On generalized sampling in shift-invariant spaces

Antonio G. García[1] Gerardo Pérez-Villalón[2]

[1]Departamento de Matemáticas
Universidad Carlos III de Madrid

[2]Departamento de Matemáticas
E.U.I.T.T, U.P.M, Madrid

Statement of the problem

Consider a shift-invariant subspace of $L^2(\mathbb{R})$ with a (stable) generator $\varphi \in L^2(\mathbb{R})$:

$$V_\varphi := \Big\{ \sum_{n\in\mathbb{Z}} a_n \varphi(t-n) \,:\, \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}$$

Given s linear-time invariant systems $\mathcal{L}_j$ defined on V_φ

Problem

Stable recovering of any $f \in V_\varphi$ from the sequence of samples

$$\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}\,,$$

or

$$\{(\mathcal{L}_j f)(rn+\varepsilon_{j,n})\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$$

The shift-invariant space V_φ

Assume the following hypotheses on the generador φ :

- The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ

A Riezs basis for a Hilbert space $\mathcal{H}$ is a sequence of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U:\mathcal{H}\to\mathcal{H}$ is a bounded bijective operator

- φ is continuous on $\mathbb{R}$
- The series $\sum_{n=-\infty}^{\infty}|\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$

The pointwise sum $f(t)=\sum_{n\in\mathbb{Z}} a_n\varphi(t-n)$ defines a continuous function on $\mathbb{R}$

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Recall that the sequence $\{\varphi(\cdot-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ if and only if $0<\|\Phi\|_0 \le \|\Phi\|_\infty < \infty$, where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2$ in $(0,1)$, and $\|\Phi\|_\infty$ its essential supremum

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V_φ as a reproducing kernel Hilbert space

The evaluation functionals are bounded in V_φ:
For each fixed $t \in \mathbb{R}$ we have

$$|f(t)|^2 \leq \frac{\|f\|^2}{\|\Phi\|_0} \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 \quad f \in V_\varphi\,,$$

Convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence which is uniform on $\mathbb{R}$.

The isomorphism $\mathcal{T}$

Let $\mathcal{T} : L^2(0,1) \longrightarrow V_\varphi$ be the isomorphism defined by $\mathcal{T}(e^{-2\pi inw}) := \varphi(t-n)$ for each $n \in \mathbb{Z}$. Then:

- For any $f \in V_\varphi$ we have

$$f(t) = \langle F, K_t \rangle_{L^2(0,1)}, \quad t \in \mathbb{R}$$

where $F = \mathcal{T}^{-1} f$ and

$$K_t(w) = \sum_{n=-\infty}^{\infty} \overline{\varphi(t-n)} e^{-2\pi inw} = \overline{Z\varphi(t,w)}$$

($Z\varphi$ is the Zak transform of φ)

- $K_{t+m}(w) = e^{-2\pi imw} K_t(w)$
- $\mathcal{T}\left[e^{-2\pi imw} F(w)\right] = f(t-m)$ where $f = \mathcal{T}(F)$

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The linear-time invariant systems $\mathcal{L}_j$

We distinguish two types of filters $\mathcal{L}$:

1. The impulse response $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$
2. The impulse response h has the form:

$$h = \sum_{k=0}^{N} c_k \delta^k (t + d_k)$$

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$\big($In this case we assume that $\varphi^{(N)}$ exists on $\mathbb{R}$, and $\sum_{n\in\mathbb{Z}} |\varphi^k(t-n)|^2$is uniformly bounded on $\mathbb{R}$ for each $k = 0, 1, \dots, N\big)$

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For any $t \in \mathbb{R}$ the sequence

$$\{(\mathcal{L}\varphi)(t+n)\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$$

An expression for the samples

For any $f \in V_\varphi$ we have

$$(\mathcal{L}f)(t) = \langle F, \overline{Z\mathcal{L}\varphi}(t, \cdot)\rangle_{L^2(0,1)} \qquad t \in \mathbb{R},$$

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In particular:

$$\begin{aligned}(\mathcal{L}_j f)(rn) &= \langle F, \overline{Z\mathcal{L}_j\varphi}(rn,\cdot)\rangle_{L^2(0,1)} \\ &= \langle F, \overline{Z\mathcal{L}_j\varphi}(0,\cdot)e^{-2\pi i rn\cdot}\rangle_{L^2(0,1)}\end{aligned}$$

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and

$$\begin{aligned}(\mathcal{L}_j f)(rn+\varepsilon_{j,n}) &= \langle F, \overline{Z\mathcal{L}_j\varphi}(rn+\varepsilon_{j,n},\cdot)\rangle_{L^2(0,1)} \\ &= \langle F, \overline{Z\mathcal{L}_j\varphi}(\varepsilon_{j,n},\cdot)e^{-2\pi i rn\cdot}\rangle_{L^2(0,1)}\end{aligned}$$

An expression for the samples

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$$(\mathcal{L}f)(t) = \langle F, \overline{Z\mathcal{L}\varphi}(t,\cdot)\rangle_{L^2(0,1)} \qquad t \in \mathbb{R}\,,$$

where $F = \mathcal{T}^{-1} f$.

Consequence

We have to study the sequences

$$\left\{a_j(\cdot)e^{2\pi i r n \cdot}\right\}_{n\in\mathbb{Z}, j=1,2,\dots,s} \quad \text{in} \quad L^2(0,1)$$

Sequences $\{a_j(\cdot)e^{2\pi i r n \cdot}\}$ in $L^2(0,1)$

Consider the sequence $\{a_j(\cdot)e^{2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$, where $a_j \in L^2(0,1)$ for any $j = 1, 2, \ldots, s$.
Let $\mathbf{A}$ be the $s \times r$ matrix function defined on $(0,1)$ as

$$\mathbf{A}(w) := \begin{bmatrix} a_1(w) & a_1(w+\frac{1}{r}) & \cdots & a_1(w+\frac{r-1}{r}) \\ a_2(w) & a_2(w+\frac{1}{r}) & \cdots & a_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ a_s(w) & a_s(w+\frac{1}{r}) & \cdots & a_s(w+\frac{r-1}{r}) \end{bmatrix}$$

(we are considering 1-periodic extensions of the functions a_j)

Sequences $\{a_j(\cdot)e^{2\pi i r n \cdot}\}$ in $L^2(0,1)$

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and its related constants

$$\alpha_{\mathbf{A}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbf{A}^*(w)\mathbf{A}(w)] \quad \beta_{\mathbf{A}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbf{A}^*(w)\mathbf{A}(w)]$$

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$\lambda_{\min}$ ($\lambda_{\max}$) the smallest (the largest) eigenvalue of the matrix $\mathbf{A}^*(w)\mathbf{A}(w)$

The result

The following result holds:

- The sequence $\{a_j(\cdot)e^{2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a Bessel sequence in $L^2(0,1)$ if and only if $a_j \in L^\infty(0,1)$ for $j = 1, \dots, s$. In this case, the optimal Bessel bound is $\beta_{\mathbf{A}}/r$.
- The sequence $\{a_j(\cdot)e^{2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbf{A}} \le \beta_{\mathbf{A}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbf{A}}/r$ and $\beta_{\mathbf{A}}/r$.

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A sequence $\{f_k\}_{k=1}^\infty$ in a Hilbert space $\mathcal{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le B\|f\|^2, \quad \text{for all } f \in \mathcal{H}$$

- The sequence $\{a_j(\cdot)e^{2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbf{A}} \le \beta_{\mathbf{A}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbf{A}}/r$ and $\beta_{\mathbf{A}}/r$.

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A sequence $\{f_k\}_{k=1}^\infty$ of elements in a Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exists constants A, $B > 0$ such that

$$A\|f\|^2 \leq \sum_{k=1}^\infty |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{H}$$

The numbers A, B are called frame bounds.

Generalized regular sampling in V_φ

Given $f \in V_\varphi$, for every $j = 1, 2, \ldots, s$ we have that

$$(\mathcal{L}_j f)(rn) = \left\langle F(\cdot), \overline{g}_j(\cdot)\, e^{-2\pi i r n \cdot} \right\rangle_{L^2(0,1)}, \quad n \in \mathbb{Z}\,.$$

where $F = \mathcal{T}^{-1} f$ and $g_j(w) = (Z\mathcal{L}_j\varphi)(0, w)$.

Assume that $g_j \in L^\infty(0,1)$, and let $[a_1(w), \ldots, a_s(w)]$ be a vector with entries in $L^\infty(0,1)$ such that

$$\big[a_1(w), \ldots, a_s(w)\big]\ \mathbf{G}(w) = [1, 0, \ldots, 0] \ \text{ a.e. in } (0,1)\,.$$

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Then,

$$F(w) = r \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn) a_j(w) e^{-2\pi i n r w}$$

in $L^2(0,1)$.

Generalized regular sampling in V_φ

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Consequence

The sequences $\{\overline{g}_j(\cdot)\, e^{-2\pi i r n \cdot}\}$ and $\{r a_j(\cdot) e^{-2\pi i r n \cdot}\}$ are dual frames in $L^2(0,1)$

Generalized regular sampling in V_φ

The isomorphism $\mathcal{T}$ gives the following sampling formula in V_φ

$$\begin{aligned}
f(t) &= r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn)\mathcal{T}[a_j(\cdot)e^{-2\pi i rn\cdot}](t)\\
&= r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn)(\mathcal{T}a_j)(t-rn)\\
&= \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn)S_j(t-rn)
\end{aligned}$$

A generalized regular sampling result

Assume that $g_j \in L^\infty(0,1)$ for each $j = 1, 2, \dots, s$. The following statements are equivalent:

1. $\alpha_{\mathbf{G}} > 0$
2. There exists a frame for V_φ having the form $\{S_j(t - rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ such that for any $f \in V_\varphi$,

$$f = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn)\, S_j(\cdot - rn) \quad \text{in } L^2(\mathbb{R})$$

3. There exist functions $a_j \in L^\infty(0,1)$, $j = 1, 2, \dots, s$, such that

$$\big[a_1(w), \dots, a_s(w)\big]\, \mathbf{G}(w) = [1, 0, \dots, 0]$$

a.e. in $(0,1)$

A generalized regular sampling result

In case the equivalent conditions are satisfied:

- $S_j = r\mathcal{T} a_j, \;\; j = 1, 2, \ldots, s$
- Convergence of the sampling series is absolute and uniform on $\mathbb{R}$
- If $r = s$ then $\{S_j(t - sn)\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ is a Riesz basis for V_φ. The functions a_j form the first row of the matrix $\mathbf{G}^{-1}$

Generalized irregular sampling in V_φ

Starting point:

- The expression of the irregular samples

$$(\mathcal{L}_j f)(rn + \varepsilon_{j,n}) = \left\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n}, \cdot)e^{-2\pi i rn\cdot} \right\rangle$$

- The sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(0,\cdot)e^{-2\pi i rn\cdot}\}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbf{G}} \le \beta_{\mathbf{G}} < \infty$
  $\big(\alpha_{\mathbf{G}}/r$ and $\beta_{\mathbf{G}}/r$ are the optimal frame bounds $\big)$

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$\big(\alpha_{\mathbf{G}}/r$ and $\beta_{\mathbf{G}}/r$ are the optimal frame bounds $\big)$

A suitable approach: Consider the sequence

$$\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi i rn\cdot}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$$

as a perturbation

Generalized irregular sampling in V_φ

Perturbation result

Let $\{f_k\}_{k=1}^\infty$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A, B, and let $\{g_k\}_{k=1}^\infty$ be a sequence in $\mathcal{H}$.
If there exists a constant $R < A$ such that

$$\sum_{k=1}^{\infty} |\langle f_k - g_k, f\rangle|^2 \leq R\|f\|^2 \quad \text{for each } f \in \mathcal{H}\,,$$

then $\{g_k\}_{k=1}^\infty$ is a frame for $\mathcal{H}$. If $\{f_k\}_{k=1}^\infty$ is a Riesz basis, then $\{g_k\}_{k=1}^\infty$ is a Riesz basis.

Generalized irregular sampling in V_φ

In our case, taking

$$f_k := \overline{(Z\mathcal{L}_j\varphi)}(0,\cdot)e^{-2\pi i r n \cdot}$$
$$g_k := \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi i r n \cdot}$$
$$f := F = \sum_{l\in\mathbb{Z}} c_l e^{-2\pi i l \cdot} \in L^2(0,1)$$

one gets

$$\sum_k |\langle f_k - g_k, f\rangle|^2 = \sum_{j=1}^{s} \|D_{\varepsilon,j}c\|^2_{\ell^2(\mathbb{Z})} \leq \|D_\varepsilon\|^2 \|F\|^2$$

Generalized irregular sampling in V_φ

$$\sum_k |\langle f_k - g_k, f\rangle|^2 = \sum_{j=1}^{s} \|D_{\varepsilon,j} c\|^2_{\ell^2(\mathbb{Z})} \le \|D_\varepsilon\|^2 \|F\|^2$$

where

$$D_{\varepsilon,j}\ c := \Big\{ \sum_{k\in\mathbb{Z}} \big[\mathcal{L}_j\varphi(rn - k + \varepsilon_{j,n}) - \mathcal{L}_j\varphi(rn-k)\big] c_k \Big\}_{n\in\mathbb{Z}}$$

for each $c = \{c_l\}_{l\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, and

$$\|D_\varepsilon\|^2 = \sum_{j=1}^{s} \|D_{\varepsilon,j}\|^2$$

A generalized irregular sampling result

Irregular sampling result

Let $\varepsilon := \{\varepsilon_{j,n}\}$ be a sequence error such that $\sum_{j=1}^{s} \|D_{\varepsilon,j}\|^2 < \alpha_{\mathbf{G}}/r$. Then, there exists a frame $\{S_{j,n}^{\varepsilon}\}$ for V_φ such that, for any $f \in V_\varphi$

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn + \varepsilon_{j,n}) S_{j,n}^{\varepsilon}(t) \quad t \in \mathbb{R}$$

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A frame algorithm can be implemented in $\ell^2(\mathbb{Z})$ to approximate $f \in V_\varphi$ from the samples $\{(\mathcal{L}_j f)(rn + \varepsilon_{j,n})\}$

A generalized irregular sampling result

A practical problem

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For any sequence ε such that $\{\varepsilon_{j,n}\}_{n\in\mathbb{Z}} \subset [\alpha_j, \beta_j] \subset [-r, r]$ for each $j = 1, 2, \ldots, s$ the following inequality holds

$$\|D_\varepsilon\|^2 \leq \sum_{j=1}^{s} \Lambda_j \Gamma_j \, ,$$

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where, for each $j = 1, 2, \dots, s$,

$$\Lambda_j := \sup_{\substack{l=0,1,\dots,r-1 \\ \{d_k\}\subset[\alpha_j,\beta_j]}} \sum_{k\in\mathbb{Z}} \left|\mathcal{L}_j\varphi(rk+l+d_k) - \mathcal{L}_j\varphi(rk+l)\right| \,,$$

$$\Gamma_j := \sup_{d\in[\alpha_j,\beta_j]} \sum_{k\in\mathbb{Z}} \left|\mathcal{L}_j\varphi(k+d) - \mathcal{L}_j\varphi(k)\right| .$$

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In Spline examples the constants Λ_j and Γ_j can be calculated in terms of δ