Regular and Irregular generalized sampling in Wavelet subspaces

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Statement of the problem

Let V_φ be a shift-invariant subspace of $L^2(\mathbb{R})$ with a (stable) generator $\varphi \in L^2(\mathbb{R})$, i.e.,

$$V_\varphi := \Big\{ \sum_{n\in\mathbb{Z}} a_n \varphi(t-n) \, : \, \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R})$$

Consider s linear-time invariant systems $\mathcal{L}_j$, $j = 1, 2, \dots, s$ defined on V_φ

The problem: Recover any function $f \in V_\varphi$ from the sequence of samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ by means of a sampling formula as

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn) S_j(t - rn) \,, \quad t \in \mathbb{R} \,,$$

where the sequence $\{S_j(t-rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for V_φ (The sampling period r necessarily satisfies $r \le s$)

A fresh approach to the problem

- We define a Fourier duality–type $\mathcal{T}_\varphi$ between $L^2(0,1)$ and V_φ
- We express the generalized samples of f in terms of a suitable frame for $L^2(0,1)$ and $F := \mathcal{T}_\varphi^{-1}(f) \in L^2(0,1)$
- We obtain its dual frames in $L^2(0,1)$
- Finally, a generalized sampling expansion for $f \in V_\varphi$ comes out by using $\mathcal{T}_\varphi$ in the frame expansion for F

The working hypotheses

Hypotheses on V_φ

Assume the following hypotheses on the generator φ :

- The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ
  Recall that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ if and only if $0 < \|\Phi\|_0 \le \|\Phi\|_\infty < \infty$, where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2$ in $(0,1)$, and $\|\Phi\|_\infty$ its essential supremum
- The generator φ is continuous on $\mathbb{R}$
- The series $\sum_{n=-\infty}^{\infty} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$
  Thus, the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n \varphi(t-n)$ defines a continuous function on $\mathbb{R}$

As a consequence, V_φ becomes a RKHS (reproducing kernel Hilbert space) and, in particular, the $L^2(\mathbb{R})$-norm convergence in V_φ implies uniform convergence on $\mathbb{R}$

Hypotheses on the systems $\mathcal{L}_j$

We consider two types of linear-time invariant systems:

1. The impulse response $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$
2. The impulse response h has the form:

$$h = \sum_{k=0}^{N} c_k \delta^{(k)}(t + d_k)$$

(In this case we assume that $\varphi^{(N)}$ exists on $\mathbb{R}$, and $\sum_{n\in\mathbb{Z}} |\varphi^k(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ for each $k = 0, 1, \dots, N$)

An expression for the samples

The isomorphism $\mathcal{T}_\varphi$

Let $\mathcal{T}_\varphi : L^2(0,1) \longrightarrow V_\varphi$ be the isomorphism defined by $\mathcal{T}_\varphi(e^{-2\pi i n w}) := \varphi(t-n)$ for each $n \in \mathbb{Z}$. Then:

- For any $f \in V_\varphi$ we have

$$f(t) = \langle F, K_t \rangle_{L^2(0,1)} \,, \quad t \in \mathbb{R}$$

where $F = \mathcal{T}_\varphi^{-1}(f)$ and

$$K_t(w) = \sum_{n=-\infty}^{\infty} \overline{\varphi(t-n)} e^{-2\pi i n w} = \overline{Z\varphi(t,w)}$$

($Z\varphi$ is the Zak transform of φ)

- $K_{t+m}(w) = e^{-2\pi i m w} K_t(w)$
- $\mathcal{T}_\varphi\big[e^{-2\pi i m w} F(w)\big] = f(t-m)$ where $f = \mathcal{T}_\varphi(F)$

An expression for the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$

Let $\mathcal{L}$ be a system as those considered here. For any $f \in V_\varphi$ we have

$$(\mathcal{L}f)(t) = \langle F, \overline{Z\mathcal{L}\varphi}(t,\cdot) \rangle_{L^2(0,1)} \qquad t \in \mathbb{R} \,,$$

where $F = \mathcal{T}_\varphi^{-1} f$. In particular, for each $j = 1, 2, \dots, s$:

$$\begin{aligned}(\mathcal{L}_j f)(rn) &= \langle F, \overline{Z\mathcal{L}_j\varphi}(rn,\cdot) \rangle_{L^2(0,1)} \\ &= \langle F, \overline{Z\mathcal{L}_j\varphi}(0,\cdot) e^{-2\pi i rn \cdot} \rangle_{L^2(0,1)} \,, \quad n \in \mathbb{Z}\end{aligned}$$

Denoting by $g_j := (Z\mathcal{L}_j\varphi)(0,\cdot)$, $j = 1, 2, \dots, s$, we consider the $s \times r$ matrix

$$\mathbf{G}(w) := \begin{bmatrix} g_1(w) & g_1(w+\frac{1}{r}) & \cdots & g_1(w+\frac{r-1}{r}) \\ g_2(w) & g_2(w+\frac{1}{r}) & \cdots & g_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ g_s(w) & g_s(w+\frac{1}{r}) & \cdots & g_s(w+\frac{r-1}{r}) \end{bmatrix}$$

(we are considering 1-periodic extensions of the functions g_j) and its related constants

$$\alpha_{\mathbf{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbf{G}^*(w)\mathbf{G}(w)] \,, \quad \beta_{\mathbf{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbf{G}^*(w)\mathbf{G}(w)]$$

$\lambda_{\min}$ ($\lambda_{\max}$) the smallest (the largest) eigenvalue of the positive semidefinite matrix $\mathbf{G}^*(w)\mathbf{G}(w)$

Two problems to solve:

- To characterize when the sequence $\{\overline{g}_j w) e^{-2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$

The sequence $\{\overline{g}_j w) e^{-2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbf{G}} \le \beta_{\mathbf{G}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbf{G}}/r$ and $\beta_{\mathbf{G}}/r$

As a consequence, for any $f \in V_\varphi$ we have the inequalities

$$0 < \frac{\alpha_{\mathbf{G}}}{r\|\Phi\|_\infty} \|f\|^2 \le \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} |(\mathcal{L}_j f)(rn)|^2 \le \frac{\beta_{\mathbf{G}}}{r\|\Phi\|_0} \|f\|^2$$

- To find its dual frames

Any sequence $\{r a_j(w) e^{-2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ where the functions $a_j \in L^\infty(0,1)$ satisfy

$$\big[a_1(w), \dots, a_s(w)\big] \, \mathbf{G}(w) = [1, 0, \dots, 0] \quad \text{a.e.}$$

is a dual frame of $\{\overline{g}_j w) e^{-2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$

Thus, for $F = \mathcal{T}_\varphi^{-1}(f)$ the following expansion holds:

$$F(w) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn) \, r a_j(w) e^{-2\pi i r n w} \quad \text{in } L^2(0,1)$$

The generalized regular sampling result

The main result: Assume that $g_j \in L^\infty(0,1)$ for each $j = 1, 2, \dots, s$ ($\Leftrightarrow \beta_G < \infty$). The following statements are equivalent:

1. $\alpha_{\mathbf{G}} > 0$
2. There exists a frame for V_φ having the form $\{S_j(t - rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ such that for any $f \in V_\varphi$,

$$f = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn) \, S_j(\cdot - rn) \quad \text{in } L^2(\mathbb{R})$$

3. There exist functions $a_j \in L^\infty(0,1)$, $j = 1, 2, \dots, s$, such that

$$\big[a_1(w), \dots, a_s(w)\big] \, \mathbf{G}(w) = [1, 0, \dots, 0]$$

a.e. in $(0,1)$

In case the equivalent conditions are satisfied:

- $S_j = r\mathcal{T}_\varphi(a_j)$, $j = 1, 2, \dots, s$
- Convergence of the sampling series is absolute and uniform on $\mathbb{R}$
- If $r = s$ then $\{S_j(t - sn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a Riesz basis for V_φ.
  The functions a_j form the first row of the matrix $\mathbf{G}^{-1}$
  In this case, the following interpolation property holds:

$$(\mathcal{L}_l S_j)(sn) = \delta_{j,l}\delta_{n,0} \quad j,l = 1, 2, \dots, s \,; \; n \in \mathbb{Z}$$

Some remarks:

- The first row of the pseudo-inverse matriz

$$\mathbf{G}^\dagger(w) = [\mathbf{G}^*(w)\mathbf{G}(w)]^{-1}\mathbf{G}^*(w)$$

gives the a_j functions defining the canonical frame dual of $\{\overline{g}_j w) e^{-2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$

- Other suitable a_j functions are given by the first row of the matrix $\mathbf{G}^\dagger(w) + \mathbf{U}(w)[\mathbf{I}_s - \mathbf{G}(w)\mathbf{G}^\dagger(w)]$, where $\mathbf{U}(w)$ is any $r \times s$ matrix function with entries in $L^\infty(0,1)$
- We can take advantage of the oversampling setting $s > r$. There exist many solutions for the a_j functions: One may use this flexibility to obtain appropriate sampling functions S_j. For instance, if the generator φ and the impulse responses of the filters $\mathcal{L}_j$ have compact support we could choose the a_j functions in order to obtain sampling functions S_j with compact support

Generalized irregular sampling

Assume that we have at our disposal the sequence of irregular samples $\{(\mathcal{L}_j f)(rn + \varepsilon_{j,n})\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$. Our starting point is:

- The expression of the irregular samples

$$(\mathcal{L}_j f)(rn + \varepsilon_{j,n}) = \big\langle F, \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n}, \cdot) e^{-2\pi i rn \cdot} \big\rangle \,, \quad n \in \mathbb{Z} \,,$$

for each $j = 1, 2, \cdots, s$.

- The sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(0,w) e^{-2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbf{G}} \le \beta_{\mathbf{G}} < \infty$

A suitable approach is to consider the sequence

$$\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n}, w) e^{-2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$$

as a perturbation of the frame $\{\overline{(Z\mathcal{L}_j\varphi)}(0,w) e^{-2\pi i r n w}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ for the error sequence $\varepsilon := \{\varepsilon_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$

For each $j = 1, 2, \cdots, s$, define on $\ell^2(\mathbb{Z})$ the linear operator

$$D_{\varepsilon,j}\, c := \Big\{ \sum_{k\in\mathbb{Z}} \big[\mathcal{L}_j\varphi(rn - k + \varepsilon_{j,n}) - \mathcal{L}_j\varphi(rn-k)\big] c_k \Big\}_{n\in\mathbb{Z}}$$

for each $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, and consider the norm

$$\|D_\varepsilon\| := \sup_{\|c\|_{\ell^2(\mathbb{Z})}=1} \sqrt{\sum_{j=1}^{s} \|D_{\varepsilon,j} c\|^2_{\ell^2(\mathbb{Z})}}$$

By using a standard result on perturbation of frames we get:

Let $\varepsilon := \{\varepsilon_{j,n}\}$ be a sequence error such that $\|D_\varepsilon\|^2 < \alpha_{\mathbf{G}}/r$. Then, there exists a frame $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ for V_φ such that, for any $f \in V_\varphi$

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn + \varepsilon_{j,n}) S^\varepsilon_{j,n}(t) \,, \quad t \in \mathbb{R}$$

Two practical problems:

- To estimate $\sum_{j=1}^{s} \|D_{\varepsilon,j}\|^2$ in terms of $\delta := \sup_{j,n} |\varepsilon_{j,n}|$. This can be done, for instance, in Spline spaces.
- The sampling functions $S^\varepsilon_{j,n}$ are impossible to determine: They depend on the error sequence. However, a frame algorithm can be implemented in $\ell^2(\mathbb{Z})$ to approximate $f \in V_\varphi$ from the samples $\{(\mathcal{L}_j f)(rn + \varepsilon_{j,n})\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$

Acknowledgements

This work is supported by Project No. BFM2003-01034 from the Spanish Ministerio de Ciencia y Tecnología.

References

The proofs of the results above exhibited can be found in the references below. See also references therein for previous and related works.

[1] A.G. García and G. Pérez-Villalón (2006). Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces. *Appl. Comput. Harmon. Anal.* **20**, 422–433.

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