Sampling in reproducing kernel Banach spaces

Antonio G. García and **Alberto Portal**
Departamento de Matemáticas, Universidad Carlos III de Madrid
Departamento de Matemática Aplicada, ETSIT, Universidad Politécnica de Madrid

Hilbert spaces
Known framework

Consider a Hilbert space $\mathcal{H}$ of functions $f:\Omega\longrightarrow\mathbb{C}$.

$\mathcal{H}$ is a reproducing kernel Hilbert space **(RKHS)** if for each $t\in\Omega$ the evaluation functional at t, $\mathcal{E}_t(f):=f(t)$ for $f\in\mathcal{H}$, is continuous on $\mathcal{H}$.

The Riesz representation theorem gives a unique function $k:\Omega\times\Omega\longrightarrow\mathbb{C}$ such that

- $\{k(\cdot,t):t\in\Omega\}\subset\mathcal{H}$, and
- $f(t)=\langle f,k(\cdot,t)\rangle_{\mathcal{H}}$, $t\in\Omega$, $f\in\mathcal{H}$.

The function k is called the **reproducing kernel** of $\mathcal{H}$.

Banach spaces
New framework

Consider a Banach space $\mathcal{B}$ of functions $f:\Omega\longrightarrow\mathbb{C}$ such that for each $f\in\mathcal{B}$, its norm $\|f\|_{\mathcal{B}}$ vanishes if and only if, as a function, $f(t)=0$ for all $t\in\Omega$.

A **reproducing kernel Banach space** on Ω is a reflexive Banach space $\mathcal{B}$ of functions on Ω **(RKBS)** for which $\mathcal{B}^*$ is isometric to a Banach space $\widetilde{\mathcal{B}}$ of functions on Ω and the point evaluation functionals are continuous on both $\mathcal{B}$ and $\widetilde{\mathcal{B}}$.

The identification $\widetilde{\mathcal{B}}$ of $\mathcal{B}^*$ is not unique [5]. Denote the chosen identification as $\mathcal{B}^*$ and define the bilinear form on $\mathcal{B}\times\mathcal{B}^*$

$$(u,v^*)_{\mathcal{B}}:=v^*(u),\quad u\in\mathcal{B},\ v^*\in\mathcal{B}^*.$$

Suppose that $\mathcal{B}$ is an RKBS on Ω. Then there exists a unique function $k:\Omega\times\Omega\longrightarrow\mathbb{C}$ such that the following statements hold:

(a) For every $t\in\Omega$, $k(\cdot,t)\in\mathcal{B}^*$ and $f(t)=\big(f,k(\cdot,t)\big)_{\mathcal{B}}$ for all $f\in\mathcal{B}$.

(b) For every $t\in\Omega$, $k(t,\cdot)\in\mathcal{B}$ and $f^*(t)=\big(k(t,\cdot),f^*\big)_{\mathcal{B}}$ for all $f^*\in\mathcal{B}^*$.

(c) $\overline{\operatorname{span}}\{k(t,\cdot):t\in\Omega\}=\mathcal{B}$ and $\overline{\operatorname{span}}\{k(\cdot,t):t\in\Omega\}=\mathcal{B}^*$.

(d) For all $t,s\in\Omega$, $k(t,s)=\big(k(t,\cdot),k(\cdot,s)\big)_{\mathcal{B}}$.

This unique function k is the **reproducing kernel** for the RKBS $\mathcal{B}$. See [5, Th. 2].

Semi-inner products

A **semi-inner-product** on a Banach space $\mathcal{B}$ is a function $[\cdot,\cdot]:\mathcal{B}\times\mathcal{B}\longrightarrow\mathbb{C}$, such that [6], for all $x_1,x_2,x_3\in\mathcal{B}$ and $\alpha\in\mathbb{C}$:

1. $[x_1+x_2,x_3]=[x_1,x_3]+[x_2,x_3]$.
2. $[\alpha x_1,x_2]=\alpha[x_1,x_2]$ and $[x_1,\alpha x_2]=\overline{\alpha}[x_1,x_2]$.
3. $[x_1,x_1]>0$ for all $x_1\neq 0$.
4. $|[x_1,x_2]|^2\le[x_1,x_1][x_2,x_2]$.

Every normed vector space $\mathcal{B}$ has a semi-inner-product that **induces its norm** [2, 4].

We assume that for all $x,y\in\mathcal{B}$ with $x\neq 0$, $\lim_{\mathbb{R}\ni t\to 0}\frac{1}{t}(\|x+ty\|_{\mathcal{B}}-\|x\|_{\mathcal{B}})$ exists and the limit is uniform on $\mathcal{S}(\mathcal{B})\times\mathcal{S}(\mathcal{B})$ where $\mathcal{S}(\mathcal{B}):=\{x\in\mathcal{B}:\|x\|_{\mathcal{B}}=1\}$. **This guarantee the uniqueness of the semi-inner-product.**

If we also assume that $\mathcal{B}$ is uniformly convex, i.e., it is reflexive and strictly convex, then **a Riesz representation theorem holds** [2]: For each $f\in\mathcal{B}^*$ there exists a unique $x\in\mathcal{B}$ such that $f(y)=[y,x]_{\mathcal{B}}$ for all $y\in\mathcal{B}$. Moreover, $\|f\|_{\mathcal{B}^*}=\|x\|_{\mathcal{B}}$.

Sampling in an RKBS: The case of L^p shift-invariant spaces

Preliminaries

A measurable function $f:\mathbb{R}\to\mathbb{C}$ belongs to $\mathcal{L}^p(\mathbb{R})$, where $1\le p\le\infty$, whenever the function $\widetilde{f}(t):=\sum_{n\in\mathbb{Z}}|f(t-n)|$ is an element of $L^p[0,1]$. In this case, we define $|f|_p:=\|\widetilde{f}\|_{L^p[0,1]}$. Endowed with this norm, the space $(\mathcal{L}^p(\mathbb{R}),|\cdot|_p)$ becomes a Banach space (see [3]).

Given a function φ in $\mathcal{L}^\infty(\mathbb{R})$, for $1\le p<\infty$ we consider the **L^p shift-invariant space**

$$V_\varphi^p:=\overline{\operatorname{span}}_{L^p(\mathbb{R})}\{\varphi(t-n)\}_{n\in\mathbb{Z}}\subset L^p(\mathbb{R}).$$

If in addition the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an **ℓ^p-Riesz basis** for V_φ^p, i.e., there exist constants $0<A\le B$ such that

$$A\|a\|_{\ell^p}\le\Big\|\sum_{n\in\mathbb{Z}}a_n\varphi(t-n)\Big\|_{L^p(\mathbb{R})}\le B\|a\|_{\ell^p}\qquad(1)$$

for all $a\in\ell^p(\mathbb{Z})$, then V_φ^p can be expressed as

$$V_\varphi^p=\Big\{\sum_{n\in\mathbb{Z}}a_n\varphi(t-n):\{a_n\}\in\ell^p(\mathbb{Z})\Big\}\subset L^p(\mathbb{R}).$$

Since V_φ^p is a closed subspace of $L^p(\mathbb{R})$, it is a uniformly Fréchet differentiable and uniformly convex Banach space [6].

Assume that the functions in V_φ^p are continuous on $\mathbb{R}$.

Thus, the shift-invariant space V_φ^p becomes a RKBS, and the **convergence in the L^p sense implies pointwise convergence** which is uniform on $\mathbb{R}$ since Hölder's inequality shows that

$$|f(t)|\le\|a\|_{\ell^p}\|\{\varphi(t-n)\}_{n\in\mathbb{Z}}\|_{\ell^q}\le A^{-1}K\|f\|_{L^p(\mathbb{R})},$$

for $f\in V_\varphi^p$ and $t\in\mathbb{R}$.

Following [3], there exists a dual function φ^* to φ (regardless p) such that

$$k(t,s):=\sum_{n\in\mathbb{Z}}\varphi(s-n)\varphi^*(t-n),$$

is the **reproducing kernel** for V_φ^p. All the spaces V_φ^p, $1<p<\infty$ have the same reproducing kernel k although they are not isomorphic.

An average sampling formula in V_φ^p

For any $f\in V_\varphi^p$, consider the sequence of samples $\{(\mathcal{C}f)(n)\}_{n\in\mathbb{Z}}$, where the **convolution system** $\mathcal{C}$ satisfies:

(a) $(\mathcal{C}f)(t):=[f*\mathrm{h}](t)$, $t\in\mathbb{R}$, with $\mathrm{h}\in\mathcal{L}^q(\mathbb{R}^d)$ and q satisfying $1/p+1/q=1$; or

(b) $(\mathcal{C}f)(t):=f(t+a)$ for some fixed $a\in\mathbb{R}$.

Note that $\{(\mathcal{C}f)(n)\}_{n\in\mathbb{Z}}\in\ell^p(\mathbb{Z})$ since the inequalities $\big\|\{\mathrm{h}*f(n)\}_{n\in\mathbb{Z}}\big\|_p\le|\mathrm{h}|_q\|f\|_p$ (see [3, p. 220]) in the first case, and $\|a*b\|_{\ell^p}\le\|a\|_{\ell^p}\|b\|_{\ell^1}$ in the second one.

Let $\mathcal{A}$ be the Wiener algebra of the functions of the form $f(x)=\sum_{n\in\mathbb{Z}}a_ne^{-2\pi inx}$ with $a\in\ell^1(\mathbb{Z})$. The space $\mathcal{A}$, normed by $\|f\|_{\mathcal{A}}:=\|a\|_1$ and with pointwise multiplication becomes a commutative Banach algebra. If $f\in\mathcal{A}$ and $f(x)\neq 0$ for every $x\in\mathbb{R}$, the function $1/f$ is also in $\mathcal{A}$ by Wiener's lemma.

Assume that $G(x):=\sum_{n\in\mathbb{Z}}(\mathcal{C}\varphi)(n)e^{-2\pi inx}$ does not vanish for any $x\in[0,1]$. Then there exists a function $S\in\mathcal{L}^\infty(\mathbb{R})\cap V_\varphi^p$ such that, for any $f\in V_\varphi^p$, the following sampling formula holds:

$$f(t)=\sum_{n\in\mathbb{Z}}(\mathcal{C}f)(n)S(t-n),\qquad t\in\mathbb{R}.\qquad(2)$$

The convergence of the series is in the L^p-sense and uniform on $\mathbb{R}$.

The sequence of reconstruction functions $\{S(\cdot-n)\}_{n\in\mathbb{Z}}$ is a ℓ^p-Riesz basis for the Banach space $(V_\varphi^p,\|\cdot\|_p)$.

As a consequence of the Corollary above, the convergence of the series in (2) is also absolute due to the unconditional character of an ℓ^p-Riesz basis expansion.

Let $\mathcal{B}$ be an s.i.p. RKBS on Ω and k its reproducing kernel. Then there exists a unique function $G:\Omega\times\Omega\longrightarrow\mathbb{C}$ such that $\{G(t,\cdot):t\in\Omega\}\subset\mathcal{B}$, $k(\cdot,t)=(G(t,\cdot))^*$, $t\in\Omega$ and

$$\begin{aligned} f(t)&=[f,G(t,\cdot)]_{\mathcal{B}}\quad\text{for all } f\in\mathcal{B},\quad t\in\Omega,\\ f^*(t)&=[k(t,\cdot),f]_{\mathcal{B}}\quad\text{for all } f\in\mathcal{B},\quad t\in\Omega.\end{aligned}$$

G is the s.i.p. kernel of the s.i.p. RKBS $\mathcal{B}$. When $G=k$, we call G an s.i.p. reproducing kernel. An s.i.p. reproducing kernel G satisfies that $G(t,s)=[G(t,\cdot),G(s,\cdot)]_{\mathcal{B}}$, $t,s\in\Omega$.

Preliminaries

- Consider a uniformly Fréchet differentiable and uniformly convex Banach space $\mathcal{B}$. Its dual $\mathcal{B}^*$ has these properties as well.
- Let $[\cdot,\cdot]_{\mathcal{B}}$ be the unique compatible semi-inner product on $\mathcal{B}$.
- Let X_d be a reflexive BK-space on $\mathbb{N}$ such that
  - If $\sum_{n=1}^\infty c_nd_n$ converges for every $c\in X_d$, then $d\in X_d^*$.
  - If $\sum_{n=1}^\infty c_nd_n$ converges for every $d\in X_d^*$, then $c\in X_d$.
  - The canonical unit vectors $\{\delta_n\}_{n=1}^\infty$ form a **Schauder basis** for both X_d and X_d^*.

Let $\{x_n^*\}_{n=1}^\infty\subset\mathcal{B}^*$ be an **X_d^*-Riesz basis** for $\mathcal{B}^*$. This means that

1. $\overline{\operatorname{span}}\{x_n^*:\ n\in\mathbb{N}\}=\mathcal{B}^*$.
2. $\sum_{n=1}^\infty c_nx_n^*$ converges in $\mathcal{B}^*$ for all $c\in X_d^*$.
3. There exist $0<A\le B<\infty$ such that

$$A\|c\|_{X_d^*}\le\Big\|\sum_{n=1}^\infty c_nx_n^*\Big\|_{\mathcal{B}^*}\le B\|c\|_{X_d^*}\quad\text{for all } c\in X_d^*.\qquad(3)$$

There exists a unique (dual) X_d-Riesz basis $\{y_n\}_{n=1}^\infty$ for $\mathcal{B}$ such that (see [6, Th. 2.15]) $[y_m,x_n]_{\mathcal{B}}=\delta_{m,n}$ for $m,n\in\mathbb{N}$, and satisfying the expansions:

$$x=\sum_{n=1}^\infty[x,x_n]_{\mathcal{B}}y_n\quad(\text{for } x\in\mathcal{B})\qquad(4)$$

$$x^*=\sum_{n=1}^\infty[y_n,x]_{\mathcal{B}}x_n^*\quad(\text{for } x^*\in\mathcal{B}^*).\qquad(5)$$

Average sampling in $\mathcal{B}_K$

- Write $\{x_n^*\}_{n\in\mathbb{N}}=\bigcup_{m=1}^M\{x_{m,n}^*\}_{n\in\mathbb{N}}$ and $\{y_n^*\}_{n\in\mathbb{N}}=\bigcup_{m=1}^M\{y_{m,n}^*\}_{n\in\mathbb{N}}$.
- For $0\le\ell\le L$, consider functions $K_\ell:\Omega\longrightarrow\mathcal{B}$ and define, for each $x\in\mathcal{B}$ and $0\le\ell\le L$, the functions

$$f_{\ell,x}(z):=[x,K_\ell(z)]_{\mathcal{B}}.$$

- We have $L+1$ transforms $\mathcal{T}_\ell:\mathcal{B}\longrightarrow\mathbb{C}^\Omega$ such that $\mathcal{T}_\ell x=f_{\ell,x}$.
- Assume that $M\le L$.
- For each $z\in\Omega$, we have

$$K_\ell(z)^*=\sum_{n=1}^\infty\sum_{m=1}^M S_{m,n}^\ell(z)x_{m,n}^*,\quad 0\le\ell\le L,$$

where $S_{m,n}^\ell(z):=[y_{m,n},K_\ell(z)]_{\mathcal{B}}=f_{\ell,y_{m,n}}(z)$.

- Suppose that there exist L sequences $\{z_n^\ell\}_{n=1}^\infty$ in Ω, with $\ell\in\{1,2,\dots,L\}$, such that

$$S_{m,n}^\ell(z_k^\ell)=a_{\ell,m}^n\delta_{n,k},\quad n,k\in\mathbb{N},\qquad(6)$$

where $1\le m\le M$, $1\le\ell\le L$ and the coefficients $a_{\ell,m}^n$ are complex numbers such that the matrices

$$A_n:=\begin{pmatrix}a_{1,1}^n & a_{1,2}^n & \cdots & a_{1,M}^n\\ a_{2,1}^n & a_{2,2}^n & \cdots & a_{2,M}^n\\ \vdots & \vdots & \ddots & \vdots\\ a_{L,1}^n & a_{L,2}^n & \cdots & a_{L,M}^n\end{pmatrix}\in\mathbb{C}^{L\times M},\qquad(n\in\mathbb{N})\qquad(7)$$

have full rank for $n\in\mathbb{N}$, i.e., $\operatorname{rank}(A_n)=M$ for every $n\in\mathbb{N}$.

- Suppose the **compatibility condition:** $\ker\mathcal{T}_0\subseteq\bigcap_{\ell=1}^L\ker\mathcal{T}_\ell$ which implies that the mapping $\mathcal{T}_0$ is one-to-one.
- Denote by $A_n^{[M]}$ any regular $M\times M$ submatrix of A_n.

We have the following results:

For every $z\in\Omega$, the sequence $\bigcup_{m=1}^M\{S_{m,n}^0(z)\}_{n\in\mathbb{N}}$ is an element of X_d^*.

Each function $f\in\mathcal{B}_{K_0}$ can be recovered from the L sequences of samples $\{f_\ell(z_n^\ell)\}_{n=1}^\infty$, $1\le\ell\le L$, by means of the following sampling formula

$$f(z)=\sum_{n=1}^\infty\mathbb{S}_n(z)^\top\Big(A_n^{[M]}\Big)^{-1}\mathbf{F}_n,\quad z\in\Omega,\qquad(8)$$

where $\mathbf{F}_n$ and $\mathbb{S}_n(z)$ denote, respectively, $\big[f_1(z_n^1),\dots,f_L(z_n^L)\big]^\top$ and $\big[S_{0,1}^n(z),\dots,S_{0,M}^n(z)\big]^\top$. The convergence of the series in (8) is uniform in subsets of Ω where the function $z\mapsto\|K(z)\|_{\mathcal{B}}$ is bounded.

An illustrative example

In theorem above, for $L=M=1$ we obtain a generalization of **Kramer sampling theorem.** Next, we give an example where $\mathcal{B}:=L^p\big[\frac{-1}{2},\frac{1}{2}\big]$ for $p\in(1,2]$, with the compatible semi-inner product

$$[f,g]_p:=\|g\|_p^{2-p}\int_{-1/2}^{1/2}f(x)\overline{g(x)}|g(x)|^{p-2}\,dx.$$

Take $X_d:=\ell^q(\mathbb{Z})$ and consider $e_n(\xi):=\mathrm{e}^{2\pi in\xi}$ and $e_n^*(\xi)=\mathrm{e}^{-2\pi in\xi}$ for $n\in\mathbb{Z}$. Easy computations show that $\|e_n^*\|_q=\|e_n\|_p=1$. See [5] for details.

For $M=L=1$ and $K(z)=K_0(z)=K_1(z)=\mathrm{e}^{2\pi iz\xi}$. we obtain the following s.i.p. RKBS

$$\mathcal{B}_K:=\Big\{f(z)=\big[F,\mathrm{e}^{2\pi iz\xi}\big]_p,\ z\in\mathbb{C},\ \text{where } F\in L^p[-1/2,1/2]\Big\},$$

endowed with the norm $\|f\|_{\mathcal{B}_K}:=\|F\|_{L^p[-1/2,1/2]}$.

We have the following sampling formula for any $f\in\mathcal{B}_K$:

$$f(z)=\operatorname{sinc}^{(2-p)/p}(iyp)\sum_{n\in\mathbb{Z}}f(n)\operatorname{sinc}\big[(z-n)-iy(p-2)\big],\qquad(9)$$

where $z=x+iy\in\mathbb{C}$. The convergence of the series in (9) is uniform on horizontal strips of $\mathbb{C}$. Observe that, if $p=2$ or $z\in\mathbb{R}$, formula (9) coincides with the cardinal series.

References

[1] A. G. García and A. Portal. Sampling in reproducing kernel Banach spaces. Preprint, 2012.

[2] J. R. Giles. Classes of semi-inner-product spaces. *Trans. Amer. Math. Soc.*, 129: 436–446, 1967.

[3] R. Q. Jia and C. A. Micchelli. Using the refinement equations for the construction of pre-waveles II: Powers of two. In *Curves and Surfaces*, P. J. Laurent, A. Le Méhauté, L. L. Schumaker (eds.), Academic Press, Boston, pp. 209–246, 1991.

[4] G. Lumer. Semi-inner-product spaces. *Trans. Amer. Math. Soc.*, 100:29–43, 1961.

[5] H. Zhang, Y. Xu, and J. Zhang. Reproducing Kernel Banach Spaces for Machine Learning. *Journal of Machine Learning Research*, 10:2741–2775, 2009.

[6] H. Zhang and J. Zhang. Frames, Riesz Bases, and Sampling Expansions in Banach Spaces via Semi-inner Products. *Appl. Comput. Harmon. Anal.*, 31:1–25, 2011.

Acknowledgments: This work has been supported by the grant MTM2009–08345 from the D.G.I. of the Spanish *Ministerio de Ciencia y Tecnología*.