Analytic Sampling and Lagrange-Type Interpolation Series

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Whittaker-Shannon-Kotelnikov sampling theorem

We consider the classical Whittaker-Shannon-Kotelnikov sampling theorem in the Paley-Wiener space

$$PW_\pi = \left\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \quad \text{supp } \widehat{f} \subseteq [-\pi, \pi] \right\}$$

where $\widehat{f}$ stands for the Fourier transform. Any function f in PW_π can be written as

$$\begin{aligned} f(z) &= \left\langle \frac{e^{iz\omega}}{\sqrt{2\pi}}, \overline{\widehat{f}} \right\rangle_{L^2[-\pi,\pi]} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(\omega) e^{iz\omega} d\omega \ , \quad z \in \mathbb{C}, \end{aligned}$$

with $\widehat{f} \in L^2[-\pi, \pi]$ and the Fourier kernel (denoted K) is given by

$$\begin{array}{rl} K : \mathbb{C} \longrightarrow & L^2[-\pi, \pi] \\ z \longmapsto & K(z) \end{array} \ , \qquad [K(z)](\omega) = \frac{e^{iz\omega}}{\sqrt{2\pi}}, \quad \omega \in [-\pi, \pi].$$

Whittaker-Shannon-Kotelnikov sampling theorem

Expanding K with respect to the orthonormal basis $\left\{e^{inx}/\sqrt{2\pi}\right\}_{n\in\mathbb{Z}}$ of $L^2[-\pi,\pi]$, for each z fixed and replacing in

$$f(z) = \left\langle K, \overline{\widehat{f}} \right\rangle_{L^2[-\pi,\pi]}$$

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Whittaker-Shannon-Kotelnikov sampling theorem.

Any function f in the Paley-Wiener space PW_π can be recovered from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ as the cardinal series

$$f(z) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}, \quad z \in \mathbb{C}.$$

The convergence in the series is absolute and uniform on horizontal strips of $\mathbb{C}$.

Kramer sampling theorem

Kramer sampling theorem.

- Let K be a complex function defined on $I \times D$, where $I \subseteq \mathbb{R}$ is a bounded interval and D is an open subset of $\mathbb{R}$.
- For every $t \in D$, $K(\cdot, t)$ are in $L^2(I)$.
- Assume that there exists a sequence of distinct real numbers $\{t_n\}_{n\in\mathbb{Z}}$, such that the sequence $\{K(x, t_n)\}$ is a orthogonal basis for $L^2(I)$.

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$$S_n(t) := \frac{\int_I K(x,t)\overline{K(x,t_n)}dx}{\int_I |K(x,t_n)|^2 dx}.$$

Contents

The Hilbert space $\mathcal{H}_K$.

Given a complex, **separable Hilbert space** $\mathbb{H}$ and a function

$$\begin{aligned} K : \ \mathbb{C} &\longrightarrow \mathbb{H} \\ z &\longmapsto K(z) \end{aligned}$$

we define a mapping between $\mathbb{H}$ and the set $\mathcal{F}(\mathbb{C},\mathbb{C}) := \{f \, : \, \mathbb{C} \longrightarrow \mathbb{C}\}$ as follows:

$$\begin{array}{rccc} \mathcal{T}_K : & \mathbb{H} & \longrightarrow & \mathcal{F}(\mathbb{C},\mathbb{C}) \\ & x & \longmapsto & \mathcal{T}_K(x) = f_x \end{array}$$

such that

$$f_x(z) = \langle K(z), x \rangle_{\mathbb{H}} \quad z \in \mathbb{C}.$$

and denote by $\mathcal{H}_K$ the linear space of all functions $f_x(z)$ in the range space of $\mathcal{T}_K$; i.e.,

$$\mathcal{H}_K := \mathcal{T}_K(\mathbb{H}) = \Big\{ f \, : \, \mathbb{C} \longrightarrow \mathbb{C} : \ f(z) = \langle K(z), x \rangle_{\mathbb{H}} \, , \ x \in \mathbb{H} \Big\}.$$

Some properties of $\mathcal{H}_K$.

- The space $\mathcal{H}_K$ endowed with the norm

$$\|f\|_{\mathcal{H}_K} := \inf \{\|x\|_{\mathbb{H}} : \ f = \mathcal{T}_K(x)\} .$$

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- The space $\mathcal{H}_K$ endowed with the norm

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  becomes a Hilbert Space.

- The mapping $\mathcal{T}_K$ is a bijective isometry from $\mathbb{H}$ to $\mathcal{H}_K$ if and only if $\{K(z) :\ z \in \mathbb{C}\}$ is complete in $\mathbb{H}$ or equivalently if and only if $\mathcal{T}_K$ is injective.

  In particular, if there exist $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ such that $\{K(z_n)\}_{n=1}^{\infty}$ is a basis in $\mathbb{H}$, then $\mathcal{T}_K$ is an antilinear isometry from $\mathbb{H}$ onto $\mathcal{H}_K$.

Some properties of $\mathcal{H}_K$.

- $\mathcal{H}_K$ is an **Hilbert space of reproducing kernel** (RKHS in short); i.e., the evaluation functionals

$$\begin{aligned} E_z :\ \mathcal{H}_K &\longrightarrow \mathbb{C} \\ f \quad &\longmapsto f(z) \end{aligned}$$

are bounded. For fixed $z \in \mathbb{C}$, for any $f \in \mathcal{H}_K$, since $f(z) = \langle K(z), x \rangle_{\mathbb{H}}$ $x \in \mathbb{H}$, using the Cauchy-Schwarz inequality we obtain

$$|f(z)| \leq \|K(z)\|_{\mathbb{H}} \|x\|_{\mathbb{H}} = C_z \|f\|_{\mathcal{H}_K}$$

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which verifies the reproducing property

$$f(\omega) = \langle f(\cdot), \kappa(\cdot, \omega) \rangle_{\mathcal{H}} \text{ for each } \omega \in \mathbb{C} \text{ and } f \in \mathcal{H}_K$$

Analyticity of the functions in $\mathcal{H}_K$.

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Characterization of the analyticity of the functions in $\mathcal{H}_K$ in terms of Riesz bases.

- A Riesz basis for $\mathbb{H}$ a separable Hilbert space is a sequence of the form $\{Ue_n\}_{n=1}^{\infty}$ where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathbb{H}$ and $U : \mathbb{H} \to \mathbb{H}$ is a bounded biyective operator.

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- If $\{x_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathbb{H}$, then there exists a unique biorthonormal (dual) Riesz basis $\{x_n^*\}_{n=1}^{\infty}$ in $\mathbb{H}$, i.e, $\langle x_n, x_m^* \rangle_{\mathbb{H}} = \delta_{n,m}$., such that the expansions

$$x = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle_{\mathbb{H}} \, x_n = \sum_{n=1}^{\infty} \langle x, x_n \rangle_{\mathbb{H}} \, x_n^*, \quad x \in \mathbb{H},$$

holds.

Analyticity of the functions in $\mathcal{H}_K$.

Suppose that a Riesz basis $\{x_n\}_{n=1}^{\infty}$ is given and let $\{x_n^*\}_{n=1}^{\infty}$ be its dual Riesz basis.

Expanding $K(z)$ for $z \in \mathbb{C}$ fixed with respect to this basis we obtain

$$K(z) = \sum_{n=1}^{\infty} \langle K(z), x_n^* \rangle_{\mathbb{H}} \, x_n$$

where the sequence of coefficients

$$S_n(z) := \langle K(z), x_n^* \rangle_{\mathbb{H}}$$

as functions in z are in $\mathcal{H}_K$. The following result holds

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Theorem

$\mathcal{H}_K$ is a RKHS of entire functions if and only if the functions $\{S_n\}_{n=1}^{\infty}$ are entire and the function $z \mapsto \|K(z)\|_{\mathbb{H}}$ is bounded on compact sets of $\mathbb{C}$.

Contents

Sampling in $\mathcal{H}_K$.

Definition

An analytic kernel $K : \mathbb{C} \longrightarrow \mathbb{H}$ is said to be an **analytic Kramer kernel** if there are sequences $\{z_n\}_{n=1}^{\infty}$in $\mathbb{C}$, $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}\setminus\{0\}$ and a Riesz basis $\{x_n\}_{n=1}^{\infty}$ for $\mathbb{H}$, such that

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Analytic Kramer sampling theorem.

Let $K : \mathbb{C} \longrightarrow \mathbb{H}$ be an analytic Kramer kernel as in above definition and $\mathcal{H}_K$ its corresponding RKHS of entire functions.

Then, any $f \in \mathcal{H}_K$ can be recovered from its samples $\{f(z_n)\}_{n=1}^{\infty}$ by means of the sampling series

$$f(z) = \sum_{n=1}^{\infty} \frac{f(z_n)}{a_n} S_n(z), \quad z \in \mathbb{C}.$$

This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Contents

Lagrange-type interpolation series

In the Whittaker-Shannon-Kotelnikov sampling formula for each f in the Paley-Wiener space PW_π, and $z \in \mathbb{C}$,

$$f(z) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)} = \sum_{n\in\mathbb{Z}} f(n) \frac{G(z)}{(z-n)G'(n)}, \quad \text{where } G(z) = \frac{\sin \pi z}{\pi}$$

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Problem

In the Analytic Kramer sampling theorem, a more difficult question concerns whether the sampling expansion

$$f(z) = \sum_{n=1}^{\infty} \frac{f(z_n)}{a_n} S_n(z), \quad z \in \mathbb{C},$$

in $\mathcal{H}_K$ can be written as a Lagrange-type interpolation series.

Lagrange-type interpolation series

A necessary and sufficient condition involves the following algebraic property:

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Definition (Zero removing property)

A space $\mathcal{A}$ of entire functions has the zero-removing property (ZR in short) if for any $g \in \mathcal{A}$ and any zero ω of g the function $g(z)/(z-\omega)$ belongs to $\mathcal{A}$.

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A space $\mathcal{H}_K$ where the ZR property holds.

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A space $\mathcal{H}_K$ where the ZR property holds.

Theorem

Let $\mathbb{H}$ be a separable Hilbert space and $\mathcal{H}_K$ the RKHS associated with a polynomial kernel $K(z) = \sum_{k=0}^{N} a_k z^k$, where $a_k \in \mathbb{H}$ and $a_N \neq 0$. Then, $\mathcal{H}_K$ has the ZR property if and only if the set $\{a_0, a_1, ...a_N\}$ is linearly independent in $\mathbb{H}$.

Lagrange-type interpolation series. (ZR Property holds-Examples)

Example 1. (The Paley-Wiener space.)

The Paley-Wiener space

$$PW_\pi = \left\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \quad \text{supp } \widehat{f} \subseteq [-\pi, \pi] \right\}$$

satisfy the ZR property. Using the classical Paley-Wiener theorem, the space PW_π also is expressable as

$$PW_\pi = \left\{ f \text{ entire function : } \quad |f(z)| \leq Ae^{\pi|z|}, \quad f\,|_{\mathbb{R}} \in L^2(\mathbb{R}) \right\}$$

From this characterization the ZR property inmediately comes out.

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From this characterization the ZR property inmediately comes out.

Example 2. (The entire functions in the Pólya class.)

The entire function $F(z)$ is said to be of Pólya class if:

- It has no zeros in the upper half-plane.
- $|F(x - iy)| \leq |F(x + iy)|$, for $y > 0$.
- $|F(x + iy)|$ is a nondecreasing function of $y > 0$, for each fixed x.

Lagrange-type interpolation series. (ZR Property does not holds-Examples)

Example 3. Let $K : \mathbb{C} \longrightarrow \mathbb{H}$ be an analytic kernel such that $K(z_0) = 0$ for some $z_0 \in \mathbb{C}$. Then all the functions in the associated space $\mathcal{H}_K$ have a zero at z_0 and the ZR property does not hold in $\mathcal{H}_K$. Let f be a nonzero entire function in $\mathcal{H}_K$ and let r denote the order of its zero z_0. The function

$$\frac{f(z)}{(z - z_0)^r}$$

is not in $\mathcal{H}_K$.

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Example 4. Consider $\mathbb{H} = L^2[-\pi, \pi]$ and $K : \mathbb{C} \longrightarrow L^2[-\pi, \pi]$ be the analytic Kramer kernel defined by:

$$[K(z)](t) := \frac{e^{iz^2 t}}{\sqrt{2\pi}}$$

Its Taylor series around $z = 0$ is given by:

$$[K(z)](t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} z^{2n}$$

Lagrange-type interpolation series. (ZR Property does not holds-Examples)

The Taylor series for any function $f(z) = \langle K(z), F \rangle_{L^2[-\pi,\pi]}$ in the space $\mathcal{H}_K$, where $F \in L^2[-\pi,\pi]$ is of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{\langle (it)^n, F \rangle}{n!} z^{2n}, \quad z \in \mathbb{C}.$$

f is an even function. Let G be a nonzero function in $L^2[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi} G = 0.$$

Since $\langle K(0), G \rangle_{L^2[-\pi,\pi]} = 0$, the entire function $g(z) = \langle K(z), G \rangle_{L^2[-\pi,\pi]}$ verifies $g(0) = 0$. Therefore,

$$\frac{g(z)}{z} = \sum_{n=0}^{\infty} \frac{\langle (it)^n, G \rangle}{n!} z^{2n-1}$$

Clearly, $g(z)/z$ does not belong to $\mathcal{H}_K$.

Lagrange-type interpolation series.

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Theorem (Lagrange-type interpolation series)

Let $\mathcal{H}_K$ be a RKHS of entire functions obtained from an analytic Kramer kernel K with respect to the sequences $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}\setminus\{0\}$, i.e., for some Riesz basis $\{x_n\}_{n=1}^{\infty}$ for $\mathbb{H}$, $K(z_n) = a_n x_n, \quad n \in \mathbb{N}$.

Then, the sampling formula

$$f(z) = \sum_{n=1}^{\infty} \frac{f(z_n)}{a_n} S_n(z),\ z \in \mathbb{C},$$

for $\mathcal{H}_K$ can be written as a Lagrange-type interpolation series

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{P(z)}{P'(z_n)(z - z_n)}.$$

where P denotes an entire function having only simple zeros at $\{z_n\}_{n=1}^{\infty}$, if and only if the space $\mathcal{H}_K$ satisfies the ZR property.

The entire function P is such that $(z - z_n)S_n(z) = \sigma_n P(z)$ for some nonzero constants $\sigma_n, \ n \in \mathbb{N}$.

Lagrange-type interpolation series.

Example (The Paley-Wiener-Levinson theorem).

- Let $\{z_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$ such that

$$\sup_{n\in\mathbb{Z}} |\text{Re } z_n - n| < \frac{1}{4} \text{ and } \sup_{n\in\mathbb{Z}} |\text{Im } z_n| < \infty.$$

- The system $\{e^{iz_n\omega}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$. Let $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ be its dual Riesz basis in $L^2[-\pi,\pi]$.
- The Fourier kernel $[K(z)](\omega) = e^{iz\omega}/\sqrt{2\pi}$, $\omega \in [-\pi,\pi]$ is an analytic Kramer kernel for the data $\{z_n\}_{n\in\mathbb{Z}}$ and $\{a_n = 1\}_{n\in\mathbb{Z}}$.

Then, for any $f \in PW_\pi$,

$$f(z) = \sum_{n=-\infty}^{\infty} f(z_n) S_n(z) = \sum_{n=-\infty}^{\infty} f(z_n) \frac{P(z)}{(z-z_n)P'(z_n)}, \qquad z \in \mathbb{C},$$

where for $n \in \mathbb{Z}$, the sampling functions are $S_n(z) = \langle K(z), g_n \rangle_{L^2[-\pi,\pi]}$ and P is the entire function having only simple zeros at $\{z_n\}_{n\in\mathbb{Z}}$.

Lagrange-type interpolation series.

Example.

Let $\mathbb{H} = L^2[0,\pi]$ be and

$$\mathcal{H}_K = \left\{ f \ : f(z) = \langle \sin(zx), F \rangle_{\mathbb{H}} \,,\ \ F \in L^2[0,\pi] \right\}$$

the RKHS corresponding to the kernel $[K(z)](x) := \sin(zx)$.

Any function $f(z) = \langle \sin(zx), F(x) \rangle$ can be expanded as the sampling formula

$$f(z) = \frac{2}{\pi} \sum_{n=1}^{\infty} f(n) \frac{(-1)^n n \sin \pi z}{z^2 - n^2}, \quad z \in \mathbb{C}$$

This sampling formula cannot be expressed as a Lagrange-Type interpolation series because the space $\mathcal{H}_K$ does not satisfy the ZR property.

Lagrange-type interpolation series.

An example involving a Sobolev space.

Let $\mathbb{H} = H^1(-\pi,\pi)$ be with its usual inner product

$$\langle f,g\rangle_1 = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx + \int_{-\pi}^{\pi} f'(x)\overline{g'(x)}dx, \quad f,g \in H^1(-\pi,\pi)$$

The sequence $\{e_n(x)\}_{n\in\mathbb{Z}} = \left\{\mathrm{e}^{inx}\right\}_{n\in\mathbb{Z}} \cup \{\sinh x\}$ forms an orthogonal basis for $H^1(-\pi,\pi)$. For a fixed $a \in \mathbb{C}\setminus\mathbb{Z}$ we defined a kernel

$$\begin{array}{rcccc} K_a & : & \mathbb{C} & \longrightarrow & H^1(-\pi,\pi) \\ & & z & \longmapsto & K_a(z) \end{array}$$

by setting

$$[K_a(z)](x) = (z-a)\mathrm{e}^{izx} + \sin\pi z \sinh x, \quad x \in (-\pi,\pi).$$

and the space $\mathcal{H}_{K_a}$ is

$$\mathcal{H}_{K_a} := \left\{f : \mathbb{C} \longrightarrow \mathbb{C} : f(z) = \left\langle K_a(z), \overline{F}\right\rangle_1, \; F \in H^1(-\pi,\pi)\right\},$$

Lagrange-type interpolation series.

An example involving a Sobolev space.

Any function $f \in \mathcal{H}_{K_a}$ can be recovered from its samples $\{f(n)\}_{n\in\mathbb{Z}} \cup \{f(a)\}$ by means of the sampling formula

$$f(z) = [1 - i(z-a)]\frac{\sin \pi z}{\sin \pi a} f(a) + \sum_{n=-\infty}^{\infty} f(n) \frac{z-a}{n-a} \frac{1+zn}{1+n^2} \operatorname{sinc}(z-n)$$

The function $f(z) = (z-a)\operatorname{sinc} z$ belongs to $\mathcal{H}_{K_a}$ since

$$(z-a)\operatorname{sinc} z = \left\langle K_a(z), \frac{1}{2\pi} \right\rangle_1 ,$$

for all $z \in \mathbb{C}$; however, $f(z)/(z-a) = \operatorname{sinc} z$ does not belong to $\mathcal{H}_{K_a}$.

“The zero-removing property and Lagrange-type interpolation series”. *Num. Fun. Anal. and Optimin., 32(8):858-876, 2011.*

Thank You!