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Matrices with orthogonal groups admitting only determinant one

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Abstract

The K-orthogonal group of an n-by-n matrix K is defined as the set of nonsingular n-by-n matrices A satisfying $A^{\mathrm{T}}KA = K$, where the superscript $^{\mathrm{T}}$ denotes transposition. These form a group under matrix multiplication. It is well-known that if K is skew-symmetric and nonsingular the determinant of every element of the K-Orthogonal group is +1, i.e., the determinant of any symplectic matrix is +1. We present necessary and sufficient conditions on a real or complex matrix K so that all elements of the K-Orthogonal group have determinant +1. These necessary and sufficient conditions can be simply stated in terms of the symmetric and skew-symmetric parts of K, denoted by K_s and K_w respectively, as follows: the determinant of every element in the K-Orthogonal group is +1 if and only if the matrix pencil $K_w - \lambda K_s$ is regular and the matrix $(K_w - \lambda_0 K_s)^{-1} K_w$ has no Jordan blocks associated to the zero eigenvalue with odd dimension, where λ_0 is any number such that $\det(K_w - \lambda_0 K_s) \neq 0$.

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1. Introduction

Every n-by-n matrix K with entries in a field $\mathbb{F}$ has an associated bilinear form $\langle x, y\rangle = x^{\mathrm{T}}Ky$, where x and y are n-by-1 column vectors with entries in $\mathbb{F}$, and the superscript $^{\mathrm{T}}$ means transposition. A nonsingular matrix A satisfying $A^{\mathrm{T}}KA = K$ represents an isometry of this bilinear form and is said to be orthogonal with respect to K or K-Orthogonal. It is easily seen that for any K the set of K-Orthogonal matrices is a group, which we call the K-Orthogonal group. If the determinant of some K-Orthogonal matrix is d, the K-Orthogonal group is said to admit determinant d. We will assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, i.e. we consider only matrices with real or complex entries.

A classical result on K-Orthogonal groups is

Theorem 1.1. *If K is nonsingular and skew-symmetric then the K-Orthogonal group only admits determinant $+1$, i.e. every K-Orthogonal matrix has determinant $+1$.*

Matrices orthogonal with respect to a nonsingular and skew-symmetric matrix K are usually called *symplectic*, and they constitute one of the most important matrix groups [2]. In fact, the classical Theorem 1.1 has received attention very recently and several proofs of it may be found in [1,6]. For every real (complex) nonsingular and skew-symmetric matrix K there exists a real (complex) nonsingular matrix S such that $S^{\mathrm{T}}KS = J_p$, where $J_p = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}$ with I_p the p-by-p identity matrix. For real matrices, this follows from [4, Corollary 2.5.14, p. 107] by performing perfect shuffle permutations of rows and columns, and, for complex matrices, from [4, Problem 26, p. 217]. This property implies that to prove Theorem 1.1, one only needs to consider the case $K = J_p$.

Theorem 1.1 raises an interesting and natural question, namely, what are necessary and sufficient conditions on an n-by-n matrix K such that if $A^{\mathrm{T}}KA = K$ then $\det A = +1$? To answer this question is the purpose of this paper. An essential ingredient in this task is the canonical form for matrix congruence [5], or more precisely the canonical form under congruence of the matrix pair (or pencil) of the unique skew-symmetric and symmetric parts of K, i.e., (K_w, K_s) (or $K_w - \lambda K_s$), where $K_w \equiv (K - K^{\mathrm{T}})/2$ and $K_s \equiv (K + K^{\mathrm{T}})/2$ [9]. The main result we prove in terms of properties of canonical forms under congruence appears in Theorem 4.9. Due to the fact that the canonical form for matrix congruence is rather complicated, necessary and sufficient conditions for the determinant of every element of the K-Orthogonal group to be $+1$ are also written in terms of more basic matrix properties. This is also presented in the main Theorem 4.9. These conditions are extremely simple in the case of nonsingular K; see Corollary 4.11. We will see that if n is even then for almost all n-by-n matrices K, every K-Orthogonal matrix has determinant $+1$. This is a mildly surprising fact by taking into account that the usual orthogonal group ($K = I$) has elements with determinant $+1$ and -1.

Subsequent to the completion of this paper, we found that the problem we deal with has been recently considered in [7]. This reference treats the problem in the context of abstract bilinear spaces and group theory for arbitrary fields of characteristic not 2. Here, we follow a fully different approach by using only matrix analytic tools to yield simple and checkable necessary and sufficient conditions in terms of basic matrix canonical forms. We think that these conditions are more useful for the Matrix Analysis Community than the abstract conditions in [7].

The paper is organized as follows: in Section 2 several basic tools are introduced. Section 3 is devoted to an exhaustive description of the canonical forms under congruence of real and complex matrices. The final Section 4 has two parts: a set of technical lemmas are proved in Subsection 4.1;

based on these lemmas the main Theorem 4.9 is proved in Subsection 4.2, and the two interesting Corollaries 4.10 and 4.11 are presented. They provide, respectively, an abstract answer to our problem and a simple characterization of nonsingular matrices with orthogonal groups admitting only determinant one.

2. Preliminaries

In this section we present four auxiliary results frequently used throughout the paper. The set of n-by-n matrices with entries in $\mathbb{F}$ is denoted by $\mathrm{M}_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For simplicity of exposition, let us define:

$$\Xi_n \equiv \{K \in \mathrm{M}_n(\mathbb{F}) | A^{\mathrm{T}} K A = K \quad \text{implies } \det(A) = +1\}. \tag{2.1}$$

In the first place, let us recall that two matrices (complex or real) K and X are *congruent*[1] if there exists a nonsingular matrix S such that $S^{\mathrm{T}} X S = K$. We will often denote K *is congruent to* X by $K \cong X$. The first important remark is that Ξ_n membership is invariant under congruence.

Proposition 2.1. *Let K and X be two congruent matrices in $\mathrm{M}_n(\mathbb{F})$. Then $K \in \Xi_n$ if and only if $X \in \Xi_n$. Furthermore, let $S^{\mathrm{T}} X S = K$ with S nonsingular, then A is an element of the K-Orthogonal group if and only if SAS^{-1} is an element of the X-Orthogonal group.*

Proof. $A^{\mathrm{T}} K A = K$ is equivalent to $(SAS^{-1})^{\mathrm{T}} X (SAS^{-1}) = X$, from which the result follows trivially. □

Proposition 2.1 suggests the main strategy in this paper: to find a matrix $S^{\mathrm{T}} K S$, where S is nonsingular, with the simplest possible form, and solve the problem for this matrix. The second relevant remark is the following proposition.

Proposition 2.2. *Let $K \in \mathrm{M}_n(\mathbb{F})$ and K_w and K_s be, respectively, the skew-symmetric and symmetric parts of K, i.e., $K_w \equiv (K - K^{\mathrm{T}})/2$ and $K_s \equiv (K + K^{\mathrm{T}})/2$. Then the K-Orthogonal group is equal to the intersection of the K_w-Orthogonal and K_s-Orthogonal groups.*

Proof. The uniqueness of the skew-symmetric and symmetric parts implies that $A^{\mathrm{T}} K A = K$ if and only if $A^{\mathrm{T}} K_w A = K_w$ and $A^{\mathrm{T}} K_s A = K_s$. □

Proposition 2.2 and Theorem 1.1 imply the following extension of the classical Theorem 1.1.

Theorem 2.3. *Let $K \in \mathrm{M}_n(\mathbb{F})$ with $\det(K - K^{\mathrm{T}}) \neq 0$. Then $K \in \Xi_n$.*

Direct sums of matrices [4, Section 0.9.2] will play a relevant role in this paper. We will use very often, without explicitly mentioning, the following two properties that relate congruence and direct sums: $A \oplus B \cong B \oplus A$ by simple permutation matrices; and $A \cong B \Rightarrow A \oplus C \cong B \oplus C$ by the identity direct summed with the same congruence matrix as for A and B. The following result shows why direct sums are important in this work.

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1 In [4, Definition 4.5.4, p. 220], two types of congruences are considered: **congruence* and $^{\mathrm{T}}$ *congruence*. In this work, we only deal with $^{\mathrm{T}}$ congruence even in the case of complex matrices.

Proposition 2.4. *Let $K \in \mathrm{M}_n(\mathbb{F})$. If K is congruent to a direct sum with an odd-dimensional direct summand then $K \notin \Xi_n$. In particular if n is an odd number then $K \notin \Xi_n$.*

Proof. Let us assume that $K \cong Y_1 \oplus Y_2$, with $Y_1 \in \mathrm{M}_{n_1}(\mathbb{F})$ and $Y_2 \in \mathrm{M}_{n_2}(\mathbb{F})$, and that n_1 is odd. Then $(-I_{n_1}) \oplus I_{n_2}$ is in the orthogonal group of $Y_1 \oplus Y_2$, and $\det((-I_{n_1}) \oplus I_{n_2}) = -1$. Proposition 2.1 implies that $K \notin \Xi_n$. □

If n is even the set of matrices K with $\det(K - K^{\mathrm{T}}) = 0$ forms an algebraic manifold of codimension 1 in the set $\mathrm{M}_n(\mathbb{F})$. Therefore, Theorem 2.3 implies that the set of matrices $K \notin \Xi_n$ has zero Lebesgue measure, i.e., matrices whose K-Orthogonal groups contain elements with determinants different from $+1$ are extremely rare from a probabilistic point of view.

3. Canonical forms under congruence

As suggested by Proposition 2.1, we look for the simplest matrix that is congruent with a given matrix K, i.e., the canonical form for matrix congruence of K. This problem was solved for real matrices in [5, Theorem II], by observing that every real matrix K can be expressed as $K_w - \rho K_s$ where K_w and K_s are, respectively, the skew-symmetric and symmetric parts of K, and $\rho = -1$. Then [9, Theorem 2(c)] is applied to the skew-symmetric/symmetric pencil $K_w - \rho K_s$ to get a canonical form under matrix congruence for this pencil, and finally $\rho = -1$ is set[2]. The same approach is valid for complex matrices by applying [9, Theorem 1(c)]. So, it can be proven that any real matrix is congruent to a unique direct sum of canonical matrix blocks of *eight* different types, and that any complex matrix is congruent to a unique direct sum of canonical matrix blocks of *six* different types. To make this paper self-contained, let us describe these blocks in detail.

3.1. The canonical form block types for real matrices

We hereby abandon the original notation of [9,5], in place of a simpler form. The notation we shall use is:

Notation		*Notation of* [9, 5]
Γ_1	$\equiv$	∞_4'
Γ_2	$\equiv$	∞_5'
Γ_3	$\equiv$	α_3'
Γ_4	$\equiv$	β_4'
Γ_5	$\equiv$	β_5'
Γ_6	$\equiv$	m_3'
Γ_7	$\equiv$	$0_3'$
Γ_8	$\equiv$	$0_4'$

For easy reference, the block types can be specified by the following constituent blocks. Unless otherwise indicated, an entry is 0.

[2] The same result is obtained if the dual pencil $K_s - \rho K_w$ is considered.

Constituent blocks:

Name	*Dimensions*
$L_k^+ = \begin{pmatrix} 1 & & \\ 1 & \ddots & \\ & \ddots & 1 \\ & & 1 \end{pmatrix}$	$(k+1) \times k$
$L_k^- = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix}$	$k \times (k+1)$
$\Delta_k = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$	$k \times k$
$\Lambda_k = \begin{pmatrix} & & & 0 \\ & & 0 & 1 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & \end{pmatrix}$	$k \times k$
$Z_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	2×2
$R = \begin{pmatrix} 1 & p \\ -p & 1 \end{pmatrix}$	2×2
$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	2×2
$T^{\pm} = \begin{pmatrix} b & a \pm 1 \\ a \pm 1 & -b \end{pmatrix}$	2×2
$U_k^{\pm} = \begin{pmatrix} & & & T^{\pm} \\ & & T^{\pm} & S \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ T^{\pm} & S & & \end{pmatrix}$	$2k \times 2k$

where $p, a, b \in \mathbb{R}$ and $p > 0$, $b \neq 0$, and $a \neq 0$.

The canonical form blocks are:

Name	*Dimensions*
$\Gamma_1 = \pm\left[\begin{pmatrix} 0 & \Delta_q \\ -\Delta_q & 0 \end{pmatrix} + \Lambda_{2q}\right]$	$2q \times 2q$
$\Gamma_2 = \begin{pmatrix} 0 & \Delta_{2q+1} + \Lambda_{2q+1} \\ -\Delta_{2q+1} + \Lambda_{2q+1} & 0 \end{pmatrix}$	$(4q+2) \times (4q+2)$
$\Gamma_3 = \begin{pmatrix} 0 & (\alpha+1)\Delta_q + \Lambda_q \\ (-\alpha+1)\Delta_q - \Lambda_q & 0 \end{pmatrix}$	$2q \times 2q$
$\Gamma_4 = \pm\begin{pmatrix} & & & R \\ & & R & Z_2 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ R & Z_2 & & \end{pmatrix}$	$2q \times 2q$
$\Gamma_5 = \begin{pmatrix} 0 & U_q^+ \\ -U_q^- & 0 \end{pmatrix}$	$4q \times 4q$
$\Gamma_6 = \begin{pmatrix} 0 & L_q^+ \\ L_q^- & 0 \end{pmatrix}$	$(2q+1) \times (2q+1)$
$\Gamma_7 = \pm\left[\Delta_{2q+1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_q \\ 0 & -\Delta_q & 0 \end{pmatrix}\right]$	$(2q+1) \times (2q+1)$
$\Gamma_8 = \begin{pmatrix} 0 & \Delta_{2q} + \Lambda_{2q} \\ \Delta_{2q} - \Lambda_{2q} & 0 \end{pmatrix}$	$4q \times 4q$

where q is any nonnegative integer and $\alpha \neq 0$ a real number.

Now, we can state Theorem II in [5].

Theorem 3.1. *Every real square matrix K is congruent to a direct sum of matrix blocks of types $\Gamma_1, \Gamma_2, \ldots, \Gamma_8$. The parameters and dimensions of the blocks appearing in this direct sum are uniquely determined by K, i.e., this direct sum is unique up to permutations of the diagonal blocks.*

The direct sum appearing in Theorem 3.1 will be called the *canonical form under congruence* of K.

3.2. Skew-symmetric parts of canonical form block types of real matrices

We have seen in Theorem 2.3 that for a matrix K with nonsingular skew-symmetric part there can be no K-Orthogonal matrix with determinant -1. A natural question for the canonical form block types is thus which ones have nonsingular skew-symmetric part. We therefore consider the following:

Block	*Singularity*
$\Gamma_1 - \Gamma_1^{\mathrm{T}} = \pm 2\begin{pmatrix} 0 & \Delta_q \\ -\Delta_q & 0 \end{pmatrix}$	Nonsingular
$\Gamma_2 - \Gamma_2^{\mathrm{T}} = 2\begin{pmatrix} 0 & \Delta_{2q+1} \\ -\Delta_{2q+1} & 0 \end{pmatrix}$	Nonsingular
$\Gamma_3 - \Gamma_3^{\mathrm{T}} = 2\begin{pmatrix} 0 & \alpha\Delta_q + \Lambda_q \\ -\alpha\Delta_q - \Lambda_q & 0 \end{pmatrix}$	Nonsingular
$\Gamma_4 - \Gamma_4^{\mathrm{T}} = \pm\begin{pmatrix} & & & R - R^{\mathrm{T}} \\ & & R - R^{\mathrm{T}} & 2Z_2 \\ & \iddots & \iddots & \\ R - R^{\mathrm{T}} & 2Z_2 & & \end{pmatrix}$	Nonsingular
$\Gamma_5 - \Gamma_5^{\mathrm{T}} = \begin{pmatrix} 0 & U_q^{+} + U_q^{-} \\ -U_q^{-} - U_q^{+} & 0 \end{pmatrix}$	Nonsingular
$\Gamma_6 - \Gamma_6^{\mathrm{T}} = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I_q \\ 0 & -I_q & 0 \end{pmatrix}$	Singular
$\Gamma_7 - \Gamma_7^{\mathrm{T}} = \pm 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_q \\ 0 & -\Delta_q & 0 \end{pmatrix}$	Singular
$\Gamma_8 - \Gamma_8^{\mathrm{T}} = 2\begin{pmatrix} 0 & \Lambda_{2q} \\ -\Lambda_{2q} & 0 \end{pmatrix}$	Singular

The (non)singularity of each block follows directly from the definition of the constituent blocks.

3.3. The canonical form block types for complex matrices

The canonical form under complex congruence is simpler than the real one. It has only six types of blocks, namely, Γ_1 without the $\pm$ sign, Γ_2, Γ_3 with $\alpha \neq 0$ a complex number, Γ_6, Γ_7 without the $\pm$ sign, and Γ_8. The singular or nonsingular property of the corresponding skew-symmetric blocks is the same as for real matrices.

4. *K*-Orthogonal groups admitting only determinant one

The results in Section 2 can be easily combined with the canonical form under congruence to prove the simple but interesting Theorem 4.1.

Theorem 4.1. *Let $K \in \mathrm{M}_n(\mathbb{F})$.*

1. *If $K \in \Xi_n$ then the canonical form of K under congruence has no Γ_6 or Γ_7 direct summands.*
2. *If the canonical form of K under congruence has no Γ_6, Γ_7 or Γ_8 direct summands then $K \in \Xi_n$.*

Proof. The first item is a consequence of Proposition 2.4 and the fact that Γ_6 and Γ_7 are the only odd-dimensional block types in the canonical form of K under congruence. The second item follows from Proposition 2.1, Theorem 2.3 and the fact that the skew-symmetric parts of block types Γ_i, $i = 1, \ldots, 5$, are nonsingular. □

Theorem 4.1 makes clear that the block type Γ_8 deserves special attention in finding necessary and sufficient conditions for $K \in \Xi_n$. In fact, we will show in Theorem 4.9 that Γ_8 does not play any role in determining if $K \in \Xi_n$ or not, but several technical Lemmas have to be established before proving this.

A simple Corollary of Theorem 4.1 solves the problem for $n = 2$.

Corollary 4.2. *Let $K \in \mathrm{M}_2(\mathbb{F})$. $K \in \Xi_2$ if and only if* $\det(K - K^{\mathrm{T}}) \neq 0$.

Proof. Simply notice that a 2×2 matrix cannot have Γ_8 blocks in its canonical form, so $K \in \Xi_n$ if and only if the canonical form of K under congruence has no Γ_6 or Γ_7 direct summands. □

The reader should notice that Corollary 4.2 may also be easily proved by elementary methods without any reference to the canonical form under congruence.

4.1. Technical lemmas

The Lemmas in this section remain valid both for complex and real matrices. It will be necessary to pay attention to the specific dimensions of the blocks of Γ_8 type appearing in this subsection. For this purpose, we will denote a Γ_8 block of dimension $4q \times 4q$ by

$$\Gamma_8(4q) \equiv \begin{pmatrix} 0 & \Delta_{2q} + \Lambda_{2q} \\ \Delta_{2q} - \Lambda_{2q} & 0 \end{pmatrix}.$$

The following congruence will be frequently used.

Lemma 4.3

$$\Gamma_8(4) \cong \begin{pmatrix} 0 & I_2 \\ I_2 & Z_2 \end{pmatrix}.$$

Proof. Interchange the rows 2 and 3 and interchange the columns 2 and 3 of $\Gamma_8(4)$. □

Next, we will see that $\Gamma_8(4) \in \Xi_4$, because $\begin{pmatrix} 0 & I_2 \\ I_2 & Z_2 \end{pmatrix} \in \Xi_4$, despite the singular skew-symmetric part of $\Gamma_8(4)$. In fact, we prove a more general result.

Lemma 4.4. *Let $S \in \mathrm{M}_p(\mathbb{F})$ be a nonsingular and skew-symmetric matrix. Then*

$$\begin{pmatrix} 0 & I_p \\ I_p & S \end{pmatrix} \in \Xi_{2p}.$$

Proof. Let us consider the symmetric and skew-symmetric parts of $\begin{pmatrix} 0 & I_p \\ I_p & S \end{pmatrix}$. These are, respectively, $\begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$. According to Proposition 2.2, $A^{\mathrm{T}}\begin{pmatrix} 0 & I_p \\ I_p & S \end{pmatrix}A = \begin{pmatrix} 0 & I_p \\ I_p & S \end{pmatrix}$ if and only if $A^{\mathrm{T}}\begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}A = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ and $A^{\mathrm{T}}\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}A = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$. Let us partition A as $\begin{pmatrix} 0 & I_p \\ I_p & S \end{pmatrix}$, so

$$\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$
$$= \begin{pmatrix} A_{21}^{\mathrm{T}} S A_{21} & A_{21}^{\mathrm{T}} S A_{22} \\ A_{22}^{\mathrm{T}} S A_{21} & A_{22}^{\mathrm{T}} S A_{22} \end{pmatrix}.$$

Hence, $A_{22}^{\mathrm{T}} S A_{22} = S$, so $\det A_{22} = 1$ by Theorem 1.1, and $A_{21}^{\mathrm{T}}(SA_{22}) = 0$, so $A_{21} = 0$, since SA_{22} is nonsingular. Therefore

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad \text{and} \quad \det A = \det(A_{11}) \det(A_{22}) = \det(A_{11}). \tag{4.1}$$

Likewise $A^{\mathrm{T}}\begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}A = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ combined with equation (4.1) yields:

$$\begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_{11}^{\mathrm{T}} A_{22} \\ A_{22}^{\mathrm{T}} A_{11} & * \end{pmatrix}.$$

Thus $1 = \det I_p = \det(A_{11}^{\mathrm{T}} A_{22}) = \det(A_{11}) \det(A_{22})$. Combining again with equation (4.1), we have $\det A = +1$, and the Lemma is proved. □

Our next result extends Lemma 4.4.

Lemma 4.5. *Let*

$$K = \begin{pmatrix} 0 & 0 & I_p \\ 0 & Y & S_{12} \\ I_p & -S_{12}^{\mathrm{T}} & S_{22} \end{pmatrix},$$

where $Y \in \mathrm{M}_q(\mathbb{F})$. *Let us express* Y *as a sum of its symmetric and skew-symmetric parts as* $Y = H + S_{11}$, *where* $H = H^{\mathrm{T}}$ *and* $S_{11} = -S_{11}^{\mathrm{T}}$, *and let assume that* $S = \begin{pmatrix} S_{11} & S_{12} \\ -S_{12}^{\mathrm{T}} & S_{22} \end{pmatrix}$ *is nonsingular and skew-symmetric, then*

$$Y \in \Xi_q \quad \textit{implies } K \in \Xi_{2p+q}.$$

Proof. Let A be a K-Orthogonal matrix. By Proposition 2.2, A is K-Orthogonal if and only if A is K_s-Orthogonal and K_w-Orthogonal. Notice that the skew symmetric part of K is $K_w = 0_p \oplus S$, and let us partition $A = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$ according to this direct sum. Then $0_p \oplus S = A^{\mathrm{T}}(0_p \oplus S)A$ implies

$$A_{22}^{\mathrm{T}} S A_{22} = S \quad \text{and} \quad \det A_{22} = 1, \tag{4.2}$$

$$A_{21}^{\mathrm{T}} S A_{22} = 0 \quad \text{and} \quad A_{21} = 0, \tag{4.3}$$

since S and SA_{22} are nonsingular, and S is skew-symmetric. Consequently,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}. \tag{4.4}$$

Let

$$A_{22} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}$$

be further partitioned with $\Lambda_{11} \in \mathrm{M}_q(\mathbb{F})$ and $\Lambda_{22} \in \mathrm{M}_p(\mathbb{F})$. Also partition $A_{12} = (B_{12} B_{13})$, where B_{12} is a $p \times q$ matrix and B_{13} is a $p \times p$ matrix. Now, let us remember that A must satisfy $A^{\mathrm{T}} K_s A = K_s$, with K_s the symmetric part of K, i.e.,

$$\begin{pmatrix} & & I_p \\ & H & \\ I_p & & \end{pmatrix} = \left(\begin{array}{c|cc} A_{11} & B_{12} & B_{13} \\ \hline 0 & \Lambda_{11} & \Lambda_{12} \\ & \Lambda_{21} & \Lambda_{22} \end{array}\right)^{\mathrm{T}} \begin{pmatrix} & & I_p \\ & H & \\ I_p & & \end{pmatrix} \left(\begin{array}{c|cc} A_{11} & B_{12} & B_{13} \\ \hline 0 & \Lambda_{11} & \Lambda_{12} \\ & \Lambda_{21} & \Lambda_{22} \end{array}\right)$$
$$= \begin{pmatrix} 0 & A_{11}^{\mathrm{T}} \Lambda_{21} & A_{11}^{\mathrm{T}} \Lambda_{22} \\ \Lambda_{21}^{\mathrm{T}} A_{11} & \Lambda_{11}^{\mathrm{T}} H \Lambda_{11} + \Lambda_{21}^{\mathrm{T}} B_{12} + B_{12}^{\mathrm{T}} \Lambda_{21} & * \\ \Lambda_{22}^{\mathrm{T}} A_{11} & * & * \end{pmatrix}.$$

Hence,

$$\Lambda_{22} = A_{11}^{-\mathrm{T}}, \tag{4.5}$$

$$\Lambda_{21} = 0, \tag{4.6}$$

$$H = \Lambda_{11}^{\mathrm{T}} H \Lambda_{11}, \tag{4.7}$$

$$\text{thus} \quad A = \left(\begin{array}{c|cc} A_{11} & \multicolumn{2}{c}{A_{12}} \\ \hline 0 & \Lambda_{11} & \Lambda_{12} \\ & 0 & A_{11}^{-T} \end{array}\right), \tag{4.8}$$

$$\text{and} \quad \det A = \det(A_{11}) \det(\Lambda_{11}) \det(A_{11})^{-1} = \det \Lambda_{11}. \tag{4.9}$$

Combining Eqs. (4.6) and (4.2) we therefore have

$$S_{11} = \Lambda_{11}^{\mathrm{T}} S_{11} \Lambda_{11}, \tag{4.10}$$

which with Eq. (4.7) implies

$$Y = \Lambda_{11}^{\mathrm{T}} Y \Lambda_{11}. \tag{4.11}$$

From Eqs. (4.9) and (4.11) we directly obtain that $Y \in \Xi_q$ implies that $K \in \Xi_{2p+q}$. □

Lemma 4.5 will be fundamental in subsequent developments. In particular, it allows us, together with Lemma 4.4, to prove the following lemma. From now on, we do not indicate explicitly the dimension in Ξ_n.

Lemma 4.6. *If* $\det(Y - Y^{\mathrm{T}}) \neq 0$ (*including* Y *as the empty matrix*), *then*

$$\Gamma_8(4) \oplus \cdots \oplus \Gamma_8(4) \oplus Y \in \Xi,$$

for any number of $\Gamma_8(4)$ *summands.*

Proof. We will prove by induction on the number of $\Gamma_8(4)$ blocks appearing in $\Gamma_8(4) \oplus \cdots \oplus \Gamma_8(4) \oplus Y$ that this matrix is congruent to a matrix with the same structure as K in Lemma

4.5 when Y is not the empty matrix, and, otherwise, that it is congruent to a matrix as the one in Lemma 4.4. We mainly focus on the case Y is not the empty matrix. The reader can check that the same procedure remains valid if Y is empty simply by ignoring row or column permutations involving Y, and by erasing the block row and block column containing Y.

Let us prove first the basic case in which there is only one $\Gamma_8(4)$ block. Notice that $\Gamma_8(4) \oplus Y \cong \begin{pmatrix} 0 & I_2 \\ I_2 & Z_2 \end{pmatrix} \oplus Y$. This is a 3-by-3 block matrix, and if we interchange block rows 2 and 3, and block columns 2 and 3, we obtain

$$\Gamma_8(4) \oplus Y \cong \begin{pmatrix} 0 & 0 & I_2 \\ 0 & Y & 0 \\ I_2 & 0 & Z_2 \end{pmatrix}. \tag{4.12}$$

For the inductive step, let us define

$$Z_{2k} = \underbrace{Z_2 \oplus \cdots \oplus Z_2}_{k}.$$

The induction hypothesis is

$$\left(\bigoplus_{j=1}^{k} \Gamma_8(4) \right) \oplus Y \cong \begin{pmatrix} 0 & 0 & I_{2k} \\ 0 & Y & 0 \\ I_{2k} & 0 & Z_{2k} \end{pmatrix}, \tag{4.13}$$

which is true for $k = 1$ by (4.12). So, by Lemma 4.3

$$\left(\bigoplus_{j=1}^{k+1} \Gamma_8(4) \right) \oplus Y \cong \begin{pmatrix} 0 & I_2 \\ I_2 & Z_2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & I_{2k} \\ 0 & Y & 0 \\ I_{2k} & 0 & Z_{2k} \end{pmatrix}.$$

This is a 5-by-5 block matrix. If in this matrix we interchange block rows 2 and 3, and block columns 2 and 3, and then block rows 3 and 4, and block columns 3 and 4, we obtain

$$\left(\bigoplus_{j=1}^{k+1} \Gamma_8(4) \right) \oplus Y \cong \begin{pmatrix} 0 & 0 & I_{2(k+1)} \\ 0 & Y & 0 \\ I_{2(k+1)} & 0 & Z_{2(k+1)} \end{pmatrix}. \tag{4.14}$$

This matrix is contained in Ξ for any nonnegative k by Lemma 4.5. In the case Y is the empty matrix we should use Lemma 4.4. $\square$

Before stating the last lemma of this section, we will first need to establish the following lemma of which it makes extensive use.

Lemma 4.7. *For some real* $4q \times 2$ *matrix* Q *of rank* 2,

$$\Gamma_8(4(q+1)) \cong \begin{pmatrix} 0 & 0 & I_2 \\ 0 & \Gamma_8(4q) & Q \\ I_2 & -Q^{\mathrm{T}} & 0 \end{pmatrix}.$$

Proof. Let

$$W_{4(q+1)} \equiv \left(\begin{array}{cc|c|c|cc} 1 & 0 & & & & \\ \hline & & I_{2q} & & & \\ \hline 0 & 0 & & & 0 & 1 \\ 0 & 1 & & & 0 & 0 \\ \hline & & & I_{2q} & & \\ \hline & & & & 1 & 0 \end{array}\right),$$

$$\Gamma_8(4(q+1)) = \left(\begin{array}{cc|cccc|cccc|cc} & & & & & & & & & & 0 & 1 \\ \hline & & & & & & & & & 0 & 1 & 1 \\ & & & & & & & & & 1 & 1 & 0 \\ & & & & & & & & \cdot^{\cdot^{\cdot}} & 1 & & \\ & & & & & & 0 & 1 & \cdot^{\cdot^{\cdot}} & & & \\ \hline & & & & & 0 & 1 & 1 & & & & \\ & & & & & 1 & 0 & & & & & \\ \hline & & & & 1 & -1 & & & & & & \\ & & & \cdot^{\cdot^{\cdot}} & & & & & & & & \\ & & 1 & \cdot^{\cdot^{\cdot}} & & & & & & & & \\ 0 & 1 & -1 & & & & & & & & & \\ \hline 1 & -1 & 0 & & & & & & & & & \end{array}\right)$$

$$\equiv \left(\begin{array}{cc|cc|cc|cc} & & & & & & 0 & 1 \\ \hline & & & & & & 1 & 1 \\ & & & & & B_{2q} & 1 & 0 \\ & & & & & & & \\ \hline & & & 0 & 1 & 1 & & \\ & & & 1 & 0 & & & \\ \hline & & & & & & & \\ & & C_{2q} & & & & & \\ 0 & 1 & & & & & & \\ \hline 1 & -1 & 0 & & & & & \end{array}\right),$$

where the partitions are in agreement with those of $W_{4(q+1)}$ above, B_{2q} and C_{2q} are the matrices shown of size $2q \times 2q$. Hence,

$$W^{\mathrm{T}}_{4(q+1)}\Gamma_8(4(q+1))W_{4(q+1)} = W^{\mathrm{T}}_{4(q+1)}\Gamma_8(4(q+1)) \left(\begin{array}{cc|c|c|cc} 1 & 0 & & & & \\ \hline & & I_{2q} & & & \\ \hline 0 & 0 & & & 0 & 1 \\ 0 & 1 & & & 0 & 0 \\ \hline & & & I_{2q} & & \\ \hline & & & & 1 & 0 \end{array}\right)$$

$$
= \left(\begin{array}{cc|ccc|c|cc}
0 & 0 & \cdots & 0 & 0 & \multirow{2}{*}{$0_{(2\times 2q)}$} & 0 & 1 \\
0 & 0 & \cdots & 0 & 1 & & 0 & 0 \\
\hline
 & & & & & & 1 & 1 \\
 & & & & & & 1 & 0 \\
\multicolumn{2}{c|}{0_{(2q\times 2)}} & & 0_{2q} & & B_{2q} & \vdots & \vdots \\
 & & & & & & 0 & 0 \\
\hline
0 & 0 & & & & & & \\
\vdots & \vdots & & C_{2q} & & 0_{2q} & \multicolumn{2}{c}{0_{(2q\times 2)}} \\
0 & 1 & & & & & & \\
\hline
1 & -1 & & \multirow{2}{*}{$0_{(2\times 2q)}$} & & 0 \quad 0 \quad \cdots & 0 & 0 \\
0 & 0 & & & & 1 \quad 1 \quad \cdots & 0 & 0
\end{array}\right)
\left(\begin{array}{cc|c|c|cc}
1 & 0 & & & & \\
\hline
 & & I_{2q} & & & \\
\hline
0 & 0 & & & 0 & 1 \\
0 & 1 & & & 0 & 0 \\
\hline
 & & & I_{2q} & & \\
\hline
 & & & & 1 & 0
\end{array}\right)
$$

$$
= \left(\begin{array}{c|ccc|cc|ccc|c}
0 & & & & 0 & 0 & & & & 1 \\
0 & & & & 1 & 0 & & & & 0 \\
\hline
 & & & & & & & & 1 & 1 \\
 & & & & & & & \iddots & 1 & \\
 & & & & & & 1 & 1 & & \\
\hline
 & & & 1 & -1 & & & & & \\
 & & \iddots & -1 & & & & & & \\
 & 1 & -1 & & & & & & & \\
\hline
1 & -1 & & & 0 & 0 & 0 & & & \\
0 & 0 & & & 0 & 1 & 1 & & &
\end{array}\right)
\left(\begin{array}{cc|c|c|cc}
1 & 0 & & & & \\
\hline
 & & I_{2q} & & & \\
\hline
0 & 0 & & & 0 & 1 \\
0 & 1 & & & 0 & 0 \\
\hline
 & & & I_{2q} & & \\
\hline
 & & & & 1 & 0
\end{array}\right)
$$

(repartitioning in accordance with $W_{4(q+1)}$)

$$
= \left(\begin{array}{cc|cccc|ccccc|cc}
 & & & & & & & & & & & 1 & 0 \\
 & & & & & & & & & & & 0 & 1 \\
\hline
 & & & & & & & & & & 1 & 1 & 0 \\
 & & & & & & & & & 1 & 1 & 0 & \\
 & & & & & & & \iddots & & \iddots & & & \\
 & & & & & & 1 & 1 & & & & 0 & \\
\hline
 & & & & & 1 & & & & & & 0 & -1 \\
 & & & & 1 & -1 & & & & & & & 0 \\
 & & & \iddots & \iddots & & & & & & & & \\
 & & 1 & -1 & & & & & & & & & \\
\hline
1 & 0 & -1 & 0 & & & 0 & & & & & & \\
0 & 1 & 0 & & & & 1 & 0 & & & & &
\end{array}\right)
= \begin{pmatrix} 0 & 0 & I_2 \\ 0 & \Gamma_8(4q) & Q \\ I_2 & -Q^{\mathrm{T}} & 0 \end{pmatrix},
$$

where Q is a real $4q \times 2$ matrix of rank 2. Hence the congruence is explicitly shown. □

We may now proceed with the last lemma of this section. This lemma extends Lemma 4.6 by including blocks of Γ_8 type of dimension larger than 4.

Lemma 4.8. *If for some real or complex matrix* Y *such that* $\det(Y - Y^{\mathrm{T}}) \neq 0$ (*including* Y *as the empty matrix*) *and for some positive integers* $n_1, \ldots, n_m$

$$\Gamma_8(4n_1) \oplus \cdots \oplus \Gamma_8(4n_m) \oplus Y \in \Xi,$$

then

$$\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus \Gamma_8(4) \oplus \cdots \oplus \Gamma_8(4) \oplus Y \in \Xi$$

for any number (*including zero*) *of* $\Gamma_8(4)$ *summands.*

Proof. We begin by proving that $\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus Y \in \Xi$, i.e., the case without $\Gamma_8(4)$ summands. By Lemma 4.7,

$$
\begin{aligned}
&\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus Y \\
&\cong \begin{pmatrix} 0 & 0 & I_2 \\ 0 & \Gamma_8(4n_1) & Q_{n_1} \\ I_2 & -Q_{n_1}^{\mathrm{T}} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 0 & I_2 \\ 0 & \Gamma_8(4n_m) & Q_{n_m} \\ I_2 & -Q_{n_m}^{\mathrm{T}} & 0 \end{pmatrix} \oplus Y \\
&\cong \left(\begin{array}{c|cc|cc|c|cc|c}
 & & I_2 & & 0_2 & & & & \\
 & & & & I_2 & \ddots & & & \\
 & & & & & & & I_2 & \\
\hline
I_2 & \Gamma_8(4n_1) & Q_{n_1} & & & & & & \\
 & -Q_{n_1}^{\mathrm{T}} & & & & & & & \\
\hline
0_2 \quad I_2 & & & \Gamma_8(4n_2) & Q_{n_2} & & & & \\
 & & & -Q_{n_2}^{\mathrm{T}} & & & & & \\
\hline
\ddots & & & & & \ddots & & & \\
\hline
 & & & & & & \Gamma_8(4n_m) & Q_{n_m} & \\
I_2 & & & & & & -Q_{n_m}^{\mathrm{T}} & & \\
\hline
 & & & & & & & & Y
\end{array}\right) \\
&\cong \left(\begin{array}{c|ccc|ccc|c}
 & & & & & I_{2m} & & \\
\hline
 & \Gamma_8(4n_1) & & & Q_{n_1} & & & \\
 & & \ddots & & & \ddots & & \\
 & & & \Gamma_8(4n_m) & & & Q_{n_m} & \\
\hline
 & -Q_{n_1}^{\mathrm{T}} & & & & & & \\
I_{2m} & & \ddots & & & & & \\
 & & & -Q_{n_m}^{\mathrm{T}} & & & & \\
\hline
 & & & & & & & Y
\end{array}\right) \\
&\cong \left(\begin{array}{c|c|c}
 & & I_{2m} \\
\hline
 & \Gamma_8(4n_1) \oplus \cdots \oplus \Gamma_8(4n_m) \oplus Y & \Omega \\
\hline
I_{2m} & -\Omega^{\mathrm{T}} &
\end{array}\right),
\end{aligned}
$$

via permutation matrices, with partitions changed when needed, and

$$\Omega = \begin{pmatrix} Q_{n_1} \oplus \cdots \oplus Q_{n_m} \\ 0 \end{pmatrix}.$$

The final congruence has skew-symmetric part $\begin{pmatrix} 0_{2m} & \\ & \Sigma \end{pmatrix}$, where if we let H and S be symmetric and skew-symmetric matrices respectively such that

$$\Gamma_8(4n_1) \oplus \cdots \oplus \Gamma_8(4n_m) \oplus Y = H + S, \qquad (4.15)$$
$$\Sigma = \begin{pmatrix} S & \Omega \\ -\Omega^{\mathrm{T}} & 0 \end{pmatrix}$$

is skew-symmetric and nonsingular. Nonsingularity can be seen by noting that each Γ_8 block has rank deficiency 2 for its skew-symmetric part (see Section 3.2), so the skew-symmetric part of $\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus Y$ has rank deficiency $2m$, and $0_{2m} \oplus \Sigma$ has the same rank deficiency, as congruence does not change rank. Hence Σ is nonsingular. Thus employing Lemma 4.5 proves that $\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus Y \in \Xi$.

Now, let us define for the sake of brevity

$$Y_\Gamma \equiv \Gamma_8(4n_1) \oplus \cdots \oplus \Gamma_8(4n_m) \oplus Y.$$

Therefore, what we have proven above is

$$\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus Y \cong \begin{pmatrix} 0 & 0 & I_{2m} \\ 0 & Y_\Gamma & \Omega \\ I_{2m} & -\Omega^{\mathrm{T}} & 0 \end{pmatrix}.$$

Let us assume that there are k summands $\Gamma_8(4)$ in $\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus \Gamma_8(4) \oplus \cdots \oplus \Gamma_8(4) \oplus Y$. Then

$$\begin{aligned}
&\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus \Gamma_8(4) \oplus \cdots \oplus \Gamma_8(4) \oplus Y \\
&\quad \cong \left(\bigoplus_{j=1}^{k} \Gamma_8(4)\right) \oplus \Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus Y \\
&\quad \cong \left(\bigoplus_{j=1}^{k} \Gamma_8(4)\right) \oplus \begin{pmatrix} 0 & 0 & I_{2m} \\ 0 & Y_\Gamma & \Omega \\ I_{2m} & -\Omega^{\mathrm{T}} & 0 \end{pmatrix} \\
&\quad \cong \begin{pmatrix} 0 & I_{2k} \\ I_{2k} & Z_{2k} \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & I_{2m} \\ 0 & Y_\Gamma & \Omega \\ I_{2m} & -\Omega^{\mathrm{T}} & 0 \end{pmatrix},
\end{aligned}$$

where (4.13)–(4.14) with Y the empty matrix has been used for the last congruence. If in this matrix we interchange block rows 2 and 3, and block columns 2 and 3, and then block rows 3 and 4, and block columns 3 and 4, we obtain

$$\begin{aligned}
&\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus \Gamma_8(4) \oplus \cdots \oplus \Gamma_8(4) \oplus Y \\
&\quad \cong \left(\begin{array}{cc|c|cc} & & & I_{2k} & 0 \\ & & & 0 & I_{2m} \\ \hline & & Y_\Gamma & 0 & \Omega \\ \hline I_{2k} & 0 & 0 & Z_{2k} & 0 \\ 0 & I_{2m} & -\Omega^{\mathrm{T}} & 0 & 0 \end{array}\right).
\end{aligned}$$

According to (4.15), the skew-symmetric part of this matrix is

$$\begin{pmatrix} 0_{2(k+m)} & 0 \\ 0 & \widetilde{\Sigma} \end{pmatrix} \quad \text{where } \widetilde{\Sigma} = \left(\begin{array}{c|cc} S & 0 & \Omega \\ \hline 0 & Z_{2k} & 0 \\ -\Omega^{\mathrm{T}} & 0 & 0 \end{array}\right).$$

$\widetilde{\Sigma}$ is nonsingular by an argument similar to that for Σ in the paragraph after (4.15). Thus employing Lemma 4.5 proves that $\Gamma_8(4(n_1+1)) \oplus \cdots \oplus \Gamma_8(4(n_m+1)) \oplus \Gamma_8(4) \oplus \cdots \oplus \Gamma_8(4) \oplus Y \in \Xi$. $\square$

4.2. Main results

The lemmas proved in Section 4.1 will allow us to prove Theorem 4.9, the main result in this paper, in terms of the canonical form under congruence of K. Furthermore, we present in Theorem 4.9 necessary and sufficient conditions on K so that the K-orthogonal group only admits determinant $+1$ in terms of more basic matrix properties that can be more easily checked. To this purpose, let us recall some very basic ideas on matrix pencils.

Let us consider the matrix pencil $A - \lambda B$ [3, Chapter XII], where A and B are n-by-n complex matrices and λ is a scalar variable. The pencil $A - \lambda B$ is *regular* if the polynomial $p(\lambda) = \det(A - \lambda B)$ does not vanish identically, i.e., $p(\lambda)$ is not the zero polynomial. In this case $p(\lambda)$ has at most n roots, and for the rest of complex numbers, μ, $p(\mu) = \det(A - \mu B) \neq 0$. It is well known, see [3, Chapter XII] or [8, Chapter VI], that if $A - \lambda B$ is *regular* there exist two nonsingular n-by-n complex matrices P and Q such that

$$P(A - \lambda B)Q = \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix} - \lambda \begin{pmatrix} I_p & 0 \\ 0 & N \end{pmatrix}, \tag{4.16}$$

where $p + q = n$, J is in Jordan canonical form, and N is in Jordan canonical form with all its eigenvalues equal to zero. J and N are unique up to permutations of their diagonal Jordan blocks. The eigenvalues of J are called the *finite* eigenvalues of $A - \lambda B$, and the matrix N reveals the Jordan structure of the *infinite* eigenvalue of the pencil. The pair of matrices $J \oplus I_q$ and $I_p \oplus N$ is called the *Weierstrass canonical form*[3] of the pencil $A - \lambda B$. If A and B are real matrices then P and Q can be chosen to be real and J would be in real Jordan canonical form, although we will not use this fact.

Now, we are in position to prove our main result. Notice that item 4 in Theorem 4.9 provides a simple way to check if the determinant of all the elements in the K-Orthogonal group of K is $+1$.

Theorem 4.9. *Let $K \in \mathrm{M}_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and K_w and K_s be, respectively, the skew-symmetric and symmetric parts of K, i.e., $K_w \equiv (K - K^{\mathrm{T}})/2$ and $K_s \equiv (K + K^{\mathrm{T}})/2$. Then the following statements are equivalent:*

1. *The determinant of every element of the K-Orthogonal group is $+1$.*
2. *The canonical form of K under congruence has no Γ_6 or Γ_7 direct summands.*
3. *The pencil $K_w - \lambda K_s$ is regular and its Weierstrass canonical form has no Jordan blocks associated to the eigenvalue zero with odd dimension.*
4. *The polynomial $p(\lambda) = \det(K_w - \lambda K_s)$ does not vanish identically, and upon choosing a λ_0 such that $p(\lambda_0) \neq 0$ the matrix $(K_w - \lambda_0 K_s)^{-1} K_w$ is found to have no Jordan blocks associated to the eigenvalue zero with odd dimension.*

[3] The general canonical form that covers regular and singular pencils, i.e., those with $\det(A - \lambda B) \equiv 0$, is called *Kronecker canonical form*. This name is frequently used for the regular case, but attending to the historical discussion in [8, p. 289] the most proper name in the regular case seems to be *Weierstrass canonical form*.

Proof. ($1 \Rightarrow 2$) This was established in Theorem 4.1.

($2 \Rightarrow 1$) According to Sections 3.2 and 3.3 the canonical form under congruence of K is of the form (i) Y or (ii) $\Gamma_8(4q_1) \oplus \cdots \oplus \Gamma_8(4q_p) \oplus Y$, where $\det(Y - Y^{\mathrm{T}}) \neq 0$ and $q_1 \geqslant \cdots \geqslant q_p \geqslant 1$. In case (ii) Y may be empty. In the case (i), $K \in \Xi_n$ by Proposition 2.1 and Theorem 2.3. In the case (ii), $K \in \Xi_n$ by using induction with Lemma 4.6 as the base case and Lemma 4.8 as the inductive case.

($2 \Leftrightarrow 3$) According to Theorems 1 (c) and 2 (c) in [9], the canonical form under congruence of K has no blocks of Γ_6 type if and only if the pencil $K_w - \lambda K_s$ has no minimal indices. This happens if and only if $K_w - \lambda K_s$ is regular [3]. Again by [9], the canonical form under congruence of K has no blocks of Γ_7 type if and only if the pencil $K_w - \lambda K_s$ has no elementary divisors λ^e associated to the zero eigenvalue with e odd. This is equivalent to the fact that the Weierstrass canonical form of $K_w - \lambda K_s$ has no Jordan blocks associated to the eigenvalue zero with odd dimension [3].

($3 \Leftrightarrow 4$) Recall that by the construction of canonical forms for congruence through matrix pencils, it is clear that $(K_w - \lambda_0 K_s)^{-1} K_w$ has no Jordan blocks associated to the eigenvalue zero with odd dimension for one value of λ_0 if and only if so for every value of λ_0 such that $p(\lambda_0) \neq 0$. Hence it is sufficient to test a single value of λ_0 to determine if K-Orthogonal matrices can have determinant values other than $+1$. Let us consider a regular complex pencil $A - \lambda B$ with Weierstrass canonical form given by (4.16), where $J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_r}(\lambda_r)$ (here $J_k(\alpha)$ is a k-by-k Jordan block with eigenvalue α), and $N = J_{m_1}(0) \oplus \cdots \oplus J_{m_s}(0)$. Then for every number $\mu_0 \neq 0$ such that $\det(A - \mu_0 B) \neq 0$ the Jordan canonical form of

$$(A - \mu_0 B)^{-1} A$$

is

$$J_{n_1}\left(\frac{\lambda_1}{\lambda_1 - \mu_0}\right) \oplus \cdots \oplus J_{n_r}\left(\frac{\lambda_r}{\lambda_r - \mu_0}\right) \oplus J_{m_1}(1) \cdots \oplus J_{m_s}(1). \tag{4.17}$$

To prove this, simply notice that from (4.16)

$$(A - \mu_0 B)^{-1} A = Q \begin{pmatrix} (J - \mu_0 I_p)^{-1} J & 0 \\ 0 & (I_q - \mu_0 N)^{-1} \end{pmatrix} Q^{-1}.$$

By using (4.17) on the pencil $K_w - \lambda K_s$ it is straightforward to prove that $3 \Leftrightarrow 4$ with the additional assumption $\lambda_0 \neq 0$: simply notice that the number and dimensions of the Jordan blocks associated to zero is the same in the Weierstrass canonical form of $K_w - \lambda K_s$ and in $(K_w - \lambda_0 K_s)^{-1} K_w$. The case $\lambda_0 = 0$ has to be dealt with separately: in this case $(K_w - \lambda_0 K_s)^{-1} K_w = K_w^{-1} K_w = I_n$, and the information on the Jordan blocks of the Weierstrass canonical form of $K_w - \lambda K_s$ is lost, but in this case K_w is nonsingular and then the pencil $K_w - \lambda K_s$ has no zero eigenvalues. As we are interested only in the zero eigenvalue this result remains valid both for real and complex matrices. □

An immediate consequence of Theorem 4.9 is a very simple yet interesting corollary that is related to results in [7].

Corollary 4.10. *Let $K \in \mathrm{M}_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The K-orthogonal group of K contains elements with determinant different from $+1$ if and only if K is congruent to a direct sum with an odd-dimensional direct summand.*

Proof. Proposition 2.4 proves that if K is congruent to a direct sum with an odd-dimensional direct summand then $K \notin \Xi_n$. To prove the opposite, suppose K is not congruent to a direct sum with an odd dimensional summand. Thus the canonical form of K under congruence must not contain Γ_6 or Γ_7 block types. Hence $K \in \Xi_n$. $\square$

Our final result is another corollary of Theorem 4.9, that provides a simple solution of the problem in the most frequent case in which K is nonsingular.

Corollary 4.11. *Let $K \in \mathrm{M}_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, be nonsingular. The determinant of all the elements in the K-Orthogonal group of K is $+1$ if and only if the matrix $K^{-1}K_w$ has no odd-dimensional Jordan blocks associated to the zero eigenvalue.*

Proof. It follows from item 4 in Theorem 4.9 simply by noticing that $K = K_w - (-1)K_s$ is nonsingular. Therefore, we can take $\lambda_0 = -1$. $\square$

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