Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces

Antonio García García

(joint work with **Héctor R. Fernández-Morales** and **Gerardo Pérez-Villalón**)

Departamento de Matemáticas
Universidad Carlos III de Madrid

1. Motivation of the problem
   - Drawbacks in classical Shannon's sampling theory

2. Generalized sampling theory in V_Φ^2
   - The shift-invariant space V_Φ^2
   - Generalized regular sampling in V_Φ^2
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Shannon sampling theorem

Any function f **band-limited** to the interval $[-1/2, 1/2]$, that is, $f(t) = \int_{-1/2}^{1/2} \widehat{f}(w) \mathrm{e}^{2\pi i t w} dw$ for each $t \in \mathbb{R}$, may be reconstructed from the sequence of samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}, \quad t \in \mathbb{R}.$$

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Its d-dimensional counterpart

Any function f band-limited to the d-dimensional cube $[-1/2, 1/2]^d$, i.e., $f(t) = \int_{[-1/2,1/2]^d} \widehat{f}(x) \mathrm{e}^{2\pi i x^\top t} dx$ for each $t \in \mathbb{R}^d$ may be reconstructed from the sequence of samples $\{f(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ as

$$f(t) = \sum_{\alpha\in\mathbb{Z}^d} f(\alpha) \frac{\sin \pi(t_1-\alpha_1)}{\pi(t_1-\alpha_1)} \cdots \frac{\sin \pi(t_d-\alpha_d)}{\pi(t_d-\alpha_d)}, \quad t \in \mathbb{R}^d.$$

Drawbacks in Shannon's theory

- It relies on the use of **low-pass ideal filters**
- The band-limited hypothesis is in contradiction with the idea of a **finite duration signal**
- The band-limiting operation generates **Gibbs oscillations**
- The **sinc function** has a very **slow decay** at infinity which makes computation in the signal domain very inefficient
- The sinc function is well-localized in the frecuency domain but it is **bad-localized** in the time domain
- In several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval

Solution:

To investigate sampling and reconstruction problems in spline spaces, wavelets spaces, and general shift-invariant spaces

$$V_\varphi^2 := \overline{\text{span}}_{L^2(\mathbb{R}^d)}\{\varphi(t-\alpha) \,:\, \alpha \in \mathbb{Z}^d\}$$
$$= \Big\{ \sum_{\alpha\in\mathbb{Z}^d} a_\alpha\, \varphi(t-\alpha) \,:\, \{a_\alpha\} \in \ell^2(\mathbb{Z}^d) \Big\} \subset L^2(\mathbb{R}^d)\,.$$

where

- The function $\varphi \in L^2(\mathbb{R}^d)$ is the generator of V_φ^2
- It is assumed that the sequence $\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^d}$ is a Riesz basis for V_φ^2

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A **Riesz basis** in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a bounded invertible operator

Statement of the problem

Consider a shift-invariant subspace V_Φ^2 in $L^2(\mathbb{R}^d)$ with r stable generators $\Phi := \{\varphi_1, \dots, \varphi_r\}$ in $L^2(\mathbb{R}^d)$, that is,

$$V_\Phi^2 = \Big\{ \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\, \varphi_k(t-\alpha) \,:\, d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2 \dots, r \Big\}$$

- Given s convolution systems $\mathcal{L}_j f = f * \mathrm{h}_j$ defined on V_Φ^2 where $\mathrm{h}_j \in L^2(\mathbb{R}^d)$ (average sampling) or linear combination of Dirac deltas (usual sampling)
- Taking samples at a lattice $M\mathbb{Z}^d$ of $\mathbb{Z}^d$ where M denotes a $d \times d$ matrix with integer entries and $\det M > 0$.

The generalized regular sampling problem consists of

The recovery of any function function $f \in V_\Phi^2$ from the sequence of samples

$$\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$$

- Intuitively, sampling at $M\mathbb{Z}^d$ we are using the sampling rate $1/r(\det M)$, so we will need $s \geq r(\det M)$ convolution systems
- We should assume a **stability condition** as: There exist two positive constants $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j f(M\alpha)|^2 \leq B\|f\|^2 \quad \text{for all} \quad f \in V_\Phi^2 .$$

- The recovery will be done by means of a **sampling formula**

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(t - M\alpha) , \quad t \in \mathbb{R}^d ,$$

where the reconstruction functions $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a **frame** for the shift-invariant space V_Φ^2

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The strategy to follow is:

- To identify the sequence of samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ as frame coefficients
- To search for the corresponding dual frames

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A sequence $\{f_k\}_{k=1}^{\infty}$ is a **frame** for a separable Hilbert space $\mathcal{H}$ if there exist constants A, $B>0$ (frame bounds) such that

$$A\|f\|^2 \le \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le B\|f\|^2 \quad \text{for all } f \in \mathcal{H}$$

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The frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ are **dual frames** if the equivalent expansions in $\mathcal{H}$ hold

$$f=\sum_{k=1}^{\infty}\langle f,f_k\rangle g_k=\sum_{k=1}^{\infty}\langle f,g_k\rangle f_k\,,\quad f\in\mathcal{H}$$

The shift-invariant space V_Φ^2

We assume the following hypotheses on the set of generators $\Phi := \{\varphi_1, \dots, \varphi_r\}$:

- The sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d, k=1,2\dots r}$ is a Riesz basis for V_Φ^2
- The functions in the shift-invariant space V_Φ^2 are continuous on $\mathbb{R}^d$.

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- The sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d, k=1,2\dots r}$ is a Riesz basis for V_Φ^2
- The functions in the shift-invariant space V_Φ^2 are continuous on $\mathbb{R}^d$. This is equivalent to say that the set of generators Φ are **continuous** on $\mathbb{R}^d$ and the series $\sum_{\alpha\in\mathbb{Z}^d} |\Phi(t-\alpha)|^2$ is **uniformly bounded** on $\mathbb{R}^d$.

  Thus, any $f \in V_\Phi^2$ is defined on $\mathbb{R}^d$ as the pointwise sum

$$f(t) = \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\, \varphi_k(t-\alpha),\ t \in \mathbb{R}^d$$

The shift-invariant space V^2_Φ

We assume the following hypotheses on the set of generators $\Phi := \{\varphi_1, \dots, \varphi_r\}$:

- The sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d, k=1,2\dots r}$ is a Riesz basis for V^2_Φ
- The functions in the shift-invariant space V^2_Φ are continuous on $\mathbb{R}^d$.

Thus, the space V^2_Φ becomes a reproducing kernel Hilbert space

$$\Downarrow$$

Convergence in V^2_Φ in the L^2-sense implies pointwise convergence which is uniform on $\mathbb{R}^d$

The isomorphism $\mathcal{T}_\Phi$

The space V_Φ^2 is the image of $L_r^2[0,1)^d$ by means of the isomorphism

$$\mathcal{T}_\Phi : L_r^2[0,1)^d \longrightarrow V_\Phi^2$$
$$\{e^{-2\pi i\alpha^\top w}\mathbf{e}_k\}_{\alpha\in\mathbb{Z}^d, k=1,2,\dots,r} \longmapsto \{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d, k=1,2,\dots,r}$$

where $\{\mathbf{e}_1,\dots,\mathbf{e}_r\}$ denotes the canonical basis of $\mathbb{R}^r$. Then, for any $\mathbf{F}=(F_1,\dots,F_r)^\top \in L_r^2[0,1)^d$ we have

$$f(t)=\mathcal{T}_\Phi\mathbf{F}(t)=\langle \mathbf{F},\mathbf{K}_t\rangle_{L_r^2[0,1)^d}\,,\quad t\in\mathbb{R}^d$$

where $\mathbf{K}_t(x):=\overline{\mathbf{Z}\Phi}(t,x)$, and $\mathbf{Z}\Phi$ denotes the **Zak transform** of Φ, i.e.,

$$(\mathbf{Z}\Phi)(t,w):=\sum_{\alpha\in\mathbb{Z}^d}\Phi(t+\alpha)\mathrm{e}^{-2\pi i\alpha^\top w}\,.$$

An expression for the samples

For any $f \in V_\Phi^2$ we have

$$\big(\mathcal{L}_j f\big)(t) = \langle \mathbf{F}, \big(\overline{\mathbf{Z}\mathcal{L}_j \Phi}\big)(t,\cdot)\rangle_{L_r^2[0,1)^d}\,, \quad \text{where } \mathbf{F} = \mathcal{T}_\Phi^{-1} f \in L_r^2[0,1)^d$$

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For any $f \in V^2_\Phi$ we have

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In particular:

$$\begin{aligned}\big(\mathcal{L}_j f\big)(M\alpha) &= \langle \mathbf{F}, \overline{\mathbf{Z}\mathcal{L}_j\Phi}(M\alpha,\cdot)\rangle_{L^2_r[0,1)^d} \\ &= \langle \mathbf{F}, \overline{\mathbf{Z}\mathcal{L}_j\Phi}(0,\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\rangle_{L^2_r[0,1)^d}\end{aligned}$$

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As a consequence,

We have to study when the sequence

$$\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}\,,$$

where $\mathbf{g}_j(x) := \mathbf{Z}\mathcal{L}_j\Phi(0,x)$, $j = 1, 2, \dots, s$, is a frame for $L^2_r[0,1)^d$

By introducing the $s \times r(\det M)$ matrix of functions

$$\mathbb{G}(x) := \begin{bmatrix} \mathbf{g}_1^\top(x) & \mathbf{g}_1^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_1^\top(x + M^{-\top} i_{\det M}) \\ \mathbf{g}_2^\top(x) & \mathbf{g}_2^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_2^\top(x + M^{-\top} i_{\det M}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_s^\top(x) & \mathbf{g}_s^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_s^\top(x + M^{-\top} i_{\det M}) \end{bmatrix},$$

and its related constants

$$A_{\mathbb{G}} := \operatorname*{ess\,inf}_{x \in [0,1)^d} \lambda_{\text{mín}}[\mathbb{G}^*(x)\mathbb{G}(x)], \quad B_{\mathbb{G}} := \operatorname*{ess\,sup}_{x \in [0,1)^d} \lambda_{\text{máx}}[\mathbb{G}^*(x)\mathbb{G}(x)],$$

where

$$\{i_1 = 0, i_2, \ldots, i_{\det M}\} = \mathbb{Z}^d \cap \{M^\top x : x \in [0,1)^d\},$$

we obtain that

The sequence $\left\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is **frame** for $L^2_r[0,1)^d$ if and only if $0 < A_{\mathbb{G}} \le B_{\mathbb{G}} < \infty$. In this case, the optimal frame bounds are $A_{\mathbb{G}}/(\det M)$ and $B_{\mathbb{G}}/(\det M)$

Moreover, assuming the existence of an $r \times s$ matrix $\mathbf{a}(x) := [\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)]$, with entries in $L^\infty[0,1)^d$, such that

$$\big[\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)\big]\ \mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}] \quad \text{a.e. in } [0,1)^d\,,$$

for $\mathbf{F} = \mathcal{T}_\Phi^{-1} f$ we get

$$\mathbf{F}(x) = (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \langle \mathbf{F}, \overline{\mathbf{Z}\mathcal{L}_j\Phi}(0,\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\rangle \mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}$$

$$= (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \big(\mathcal{L}_j f\big)(M\alpha)\,\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x} \quad \text{in} \quad L^2_r[0,1)^d$$

Finally, by using the isomorphism $\mathcal{T}_\Phi$ we obtain

$$\begin{aligned} f(t) &= (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(M\alpha)\mathcal{T}_\Phi\big(\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big)(t) \\ &= (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(M\alpha)(\mathcal{T}_\Phi\mathbf{a}_j)(t-M\alpha) \\ &= \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(M\alpha)S_{j,\mathbf{a}}(t-M\alpha)\,, \quad t\in\mathbb{R}^d \end{aligned}$$

where we have used the **shifting property**

$$\mathcal{T}_\Phi\big[\mathbf{F}(\cdot)e^{-2\pi i\alpha^\top\cdot}\big](t) = \mathcal{T}_\Phi\mathbf{F}(t-\alpha),$$

and that the space V_Φ^2 is a RKHS.

The main result

Assume that the functions $\mathbf{g}_j \in L_r^\infty[0,1)^d$, $j = 1, 2, \dots, s$ ($\iff B_{\mathbb{G}} < \infty$). The following statements are equivalents:

(a) $A_{\mathbb{G}} > 0$

(b) There exists an $r \times s$ matrix $\big[\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)\big]$ with entries $\mathbf{a}_j \in L_r^\infty[0,1)^d$ and satisfying

$$\big[\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)\big]\mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M - 1)r \times r}] \quad \text{a.e. in } [0,1)^d$$

(c) There exists a frame $\{S_j(\cdot - M\alpha)\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ for V_Φ^2 such that for any $f \in V_\Phi^2$

$$f = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d)$$

(d) There exists a frame $\{S_{j,\alpha}(\cdot)\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ for V_Φ^2 such that

$$f = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\alpha}(t)\,, \quad f \in V_\Phi^2$$

Some final comments

- All the matrices $\left[\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)\right]$ in the former result correspond to the first r rows of the matrices of the form

$$\mathbb{G}^{\dagger}(x) + \mathbb{U}(x)\left[\mathbb{I}_s - \mathbb{G}(x)\mathbb{G}^{\dagger}(x)\right],$$

  where $\mathbb{U}(x)$ is any $r(\det M) \times s$ matrix with entries in $L^\infty[0,1)^d$, and $\mathbb{G}^{\dagger}(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$
- If $s = r(\det M)$ we are in the Riesz bases setting
- The reconstruction functions $S_{j,\mathbf{a}} = (\det M)\mathcal{T}_{\Phi}(\mathbf{a}_j)$, $j = 1, 2, \dots, s$, are determined from the Fourier coefficients of the components of $\mathbf{a}_j(x) := [a_{1,j}(x), a_{2,j}(x), \dots, a_{r,j}]^\top$, $j = 1, 2, \dots, s$. If $\widehat{a}_{k,j}(\alpha) := \int_{[0,1)^d} a_{k,j}(x) \mathrm{e}^{2\pi i \alpha^\top x} dx$,

$$S_{j,\mathbf{a}}(t) = (\det M) \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} \widehat{a}_{k,j}(\alpha)\varphi_k(t-\alpha), \quad t \in \mathbb{R}^d.$$

A generalized sampling formula as a filter bank:

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha)$$

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$$S_{j,\mathbf{a}}(t) = \sum_{\beta \in \mathbb{Z}^d} \sum_{k=1}^{r} \widehat{a}_{k,j}(\beta) \varphi_k(t - \beta)$$

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\begin{aligned}
f(t) &= \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha) \\
&= \sum_{k=1}^{r} \sum_{\gamma\in\mathbb{Z}^d} \left\{ \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) \widehat{a}_{k,j}(\gamma - M\alpha) \right\} \varphi_k(t-\gamma)
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&= \sum_{k=1}^{r} \sum_{\gamma \in \mathbb{Z}^d} d_k(\gamma)\, \varphi_k(t - \gamma)
\end{aligned}
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A generalized sampling formula as a filter bank:

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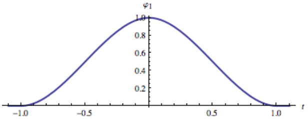
$$= \sum_{k=1}^{r} \sum_{\gamma \in \mathbb{Z}^d} d_k(\gamma)\, \varphi_k(t - \gamma)$$

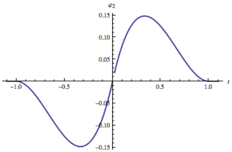
Oversampling, that is, $s > r(\det M)$, allows reconstruction functions $S_{j,\mathbf{a}}$ with prescribed properties

An illustrative example

Consider the Hermite cubic splines

$$\varphi_1(t) = \begin{cases} (t+1)^2(1-2t), & t \in [-1,0] \\ (1-t)^2(1+2t), & t \in [0,1] \\ 0, & |t| > 1 \end{cases}, \qquad \varphi_2(t) = \begin{cases} (t+1)^2 t, & t \in [-1,0] \\ (1-t)^2 t, & t \in [0,1] \\ 0, & |t| > 1 \end{cases}$$





which are stable generators for the subspace $V^2_{\varphi_1,\varphi_2} \subset L^2(\mathbb{R})$

Consider the sampling period $M = 1$ and the systems defined by

$$\mathcal{L}_1 f(t) := \int_t^{t+1/3} f(u)du\,, \quad \mathcal{L}_2 f(t) := \mathcal{L}_1 f\big(t+\frac{1}{3}\big), \quad \mathcal{L}_3 f(t) := \mathcal{L}_1 f\big(t+\frac{2}{3}\big)\,.$$

In this case, we have a 3×2 matrix $\mathbb{G}(x)$ with trigonometric polynomial entries; we can try to search for a 2×3 matrix $[\mathbf{a}_1(x), \mathbf{a}_2(x), \mathbf{a}_3(x)]$, with trigonometric polynomial entries, such that

$$[\mathbf{a}_1(x), \mathbf{a}_2(x), \mathbf{a}_3(x)]\,\mathbb{G}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\,.$$

Thus, we obtain in $V^2_{\varphi_1,\varphi_2}$ the following sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{3} \mathcal{L}_j f(n)\, S_{j,\mathbf{a}}(t-n)\,, \quad t \in \mathbb{R}\,,$$

where the sampling functions are given by

$$
\begin{aligned}
S_{1,\mathbf{a}}(t) &:= \frac{85}{44}\varphi_1(t) + \frac{1}{11}\varphi_1(t-1) + \frac{85}{4}\varphi_2(t) - \varphi_2(t-1)\,,\\
S_{2,\mathbf{a}}(t) &:= \frac{-23}{44}\varphi_1(t) - \frac{23}{44}\varphi_1(t-1) - \frac{23}{4}\varphi_2(t) + \frac{23}{4}\varphi_2(t-1)\,,\\
S_{3,\mathbf{a}}(t) &:= \frac{1}{11}\varphi_1(t) + \frac{85}{44}\varphi_1(t-1) + \varphi_2(t) - \frac{85}{4}\varphi_2(t-1)\,, \quad t \in \mathbb{R}\,.
\end{aligned}
$$

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