot\|$ and by $\langle\cdot,\cdot\rangle$ the norm and the scalar product of $L^2(\mathbb{R})$, by $\|\cdot\|_\infty$ the norm of $L^\infty(\mathbb{R})$, by $\widehat{f}(w) = \int_{-\infty}^{\infty} f(t)e^{-itw}dt$ the Fourier transform of f, and by $\sum_k a_k$ the series $\sum_{k=-\infty}^{\infty} a_k$.

The following Theorem provides an integral representation for the error (8) in the frequency domain. This Theorem and its Corollaries are inspired by the work of Blu and Unser [4]. Some advantages of this kind of representation will become apparent later. In [4, 5, 26] the reader can find an ampler discussion about the significance and advantages of this type of integral representation, specially in signal processing applications. A representation for the error $\left\|f - \sum_k c_{J,k}^{\text{approx}} \varphi_{J,k}\right\|$ can be obtained from Theorem 1 in [4]. However, we are interested in the error $\left\|\{c_{J,k} - c_{J,k}^{\text{approx}}\}_{k\in\mathbb{Z}}\right\|_{\ell^2(\mathbb{Z})}$, since it is significantly smaller than $\left\|f - \sum_k c_{J,k}^{\text{approx}} \varphi_{J,k}\right\|$ when the order of the approximation formula (7) is strictly greater than the order of the wavelet system, $L > N$ in our notation.

Theorem 1 *Assume that the sequence of weights $\{\alpha_n\}_{n\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z})$ and let $r \in \mathbb{N}$. Let $\mathcal{E}(f)$ the error defined by (8). Then, for any $f \in W_2^r$,*

$$\mathcal{E}(f) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(w)|^2 E(Tw)dw\right]^{1/2} + d(f,T), \tag{9}$$

where

$$E(w) := \left|\widehat{\widetilde{\varphi}}(w) - \widehat{u}(w)\sum_n \alpha_n e^{-iw(Bn+\tau)}\right|^2$$

and

$$|d(f,T)| \leq C\ \|f^{(r)}\|\ T^r,$$

with $C := \|E\|_\infty^{1/2}\pi^{-r}\big[1 + \big(\sum_{k\neq 0}(2|k|-1)^{-2r}\big)^{1/2}\big]$.

Moreover, the correction term $d(f,T)$ vanishes when f is band-limited to an interval of length less than $2\pi/T$.

Proof. First we prove that

$$c_{J,k}^{\text{approx}} = \frac{1}{2\pi}\big\langle \widehat{f}, (\Lambda_{J,k})^\wedge\big\rangle, \quad \text{where } \Lambda(t) := \sum_n \alpha_n\, u(t - Bn - \tau). \tag{10}$$

Assume that the average function $u(t)$ is a bounded compactly supported function. Since we have assumed that $\{\alpha_n\} \in \ell^1$, by using [14, Theorem 2.1] we obtain that $\sum_n |\alpha_n\, u(t-n)| \in L^2(\mathbb{R})\cap L^1(\mathbb{R})$. Then $\Lambda \in L^2(\mathbb{R})\cap L^1(\mathbb{R})$. Besides, since $f \in L^2(\mathbb{R})$, we have that for $k \in \mathbb{Z}$, $\int_{-\infty}^{\infty} |f(t)| \sum_n |\alpha_n u(t/T - Bn - k - \tau)|dt < \infty$. Then, from (6) and the Lebesgue dominated convergence theorem, we obtain

$$c_{J,k}^{\text{approx}} = \int_{-\infty}^{\infty} f(t)\sum_n \alpha_n \frac{1}{\sqrt{T}} u\Big(\frac{t}{T} - Bn - k - \tau\Big)dt = \langle f, \Lambda_{J,k}\rangle.$$

Having in mind that $f, \Lambda \in L^2(\mathbb{R})$, we deduce (10) in this case. Assume now that $u = \delta$. Notice that since we have assumed that $f \in W_2^r$, we have $\widehat{f} \in L^1(\mathbb{R})$. Indeed,

$$\int_{-\infty}^{\infty} |\widehat{f}(w)|dw = \int_{-\infty}^{\infty} (1+|w|^2)^{-r/2}(1+|w|^2)^{r/2}|\widehat{f}(w)|dw$$
$$\leq \left[\int_{-\infty}^{\infty}(1+|w|^2)^{-r}dw \int_{-\infty}^{\infty}(1+|w|^2)^{r}|\widehat{f}(w)|^2dw\right]^{1/2} < \infty.$$

By using the inverse Fourier transform and the dominated convergence theorem, having in mind that $\sum_n \int_{-\infty}^{\infty} |\alpha_n \widehat{f}(w)| dw \le \|\{\alpha_n\}\|_{\ell^1(\mathbb{Z})} \|\widehat{f}\|_{L^1(\mathbb{R})} < \infty$, we obtain

$$\begin{aligned} c_{J,k}^{\text{approx}} &= \sqrt{T} \sum_n \alpha_n f\big(T[Bn + k + \tau]\big) = \sum_n \alpha_n \frac{\sqrt{T}}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(w) e^{iwT(Bn+k+\tau)} dw \\ &= \frac{\sqrt{T}}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(w) \sum_n \alpha_n e^{iwT(Bn+k+\tau)} dw = \frac{1}{2\pi} \langle \widehat{f}, (\Lambda_{J,k})^\wedge \rangle. \end{aligned}$$

This proves (10).

From (10) we deduce the following representation for the sequence of errors

$$e_k^f := c_{J,k} - c_{J,k}^{\text{approx}} = \frac{1}{2\pi} \big\langle \widehat{f}, (\widetilde{\varphi}_{J,k} - \Lambda_{J,k})^\wedge \big\rangle .$$

We denote

$$D := (\widetilde{\varphi} - \Lambda)^\wedge.$$

Notice that $\|D\|_\infty < \infty$. Indeed, $\widetilde{\varphi} \in L^1(\mathbb{R})$ and when $u = \delta$ we have $\|\widehat{\Lambda}\|_\infty \le \|\{\alpha_n\}_n\|_{\ell^1} \le \infty$ and otherwise $\Lambda \in L^1(\mathbb{R})$.

Denoting $b := 2\pi/T$, we have

$$\begin{aligned} e_k^f &= \frac{\sqrt{T}}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(w)\, \overline{D}(Tw)\, e^{ikTw} dw = \frac{\sqrt{T}}{2\pi} \sum_n \int_0^b \widehat{f}(w + bn)\, \overline{D}(Tw + 2\pi n)\, e^{ikTw} dw \\ &= \frac{\sqrt{T}}{2\pi} \int_0^b \sum_n \widehat{f}(w + bn)\, \overline{D}(Tw + 2\pi n)\, e^{ikTw} dw, \end{aligned}$$

where we have used Lebesgue dominated convergence theorem, having in mind that

$$\int_0^b \sum_n \big|\widehat{f}(w + bn)\, \overline{D}(Tw + 2\pi n)\big| dw \le \|D\|_\infty \|\widehat{f}\,\|_{L^1(\mathbb{R})} < \infty.$$

Therefore, the sequence of errors e_k^f are the Fourier coefficients of the b-periodic function $T^{-1/2} \sum_n \widehat{f}(w + bn)\, \overline{D}(Tw + 2\pi n)$ in the system $\{e^{-ikTw}\}_{k\in\mathbb{Z}}$. This function belongs to $L^2(0, b)$ since

$$\int_0^b \Big| \sum_n \widehat{f}(w + bn)\, \overline{D}(Tw + 2\pi n) \Big|^2 dw \le \|D\|_\infty^2 \int_0^b \Big| \sum_n |\widehat{f}(w + bn)| \Big|^2 dw$$

and

$$\begin{aligned} \int_0^b \Big| \sum_n |\widehat{f}(w + bn)| \Big|^2 dw &\le \int_0^b \Big| \sum_n |\widehat{f}(w + bn)| [1 + |w + bn|^2]^{\frac{r}{2}} [1 + |w + bn|^2]^{-\frac{r}{2}} \Big|^2 dw \\ &\le \int_0^b \sum_n \big|\widehat{f}(w + bn)[1 + |w + bn|^2]^{\frac{r}{2}}\big|^2 \sum_n [1 + |w + bn|^2]^{-r} dw \\ &\le \operatorname{supp}_{w\in[0,b]} \Big\{ \sum_n [1 + |w + bn|^2]^{-r} \Big\} \int_{-\infty}^{\infty} \big|\widehat{f}(w)(1 + |w|^2)^{r/2}\big|^2 dw < \infty. \end{aligned} \tag{11}$$

Hence, by using Parseval's identity we obtain

$$\mathcal{E}^2(f) = \big\|\{e_k^f\}\big\|_{\ell^2(\mathbb{Z})}^2 = \frac{1}{2\pi}\Big\|\sum_n \widehat{f}(w+bn)\,\overline{D}(Tw+2\pi n)\Big\|_{L^2(0,b)}^2. \tag{12}$$

Now we follow the same steps as in the proof of Theorem 1 in [4]. First observe that

$$\mathcal{E}^2(f) = \frac{1}{2\pi}\int_0^b \sum_n \Big|\widehat{f}(w+bn)\,\overline{D}(Tw+2\pi n)\Big|^2 dw + \gamma(f,T)$$

where

$$\gamma(f,T) := \frac{1}{2\pi}\int_0^b \sum_{n,m\in\mathbb{Z}, n\neq m} \widehat{f}(w+bn)\,\overline{D}(Tw+2\pi n)\overline{\widehat{f}}(w+bm)\,D(Tw+2\pi m)dw \tag{13}$$

vanishes when f is band-limited to an interval of length $b = 2\pi/T$. The series in the definition $\Lambda(t) := \sum_n \alpha_n\, u(t-Bn-\tau)$ converges in $L^1(\mathbb{R})$ when u is a bounded compactly supported function. Then $\widehat{\Lambda}(w) = \sum_n \alpha_n\,\widehat{u}(w)e^{-iw(Bn+\tau)}$ and thus

$$\mathcal{E}^2(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(w)|^2 E(Tw)dw + \gamma(f,T), \tag{14}$$

where

$$E(w) = \big|D(w)\big|^2 = \big|(\widetilde{\varphi}-\Lambda)^\wedge(w)\big|^2 = \big|\widehat{\widetilde{\varphi}}(w) - \widehat{u}(w)\sum_n \alpha_n e^{-iw(Bn+\tau)}\big|^2.$$

For the bandpass components

$$\widehat{f_k}(w) := \begin{cases} \widehat{f}(w), & \frac{(2k-1)\pi}{T} \le w < \frac{(2k+1)\pi}{T} \\ 0, & \text{elsewhere} \end{cases}$$

the error formula (14) holds with correction term $\gamma(f_k, T) = 0$, i.e.,

$$\mathcal{E}^2(f_k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f_k}(w)|^2 E(Tw)dw, \quad k\in\mathbb{Z}.$$

Besides, $f = \sum_k f_k$ and then $e_n^f = \sum_k e_n^{f_k}$, $n\in\mathbb{Z}$. By using Minkowski's inequality, we obtain

$$\mathcal{E}(f) = \Big[\sum_n |e_n^f|^2\Big]^{1/2} = \Big[\sum_n \big|\sum_k e_n^{f_k}\big|^2\Big]^{1/2} \le \sum_k \Big[\sum_n \big|e_n^{f_k}\big|^2\Big]^{1/2} = \mathcal{E}(f_0) + \sum_{k\neq 0}\mathcal{E}(f_k).$$

Analogously, we obtain $\mathcal{E}(f_0) \le \mathcal{E}(f) + \sum_{k\neq 0}\mathcal{E}(f_k)$, and then

$$\big|\mathcal{E}(f) - \mathcal{E}(f_0)\big| \le \sum_{k\neq 0}\mathcal{E}(f_k).$$

Having in mind that $|w| \geq (2|k|-1)\pi/T$, for $w \in \operatorname{supp} \widehat{f}_k$, we obtain that for $k \neq 0$,

$$\begin{aligned}\mathcal{E}(f_k) &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}_k(w)|^2 E(Tw)dw\right]^{1/2} \leq \left[\frac{\|E\|_\infty}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}_k(w)|^2 dw\right]^{1/2} \\ &\leq \frac{T^r}{(2|k|-1)^r\pi^r}\left[\frac{\|E\|_\infty}{2\pi}\int_{-\infty}^{\infty} \left|w^r\widehat{f}_k(w)\right|^2 dw\right]^{1/2}.\end{aligned}$$

By using Cauchy-Schwarz inequality and that $\sum_{k\neq 0}\int_{-\infty}^{\infty} |w^r\widehat{f}_k(w)|^2 dw \leq \|w^r\widehat{f}(w)\|^2$, we obtain

$$\left|\mathcal{E}(f) - \mathcal{E}(f_0)\right| \leq \sum_{k\neq 0}\mathcal{E}(f_k) \leq \sqrt{\frac{\|E\|_\infty}{2\pi}}\frac{T^r}{\pi^r}\Big[\sum_{k\neq 0}\frac{1}{(2|k|-1)^{2r}}\Big]^{1/2}\|w^r\widehat{f}(w)\|. \tag{15}$$

On the other hand, having in mind that $|w| \geq \pi/T$, for $w \in \operatorname{supp}(\widehat{f} - \widehat{f}_0)$, we obtain

$$\begin{aligned}&\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(w)|^2\, E(Tw)dw\right]^{\frac{1}{2}} \\ &\leq \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}_0(w)|^2\, E(Tw)dw\right]^{\frac{1}{2}} + \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(w) - \widehat{f}_0(w)|^2\, E(Tw)dw\right]^{\frac{1}{2}} \\ &\leq \mathcal{E}(f_0) + \frac{T^r}{\pi^r}\left[\frac{\|E\|_\infty}{2\pi}\int_{-\infty}^{\infty} |w|^{2r}|\widehat{f}(w) - \widehat{f}_0(w)|^2 dw\right]^{\frac{1}{2}} \leq \mathcal{E}(f_0) + \frac{T^r}{\pi^r}\sqrt{\frac{\|E\|_\infty}{2\pi}}\,\|w^r\widehat{f}(w)\|.\end{aligned}$$

Analogously, $\mathcal{E}(f_0) \leq \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(w)|^2\, E(Tw)dw\right]^{1/2} + \frac{T^r}{\pi^r}\sqrt{\frac{\|E\|_\infty}{2\pi}}\|w^r\widehat{f}(w)\|$, and then

$$\left|\mathcal{E}(f_0) - \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(w)|^2\, E(Tw)dw\right]^{1/2}\right| \leq \frac{T^r}{\pi^r}\sqrt{\frac{\|E\|_\infty}{2\pi}}\;\|w^r\widehat{f}(w)\|. \tag{16}$$

From (15) and (16) we deduce the estimation given in the theorem. □

For the case of point samples $u = \delta$, and for functions f band-limited to the interval $(-\pi/T, \pi/T)$ the error formula (9) was given by X.G. Xia, C.C.J. Kuo, and Z. Zhang in [26] (in this case $d(f,t) = 0$).

Notice that the error formula (9) can be specially useful in applications where some partial knowledge of the signal spectrum is available.

The first term of the error formula (9), $\big[(1/2\pi)\int_{-\infty}^{\infty} |\widehat{f}(w)|^2 E(Tw)dw\big]^{1/2}$, also gives significant results when the sampling step $T = 2^{-J}$ is not small. One reason for that assertion is that if f is band-limited to an interval of length less than $2\pi/T$, the correction term $d(f,T)$ vanishes.

Let $f_\rho(t) := f(t-\rho)$. The approximation error $\mathcal{E}(f_\rho) = \big\|\{c_{J,k}^{\text{approx}} - c_{J,k}\}_k\big\|_{\ell^2(\mathbb{Z})}$ varies with ρ, but it is T-periodic, since $c_{J,k}^{\text{approx}} - c_{J,k}$ for $f_{\rho+T}$ is equal to $c_{J,k-1}^{\text{approx}} - c_{J,k-1}$ for f_ρ. A version of the error which is independent of the sampling phase ρ is given by

$$\mathbb{E}^2(f) := \frac{1}{T}\int_0^T \mathcal{E}^2(f_\rho)\, d\rho. \tag{17}$$

The following Corollary shows that for this version of the error, which considers irrelevant sampling phase, the correction term $d(f,T)$ vanishes, showing again the significance of the first term of the error formula (9).

Corollary 1 *Assume that* $\{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ *and* $r \in \mathbb{N}$. *Let* $\mathbb{E}(f)$ *the error defined by* (17). *For any* $f \in W_2^r$,

$$\mathbb{E}(f) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(w)|^2 E(Tw)dw\right]^{1/2}.$$

Proof. We use the notation of the proof of Theorem 1. From (14) we have

$$\int_0^T \mathcal{E}^2(f_\rho)d\rho = \frac{1}{2\pi}\int_0^T\int_{-\infty}^{\infty} |\widehat{f}_\rho(w)|^2 E(Tw)dwd\rho + \int_0^T \gamma(f_\rho,T)d\rho.$$

We can check that $\int_0^T \gamma(f_\rho,T)d\rho = 0$, by integrating in (13), and by interchanging the order of the series and integrals, which can be done since (see (11))

$$\int_0^T\int_0^b \sum_{n,m\in\mathbb{Z},n\neq m} |\widehat{f}_\rho(w+bn)\,\overline{D}(Tw+2\pi n)\overline{\widehat{f}}_\rho(w+bm)\,D(Tw+2\pi m)|dwd\rho$$
$$\leq T\,\|D\|_\infty^2 \int_0^b \Big(\sum_{n\in\mathbb{Z}} |\widehat{f}(w+bn)|\Big)^2 dw < \infty.$$

Now, using again that $|\widehat{f}_\rho| = |\widehat{f}\,|$ the result follows. □

The following corollary gives a simple asymptotic expression for the error, which allows to compute the error with enough accuracy when T is small enough. For greater values of T, it is better to use the theorem.

Corollary 2 *Let* $L \in \mathbb{N}$. *Assume that*

$$G(w) := \widehat{\widetilde{\varphi}}(w) - \widehat{u}(w)\sum_n \alpha_n e^{-iw(Bn+\tau)}$$

satisfies $G^{(l)}(0) = 0$ *for* $l = 0,1,\ldots,L-1$. *Then, for any* $f \in W_2^{L+1}$, *we have*

$$\mathcal{E}(f) = \frac{|G^{(L)}(0)|}{L!}\,\|f^{(L)}\|\,T^L + \mathcal{O}(T^{L+1}), \quad \text{as } T = 2^{-J} \to 0. \tag{18}$$

Proof. By using that $E^{(n)}(w) = \sum_{k=0}^n \binom{n}{k} G^{(k)}(w)\overline{G}^{(n-k)}(w)$ we obtain that $E^{(n)}(0) = 0$ for $n = 0,1,\ldots,2L-1$ and $E^{(2L)}(0) = \binom{2L}{L}|G^{(L)}(0)|^2$. Besides $E^{(2L+1)}(0) = 0$ since $E(w)$ is even. Then

$$E(Tw) = \frac{1}{L!^2}|G^{(L)}(0)|^2(Tw)^{2L} + \xi(w),$$

where $|\xi(w)| \le \|E^{(2L+2)}\|_\infty |Tw|^{2L+2}/(2L+2)!$. Using the notation of theorem 1, we obtain

$$\begin{aligned}
\mathcal{E}(f) &\le \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(w)|^2 \Big\{ |G^{(L)}(0)|^2 \frac{(Tw)^{2L}}{L!^2} + \xi(w) \Big\} dw \right]^{1/2} + d(f,T) \\
&\le \frac{|G^{(L)}(0)|}{L!} \|f^{(L)}\| T^L + \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(w)|^2 |\xi(w)| dw \right]^{1/2} + d(f,T) \\
&\le \frac{|G^{(L)}(0)|}{L!} \|f^{(L)}\| T^L + K \|f^{(L+1)}\| T^{L+1} + C T^{L+1} \|f^{(L+1)}\|,
\end{aligned}$$

where $K = (\|E^{(2L+2)}\|_\infty/(2L+2)!)^{1/2}$. □

3 Approximation order and Moments

As it was expected, approximation rule (6) satisfies

$$\mathcal{E}(f) = \left\| \{c_{J,k} - c_{J,k}^{\text{approx}}\}_{k\in\mathbb{Z}} \right\|_{\ell^2(\mathbb{Z})} = \mathcal{O}(T^L) \quad \text{as} \quad T = 2^{-J} \mapsto 0,$$

if and only if the approximation formula (7) has order L, i.e., it is exact for polynomials of degree at most $L-1$. Indeed, the function $G(w)$ in Corollary 2 is the Fourier transform of $\widetilde{\varphi}(t) - \sum_n \alpha_n u(t - Bn - \tau)$, and thus the condition $G^{(l)}(0) = 0$, $l = 0, 1, \ldots, L-1$, can be written as

$$\int_{-\infty}^{\infty} t^l \widetilde{\varphi}(t) dt = \int_{-\infty}^{\infty} t^l \sum_n \alpha_n u(t - Bn - \tau) dt, \quad l = 0, 1, \ldots, L-1. \tag{19}$$

Estimations showing that in this case, for any $k \in \mathbb{Z}$, $c_{J,k} - c_{J,k}^{\text{approx}} = \mathcal{O}(T^L)$ can be found in [10, 16, 21, 26, 27] for the case of point samples and in [10] for the case $u = \chi_{[-1/2,1/2)}$.

Let $\widetilde{h}_n$ be the refinement coefficients of the dual scaling function $\widetilde{\varphi}$, i.e.

$$\widetilde{\varphi}(t) = \sum_n \widetilde{h}_n \, \widetilde{\varphi}(2t - n). \tag{20}$$

In order to determine a translation parameter τ and weights α_n satisfying (19) we need the value of the moments

$$\widetilde{\varphi}_{[l]} := \int_{-\infty}^{\infty} t^l \widetilde{\varphi}(t) dt.$$

As Sweldens and Piessens [21] observed, the moments $\widetilde{\varphi}_{[l]}$ can be calculated from the refinement coefficients $\widetilde{h}_n$ by means of the recursion relation

$$\widetilde{\varphi}_{[l]} = \frac{1}{2^{l+1} - 2} \sum_{i=1}^{l} \binom{l}{i} \widetilde{h}_{[i]} \widetilde{\varphi}_{[l-i]}, \quad l = 1, 2, \ldots, \tag{21}$$

where $\widetilde{h}_{[i]}$ are the moments of $\{\widetilde{h}_n\}_{n\in\mathbb{Z}}$, $\widetilde{h}_{[i]} := \sum_n \widetilde{h}_n \, n^i$. This recursion relation can be easily obtained from the scale equation (20). Notice that if the average function u is a refinable function we can calculate its moments, $u_{[l]} = \int_{-\infty}^{\infty} t^l \, u(t) dt$ in the same way.

The order condition (19) can be written in terms of the moments. Indeed, (19) is equivalent to

$$\int_{-\infty}^{\infty}(t-\tau)^l\widetilde{\varphi}(t)dt = \int_{-\infty}^{\infty}(t-\tau)^l\sum_n \alpha_n u(t-Bn-\tau)dt = \int_{-\infty}^{\infty}\sum_n (t+Bn)^l\,\alpha_n u(t)dt$$

for $l=0,1,\ldots,L-1$, which can be written as

$$\sum_{r=0}^{l}\binom{l}{r}(-\tau)^r\widetilde{\varphi}_{[l-r]} = \sum_{r=0}^{l}\binom{l}{r}\alpha_{[r]}u_{[l-r]}B^r, \quad l=0,1,\ldots,L-1. \tag{22}$$

Notice that since we have assumed the normalization $\int\widetilde{\varphi} = \int u = 1$, the condition $\alpha_{[0]} = \sum_n \alpha_n = 1$ is necessary and sufficient to have at least order 1. Moreover, for a fixed translation parameter τ, we can compute recursively the moments $\alpha_{[l]}$ of a sequence of weights $\{\alpha_n\}$ that gives approximation order L,

$$\alpha_{[0]} = 1, \quad \alpha_{[l]} = \frac{1}{B^l}\left[\sum_{r=0}^{l}\binom{l}{r}(-\tau)^r\widetilde{\varphi}_{[l-r]} - \sum_{r=0}^{l-1}\binom{l}{r}\alpha_{[r]}u_{[l-r]}B^r\right], \quad l=0,1,\ldots,L-1. \tag{23}$$

Solving a Vandermonde system, we can find a finite sequence of L weights α_n having these moments. Therefore, for any fixed τ, we can get an L-point formula giving order L. In Section 4 we will give a method to compute τ in order to get an $(L-1)$-point formula of order L.

For fixed parameters τ and B, and weights α_n, the term $|G^{(L)}(0)|$ in the asymptotic formula (18) can be easily calculated from the moments. Indeed, recalling that $G(w)$ is the Fourier transform of $\widetilde{\varphi}(t) - \sum_n \alpha_n u(t-Bn-\tau)$ we obtain

$$\left|G^{(L)}(0)\right| = \left|\widetilde{\varphi}_{[L]} - \sum_n \alpha_n \sum_{r=0}^{L}\binom{L}{r}(Bn+\tau)^r u_{[L-r]}\right|. \tag{24}$$

3.1 One-Point formula

The second order conditions are (see (22))

$$\sum_n \alpha_n = 1, \quad \widetilde{\varphi}_{[1]} = u_{[1]} + B\,\alpha_{[1]} + \tau. \tag{25}$$

We consider now, the formulas with a unique nonzero weight α_n. We assume, without loss of generality, that this is α_0. In this case, the second order conditions (25), read as $\alpha_0 = 1, \quad \tau = \widetilde{\varphi}_{[1]} - u_{[1]}$. Hence, there exists exactly one of such formulas giving second order

$$c_{J,k} \;\approx\; c_{J,k}^{\text{approx}} \;=\; \frac{1}{\sqrt{T}}\int_{-\infty}^{\infty} f(x)u\Big(\frac{x}{T} - k + u_{[1]} - \widetilde{\varphi}_{[1]}\Big)dt, \tag{26}$$

that we call, in future references, 1-Point formula. From Corollary 2, the error for this formula satisfies

$$\mathcal{E}(f) = \frac{|G^{(2)}(0)|}{2}\|f^{(2)}\|T^2 + \mathcal{O}(T^3), \quad f\in W_2^3,$$

where $G^{(2)}(0) = \widetilde{\varphi}_{[2]} - \widetilde{\varphi}_{[1]}^2 - \left[u_{[2]} - u_{[1]}^2\right]$.

Sweldens and Piessens [21] proved that if $\widetilde{\varphi}$ is an orthogonal scaling function then $\widetilde{\varphi}_{[2]} = \widetilde{\varphi}_{[1]}^2$. Thus, the accuracy of the approximation of the 1-Point formula depends on the central moment of second order of the average function u. In the case $u = a^{-1}\chi_{[-a/2,a/2]}$ we have $|G^{(2)}(0)|/2 = a^2/24$. In the case $u = \delta$, we have $|G^{(2)}(0)| = 0$, and then, the 1-Point formula gives order 3. In more detail, from Corollary 2 and (24) we obtain that for this case,

$$\mathcal{E}(f) = \frac{\widetilde{\varphi}_{[3]} - \widetilde{\varphi}_{[1]}^3}{6} \|f^{(3)}\| \, T^3 + \mathcal{O}(T^4), \quad f \in W_2^4.$$

3.2 Symmetric formulas

When the dual scaling functions φ, $\widetilde{\varphi}$ and the average function u are symmetric functions, in order to take advantage of the symmetry, we take coefficients α_n symmetric and $\tau = 0$. Thus, the order condition (19) is satisfied for l odd, since $\alpha_{[k]} = u_{[k]} = 0$ for k odd. The recursion relation (23) for order $L = 2M$ reads as

$$\alpha_{[0]} = 1, \quad \alpha_{[2m]} = \frac{1}{B^{2m}}\left[\widetilde{\varphi}_{[2m]} - \sum_{r=0}^{m-1}\binom{2m}{2r}\alpha_{[2r]}u_{[2m-2r]}B^{2r}\right], \quad m = 0, 1, \ldots, M-1. \quad (27)$$

The solution of the system

$$2\begin{bmatrix} 1/2 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2^2 & \cdots & (M-1)^2 \\ 0 & 1 & 2^4 & \cdots & (M-1)^4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2^{2(M-1)} & \cdots & (M-1)^{2(M-1)} \end{bmatrix}\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix} = \begin{bmatrix} \alpha_{[0]} \\ \alpha_{[2]} \\ \vdots \\ \alpha_{[2(M-1)]} \end{bmatrix} \quad (28)$$

gives us a symmetric sequence with $L - 1 = 2M - 1$ points, $\alpha_{-(M-1)}, \ldots, \alpha_0, \ldots \alpha_{M-1}$, with these moments, and giving $L = 2M$ order.

4 Determination of the translation parameter

Now we consider the calculation of translation parameters τ for which there exists an $(L-1)$-Point formula of order L (see Section 3.2 for the symmetric case). First notice that we can write the L order condition (22) in matrix form as

$$\begin{bmatrix} P_0(\tau) \\ P_1(\tau) \\ \vdots \\ P_{L-1}(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}u_{[1]} & 1 & 0 & \cdots & 0 \\ \binom{2}{0}u_{[2]} & \binom{2}{1}u_{[1]} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{L-1}{0}u_{[L-1]} & \binom{L-1}{1}u_{[L-2]} & \binom{L-1}{2}u_{[L-3]} & \cdots & 1 \end{bmatrix} D \begin{bmatrix} \alpha_{[0]} \\ \alpha_{[1]} \\ \vdots \\ \alpha_{[L-1]} \end{bmatrix} \quad (29)$$

where $P_l(\tau)$ is the polynomial $P_l(\tau) := \sum_{r=0}^{l} \binom{l}{r} \widetilde{\varphi}_{[r]}(-\tau)^{l-r}$ and D is the diagonal matrix $D := \operatorname{diag}(B^0, B^1, \dots, B^{L-1})$. If the nonzero weights are $\alpha_0, \dots, \alpha_{L-2}$, the vector of moments can be expressed as

$$\begin{bmatrix} \alpha_{[0]} \\ \alpha_{[1]} \\ \vdots \\ \alpha_{[L-1]} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2^1 & \cdots & (L-1)^1 \\ 0 & 1 & 2^2 & \cdots & (L-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2^{L-1} & \cdots & (L-1)^{L-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{L-2} \\ 0 \end{bmatrix}.$$

Denoting by W the $(L \times L)$-matrix in (29), and by V the above Vandermonde matrix, we have that

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{L-2} \\ 0 \end{bmatrix} = (WDV)^{-1} \begin{bmatrix} P_0(\tau) \\ P_1(\tau) \\ \vdots \\ P_{L-1}(\tau) \end{bmatrix}.$$

The last equation of the above system is a polynomial equation of degree $L-1$ with one variable τ. Any real root of this equation gives us, by means of the above equality, the coefficients $\alpha_0, \dots, \alpha_{L-2}$ of an $(L-1)$-Point formula of order L. When L is odd, the existence of a real root of this equation is not guaranteed. If there is not such a root, we could try with another value for B.

4.1 Construction for large order

When the order L is large, the Vandermonde Matrix V is ill-conditioned. In the point sample case, Sweldens and Piessens [21] overcame this difficulty by using Chebyshev polynomials (see also [13]). The same technique also works in the case of average samples. Specifically, we use a scaled version of the Chebyshev polynomials of the first kind

$$T_l^*(t) := T_l\Big(\frac{2}{b-a}[t-a] - 1\Big)$$

instead of the monomials t^l, i.e. we compute $\{\alpha_n\}$ and τ satisfying

$$\int_{-\infty}^{\infty} T_l^*(t-\tau)\widetilde{\varphi}(t)dt = \int_{-\infty}^{\infty} T_l^*(t) \sum_n \alpha_n u(t-Bn)dt, \quad l = 0, 1, \dots, L-1, \tag{30}$$

which is equivalent to the order condition (19). We choose an interval $[a, b]$ containing the supports of the integrals in the above equality. The equations (30) can be written as

$$\begin{bmatrix} \mathcal{P}_0^*(\tau) \\ \mathcal{P}_1^*(\tau) \\ \vdots \\ \mathcal{P}_{L-1}^*(\tau) \end{bmatrix} = A \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{L-2} \\ 0 \end{bmatrix}$$

where $\mathcal{P}_l^*(\tau) := \displaystyle\int_{-\infty}^{\infty} T_l^*(t-\tau)\widetilde{\varphi}(t)dt$, and $A_{l,n} := \int_{-\infty}^{\infty} T_l^*(t)\, u(t-Bn)dt, \quad l,n = 0,\dots,L-1$. We can follow the same steps than before to compute τ and the coefficients α_n, but using the matrix A instead of WDV, thereby obtaining a better condition number.

For example, when $\widetilde{\varphi}$ is the orthogonal Daubechies scaling function $u =_6\phi$ (in the notation of [9]), the average function is $u = \chi_{[-1/2,1/2]}$, $L = 12$ and $B = 1$, the matrix WDV has condition number $31 \cdot 10^{13}$, but for $[a,b] = [-0.5, 11.5]$ (the support of $u =_6\phi$ is $[0,11]$), the condition number of A is 1006 (in the L^∞-norm).

We could compute the coefficients of $\mathcal{P}_l^*(\tau)$ from the moments $\widetilde{\varphi}_{[n]}$ and the Chebyshev coefficients but the conditioning of the method is bad. In order to compute $\mathcal{P}_l^*(\tau)$ directly from the refinement coefficients $\widetilde{h}_n$, first notice that

$$\mathcal{P}_i^*(\tau) = \int_{-\infty}^{\infty} T_i\Big(\frac{t-\tau-a/2-b/2}{(b-a)/2}\Big)\widetilde{\varphi}(t)dt, \tag{31}$$

and

$$\mathcal{P}_l^*(\tau) = \int_{-\infty}^{\infty} T_l^*(t-\tau)\sum_k \widetilde{h}_k\widetilde{\varphi}(2t-k)dt = \frac{1}{2}\sum_k \widetilde{h}_k \int_{-\infty}^{\infty} T_l\Big(\frac{t+k-2\tau-a-b}{b-a}\Big)\widetilde{\varphi}(t)dt. \tag{32}$$

On the other hand, there exist coefficients $w_i^{(l)}(\xi)$ such that

$$T_l\Big(\frac{x-\xi-a/2-b/2}{b-a}\Big) = \sum_{i=0}^{l} w_i^{(l)}(\xi) T_i\Big(\frac{2x}{b-a}\Big). \tag{33}$$

The coefficients $w_i^{(l)}(\xi)$ are polynomials of degree $l-i$ and can be computed using the algorithm given by D. Huybrechs and S. Vandewalle in [13, Appendix A]. For this case, and denoting $w_i^{(l)}(\xi)$ by $w_i^{(l)}$ and

$$\beta = \frac{\xi + a/2 + b/2}{a-b}$$

this algorithm reads as:

$w_0^{(0)} \leftarrow 1,\ w_0^{(1)} \leftarrow \beta,\ w_1^{(1)} \leftarrow \frac{1}{2},\ w_0^{(2)} \leftarrow 2\beta^2 - \frac{3}{4},\ w_1^{(2)} \leftarrow 2\beta,\ w_2^{(2)} \leftarrow \frac{1}{4}$
for $p \leftarrow 2, 3, \dots$ **do**
  $w_0^{(p+1)} \leftarrow 2\beta d_0^p + \frac{1}{2}w_1^p - w_0^{(p-1)}\ ,\quad w_1^{(p+1)} \leftarrow w_0^{(p)} + 2\beta w_1^{(p)} + \frac{1}{2}w_2^p - w_1^{(p-1)}$
  for $i \leftarrow 2, 3, \dots, p-1$ **do**
    $w_i^{(p+1)} \leftarrow \frac{1}{2}w_{i-1}^{(p)} + 2\beta w_i^{(p)} + \frac{1}{2}w_{i+1}^{(p)} - w_i^{(p-1)}$
  end for
  $w_p^{(p+1)} \leftarrow \frac{1}{2}w_{p-1}^{(p)} + 2\beta w_p^{(p)},\quad w_{p+1}^{(p+1)} \leftarrow \frac{1}{2}w_p^{(p)}$
end for.

For $x = t-\tau-a/2-b/2$ and $\xi = \tau - k$, the equality (33) gives

$$T_l\Big(\frac{x+k-2\tau-a-b}{b-a}\Big) = \sum_{i=0}^{l} w_i^{(l)}(\tau-k) T_i\Big(\frac{x-\tau-a/2-b/2}{(b-a)/2}\Big).$$

Then, from (32) and (31) we obtain

$$\mathcal{P}_l^*(\tau) = \frac{1}{2} \sum_k \widetilde{h}_k \sum_{i=0}^{l} w_i^{(l)}(\tau - k)\mathcal{P}_i^*(\tau).$$

Finally, having in mind that $w_l^{(l)}(\xi) = 2^{-l}$ and $\sum_k \widetilde{h}_k = 2$, we obtain the formula

$$\mathcal{P}_0^*(\tau) = 1, \quad \mathcal{P}_l^*(\tau) = \frac{2^{l-1}}{2^l - 1} \sum_{i=0}^{l-1} \sum_k \widetilde{h}_k w_i^{(l)}(\tau - k)\mathcal{P}_i^*(\tau), \quad l = 1, \dots, L-1$$

which allows us to compute recursively $\mathcal{P}_l^*(\tau)$ from $\widetilde{h}_k$.

5 Examples

As an example, we consider the scaling function $\varphi(t) = (1 - |t|)\chi_{[-1,1]}(t)$, the B-spline of degree 1, and the biorthogonal symmetric scaling function $\widetilde{\varphi}$ defined by its refinement coefficients $\widetilde{h}_0 = \frac{3}{2}$, $\widetilde{h}_1 = \widetilde{h}_{-1} = \frac{1}{2}$, $\widetilde{h}_2 = \widetilde{h}_{-2} = -\frac{1}{4}$, denoted in [7, 8] by ${}_{2,2}\phi$. The approximation order is $N = 2$.

We obtain formulas to compute the wavelet coefficients in this symmetric biorthogonal system from three types of samples and we estimate the corresponding errors.

For the first two types of samples the average functions are symmetric, and then we take $\tau = 0$ and symmetric coefficients. Besides, we take $B = 1$, which is clearly the best choice, having in mind the support of $\widetilde{\varphi}$ and u. For the third type of samples the support of the average function is $[0, 5]$ and the choice of τ and B is not so clear. In order to choose among the possible values we use Corollary 2.

Point Samples, $u = \delta$: We take $\tau = 0$ and $B = 1$. For $M = 2$ and $M = 3$, (27) and (28) give respectively the coefficients

$$\alpha_0 = 7/6, \quad \alpha_{-1} = \alpha_1 = -1/12,$$
$$\alpha_0 = 139/120, \quad \alpha_{-1} = \alpha_1 = -7/90, \quad \alpha_{-2} = \alpha_2 = -1/720.$$

The corresponding 3-point and 5-point formulas give respectively order 4 and 6. For the 1-point (26), 3-point and 5-point formulas, Corollary 2 gives respectively the estimations

$$\frac{1}{12}\|f^{(2)}\|T^2 + \mathcal{O}(T^3), \quad \frac{1}{720}\|f^{(4)}\|T^4 + \mathcal{O}(T^5), \quad \frac{1}{2880}\|f^{(6)}\|T^6 + \mathcal{O}(T^7)$$

for the errors $\mathbb{E}(f)$. Notice that the one-point formula gives only order 2, since the scaling function is not orthogonal.

In Table 1 we give the value of $\mathbb{E}(f)$ for the function $f(t) = e^{-t^2}$ and for $J = 0, 1, 2, 3, 4$, computed with the integral expression (9) (to compute $E(Tw)$, we use $\widehat{\widetilde{\varphi}}(w) = \prod_{n=1}^{\infty} \frac{1}{2} \sum \widetilde{h}_n e^{-inw/2^n}$). For greater values of J the error $\mathbb{E}(f)$ can be computed with 3 digits of precision by means of the above asymptotic formulas. This error is not only significant for the function e^{-t^2}, but also for any function with a similar frequency localization.

J	T	1-point	3-point	5-point
0	1.00e-00	9.85e-02	2.01e-02	1.64e-02
1	5.00e-01	3.56e-02	1.14e-03	4.85e-04
2	2.50e-01	9.79e-03	6.50e-05	9.08e-06
3	1.25e-01	2.50e-03	3.93e-06	1.49e-07
4	6.25e-02	6.30e-04	2.44e-07	2.35e-09

Table 1: Error $\mathbb{E}(e^{-t^2})$ using point samples

J	T	1-point	3-point	5-point
0	1.00e-00	1.71e-01	3.28e-02	1.54e-02
1	5.00e-01	5.55e-02	3.83e-03	7.65e-04
2	2.50e-01	1.48e-02	2.86e-04	1.69e-05
3	1.25e-01	3.77e-03	1.87e-05	2.88e-07
4	6.25e-02	9.45e-04	1.18e-06	4.61e-09

Table 2: Error $\mathbb{E}(e^{-t^2})$ using the average function $u = \chi_{[-1/2,1/2)}$

Average Samples, $u = \chi_{[-1/2,1/2)}$: For this case and for $M = 2$ and 3, formulas (27) and (28) give respectively the coefficients

$$\alpha_0 = 5/4, \quad \alpha_{-1} = \alpha_1 = -1/8,$$
$$\alpha_0 = 413/320, \quad \alpha_{-1} = \alpha_1 = -73/480, \quad \alpha_{-2} = \alpha_2 = 13/1920.$$

The corresponding 3-point and 5-point formulas give respectively order 4 and 6. For the 1-point (26), 3-point and 5-point formulas, Corollary 2 gives respectively the estimations

$$\frac{1}{8}\|f^{(2)}\|T^2 + \mathcal{O}(T^3), \quad \frac{13}{1920}\|f^{(4)}\|T^4 + \mathcal{O}(T^5), \quad \frac{661}{967680}\|f^{(6)}\|T^6 + \mathcal{O}(T^7)$$

for the errors $\mathbb{E}(f)$.

In table 2 we give the value of $\mathbb{E}(f)$ for the function $f(t) = e^{-t^2}$, for these formulas, and for $J = 0, 1, 2, 3, 4$. For greater values of J the error $\mathbb{E}(f)$ can be computed with 3 digits of precision with the above asymptotic formulas.

Change of wavelet coefficients: Assume now that the samples are the coefficients in the Daubechies orthogonal system of order 3, i.e. we consider the average function $u =_3\phi$ (in the notation of [9]) whose refinement coefficients are

$$\mathrm{u}_0 = \frac{1}{16}\big(1 + \sqrt{10} + \gamma\big), \ \mathrm{u}_1 = \frac{1}{16}\big(5 + \sqrt{10} + 3\gamma\big), \ \mathrm{u}_2 = \frac{1}{8}\big(5 - \sqrt{10} + \gamma\big),$$
$$\mathrm{u}_3 = \frac{1}{8}\big(5 - \sqrt{10} - \gamma\big), \ \mathrm{u}_4 = \frac{1}{16}\big(5 + \sqrt{10} - 3\gamma\big), \ \mathrm{u}_5 = \frac{1}{16}\big(1 + \sqrt{10} - \gamma\big),$$

J	T	1-point	2-point	3-point	4-point	5-point
0	1.00e-00	1.46e-01	3.56e-02	5.58e-02	5.45e-02	1.38e-02
1	5.00e-01	3.81e-02	6.41e-03	3.74e-03	3.66e-03	1.77 e-03
2	2.50e-01	9.89e-03	1.11e-03	1.44e-04	1.40e-04	3.95 e-07
3	1.25e-01	2.51e-03	1.56e-04	4.97e-06	4.58e-06	6.73 e-07
4	6.25e-02	6.30e-04	2.00e-06	1.84e-07	1.45e-07	1.07e-08

Table 3: Error $\mathbb{E}(e^{-t^2})$ using the wavelet coefficients in the Daubechies wavelets of order 3

where $\gamma = \sqrt{5+2\sqrt{10}}$. The approximation formula (7) used in this case is

$$\int_{-2}^{2} g(t)\ {}_{2,2}\phi(t)\,dt \ \approx\ \sum_n \alpha_n \int_0^5 g(t+Bn+\tau)\ {}_3\phi(t)\,dt.$$

The translation parameter of the 1-point formula is $\tau = \widetilde{\varphi}_{[1]} - u_{[1]} = (\gamma-5)/2 \approx -0.8174$. In order to obtain a 2-point formula with order 3, we follow the method in Section 4. For $B = 1$, we get two possible translation parameters $\tau = -3+\gamma/2 \pm \sqrt{15}/6$. We select $\tau = -3+\gamma/2-\sqrt{15}/6$ since it has a more favourable asymptotic formula (18), specifically $|G^{(3)}(0)| \approx 0.11$ versus $|G^{(3)}(0)| \approx 1.8$. The choice $B = 2$ leads to a less favourable asymptotic estimation, which is not surprising, if we take a look at the graphs of ${}_3\phi$ and ${}_{2,2}\phi$ [8, p.197 and p.273].

Following the same steps (with $B = 1$), we obtain respectively the translation parameters $\tau = -1.884726066187672$, $\tau = -1.889656917609170$ and $\tau = -2.987567895826448$ for the 3-point, 4-point and 5-point formulas. For the 1-point, 2-point, 3-point, 4-point and 5-point corresponding formulas, Corollary 2 gives respectively the estimations

$$\frac{1}{12}\|f^{(2)}\|T^2+\mathcal{O}(T^3),\quad 0.0198\,\|f^{(3)}\|T^3+\mathcal{O}(T^4),\quad 0.000636\,\|f^{(4)}\|T^4+\mathcal{O}(T^5)$$
$$0.0044351\,\|f^{(5)}\|T^5+\mathcal{O}(T^6)\quad \text{and}\quad 0.0015898\,\|f^{(6)}\|T^6+\mathcal{O}(T^7),$$

for the errors $\mathbb{E}(f)$.

In Table 3 we give the value of $\mathbb{E}(f)$ for the function $f(t) = e^{-t^2}$, for these formulas, and for $J = 0,1,2,3,4$. The errors for the one-point formula is very similar to the one in the case of point samples, since it has the same asymptotic estimation $T^2\|f^{(2)}\|/12+\mathcal{O}(T^3)$. For the values of T in the Table, there is only a very small improvement between 3-point and 4 point formulas, which was expected since the above estimations.

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