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First order spectral perturbation theory of square singular matrix pencils ☆

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Abstract

Let $H(\lambda) = A_0 + \lambda A_1$ be a square singular matrix pencil, and let $\lambda_0 \in \mathbb{C}$ be an eventually multiple eigenvalue of $H(\lambda)$. It is known that arbitrarily small perturbations of $H(\lambda)$ can move the eigenvalues of $H(\lambda)$ anywhere in the complex plane, i.e., the eigenvalues are discontinuous functions of the entries of A_0 and A_1. Therefore, it is not possible to develop an eigenvalue perturbation theory for arbitrary perturbations of $H(\lambda)$. However, if the perturbations are restricted to lie in an appropriate set then the eigenvalues change continuously. We prove that this set of perturbations is generic, i.e., it contains almost all pencils, and present sufficient conditions for a pencil to be in this set. In addition, for perturbations in this set, explicit first order perturbation expansions of λ_0 are obtained in terms of the perturbation pencil and bases of the left and right null spaces of $H(\lambda_0)$, both for simple and multiple eigenvalues. Infinite eigenvalues are also considered. Finally, information on the eigenvectors of the generically regular perturbed pencil is presented. We obtain, as corollaries, results for regular pencils that are also new.

AMS classification: 15A18; 15A22; 65F15; 65F22

Keywords: Eigenvalues; Eigenvectors; Perturbation; Puiseux expansions; Singular pencils

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1. Introduction

Let $A_0, A_1 \in \mathbb{C}^{m\times n}$ be two matrices. The matrix pencil $A_0 + \lambda A_1$ is called *singular* if (i) $m \neq n$, or (ii) $m = n$ and $\det(A_0 + \lambda A_1) = 0$ for all λ. Otherwise, the pencil is called *regular*. The matrix pencil $A_0 + \lambda A_1$ can be considered as a matrix polynomial or as a λ-matrix. The rank of a matrix polynomial is the dimension of its larger minor that is not equal to the zero polynomial in λ [9]. This definition applied on a pencil $A_0 + \lambda A_1$ is frequently known as the *normal rank* of the pencil, and it is denoted by $\mathrm{nrank}(A_0 + \lambda A_1)$. A complex number λ_0 is called an *eigenvalue* of the pencil $A_0 + \lambda A_1$ if

$$\mathrm{rank}(A_0 + \lambda_0 A_1) < \mathrm{nrank}(A_0 + \lambda A_1). \tag{1}$$

This definition was introduced in [19] and it reduces to the usual definition of eigenvalue in the case of regular pencils and matrices. Note that the left hand side of (1) is the rank of a constant matrix, while the right hand side is the rank of a λ-matrix. According to (1), the eigenvalues of a pencil are precisely the zeros of its invariant polynomials, or, equivalently, the zeros of its elementary divisors [9, Chapter VI]. The eigenvalue λ_0 of $A_0 + \lambda A_1$ is *simple* if $A_0 + \lambda A_1$ has only one elementary divisor associated to λ_0 and this elementary divisor has degree one. Otherwise λ_0 is a multiple eigenvalue of $A_0 + \lambda A_1$. It is said that the pencil $A_0 + \lambda A_1$ has an *infinite* eigenvalue if zero is an eigenvalue of the dual pencil $A_1 + \lambda A_0$. This definition allows us to focus on finite eigenvalues, and to obtain perturbation results for the infinite eigenvalue from the results corresponding to the zero eigenvalue of the dual pencil.

It is well known that most singular pencils, square or rectangular, do not have eigenvalues [2, Section 7]. However, when they exist, the eigenvalues of singular matrix pencils play a relevant role in a number of applications, as for instance differential-algebraic equations [27], and control theory [22]. In particular, the eigenvalues of certain singular pencils are the uncontrollable and unobservable modes of time-invariant linear systems [4].

It was pointed out in [28] that the eigenvalues of singular pencils are discontinuous functions of matrix entries. For instance the pencil $A_0 + \lambda A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ has only one eigenvalue equal to $\lambda_0 = 0$. However the perturbed pencil

$$\widehat{A}_0 + \lambda \widehat{A}_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \left(\begin{bmatrix} 6 & -3 \\ -10 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} \lambda + 6\epsilon & \epsilon(\lambda - 3) \\ \epsilon(\lambda - 10) & 0 \end{bmatrix}, \tag{2}$$

satisfies $\det(\widehat{A}_0 + \lambda \widehat{A}_1) = -\epsilon^2\,(\lambda - 3)(\lambda - 10)$, and, therefore it is regular and has two eigenvalues, 3 and 10, for any $\epsilon \neq 0$. Note, that if the previous example is modified by replacing -3 and -10, by any pair of numbers $-a$ and $-b$, then the eigenvalues of $\widehat{A}_0 + \lambda \widehat{A}_1$ are a and b. So, arbitrarily small perturbations may place the eigenvalues anywhere in the complex plane. The situation is even worse in the case of rectangular pencils. For instance, the pencil $A_0 + \lambda A_1 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has again only one eigenvalue equal to $\lambda_0 = 0$, but the perturbed pencil

$$\widehat{A}_0 + \lambda \widehat{A}_1 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 1 & 2+\lambda & 3 \\ 4 & 5 & 6 \end{bmatrix} \tag{3}$$

has no eigenvalues for any nonzero $\epsilon \neq 1/2$, because in this case $\mathrm{rank}(\widehat{A}_0 + \lambda_0 \widehat{A}_1) = \mathrm{nrank}(\widehat{A}_0 + \lambda \widehat{A}_1) = 2$ for all numbers λ_0.

The examples in the previous paragraph show that arbitrarily small perturbations may completely change or destroy the eigenvalues of a singular pencil. This means that we cannot expect a reasonable eigenvalue perturbation theory for arbitrary perturbations of singular pencils, and that we need to restrict the set of allowable perturbations before developing such a theory. In this context, square and rectangular pencils are very different from each other, because given a square singular pencil $A_0 + \lambda A_1$ almost all small perturbations make the perturbed pencil regular and, in addition, some of the eigenvalues of the perturbed pencil are very close to the eigenvalues of $A_0 + \lambda A_1$, in the case this pencil has eigenvalues. This was observed in [28]. Therefore, for a square singular pencil that has eigenvalues, one can expect to develop an eigenvalue perturbation theory for almost all small perturbations. The situation is the opposite for rectangular pencils, because given any rectangular pencil almost all small perturbations produce a pencil that does not have eigenvalues. The reason is that, generically, rectangular pencils do not have eigenvalues [2, Corollary 7.1]. Therefore, an eigenvalue perturbation theory for a rectangular pencil that has eigenvalues is only possible for very special perturbations that lie in a particular manifold in the set of pencils. A consequence of the previous discussion is that the study of the variation of the eigenvalues of a singular pencil for almost all small perturbations only makes sense for square pencils.

The main goal of this paper is, given a complex square singular pencil $H(\lambda) = A_0 + \lambda A_1$ that has eigenvalues, to find sufficient conditions on the pencil $M(\lambda) = B_0 + \lambda B_1$ allowing the existence of a first order eigenvalue perturbation theory for the eigenvalues of

$$H(\lambda) + \epsilon M(\lambda), \tag{4}$$

in terms of the small parameter ϵ, and to develop such a perturbation theory. These sufficient conditions on $M(\lambda) = B_0 + \lambda B_1$ will imply that the pencil (4) is regular for all $\epsilon \neq 0$ small enough, and they are *generic*, i.e., they hold for all pencils except those in an algebraic manifold of codimension larger than zero. This implies that they hold for all pencils except those in a subset of zero Lebesgue measure in the set of pencils. Under these generic conditions, we obtain first order perturbation expansions for those eigenvalues of (4) whose limits as ϵ tends to zero are the eigenvalues of the unperturbed pencil $H(\lambda)$. This is done both for simple and multiple eigenvalues. To our knowledge, this is the first time that first order perturbation expansions have been obtained for eigenvalues of *singular* matrix pencils. It is worth noticing that these expansions remain valid when $H(\lambda)$ is regular, and the ones we obtain in this case for multiple eigenvalues in terms of the Weierstrass canonical form are also new.

More precisely, let λ_0 be a finite eigenvalue of $H(\lambda)$ with elementary divisors $(\lambda - \lambda_0)^{m_1}, \ldots, (\lambda - \lambda_0)^{m_g}$, or, equivalently, with Jordan blocks of dimensions $m_1, \ldots, m_g$ in the Kronecker canonical form of $H(\lambda)$ [9, Chapter XII]. Then, we will prove that generically there are $m_1 + \cdots + m_g$ eigenvalues of $H(\lambda) + \epsilon M(\lambda)$ with expansions

$$\lambda(\epsilon) = \lambda_0 + c\epsilon^{1/p} + \mathrm{o}(\epsilon^{1/p}), \tag{5}$$

where $p = m_1$ for m_1 of these expansions, $p = m_2$ for m_2 of these expansions, …, $p = m_g$ for m_g of these expansions. In addition, we will find explicit expressions for the leading coefficients c of the expansions (5). Notice that the generic exponents of these expansions are determined by the degrees of the elementary divisors of λ_0 in the same way as in the regular case [13]. In particular, if the eigenvalue λ_0 is simple then $p = 1$ and one can write $\lambda(\epsilon) = \lambda_0 + c\epsilon + \mathrm{O}(\epsilon^2)$, because in this case $\lambda(\epsilon)$ is a usual power series in ϵ, convergent in a neighborhood of $\epsilon = 0$. All the series in (5) are convergent for ϵ small enough, and are called *Puiseux expansions* when they contain fractional exponents.

We will prove that the coefficients c of the expansions (5) are determined by $M(\lambda_0)$ and certain bases of the left and right null spaces of the matrix $H(\lambda_0)$. In the case of multiple eigenvalues these bases have to be carefully selected and normalized in a nontrivial way. This difficulty is not related to the fact that $H(\lambda)$ is singular, and it also appears in the perturbation theory of multiple eigenvalues of matrices and regular pencils [25,14,13,16]. However, in the most frequent case of λ_0 being a simple eigenvalue, normalization is not needed, any bases can be used, and the perturbation result takes a neat form: let us denote by W (resp. Z) a matrix whose rows (resp. columns) form any basis of the left (resp. right) null space of $H(\lambda_0)$, then the pencil $WM(\lambda_0)Z + \zeta WA_1Z$ is generically regular and has only one finite eigenvalue, and, if this eigenvalue is denoted by ξ, there is a unique eigenvalue $\lambda(\epsilon)$ of $H(\lambda) + \epsilon M(\lambda)$ such that

$$\lambda(\epsilon) = \lambda_0 + \xi\epsilon + \mathrm{O}(\epsilon^2),$$

as ϵ tends to zero. It should be remarked that in the simple case the generic conditions are precisely that $WM(\lambda_0)Z + \zeta WA_1Z$ is regular and has only one finite eigenvalue. If $H(\lambda)$ is regular then $WM(\lambda_0)Z + \zeta WA_1Z$ is 1×1, and it is regular with only one finite eigenvalue for all perturbations $M(\lambda)$. Therefore, in the regular case, $\xi = -(WM(\lambda_0)Z)/WA_1Z$ and we recover a well known result (see, for instance [18, Theorem VI.2.2]).

A *generic* perturbation theory for eigenvectors of singular pencils cannot be developed, because eigenvectors are not defined in singular pencils, even for simple eigenvalues. The correct concept to use in singular pencils is *reducing subspace* [23]. Taking into account that the perturbed pencil (4) is generically regular, it has no reducing subspaces, and, therefore, neither a *generic* perturbation theory for reducing subspaces is possible. However, when (4) is regular, its eigenvectors are perfectly defined, and it is natural to ask how are these eigenvectors related to properties of the unperturbed pencil $H(\lambda)$ when ϵ is close to zero. We have also answered this question up to first order in ϵ.

Perturbation theory of eigenvalues of singular pencils has been studied in a few previous works. Sun [19,20] considers $n \times n$ square singular pencils $A_0 + \lambda A_1$ that are strictly equivalent to diagonal pencils and such that $\mathrm{nrank}(A_0 + \lambda A_1) = n - 1$, and develops finite perturbation bounds of Gerschgorin, Hoffman–Wielandt, and Bauer–Fike type in a probabilistic sense, i.e., assuming that the perturbation pencils satisfy a certain random distribution. So, the perturbation pencils can be considered generic. Compared with the results in [19,20], the perturbation expansions we present in this work do not assume any special structure on the unperturbed pencil, and are not of a probabilistic nature, but they are only valid up to first order.

Demmel and Kågström [3] study very specific *non-generic* perturbations of square and rectangular singular pencils, and present bounds for the variation of eigenvalues and reducing subspaces. These particular perturbations are very useful to bound the errors in the algorithms computing the generalized Schur form (GUPTRI) of singular pencils [21,5,6]. Finally, Stewart [17] considers only rectangular pencils and certain specific *non-generic* perturbations that may appear in practice.

A common feature in [3,17,19,20] is that the original problem is reduced to an eigenvalue perturbation problem of a regular pencil by using the fact that perturbations with specific properties are considered. We will also follow this approach, using the Smith canonical form of matrix polynomials [9,10] to transform the original perturbation problem for the singular pencil into a regular perturbation problem, and, then, applying the perturbation theory for regular problems presented in [13]. In addition, considerable algebraic work will be performed to present the perturbation expansions in terms of intrinsic spectral magnitudes of singular pencils, i.e., null spaces associated with eigenvalues, reducing subspaces, and the Kronecker canonical form.

The paper is organized as follows: we review in Section 2 basic properties of matrix pencils, and identify the bases of the null spaces of $H(\lambda_0)$ which contain the information to derive the Puiseux expansions (5). In Section 3, we present sufficient generic conditions for the existence of a first order eigenvalue perturbation theory of square singular pencils, and we show how to transform this perturbation problem into a regular one. In Section 4, we establish a connection between the local Smith form and the Kronecker form of pencils. Section 5 presents the announced eigenvalue expansions: Theorem 2 for multiple finite eigenvalues, Corollary 1 for the infinite eigenvalue, and Theorem 3 for the normalization-free result for simple eigenvalues. Finally, in section 6 we study the eigenvectors of the perturbed pencil.

2. Preliminaries

In this section we briefly review the Kronecker canonical form of a pencil, the Smith canonical form of matrix polynomials, reducing subspaces of singular pencils, and analyze the structure of null spaces associated with eigenvalues of singular pencils. Simultaneously, some notation is established. Although all the concepts we define are valid for rectangular pencils, we restrict ourselves to square pencils. Unless otherwise specified, we use the general convention of taking *row vectors* when we refer to *left* null spaces of matrices. In addition, we denote by $A(i_1, i_2, \ldots, i_k)$ the $k \times k$ principal submatrix of A containing the rows and columns indexed by $i_1, i_2, \ldots, i_k$. Given any scalar function $f(\lambda)$, we denote by $f(\lambda) \not\equiv 0$ that $f(\lambda)$ is not identically zero, i.e., that there exists at least one number μ such that $f(\mu) \neq 0$.

2.1. The Kronecker canonical form

Let $A_0, A_1 \in \mathbb{C}^{n\times n}$, and $H(\lambda) = A_0 + \lambda A_1$ be a matrix pencil with normal rank r. Let λ_0 be a finite eigenvalue of $H(\lambda)$. Then, there exist two nonsingular $n \times n$ matrices P and Q [9, Chapter XII] such that

$$\begin{aligned} PH(\lambda)Q = K_H(\lambda) &= \mathrm{diag}(\lambda I - J_{\lambda_0}, \lambda I - \widehat{J}, I - \lambda J_\infty) \\ &\oplus \mathrm{diag}(L_{\varepsilon_1}(\lambda), \ldots, L_{\varepsilon_d}(\lambda), L_{\eta_1}^{\mathrm{T}}(\lambda), \ldots, L_{\eta_d}^{\mathrm{T}}(\lambda)), \end{aligned} \tag{6}$$

where $J_{\lambda_0} \in \mathbb{C}^{a\times a}$ is a direct sum of g Jordan blocks associated with λ_0. Analogously to the regular case, the dimension a is said to be the *algebraic multiplicity* of λ_0 as an eigenvalue of $H(\lambda)$, and g its *geometric multiplicity*. By a $k \times k$ *Jordan block* associated with λ_0 we understand a $k \times k$ matrix of the form

$$J_k(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda_0 \end{bmatrix}.$$

The matrix $\widehat{J}$ in (6) is a direct sum of Jordan blocks associated with the remaining finite eigenvalues of $H(\lambda)$, and J_∞ is a direct sum of Jordan blocks

$$\begin{bmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 0 \end{bmatrix}$$

associated with the infinite eigenvalue. It is worth noticing that the matrices P and Q appearing in (6) are not unique, and the way in which they are not unique is much more complicated than in regular pencils. This is related to the definition of reducing subspaces (see Section 2.3).

The $\rho \times \rho$ matrix pencil $\mathrm{diag}(\lambda I - J_{\lambda_0}, \lambda I - \widehat{J}, I - \lambda J_\infty)$ is regular, and the blocks $L_{\varepsilon_i}(\lambda)$ and $L^{\mathrm{T}}_{\eta_j}(\lambda)$, with respective dimensions $\varepsilon_i \times (\varepsilon_i + 1)$ and $(\eta_j + 1) \times \eta_j$, are called, respectively, *right* and *left singular blocks*. Both are given by

$$L_\sigma(\lambda) = \begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \end{bmatrix} \in \mathbb{C}^{\sigma \times (\sigma+1)},$$

where σ is said to be the *minimal index* of L_σ. More specifically, ε_i are the *column minimal indices*, and η_i are the *row minimal indices*. The sums

$$\varepsilon = \varepsilon_1 + \cdots + \varepsilon_d, \quad \eta = \eta_1 + \cdots + \eta_d \tag{7}$$

of the minimal indices satisfy

$$\varepsilon + \eta = r - \rho.$$

Another equation to bear in mind is

$$r = n - d.$$

The right-hand side $K_H(\lambda)$ of (6) is known as the *Kronecker canonical form* (hereafter, KCF) of $H(\lambda)$, and is unique up to permutation of its diagonal blocks. If $H(\lambda)$ is regular, then $K_H(\lambda) = \mathrm{diag}(\lambda I - J_{\lambda_0}, \lambda I - \widehat{J}, I - \lambda J_\infty)$ with no rectangular, singular blocks. This canonical form for regular pencils is the so-called *Weierstrass canonical form* [18, Section VI.1.2].

2.2. *The Smith canonical form*

Given an arbitrary $n \times n$ complex matrix pencil $H(\lambda)$ with normal rank r, there exist two matrix polynomials $U(\lambda)$ and $V(\lambda)$ with dimensions $n \times n$ and nonzero constant determinants, such that

$$U(\lambda)H(\lambda)V(\lambda) = \mathrm{diag}(h_1(\lambda), \ldots, h_r(\lambda), 0, \ldots, 0), \tag{8}$$

where $h_i(\lambda)$ are nonzero monic polynomials satisfying $h_i(\lambda) | h_{i+1}(\lambda)$, i.e., $h_i(\lambda)$ divides $h_{i+1}(\lambda)$, for $i = 1, \ldots, r-1$ [9, Chapter VI,10, Chapter S1]. These polynomials are called the *invariant polynomials* of $H(\lambda)$, and the diagonal matrix in the right hand side of (8) is called the *Smith canonical form* of $H(\lambda)$. This form is unique. If each

$$h_i(\lambda) = (\lambda - \lambda_1)^{\nu_{i1}} \cdots (\lambda - \lambda_q)^{\nu_{iq}}, \quad \text{for } i = 1, \ldots, r \tag{9}$$

is decomposed in powers of different irreducible factors, then those factors among $(\lambda - \lambda_1)^{\nu_{11}}, \ldots, (\lambda - \lambda_q)^{\nu_{1q}}, \ldots, (\lambda - \lambda_1)^{\nu_{r1}}, \ldots, (\lambda - \lambda_q)^{\nu_{rq}}$ with $\nu_{ij} > 0$ are called the *elementary divisors of* $H(\lambda)$. Obviously the roots of the elementary divisors are the finite eigenvalues of $H(\lambda)$ according to (1). It is well known that for each elementary divisor $(\lambda - \lambda_j)^{\nu_{ij}}$ of $H(\lambda)$ there exists a Jordan block of dimension ν_{ij} associated with the finite eigenvalue λ_j in the KCF of $H(\lambda)$, and vice versa.

The matrices $U(\lambda)$ and $V(\lambda)$ in (8) are not unique. For instance, the last d columns (resp. rows) of $V(\lambda)$ (resp. $U(\lambda)$) can be multiplied on the right (resp. left) by a matrix polynomial with nonzero constant determinant and the right hand side of (8) remains the same. Other types of non-uniqueness are also possible.

2.3. Reducing subspaces

Let us consider the pencil $A_0 + \lambda A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ given in KCF, and having only one simple eigenvalue equal to $\lambda_0 = 0$. At first glance, it is tempting to say that $[1, 0]^{\mathrm{T}}$ is the right eigenvector associated with $\lambda_0 = 0$. But note that

$$\begin{bmatrix} 1/\alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$$

for every pair of numbers α, β such that $\alpha \neq 0$. This example shows that eigenvectors cannot be defined in singular pencils, and that very different pairs of matrices P and Q may lead to the KCF (6). Of course, the difficulties appearing in this example are related with the fact that the null space associated with the zero eigenvalue has dimension two although this eigenvalue is simple.

The correct concept to use in singular pencils is *reducing subspace*. It was introduced in [23]. A subspace $\mathscr{X} \subset \mathbb{C}^n$ is a reducing subspace of the $n \times n$ pencil $H(\lambda) = A_0 + \lambda A_1$ if $\dim(A_0\mathscr{X} + A_1\mathscr{X}) = \dim(\mathscr{X}) - \#(L_{\varepsilon_i}$ blocks in the KCF of $A_0 + \lambda A_1)$, where # stands for "number of". In terms of the KCF (6) every reducing subspace is spanned by all the columns of Q corresponding to the blocks $L_{\varepsilon_1}(\lambda), \ldots, L_{\varepsilon_d}(\lambda)$ plus the columns of Q corresponding to some blocks of the regular part $\operatorname{diag}(\lambda I - J_{\lambda_0}, \lambda I - \widehat{J}, I - \lambda J_\infty)$ of (6). These columns corresponding to the regular part are not necessarily present. It should be noticed that the columns of Q corresponding to the left singular blocks $L_{\eta_j}^{\mathrm{T}}(\lambda)$ are never in a reducing subspace of $A_0 + \lambda A_1$. The *minimal reducing subspace*, $\mathscr{R}$, is the one spanned only by the columns of Q corresponding to the blocks $L_{\varepsilon_1}(\lambda), \ldots, L_{\varepsilon_d}(\lambda)$. $\mathscr{R}$ is the only reducing subspace that is a subset of any other reducing subspace. We will also use the *row minimal reducing subspace*[1] of $A_0 + \lambda A_1$. This subspace is spanned by the rows of P corresponding to the blocks $L_{\eta_1}^{\mathrm{T}}(\lambda), \ldots, L_{\eta_d}^{\mathrm{T}}(\lambda)$ in (6) and will be denoted by $\mathscr{R}_T$.

Reducing subspaces play in singular pencils a role analogous to deflating subspaces in regular pencils. In addition, reducing subspaces can be determined from the GUPTRI form of a pencil [4]. This canonical form can be stably computed [21,5,6], while this is not possible for the KCF. This is one of the reasons why reducing subspaces are very important from an applied point of view [4].

2.4. Null spaces associated with eigenvalues

Given a finite eigenvalue λ_0 of the $n \times n$ singular pencil $H(\lambda) = A_0 + \lambda A_1$, the left (or row) and right null spaces of the matrix $H(\lambda_0)$ will be essential in the eigenvalue perturbation theory of singular pencils, as they are in regular pencils. Let us denote these subspaces, respectively, by $\mathscr{N}_T(H(\lambda_0))$ and $\mathscr{N}(H(\lambda_0))$, where the subscript T in the left null space stands for the fact that its elements are row vectors. We will need to consider also the intersections of these subspaces with the minimal reducing subspaces, i.e., $\mathscr{N}_T(H(\lambda_0)) \cap \mathscr{R}_T$ and $\mathscr{N}(H(\lambda_0)) \cap \mathscr{R}$. To this purpose, let us group the columns of the matrix Q in (6) into blocks corresponding to the blocks of $K_H(\lambda)$ as follows:

$$Q = [Q_{\lambda_0} | \widehat{Q} | Q_\infty | Q_{\varepsilon_1} | \cdots | Q_{\varepsilon_d} | Q_{\eta_1} | \cdots | Q_{\eta_d}] \tag{10}$$

and the rows of P as

[1] We do not term this reducing subspace as *left* to avoid confusion with Ref. [3–5], where $\mathscr{X}$ and $A_0\mathscr{X} + A_1\mathscr{X}$ are called, respectively, right and left reducing subspaces of $A_0 + \lambda A_1$ whenever they satisfy $\dim(A_0\mathscr{X} + A_1\mathscr{X}) = \dim(\mathscr{X}) - \#(L_{\varepsilon_i}$ blocks in the KCF of $A_0 + \lambda A_1)$.

$$P^{\mathrm{T}} = [P_{\lambda_0}^{\mathrm{T}} | \widehat{P}^{\mathrm{T}} | P_{\infty}^{\mathrm{T}} | P_{\varepsilon_1}^{\mathrm{T}} | \cdots | P_{\varepsilon_d}^{\mathrm{T}} | P_{\eta_1}^{\mathrm{T}} | \cdots | P_{\eta_d}^{\mathrm{T}}]. \tag{11}$$

Thus, for instance, $P_{\varepsilon_i} H(\lambda)\, Q_{\varepsilon_i} = L_{\varepsilon_i}(\lambda)$ and $P_{\eta_j} H(\lambda) Q_{\eta_j} = L_{\eta_j}^{\mathrm{T}}(\lambda)$. From these partitions, let us define the vector polynomials

$$\pi_i(\lambda) \equiv [1 \quad \lambda \quad \cdots \quad \lambda^{\eta_i}] P_{\eta_i}, \quad \psi_i(\lambda) \equiv Q_{\varepsilon_i} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{\varepsilon_i} \end{bmatrix}, \qquad i = 1, \ldots, d. \tag{12}$$

These vector polynomials satisfy some properties that are summarized in Lemma 1. The definition of *minimal bases* appears in [8, Section 2]. It will be used just to prove the second item in Lemma 2, and those readers not interested in technical details may skip this concept.

Lemma 1. *Let* $\{\pi_1(\lambda), \ldots, \pi_d(\lambda)\}$ *and* $\{\psi_1(\lambda), \ldots, \psi_d(\lambda)\}$ *be, respectively, the row and column vector polynomials defined in* (12). *Then*

1. *If the number* μ *is not an eigenvalue of the square singular pencil* $H(\lambda)$, *then* $\{\pi_1(\mu), \ldots, \pi_d(\mu)\}$ *and* $\{\psi_1(\mu), \ldots, \psi_d(\mu)\}$ *are, respectively, bases of the left and right null spaces of the matrix* $H(\mu)$. *In addition, these null spaces are, respectively, subsets of* $\mathscr{R}_T$ *and* $\mathscr{R}$.
2. *If the number* λ_0 *is an eigenvalue of the square singular pencil* $H(\lambda)$, *then* $\{\pi_1(\lambda_0), \ldots, \pi_d(\lambda_0)\}$ *and* $\{\psi_1(\lambda_0), \ldots, \psi_d(\lambda_0)\}$ *are, respectively, bases of* $\mathscr{N}_T\ (H(\lambda_0)) \cap \mathscr{R}_T$ *and* $\mathscr{N}\ (H(\lambda_0)) \cap \mathscr{R}$.
3. $\{\pi_1(\lambda), \ldots, \pi_d(\lambda)\}$ *and* $\{\psi_1(\lambda), \ldots, \psi_d(\lambda)\}$ *are, respectively, minimal bases of the left and right null spaces (over the field of rational functions in* λ*) of the matrix polynomial* $H(\lambda)$.

Proof. The first two items follow trivially from (6). For the third one: it is easy to prove that the considered sets are bases. The fact that they are minimal is a simple consequence of the theory of singular pencils, see [7, Lemma 2.4]. □

The subspaces considered in Lemma 1 admit many other bases. Lemma 2 shows some more that will appear in the next sections.

Lemma 2. *Let* $H(\lambda)$ *be an* $n \times n$ *singular pencil with Smith normal form given by* (8), *and set* $d = n - r$. *Then*

1. *If the number* μ *is not an eigenvalue of* $H(\lambda)$, *then the last* d *rows of* $U(\mu)$ *and the last* d *columns of* $V(\mu)$ *are, respectively, bases of the left and right null spaces of the matrix* $H(\mu)$.
2. *If the number* λ_0 *is an eigenvalue of* $H(\lambda)$, *then the last* d *rows of* $U(\lambda_0)$ *and the last* d *columns of* $V(\lambda_0)$ *are, respectively, bases of* $\mathscr{N}_T\ (H(\lambda_0)) \cap \mathscr{R}_T$ *and* $\mathscr{N}\ (H(\lambda_0)) \cap \mathscr{R}$.
3. *The last* d *rows of* $U(\lambda)$ *and the last* d *columns of* $V(\lambda)$ *are, respectively, bases of the left and right null spaces* (*over the field of rational functions in* λ) *of the matrix polynomial* $H(\lambda)$.

Proof. The matrix polynomials $U(\lambda)$ and $V(\lambda)$ have nonzero constant determinant, therefore for any number μ the rows and columns of the constant matrices $U(\mu)$ and $V(\mu)$ are linearly independent. The first item follows directly from combining this fact with (8). The third item is trivial. To prove the second item, we need to work a little bit more. We only prove the statement for the last d columns of $V(\lambda_0)$, the one for the rows of $U(\lambda_0)$ is similar. By item 3 and Lemma 1, the last d columns of $V(\lambda)$ are linear combinations of $\{\psi_1(\lambda), \ldots, \psi_d(\lambda)\}$ with *polynomial coefficients*, because $\{\psi_1(\lambda), \ldots, \psi_d(\lambda)\}$ is a minimal basis of the right null space of $H(\lambda)$ [8, p. 495]. If these linear combinations are evaluated at $\lambda = \lambda_0$, we get that the last d columns of $V(\lambda_0)$ are linear combinations of $\{\psi_1(\lambda_0), \ldots, \psi_d(\lambda_0)\}$. As both sets are linearly independent, they span the same subspace of $\mathbb{C}^n$. $\square$

Note that for any eigenvalue λ_0 of $H(\lambda)$, it follows from (6) or (8) that

$$\dim \mathcal{N}_T(H(\lambda_0)) = \dim \mathcal{N}(H(\lambda_0)) = d + g, \tag{13}$$

where g is the geometric multiplicity of λ_0, i.e., the number of Jordan blocks associated with λ_0 in the KCF of $H(\lambda)$. At present, we have only determined bases of $\mathcal{N}_T(H(\lambda_0)) \cap \mathcal{R}_T$ and $\mathcal{N}(H(\lambda_0)) \cap \mathcal{R}$. Now we complete these bases to get bases of the whole subspaces $\mathcal{N}_T(H(\lambda_0))$ and $\mathcal{N}(H(\lambda_0))$. It is essential to remark that any bases of $\mathcal{N}_T(H(\lambda_0)) \cap \mathcal{R}_T$ and $\mathcal{N}(H(\lambda_0)) \cap \mathcal{R}$ can be used in the perturbation expansions that we present, but that for *multiple eigenvalues* very particular vectors have to be added to get the bases of $\mathcal{N}_T(H(\lambda_0))$ and $\mathcal{N}(H(\lambda_0))$ that we need. These vectors are related with the KCF (6), and are described in the rest of this section.

Let us specify more the spectral structure associated with the finite eigenvalue λ_0 in the KCF (6) of the singular pencil $H(\lambda)$. Let the matrix J_{λ_0} be of the form

$$J_{\lambda_0} = \operatorname{diag}(J_{n_1}^1(\lambda_0), \ldots, J_{n_1}^{r_1}(\lambda_0), \ldots, J_{n_q}^1(\lambda_0), \ldots, J_{n_q}^{r_q}(\lambda_0)), \tag{14}$$

where, for each $i = 1, \ldots, q$, the matrices $J_{n_i}^k(\lambda_0),\ k = 1, \ldots, r_i$ are Jordan blocks of dimension $n_i \times n_i$ associated with λ_0. We assume the Jordan blocks $J_{n_i}^k(\lambda_0)$ to be ordered so that

$$n_1 < n_2 < \cdots < n_q. \tag{15}$$

The dimensions n_i are usually called the *partial multiplicities* for λ_0, and we will refer to the partition (14) as the *spectral structure* of λ_0 in $H(\lambda)$.

Let a be the algebraic multiplicity of λ_0 and let P_{λ_0} (resp. Q_{λ_0}) be the matrix appearing in (11) (resp. (10)), i.e., the matrix whose rows (resp. columns) are the first a rows of the matrix P (resp. the first a columns of the matrix Q) in (6), and partition

$$P_{\lambda_0} = \left[\begin{array}{c} Y_{n_1}^1 \\ \hline \vdots \\ \hline Y_{n_1}^{r_1} \\ \hline \vdots \\ \hline Y_{n_q}^1 \\ \hline \vdots \\ \hline Y_{n_q}^{r_q} \end{array}\right], \quad Q_{\lambda_0} = \left[\begin{array}{c|c|c|c|c|c|c} X_{n_1}^1 & \cdots & X_{n_1}^{r_1} & \cdots & X_{n_q}^1 & \cdots & X_{n_q}^{r_q} \end{array}\right], \tag{16}$$

conformally with (14). We denote by x_i^k the first column of $X_{n_i}^k$, and by y_i^k the last row of $Y_{n_i}^k$. With this choice, each x_i^k is an element of $\mathcal{N}(H(\lambda_0))$ but not an element of $\mathcal{R}$, and each y_i^k is an element of $\mathcal{N}_T(H(\lambda_0))$ but not an element of $\mathcal{R}_T$. Now, for each $i = 1, \ldots, q$ we build up matrices

$$L_i = \begin{bmatrix} y_i^1 \\ \vdots \\ y_i^{r_i} \end{bmatrix}, \quad R_i = \begin{bmatrix} x_i^1 \cdots x_i^{r_i} \end{bmatrix}$$

and

$$W_i = \begin{bmatrix} L_i \\ \vdots \\ L_q \end{bmatrix}, \quad Z_i = \begin{bmatrix} R_i \cdots R_q \end{bmatrix}. \tag{17}$$

In this setting, the two quantities

$$a = \sum_{i=1}^{q} r_i n_i, \quad g = \sum_{i=1}^{q} r_i,$$

are, respectively, the algebraic and geometric multiplicities of λ_0. Finally, for each $j = 1, \ldots, q$, we define

$$f_j = \sum_{i=j}^{q} r_i, \quad f_{q+1} = 0, \tag{18}$$

so $W_i \in \mathbb{C}^{f_i \times n}$ and $Z_i \in \mathbb{C}^{n \times f_i}$. In particular, $f_1 = g$.

If $H(\lambda)$ is regular, the matrices W_1 and Z_1 contain, respectively, bases of left and right eigenvectors associated with λ_0, i.e., bases of the left and right null spaces of $H(\lambda_0)$. When $H(\lambda)$ is singular, we need to add to W_1 and Z_1, respectively, bases of $\mathscr{N}_T(H(\lambda_0)) \cap \mathscr{R}_T$ and $\mathscr{N}(H(\lambda_0)) \cap \mathscr{R}$ to get the bases of $\mathscr{N}_T(H(\lambda_0))$ and $\mathscr{N}(H(\lambda_0))$ we need.

3. Existence of expansions

This section is devoted to characterize generic perturbations, $M(\lambda) = B_0 + \lambda B_1$, for which all the eigenvalues of the perturbed pencil (4) are power series of ϵ (eventually with rational exponents), and such that by taking the limits of these series as ϵ tends to zero *all* the eigenvalues, finite or infinite, of the square singular pencil $H(\lambda) = A_0 + \lambda A_1$ are obtained, together with some numbers (or infinities) that are fully determined by $M(\lambda)$ and are not eigenvalues of $H(\lambda)$. In the process, we will show how to transform the original perturbation problem for the singular pencil $H(\lambda)$ into a regular perturbation problem.

The Smith canonical form (8) will be fundamental in this section. For the sake of simplicity let us partition (8) into blocks as

$$U(\lambda)\, H(\lambda)\, V(\lambda) \equiv \begin{bmatrix} U_1(\lambda) \\ U_2(\lambda) \end{bmatrix} H(\lambda)[V_1(\lambda)\ V_2(\lambda)] \equiv \begin{bmatrix} D_S(\lambda) & 0 \\ 0 & 0_{d\times d} \end{bmatrix}, \tag{19}$$

where $D_S(\lambda) = \operatorname{diag}(h_1(\lambda), \ldots, h_r(\lambda))$, and the dimensions of $U_1(\lambda)$ and $V_1(\lambda)$ are chosen accordingly. We will see that the generic conditions on the perturbations are related to the block partitioned matrix (19). The next lemma expresses in several equivalent ways these conditions.

Lemma 3. *Let $H(\lambda) = A_0 + \lambda A_1$ be an $n \times n$ singular pencil with Smith canonical form given by* (19), *and $M(\lambda) = B_0 + \lambda B_1$ be another $n \times n$ pencil. Then the following statements are equivalent*

1. $\det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda)) \not\equiv 0$.
2. *There exists a number* μ, *that is not an eigenvalue of* $H(\lambda)$, *and such that*
$$\det(\widetilde{U}_2\ M(\mu)\ \widetilde{V}_2) \neq 0,$$
for any pair of matrices $\widetilde{U}_2 \in \mathbb{C}^{d\times n}$ *and* $\widetilde{V}_2 \in \mathbb{C}^{n\times d}$ *whose, respectively, rows and columns are bases of* $\mathcal{N}_T\ (H(\mu))$ *and* $\mathcal{N}\ (H(\mu))$.
3. *There exists a number* μ, *that is not an eigenvalue of* $A_1 + \lambda A_0$, *and such that*
$$\det(\widetilde{U}_2\ (B_1 + \mu B_0)\ \widetilde{V}_2) \neq 0,$$
for any pair of matrices $\widetilde{U}_2 \in \mathbb{C}^{d\times n}$ *and* $\widetilde{V}_2 \in \mathbb{C}^{n\times d}$ *whose, respectively, rows and columns are bases of* $\mathcal{N}_T\ (A_1 + \mu A_0)$ *and* $\mathcal{N}\ (A_1 + \mu A_0)$.

Proof. Note that $p(\lambda) = \det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda))$ is a polynomial in λ, therefore it is not the zero polynomial if and only if $p(\mu) \neq 0$ for some μ. Note also that $p(\mu) \neq 0$ for some μ if and only if $p(\mu) \neq 0$ for some μ that is not an eigenvalue of $H(\lambda)$. Thus, the first statement is equivalent to the existence of μ that is not an eigenvalue of $H(\lambda)$, and such that $\det(U_2(\mu)\ M(\mu)\ V_2(\mu)) \neq 0$, and the equivalence with the second statement follows from Lemma 2, because $\widetilde{V}_2 = V_2(\mu)S$ and $\widetilde{U}_2 = T\ U_2(\mu)$ with S and T nonsingular matrices. The equivalence between the second and third statements follows from the facts that μ can be taken different from zero, the null spaces of $A_0 + \mu A_1$ and $(1/\mu)A_0 + A_1$ are equal, and $\widetilde{U}_2\ (B_0 + \mu B_1)\ \widetilde{V}_2$ is nonsingular if and only if $\widetilde{U}_2\ ((1/\mu)\ B_0 + B_1)\ \widetilde{V}_2$ is nonsingular. □

Let us note that once the pencil $H(\lambda)$ and the partition (19) are fixed, $\det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda)) \not\equiv 0$ is a generic condition on the set of perturbation pencils $B_0 + \lambda B_1$, because it does not hold only on the algebraic manifold defined by equating to zero all the coefficients of the polynomial $p(\lambda) = \det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda))$. These coefficient are multivariate polynomials in the entries of B_0 and B_1. Notice also that the third item in Lemma 3 means that the condition holds simultaneously for the dual pencils.

Theorem 1 below maps the original singular perturbation problem for the eigenvalues of (4) into a regular perturbation problem for the roots of a certain polynomial. Some interesting conclusions are obtained from combining this fact with classical results of Algebraic Function Theory (see, for instance [11, Chapter 12]).

Theorem 1. *Let* $H(\lambda)$ *be an* $n \times n$ *singular pencil with Smith canonical form given by* (19), *and* $M(\lambda)$ *be another* $n \times n$ *pencil such that* $\det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda)) \not\equiv 0$. *Then*

1. *There exists a constant* $b > 0$ *such that the pencil* $H(\lambda) + \epsilon M(\lambda)$ *is regular whenever* $0 < |\epsilon| < b$.
2. *For* $0 < |\epsilon| < b$ *the finite eigenvalues of* $H(\lambda) + \epsilon M(\lambda)$ *are the roots of a polynomial*, $p_\epsilon(\lambda)$, *in* λ *whose coefficients are polynomials in* ϵ. *In addition, when* $\epsilon = 0$,
$$p_0(\lambda) = \det(D_S(\lambda))\ \det(U_2(\lambda)\, M(\lambda)\, V_2(\lambda)). \tag{20}$$
3. *Let* ϵ *be such that* $0 < |\epsilon| < b$. *Then the* n *eigenvalues,*[2] $\{\lambda_1(\epsilon), \ldots, \lambda_n(\epsilon)\}$, *of* $H(\lambda) + \epsilon M(\lambda)$ *can be expanded as (fractional) power series in* ϵ. *Some of these series may have*

[2] It is well known that any $n \times n$ regular pencil has exactly n eigenvalues, if finite and infinite eigenvalues are counted [18, Chapter VI].

terms with negative exponents and tend to ∞ *as* ϵ *tends to zero. The rest of the series converge in a neighborhood of* $\epsilon = 0$.

4. *If the finite eigenvalues of* $H(\lambda)$ *are* $\{\mu_1, \ldots, \mu_k\}$, *where common elements are repeated according to their algebraic multiplicity, then there exists a subset* $\{\lambda_{i_1}(\epsilon), \ldots, \lambda_{i_k}(\epsilon)\}$ *of* $\{\lambda_1(\epsilon), \ldots, \lambda_n(\epsilon)\}$ *such that*

$$\lim_{\epsilon\to 0} \lambda_{i_j}(\epsilon) = \mu_j, \quad j = 1, \ldots, k.$$

5. *If the pencil* $H(\lambda)$ *has an infinite eigenvalue with algebraic multiplicity* p, *then there exist* $\{\lambda_{l_1}(\epsilon), \ldots, \lambda_{l_p}(\epsilon)\}$ *such that*

$$\lim_{\epsilon\to 0} \lambda_{l_j}(\epsilon) = \infty, \quad j = 1, \ldots, p.$$

Proof. Let us partition $U(\lambda)\ M(\lambda)\ V(\lambda)$ conformally with (19) as

$$U(\lambda)\ M(\lambda)\ V(\lambda) = \begin{bmatrix} B_{11}(\lambda) & B_{12}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) \end{bmatrix}.$$

This means that $B_{22}(\lambda) = U_2(\lambda)\ M(\lambda)\ V_2(\lambda)$. Thus

$$\det(H(\lambda) + \epsilon M(\lambda)) = C\ \det \begin{bmatrix} D_S(\lambda) + \epsilon B_{11}(\lambda) & \epsilon B_{12}(\lambda) \\ \epsilon B_{21}(\lambda) & \epsilon B_{22}(\lambda) \end{bmatrix},$$

where C is the nonzero constant $C = 1/\det(U(\lambda)\ V(\lambda))$. Then

$$\det(H(\lambda) + \epsilon M(\lambda)) = C\ \epsilon^d\ \det \begin{bmatrix} D_S(\lambda) + \epsilon B_{11}(\lambda) & B_{12}(\lambda) \\ \epsilon B_{21}(\lambda) & B_{22}(\lambda) \end{bmatrix}.$$

Let us define the polynomial in λ

$$p_\epsilon(\lambda) \equiv \det \begin{bmatrix} D_S(\lambda) + \epsilon B_{11}(\lambda) & B_{12}(\lambda) \\ \epsilon B_{21}(\lambda) & B_{22}(\lambda) \end{bmatrix},$$

whose coefficients are polynomials in ϵ, and write

$$\det(H(\lambda) + \epsilon M(\lambda)) = C\ \epsilon^d\ p_\epsilon(\lambda). \tag{21}$$

It is obvious that when $\epsilon = 0$

$$p_0(\lambda) = \det(D_S(\lambda))\ \det(B_{22}(\lambda)). \tag{22}$$

We know that $\det(D_S(\lambda)) \not\equiv 0$, and, therefore, $\det(B_{22}(\lambda)) \not\equiv 0$ implies that $H(\lambda) + \epsilon M(\lambda)$ is regular in a punctured disk $0 < |\epsilon| < b$. This is obvious by continuity: if $\det(D_S(\mu))\ \det(B_{22}(\mu)) \neq 0$ for some fixed number μ, then $p_\epsilon(\mu) \neq 0$ for ϵ small enough, since $p_\epsilon(\mu)$ is continuous as a function of ϵ. In addition, whenever $0 < |\epsilon| < b$, Eq. (21) implies that z is a finite eigenvalue of $H(\lambda) + \epsilon M(\lambda)$ if and only if $p_\epsilon(z) = 0$. So, the first and second items in Theorem 1 are proved.

Notice that we have reduced the original perturbation eigenvalue problem to the study of the variation of the roots of $p_\epsilon(\lambda)$ as ϵ tends to zero. But since the coefficients are polynomials in ϵ, this is a classical problem solved by Algebraic Function Theory, see for instance [11, Sections 12.1–12.3]. In particular the third item is a consequence of this theory (for infinite eigenvalues similar arguments can be applied to zero eigenvalues of dual pencils). We just comment that if the degree of $p_\epsilon(\lambda)$ in λ is δ_1 and the degree of $\det(D_S(\lambda))\ \det(B_{22}(\lambda))$ is $\delta_2 < \delta_1$, then $\delta_1 - \delta_2$ roots of $p_\epsilon(\lambda)$ tend to infinity when ϵ tends to zero. The fourth item is again a consequence of Algebraic Function Theory and (22), since those roots that remain finite have as limits the roots of $\det(D_S(\lambda))\ \det(B_{22}(\lambda))$, and the roots of $\det(D_S(\lambda))$ are precisely the finite eigenvalues of $H(\lambda)$.

The last item can be proved by applying the previous results to the zero eigenvalue of the dual pencil of $H(\lambda) + \epsilon M(\lambda)$, and taking into account that $\lambda_i(\epsilon)$ is an eigenvalue of $H(\lambda) + \epsilon M(\lambda)$ if and only if $1/\lambda_i(\epsilon)$ is an eigenvalue of the dual pencil. □

Theorem 1 gives a sufficient condition for the *simultaneous* existence of perturbation expansions for *all* the eigenvalues of $H(\lambda) + \epsilon M(\lambda)$. Some of these expansions have as limits the roots of $\det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda))$ that are fully determined by the perturbation $M(\lambda)$, the rest of the expansions have as limits the eigenvalues of $H(\lambda)$. The condition $\det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda)) \not\equiv 0$ can be relaxed if we are only interested in the existence of some of these expansions. In addition, Theorem 1 is a very simple result that does not say which are these expansions, or which are their leading exponents and coefficients. We will get this information in Section 5, at the cost of imposing more specific assumptions. The main point of Theorem 1 and its proof is that, generically, first order perturbation theory of eigenvalues of square singular pencils is just a usual perturbation problem for the roots of a polynomial whose coefficients are polynomials in the perturbation parameter.

Example 1. Let us apply the results of this section to the first example (2) in the Introduction. Note that in this case $H(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$,

$$M(\lambda) = \begin{bmatrix} 6 & -3 \\ -10 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{23}$$

and $U(\lambda)$ and $V(\lambda)$ are the 2×2 identity matrix. Therefore, $U_2(\lambda)\ M(\lambda)\ V_2(\lambda) = 0$ is the (2, 2)-entry of the perturbation, and Theorem 1 cannot be applied. If the perturbation $M(\lambda)$ is modified by setting $M_{22}(\lambda) = c_{22} + \lambda d_{22} \neq 0$, then the reader can check that the limits as ϵ tends to zero of the roots of $\det(H(\lambda) + \epsilon M(\lambda)) = 0$ are precisely the roots of $p_0(\lambda) = \lambda(c_{22} + \lambda d_{22})$, that is (20) for this example. So, for ϵ small enough there is always a root close to zero. Other interesting observations that can be easily checked are: (i) if $d_{22} = 0$ then one of the roots tends to infinity; (ii) if $c_{22} = 0$ both roots approach to zero as $\pm\, c\, \epsilon^{1/2} + o(\epsilon^{1/2})$. In this last case the perturbation makes the simple eigenvalue $\lambda = 0$ of the pencil $\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ to behave as a double eigenvalue from the point of view of perturbations. The theory that we will develop does not cover this kind of nongeneric situations. Finally, note that the perturbation $M(\lambda) = \begin{bmatrix} 6 & \lambda \\ \lambda - 3 & 0 \end{bmatrix}$ does not satisfy the assumption of Theorem 1, however $\det\,(H(\lambda) + \epsilon M(\lambda)) = -\epsilon^2\, \lambda(\lambda - 3)$ and $\lambda_0 = 0$ is a simple eigenvalue for any value of ϵ. Therefore, the generic assumption $\det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda)) \not\equiv 0$ in Theorem 1 is sufficient but not necessary for the existence of expansions.

4. From Kronecker to local Smith form

The results in Section 3 show that the Smith canonical form plays a relevant role in the generic perturbation theory of eigenvalues of square singular pencils. However, the Smith normal form does not reveal all the spectral features of singular pencils, this is only done by the KCF. In fact, it is easy to devise examples of pencils with the same Smith canonical form, but different KCFs. The purpose of this section is to relate the matrices transforming a pencil into its KCF with the matrices transforming the same pencil into a simplified version of its Smith canonical form. This simplified version is called *local Smith form* [10, p. 331], and reveals the normal rank of the pencil, and the elementary divisors corresponding to only one eigenvalue λ_0 of the pencil.

Let $H(\lambda)$ be a square matrix pencil with Smith canonical form given by (8), and λ_0 one of its finite eigenvalues with spectral structure given by (14). Then $U(\lambda)$ and/or $V(\lambda)$ (only one is necessary) can be multiplied by inverses of diagonal matrix polynomials whose diagonal entries satisfy $q_i(\lambda_0) \neq 0$, to obtain two $n \times n$ matrices $\mathscr{P}(\lambda)$ and $\mathscr{Q}(\lambda)$, whose entries are rational functions with nonzero denominators at λ_0, $\det(\mathscr{P}(\lambda)) = 1/p(\lambda)$, $\det(\mathscr{Q}(\lambda)) = 1/q(\lambda)$, where $p(\lambda)$ and $q(\lambda)$ are polynomials satisfying $p(\lambda_0) \neq 0$ and $q(\lambda_0) \neq 0$, and such that

$$\mathscr{P}(\lambda)H(\lambda)\mathscr{Q}(\lambda) = \Delta(\lambda) \tag{24}$$

with

$$\Delta(\lambda) = \begin{bmatrix} D(\lambda) & & \\ & I & \\ & & 0_{d\times d} \end{bmatrix}, \quad d = n - \operatorname{nrank}(H(\lambda)), \tag{25}$$

where $D(\lambda)$ is the $g \times g$ matrix

$$D(\lambda) = \operatorname{diag}(\overbrace{(\lambda-\lambda_0)^{n_1}, \dots, (\lambda-\lambda_0)^{n_1}}^{r_1}, \dots, \overbrace{(\lambda-\lambda_0)^{n_q}, \dots, (\lambda-\lambda_0)^{n_q}}^{r_q}).$$

The matrix $\Delta(\lambda)$ is the *local Smith form of* $H(\lambda)$ *at* λ_0 and is unique up to permutation of the diagonal entries. Notice that if $H(\lambda)$ is regular, no zeros appear on the main diagonal of $\Delta(\lambda)$.

The matrices $\mathscr{P}(\lambda)$ and $\mathscr{Q}(\lambda)$ in (24) are not unique. In this subsection, we relate the Kronecker and the local Smith forms by showing that one can transform the constant matrices P and Q in the KCF (6) to obtain specific rational matrices $\mathscr{P}(\lambda)$ and $\mathscr{Q}(\lambda)$ satisfying (24). The procedure will be the following:

(i) Transform $H(\lambda)$ into its KCF $K_H(\lambda)$ by means of P and Q as in (6).

(ii) Transform $K_H(\lambda)$ into $\Delta(\lambda)$ by means of rational matrices $\mathscr{P}_1(\lambda)$ and $\mathscr{Q}_1(\lambda)$, such that $\det(\mathscr{P}_1(\lambda)) = 1/p(\lambda)$, $\det(\mathscr{Q}_1(\lambda)) = 1/q(\lambda)$, where $p(\lambda)$ and $q(\lambda)$ are polynomials satisfying $p(\lambda_0) \neq 0$ and $q(\lambda_0) \neq 0$:

$$\mathscr{P}_1(\lambda)K_H(\lambda)\mathscr{Q}_1(\lambda) = \Delta(\lambda).$$

(iii) Set $\mathscr{P}(\lambda) = \mathscr{P}_1(\lambda)P$ and $\mathscr{Q}(\lambda) = Q\mathscr{Q}_1(\lambda)$.

These matrices evaluated at λ_0, i.e., $\mathscr{P}(\lambda_0)$ and $\mathscr{Q}(\lambda_0)$, are related to the matrices W_1 and Z_1 defined in (17).

Let us begin by specifying the λ-dependent transformations to be used in stage (ii).

Lemma 4. *Let λ_0 be a complex number. Then*

(a) *For each positive integer k we have*

$$P_k(\lambda-\lambda_0)(\lambda I_k - J_k(\lambda_0))Q_k(\lambda-\lambda_0) = \operatorname{diag}((\lambda-\lambda_0)^k, 1, \dots, 1),$$

where the matrices

$$P_k(\lambda) = \begin{bmatrix} \lambda^{k-1} & \dots & \lambda & 1 \\ \vdots & \iddots & \iddots & \\ \lambda & 1 & & \\ 1 & & & \end{bmatrix} \in \mathbb{C}^{k\times k}, \quad Q_k(\lambda) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \lambda & 0 & \dots & -1 \\ \vdots & \vdots & \iddots & \\ \lambda^{k-1} & -1 & & \end{bmatrix} \in \mathbb{C}^{k\times k}$$

are matrix polynomials with nonzero constant determinants equal to ± 1, i.e., nonsingular for all λ.

(b) *For each* $\lambda_i \neq \lambda_0$ *and each positive integer* k *there exist two* $k \times k$ *matrices* $\widehat{P}_k^i(\lambda)$ *and* $\widehat{Q}_k^i(\lambda)$, *such that one of them is a polynomial matrix with nonzero constant determinant, and the other has rational entries whose denominators are* $(\lambda - \lambda_i)^k$ *or* 1, *and determinant* $\pm 1/(\lambda - \lambda_i)^k$, *and*

$$\widehat{P}_k^i(\lambda)(\lambda I_k - J_k(\lambda_i))\widehat{Q}_k^i(\lambda) = I_k.$$

(c) *For each positive integer* k *we have*

$$(I - \lambda J_k(0))Q_k^\infty(\lambda) = I_k,$$

where the matrix

$$Q_k^\infty(\lambda) = \begin{bmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^{k-1} \\ & 1 & \lambda & \dots & \lambda^{k-2} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \lambda \\ & & & & 1 \end{bmatrix} \in \mathbb{C}^{k \times k}$$

has nonzero constant determinant equal to 1.

(d) *For each positive integer* σ *we have*

$$L_\sigma(\lambda)C_{\sigma+1}(\lambda) := B_\sigma := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{C}^{\sigma \times (\sigma+1)},$$

where the matrix

$$C_{\sigma+1}(\lambda) = \begin{bmatrix} 1 & & & & \\ \lambda & -1 & & & \\ \lambda^2 & -\lambda & -1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \lambda^\sigma & -\lambda^{\sigma-1} & \dots & -\lambda & -1 \end{bmatrix} \in \mathbb{C}^{(\sigma+1)\times(\sigma+1)}$$

has nonzero constant determinant equal to ± 1.

Proof. Items (a), (c) and (d) can be easily checked. To prove (b), notice that a transformation analogous to the one described in (a) transforms each block $(\lambda I_k - J_k(\lambda_i))$ into $\mathrm{diag}((\lambda - \lambda_i)^k, 1, \dots, 1)$, for $\lambda_i \neq \lambda_0$. Multiplying on the right by $\mathrm{diag}((\lambda - \lambda_i)^{-k}, 1, \dots, 1)$ leads to I_k. Notice that $\det(\mathrm{diag}((\lambda - \lambda_i)^{-k}, 1, \dots, 1)) = (\lambda - \lambda_i)^{-k}$. $\square$

We may now specify the mentioned matrices $\mathscr{P}(\lambda_0)$ and $\mathscr{Q}(\lambda_0)$. This involves the minimal reducing subspaces defined in Section 2.3, and the left and right null spaces, i.e., $\mathscr{N}_T(H(\lambda_0))$ and $\mathscr{N}(H(\lambda_0))$, associated with a finite eigenvalue λ_0. These null subspaces were studied in Section 2.4.

Lemma 5. *Let* $H(\lambda)$ *be an* $n \times n$ *singular pencil with KCF given by* (6), *with minimal reducing subspace* $\mathscr{R}$, *and row minimal reducing subspace* $\mathscr{R}_T$. *Let* λ_0 *be a finite eigenvalue of* $H(\lambda)$ *with spectral structure* (14). *Let* W_1 *and* Z_1 *be the matrices defined in* (17), *denote by* $W_{\mathscr{R}} \in \mathbb{C}^{d\times n}$ *a matrix whose rows form any basis of* $\mathscr{N}_T(H(\lambda_0)) \cap \mathscr{R}_T$, *and by* $Z_{\mathscr{R}} \in \mathbb{C}^{n\times d}$ *a matrix whose columns form any basis of* $\mathscr{N}(H(\lambda_0)) \cap \mathscr{R}$. *Let* $\Delta(\lambda)$ *be the local Smith form of* $H(\lambda)$ *at* λ_0 *defined* *in* (25). *Then, there exist two matrices* $\mathscr{P}(\lambda)$ *and* $\mathscr{Q}(\lambda)$, *whose entries are rational functions with nonzero denominators at* λ_0, $\det(\mathscr{P}(\lambda)) = 1/p(\lambda)$, $\det(\mathscr{Q}(\lambda)) = 1/q(\lambda)$, *where* $p(\lambda)$ *and* $q(\lambda)$ *are polynomials satisfying* $p(\lambda_0) \neq 0$ *and* $q(\lambda_0) \neq 0$, *and such that*

$$\mathscr{P}(\lambda)H(\lambda)\mathscr{Q}(\lambda) = \Delta(\lambda)$$

and

$$\mathscr{P}(\lambda_0) = \begin{bmatrix} W_1 \\ * \\ W_{\mathscr{R}} \end{bmatrix}, \quad \mathscr{Q}(\lambda_0) = \begin{bmatrix} Z_1 & * & Z_{\mathscr{R}} \end{bmatrix}, \tag{26}$$

where the rows and columns denoted with $*$ *are not specified.*

Proof. First, we collect in two block diagonal matrices $\widehat{P}(\lambda)$ and $\widehat{Q}(\lambda)$ all transformations $\widehat{P}_k^i(\lambda)$ and $\widehat{Q}_k^i(\lambda)$, as in Lemma 4(b), corresponding to Jordan blocks associated with finite eigenvalues $\lambda_i \neq \lambda_0$. We also build up a block diagonal matrix $Q_\infty(\lambda)$ of dimension $a_\infty \times a_\infty$ (a_∞ is the algebraic multiplicity of the infinite eigenvalue) which includes all matrices $Q_k^\infty(\lambda)$ from Lemma 4(c) corresponding to Jordan blocks associated with the infinite eigenvalue.

We now set

$$\mathscr{P}_0(\lambda) = \operatorname{diag}(P_{n_1}(\lambda-\lambda_0), \ldots, P_{n_q}(\lambda-\lambda_0), \widehat{P}(\lambda), I_{a_\infty}, I_\varepsilon, C^{\mathrm{T}}_{\eta_1+1}(\lambda), \ldots, C^{\mathrm{T}}_{\eta_d+1}(\lambda)),$$

with ε given by (7), and

$$\begin{aligned}\mathscr{Q}_0(\lambda) = \operatorname{diag}(&Q_{n_1}(\lambda-\lambda_0), \ldots, Q_{n_q}(\lambda-\lambda_0), \widehat{Q}(\lambda), Q_\infty(\lambda),\\ &C_{\varepsilon_1+1}(\lambda), \ldots, C_{\varepsilon_d+1}(\lambda), I_\eta),\end{aligned}$$

where η is given by (7), the diagonal blocks $P_{n_i}(\cdot)$ and $Q_{n_i}(\cdot)$ are as defined in Lemma 4(a), and C_{ε_j+1}, $C^{\mathrm{T}}_{\eta_j+1}$ are as in Lemma 4(d). Then

$$\begin{aligned}\mathscr{P}_0(\lambda)K_H(\lambda)\mathscr{Q}_0(\lambda) = \operatorname{diag}(&\overbrace{(\lambda-\lambda_0)^{n_1}, 1, \ldots, 1}^{n_1}, \ldots, \overbrace{(\lambda-\lambda_0)^{n_q}, 1, \ldots, 1}^{n_q}, I,\\ &B_{\varepsilon_1}, \ldots, B_{\varepsilon_d}, B^{\mathrm{T}}_{\eta_1}, \ldots, B^{\mathrm{T}}_{\eta_d}),\end{aligned}$$

and each $\operatorname{diag}((\lambda-\lambda_0)^{n_i}, 1, \ldots, 1)$ is repeated r_i times along the diagonal. So there are $g = \sum_{i=1}^q r_i$ of these blocks.

A final permutation of the rows and columns of this matrix leads to the Smith local form at λ_0. This permutation moves each first row and each first column corresponding to a diagonal block $\operatorname{diag}((\lambda-\lambda_0)^{n_i}, 1, \ldots, 1))$, to the first g rows and columns of $\Delta(\lambda)$. On the other hand, the last d null rows (resp. the last d null columns) of $\Delta(\lambda)$ come from the first row (resp. the first column) of each one of the d singular blocks $B^{\mathrm{T}}_{\eta_j}$ (resp. B_{ε_i}) above. If we denote by Π_l and Π_r the corresponding left and right permutation matrices, then we define

$$\mathscr{P}_1(\lambda) = \Pi_l\mathscr{P}_0(\lambda), \quad \mathscr{Q}_1(\lambda) = \mathscr{Q}_0(\lambda)\Pi_r$$

and

$$\widetilde{\mathscr{P}}(\lambda) = \mathscr{P}_1(\lambda)P, \quad \widetilde{\mathscr{Q}}(\lambda) = Q\mathscr{Q}_1(\lambda).$$

The matrices $\widetilde{\mathscr{P}}(\lambda_0)$ and $\widetilde{\mathscr{Q}}(\lambda_0)$ are as the ones described in (26) for a specific choice of $W_{\mathscr{R}}$ and $Z_{\mathscr{R}}$. To see this, we need only to keep track of the rows of P (resp., of the columns of Q) after multiplying on the left by $\mathscr{P}_1(\lambda_0)$ (resp., on the right by $\mathscr{Q}_1(\lambda_0)$). First, notice that, for each $k = n_1, \ldots, n_q$, the permutation matrix $P_k(0)$ includes a transposition of rows, whereas multiplication on the right by $Q_k(0)$ keeps the first column fixed. Therefore, using the notation in the paragraph after Eq. (16), multiplication on the left by $\mathscr{P}_0(\lambda_0)$ moves each row vector y_i^j to the first row in its corresponding block, while multiplying by $\mathscr{Q}_0(\lambda_0)$ on the right leaves the column vectors x_i^j unchanged. The final multiplication by Π_l and Π_r leads the vectors y_i^j (resp., x_i^j) to the first g rows of $\widetilde{\mathscr{P}}(\lambda_0)$ (resp., to the first g columns of $\widetilde{\mathscr{Q}}(\lambda_0)$). Therefore, we obtain that

$$\widetilde{\mathscr{P}}(\lambda_0) = \begin{bmatrix} W_1 \\ \hline * \end{bmatrix}, \quad \widetilde{\mathscr{Q}}(\lambda_0) = \begin{bmatrix} Z_1 & | & * \end{bmatrix}.$$

As to the last d rows of $\widetilde{\mathscr{P}}(\lambda_0)$, take the rows of P corresponding to some block P_{η_i} appearing in (11). Multiplication on the left by $\mathscr{P}_0(\lambda_0)$, restricted to these rows, gives the product

$$C^{\mathrm{T}}_{\eta_i+1}(\lambda_0)\ P_{\eta_i} = \begin{bmatrix} \pi_i(\lambda_0) \\ * \\ \vdots \\ * \end{bmatrix},$$

according to (12) (the entries denoted with $*$ have no significance in our argument). The final permutation Π_l moves the rows $\pi_1(\lambda_0), \ldots, \pi_d(\lambda_0)$ to the last d rows in $\widetilde{\mathscr{P}}(\lambda_0)$. A similar argument with the columns of Q gives the corresponding result for $\widetilde{\mathscr{Q}}(\lambda_0)$. We have thus obtained that

$$\widetilde{\mathscr{P}}(\lambda_0) = \begin{bmatrix} W_1 \\ * \\ \pi_1(\lambda_0) \\ \vdots \\ \pi_d(\lambda_0) \end{bmatrix}, \quad \widetilde{\mathscr{Q}}(\lambda_0) = \begin{bmatrix} Z_1 & * & \psi_1(\lambda_0) & \cdots & \psi_d(\lambda_0) \end{bmatrix}$$

with the polynomial vectors $\pi_i(\lambda)$ and $\psi_i(\lambda)$ as defined in (12). These matrices are of the type appearing in (26) by Lemma 1. Finally, to obtain any basis $W_{\mathscr{R}}$ of $\mathscr{N}_T(H(\lambda_0)) \cap \mathscr{R}_T$, and any basis $Z_{\mathscr{R}}$ of $\mathscr{N}(H(\lambda_0)) \cap \mathscr{R}$, we multiply by block diagonal matrices

$$\mathscr{P}(\lambda) = \operatorname{diag}(I_{n-d}, E)\ \mathscr{P}_1(\lambda)\ P, \quad \widetilde{\mathscr{Q}}(\lambda) = Q\ \mathscr{Q}_1(\lambda)\ \operatorname{diag}(I_{n-d}, F),$$

where E and F are constant $d \times d$ nonsingular matrices. □

5. Puiseux expansions for eigenvalues of perturbed pencils

Given a finite eigenvalue λ_0 of an arbitrary square pencil $H(\lambda)$, regular or singular, we now turn to our central problem, namely that of obtaining, under certain generic conditions on the perturbation pencils $M(\lambda)$, first order perturbation expansions in terms of the parameter ϵ for those eigenvalues of the perturbed pencil (4) whose limit is λ_0 as ϵ tends to zero. The leading coefficients of these first order perturbation expansions will be shown to be the finite eigenvalues of certain auxiliary regular matrix pencils constructed by using $M(\lambda_0)$ and bases of the left and right null spaces of $H(\lambda_0)$. For simple eigenvalues we will see that any of these bases can be used, but for multiple eigenvalues very specific bases, normalized in a nontrivial way, have to be used to construct the auxiliary pencils. In Section 5.1 we define these auxiliary pencils and prove some of their basic properties. In Section 5.2 we present the perturbation expansions for finite eigenvalues. The expansions for the infinite eigenvalues, obtained from the expansions of the zero eigenvalue of the dual pencil, are presented in Section 5.3. Finally, the expansions for simple eigenvalues are studied in Section 5.4.

5.1. The auxiliary pencils

Let us recall some matrices previously introduced. Given a finite eigenvalue λ_0 of the square pencil $H(\lambda)$ with Kronecker form (6) and spectral structure (14) for λ_0, we consider the matrices W_i and Z_i, $i = 1, \ldots, q$, defined in (17). Let us denote by $W_{\mathscr{R}} \in \mathbb{C}^{d\times n}$ a matrix whose rows form any basis of $\mathscr{N}_T(H(\lambda_0)) \cap \mathscr{R}_T$, and by $Z_{\mathscr{R}} \in \mathbb{C}^{n\times d}$ a matrix whose columns form any basis of $\mathscr{N}(H(\lambda_0)) \cap \mathscr{R}$, where $\mathscr{R}_T$ and $\mathscr{R}$ are the minimal reducing subspaces of $H(\lambda)$ (see Section 2.3), and $\mathscr{N}_T(H(\lambda_0))$ and $\mathscr{N}(H(\lambda_0))$ are the left and right null spaces of $H(\lambda_0)$. We denote by Φ_1 the $(g+d)\times(g+d)$ matrix

$$\Phi_1 = \begin{bmatrix} W_1 \\ W_{\mathscr{R}} \end{bmatrix} M(\lambda_0) \begin{bmatrix} Z_1 & Z_{\mathscr{R}} \end{bmatrix}. \tag{27}$$

Remember that the rows of $\begin{bmatrix} W_1 \\ W_{\mathscr{R}} \end{bmatrix}$ are a basis of $\mathscr{N}_T(H(\lambda_0))$, and the columns of $[Z_1 \quad Z_{\mathscr{R}}]$ are a basis of $\mathscr{N}(H(\lambda_0))$. We now recall the dimensions f_j defined in (18) and, for each $j = 1, \ldots, q$, define

$$\Phi_j = \Phi_1(g - f_j + 1, \ldots, g, g+1, \ldots, g+d) = \begin{bmatrix} W_j \\ W_{\mathscr{R}} \end{bmatrix} M(\lambda_0) \begin{bmatrix} Z_j & Z_{\mathscr{R}} \end{bmatrix} \tag{28}$$

as the $(f_j + d) \times (f_j + d)$ lower right principal submatrix of Φ_1. Finally, we define

$$\Phi_{q+1} = \Phi_1(g+1, \ldots, g+d) = W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}}. \tag{29}$$

Notice that each Φ_j is nested as a lower right principal submatrix of Φ_{j-1}. Note also that if $H(\lambda)$ is regular, then Φ_j is just

$$\Phi_j = W_j\, M(\lambda_0) Z_j. \tag{30}$$

These notations are illustrated with the following example.

Example 2. Consider the pencil

$$H(\lambda) = \begin{bmatrix} \lambda-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda-1 & -1 & 0 & 0 & 0 \\ 0 & 0 & \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

which is already in Kronecker form. It has only one finite eigenvalue $\lambda_0 = 1$ with algebraic multiplicity 3, and one left and one right singular block with row and column minimal indices equal to 1. According to our notation in (14) and in Section 2.1, we have

$$r_1 = 1, \quad n_1 = 1, \quad r_2 = 1, \quad n_2 = 2, \quad d = 1, \quad \varepsilon_1 = \eta_1 = 1.$$

If we take the perturbation pencil

$$M(\lambda) = \begin{bmatrix} 2 & 1 & -1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 3 & 1 & 4 \\ 3 & 2 & 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 5 & 1 & 0 \\ 0 & 3 & 1 & 1 & 1 & 2 \\ 5 & 1 & 0 & 0 & -2 & -2 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 2 & 3 & 1 & 2 & 0 \\ 1 & 1 & -1 & 3 & 2 & 1 \\ 1 & 0 & 0 & 2 & 3 & 1 \\ -4 & 1 & 2 & 6 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 & -2 \\ -5 & -1 & 2 & 1 & -3 & 0 \end{bmatrix},$$

then the matrix Φ_1 is

$$\Phi_1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & 1 & 1 \end{array}\right] M(1) \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \\ 0 & 0 & \\ \hline & & 1 \\ & & 1 \\ & & 0 \end{array}\right] = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 2 & 6 \\ 0 & 3 & 1 \end{bmatrix},$$

where Lemma 1 has been used to construct the matrices $W_{\mathscr{R}}$ and $Z_{\mathscr{R}}$. In addition,

$$\Phi_2 = \begin{bmatrix} 2 & 6 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \Phi_3 = 1.$$

Associated with the matrices Φ_j, $j = 1, \ldots, q$, we define

$$E_j = \operatorname{diag}(I_{r_j}, 0_{(f_{j+1}+d)\times(f_{j+1}+d)}), \quad j = 1, \ldots, q. \tag{31}$$

Notice that E_j is a $(f_j + d) \times (f_j + d)$ matrix. The pencils needed in the perturbation expansions below are $\Phi_j + \zeta E_j$, $j = 1, \ldots, q$. Some properties of these pencils are presented in the simple Lemma 6.

Lemma 6. *Let Φ_j and E_j, $j = 1, \ldots, q$, be the matrices defined, respectively, in (28) and (31), and Φ_{q+1} be the matrix defined in (29). If the matrix Φ_{j+1} is nonsingular then*

1. *The pencil $\Phi_j + \zeta E_j$ is regular and has exactly r_j finite eigenvalues.*
2. *The finite eigenvalues of $\Phi_j + \zeta E_j$ are minus the eigenvalues of the Schur complement of Φ_{j+1} in Φ_j.*
3. *If, in addition, Φ_j is nonsingular then the r_j finite eigenvalues of $\Phi_j + \zeta E_j$ are all different from zero.*

Proof. Let us express

$$\Phi_j = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & \Phi_{j+1} \end{bmatrix}.$$

Thus

$$\Phi_j + \zeta E_j = \begin{bmatrix} C_{11} + \zeta I_{r_j} & C_{12} \\ C_{21} & \Phi_{j+1} \end{bmatrix}$$

and

$$(\Phi_j + \zeta E_j) \begin{bmatrix} I_{r_j} & 0 \\ -\Phi_{j+1}^{-1} C_{21} & I \end{bmatrix} = \begin{bmatrix} C_{11} - C_{12}\Phi_{j+1}^{-1} C_{21} + \zeta I_{r_j} & C_{12} \\ 0 & \Phi_{j+1} \end{bmatrix}.$$

The pencil in the right-hand side of the previous equation is strictly equivalent to $(\Phi_j + \zeta E_j)$ and the first two items follow easily. If Φ_j is nonsingular, then $C_{11} - C_{12}\Phi_{j+1}^{-1} C_{21}$ is nonsingular and all the finite eigenvalues must be different from zero. □

5.2. First order expansions for finite eigenvalues

We are now in the position of proving Theorem 2, the main result in this paper. The proof of this theorem has two parts: the first one uses the local Smith form to transform the original eigenvalue perturbation problem of a pencil that may be singular into a regular perturbation problem. The second part applies to this regular perturbation problem the techniques developed in [13] to obtain the first order perturbation expansions for eigenvalues. This second part is not presented here, since it is long and amounts to repeating the arguments in [13, pp. 798–801] in a different situation.[3] After the proof of Theorem 2, we discuss the genericity conditions imposed on the perturbations and compare these conditions with that in Theorem 1. Note that if the pencil $H(\lambda)$ is regular then the matrix Φ_{q+1} does not exist, and conditions on this matrix are not needed. Note also that the results in Lemma 6 on the pencil $\Phi_j + \zeta E_j$ are implicitly referred to in the statement of Theorem 2.

Theorem 2. *Let $H(\lambda)$ be an arbitrary $n \times n$ matrix pencil (singular or not) with Kronecker form (6), and $M(\lambda)$ another pencil with the same dimension. Let λ_0 be a finite eigenvalue of $H(\lambda)$ with spectral structure given by (14) and (15). Let Φ_j and E_j, $j = 1, \ldots, q$, be the matrices defined in (28) and (31), and Φ_{q+1} be the matrix defined in (29). If $\det \Phi_{j+1} \neq 0$ for some $j \in \{1, 2, \ldots, q\}$, let $\xi_1, \ldots, \xi_{r_j}$ be the r_j finite eigenvalues of the pencil $\Phi_j + \zeta E_j$, and $(\xi_r)_s^{1/n_j}$, $s = 1, \ldots, n_j$, be the n_j determinations of the n_jth root. Then, in a neighborhood of $\epsilon = 0$, the pencil $H(\lambda) + \epsilon M(\lambda)$ has $r_j n_j$ eigenvalues satisfying*

$$\lambda_j^{rs}(\epsilon) = \lambda_0 + (\xi_r)_s^{1/n_j}\, \epsilon^{1/n_j} + o(\epsilon^{1/n_j}), \quad r = 1, 2, \ldots, r_j,\ s = 1, 2, \ldots, n_j, \tag{32}$$

where ϵ^{1/n_j} is the principal determination[4] of the n_jth root of ϵ. Moreover, the pencil $H(\lambda) + \epsilon M(\lambda)$ is regular in the same neighborhood for $\epsilon \neq 0$. If, in addition, $\det \Phi_j \neq 0$, then all ξ_r in (32) are nonzero, and (32) are all the expansions near λ_0 with leading exponent $1/n_j$.

Proof. The proof is based on the local Smith form in Lemma 5. We restrict ourselves to the case $\lambda_0 = 0$. If $\lambda_0 \neq 0$, we just make a shift $\mu = \lambda - \lambda_0$ in the local Smith form: $\mathscr{P}(\lambda - \lambda_0 + \lambda_0)H(\lambda - \lambda_0 + \lambda_0)\mathscr{Q}(\lambda - \lambda_0 + \lambda_0) = \Delta(\lambda - \lambda_0 + \lambda_0)$, define $\widetilde{\mathscr{P}}(\mu) := \mathscr{P}(\mu + \lambda_0)$, $\widetilde{\mathscr{Q}}(\mu) := \mathscr{Q}(\mu+\lambda_0)$, $\widetilde{H}(\mu) := H(\mu+\lambda_0)$, and $\widetilde{\Delta}(\mu) := \Delta(\mu+\lambda_0)$, and, finally, consider $\widetilde{\mathscr{P}}(\mu)\widetilde{H}(\mu)\widetilde{\mathscr{Q}}(\mu) = \widetilde{\Delta}(\mu)$. Note that $\widetilde{\mathscr{P}}(0) = \mathscr{P}(\lambda_0)$ and $\widetilde{\mathscr{Q}}(0) = \mathscr{Q}(\lambda_0)$, and that these matrices are given by (26).

Assuming that $\lambda_0 = 0$, we consider the transformation to the local Smith form at $\lambda_0 = 0$,

$$\mathscr{P}(\lambda)(H(\lambda) + \epsilon M(\lambda))\mathscr{Q}(\lambda) = \Delta(\lambda) + \epsilon\mathscr{P}(\lambda)M(\lambda)\mathscr{Q}(\lambda) \equiv \widehat{\Delta}(\lambda) + G(\lambda, \epsilon), \tag{33}$$

where

$$\widehat{\Delta}(\lambda) = \begin{bmatrix} D(\lambda) & & \\ & 0 & \\ & & 0_{d\times d} \end{bmatrix} \quad \text{and} \quad G(\lambda, \epsilon) = \begin{bmatrix} \epsilon G_{11}(\lambda) & \epsilon G_{12}(\lambda) & \epsilon G_{13}(\lambda) \\ \epsilon G_{21}(\lambda) & I + \epsilon G_{22}(\lambda) & \epsilon G_{23}(\lambda) \\ \epsilon G_{31}(\lambda) & \epsilon G_{32}(\lambda) & \epsilon G_{33}(\lambda) \end{bmatrix}$$

[3] The arguments in [13] are based on the *Newton Polygon*. The reader can find information on the Newton Polygon in [13] and the references therein, and also in the general Refs. [1,11]. Also, see the survey [15].

[4] In fact, it is easy to see that any determination of the root can be used.

are partitioned conformally, and $[G_{ij}(\lambda)]_{i,j=1}^{3} = \mathscr{P}(\lambda)M(\lambda)\mathscr{Q}(\lambda)$. Therefore, if $H(\lambda) + \epsilon M(\lambda)$ is regular, its finite eigenvalues are the roots of

$$f(\lambda, \epsilon) = \det(H(\lambda) + \epsilon M(\lambda)) = \delta(\lambda)\epsilon^{d} \widetilde{f}(\lambda, \epsilon),$$

where

$$\widetilde{f}(\lambda, \epsilon) = \det(\widehat{\Delta}(\lambda) + \widetilde{G}(\lambda, \epsilon))$$

and

$$\widetilde{G}(\lambda, \epsilon) = \begin{bmatrix} \epsilon G_{11}(\lambda) & \epsilon G_{12}(\lambda) & G_{13}(\lambda) \\ \epsilon G_{21}(\lambda) & I + \epsilon G_{22}(\lambda) & G_{23}(\lambda) \\ \epsilon G_{31}(\lambda) & \epsilon G_{32}(\lambda) & G_{33}(\lambda) \end{bmatrix}.$$

In addition, the function $\delta(\lambda)$ is given by $\delta(\lambda) = p(\lambda)q(\lambda)$ where, $\det(\mathscr{P}(\lambda)) = 1/p(\lambda)$ and $\det(\mathscr{Q}(\lambda)) = 1/q(\lambda)$. So $\delta(\lambda)$ is a polynomial such that $\delta(0) \neq 0$ and that does not depend on the perturbation $M(\lambda)$. These facts imply that for $\epsilon \neq 0$, the pencil $H(\lambda) + \epsilon M(\lambda)$ is regular if and only if $\widetilde{f}(\lambda, \epsilon) \not\equiv 0$, and that, in this case, the eigenvalues of $H(\lambda) + \epsilon M(\lambda)$ whose limit is $\lambda_0 = 0$ as ϵ tends to zero are those zeros, $\lambda(\epsilon)$, of $\widetilde{f}(\lambda, \epsilon)$ whose limit is 0. Obviously (see (21)), $\widetilde{f}(\lambda, \epsilon)$ is a rational function in λ, where the coefficients of the numerator are polynomials in ϵ, and the denominator is precisely $\delta(\lambda)$. So, $\widetilde{f}(\lambda, \epsilon)$ can be also seen as a polynomial in ϵ whose coefficients are rational functions in λ. Let us study more carefully the function $\widetilde{f}(\lambda, \epsilon)$.

In the first place, note that according to Lemma 5 and the definitions (27) and (29),

$$\Phi_1 = \begin{bmatrix} G_{11}(0) & G_{13}(0) \\ G_{31}(0) & G_{33}(0) \end{bmatrix} \quad \text{and} \quad \Phi_{q+1} = G_{33}(0). \tag{34}$$

In the second place, we rename the dimensions of the Jordan blocks associated with $\lambda_0 = 0$

$$\{\underbrace{n_1, \ldots, n_1}_{r_1}, \ldots, \underbrace{n_q, \ldots, n_q}_{r_q}\} \equiv \{m_1, \ldots, m_g\}.$$

We now make use of the Lemma in [13, p. 799], on determinants of the type $\det(D + G)$ with D diagonal, to expand $\widetilde{f}(\lambda, \epsilon)$ as

$$\widetilde{f}(\lambda, \epsilon) = \det \widetilde{G}(\lambda, \epsilon) + \sum \lambda^{m_{\nu_1}} \cdots \lambda^{m_{\nu_r}} \det \widetilde{G}(\lambda, \epsilon)(\{\nu_1, \ldots, \nu_r\}'), \tag{35}$$

where for any matrix C, $C(\{\nu_1, \ldots, \nu_r\}')$ denotes the matrix obtained by removing from C the rows and columns with indices $\nu_1, \ldots, \nu_r$. The sum runs over all $r \in \{1, \ldots, g\}$ and all $\nu_1, \ldots, \nu_r$ such that $1 \leqslant \nu_1 < \cdots < \nu_r \leqslant g$. Finally, note that

$$\det \widetilde{G}(\lambda, \epsilon) = \epsilon^{g}(\det \Phi_1 + Q_0(\lambda, \epsilon)), \tag{36}$$

for $Q_0(\lambda, \epsilon)$ rational with $Q_0(0, 0) = 0$, and

$$\det \widetilde{G}(\lambda, \epsilon)(\{\nu_1, \ldots, \nu_r\}') = \epsilon^{g-r}(\det \Phi_1(\{\nu_1, \ldots, \nu_r\}') + Q_{\nu_1,\ldots,\nu_r}(\lambda, \epsilon)), \tag{37}$$

with $Q_{\nu_1,\ldots,\nu_r}$ rational and $Q_{\nu_1,\ldots,\nu_r}(0, 0) = 0$. From now on, it suffices to repeat the arguments in [13, pp. 799–800]. The only remark to be made is that Eqs. (35)–(37) show that $\widetilde{f}(\lambda, \epsilon) \not\equiv 0$, since $\det \Phi_{j+1} = \det \Phi_1(\{1, \ldots, \sum_{i=1}^{j} r_i\}') \neq 0$ is the coefficient of $\epsilon^{f_{j+1}} \lambda^{r_1 n_1 + \cdots + r_j n_j}$ in the two variable Taylor expansion of $\widetilde{f}(\lambda, \epsilon)$ (f_{j+1} was defined in (18)). $\square$

Obviously, the assumption $\det \Phi_{j+1} \neq 0$ in Theorem 2 is a generic condition on the set of perturbations $M(\lambda) = B_0 + \lambda B_1$, because if $H(\lambda)$ is fixed then $\det \Phi_{j+1}$ is a multivariate polynomial in the entries of B_0 and B_1. However, we should stress that the assumption $\det \Phi_{j+1} \neq 0$ is different from the assumption $\det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda)) \not\equiv 0$ in Theorem 1. The reason is that Theorem 2 deals with only one eigenvalue of the unperturbed pencil $H(\lambda)$, while Theorem 1 deals simultaneously with all the eigenvalues of $H(\lambda)$. In addition, Theorem 1 only establishes the existence of expansions, while expansions with specific first order terms are developed in Theorem 2. Note also that although the algebraic multiplicity of λ_0 in $H(\lambda)$ is $r_1 n_1 + \cdots + r_q n_q$, the condition $\det \Phi_{j+1} \neq 0$ in Theorem 2 only guarantees the existence of $r_j n_j$ expansions with the leading exponents and coefficients in (32). To finish this discussion, we point out that $\det \Phi_{q+1} \neq 0$ implies $\det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda)) \not\equiv 0$. This follows easily from (29) and Lemma 2. Therefore $\det \Phi_{q+1} \neq 0$ *for only one* eigenvalue guarantees the existence of expansions for *all eigenvalues*, although not necessarily of type (32).

Theorem 2 is illustrated with the following example.

Example 3. We continue with Example 2. The fact that $\det \Phi_3 \neq 0$ guarantees the existence of two expansions with leading exponent $1/2$ and limit 1 as ϵ tends to zero. To obtain the leading coefficients of these expansions, we must solve

$$\det \begin{bmatrix} 2+\zeta & 6 \\ 3 & 1 \end{bmatrix} = 0.$$

The two square roots of its solution $\xi = 16$ provide the leading coefficients of the expansions with leading exponent $1/2$:

$$\lambda_1(\epsilon) = 1 + 4\epsilon^{1/2} + \mathrm{o}(\epsilon^{1/2}),$$
$$\lambda_2(\epsilon) = 1 - 4\epsilon^{1/2} + \mathrm{o}(\epsilon^{1/2}).$$

In a similar way $\det \Phi_2 \neq 0$ guarantees the existence of one expansion with leading exponent 1 and limit 1 as ϵ tends to zero. The leading coefficient of the expansion is the root of

$$\det \begin{bmatrix} 2+\zeta & 3 & 5 \\ 4 & 2 & 6 \\ 0 & 3 & 1 \end{bmatrix} = 0,$$

so

$$\lambda_3(\epsilon) = 1 + \epsilon + \mathrm{o}(\epsilon).$$

For the purpose of comparison, we have computed the eigenvalues of the pencil $H(\lambda) + \epsilon M(\lambda)$, for $\epsilon = 10^{-4},\ 10^{-6},\ 10^{-8}$, solving the polynomial equation $\det(H(\lambda) + \epsilon M(\lambda)) = 0$ in the variable precision arithmetic of MATLAB 7.0 with 64 decimal digits of precision, and rounding the results to ten digits. The three roots closest to 1 are

	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-8}$
λ_1	1.053399042	1.004079394	1.000400768
λ_2	0.9657365454	0.9960738628	0.9996007623
λ_3	1.000099915	1.000001000	1.000000010

The reader can check that the results coincide with the ones predicted by the perturbation theory, up to the corresponding order.

5.3. The infinite eigenvalue

Although infinite eigenvalues have been excluded from our previous analysis, they can be easily included by considering the zero eigenvalue of the dual pencil

$$H^d(\lambda) + \epsilon M^d(\lambda) := A_1 + \lambda A_0 + \epsilon(B_1 + \lambda B_0).$$

From the KCF several properties can be easily checked: if $\mu(\epsilon) \neq 0$ is an eigenvalue of $H^d(\lambda) + \epsilon M^d(\lambda)$, then $\lambda(\epsilon) = \mu(\epsilon)^{-1}$ is an eigenvalue of the original pencil $H(\lambda) + \epsilon M(\lambda)$. Conversely, each finite eigenvalue $\lambda(\epsilon)$ of $H(\lambda) + \epsilon M(\lambda)$ is $\mu(\epsilon)^{-1}$ for some eigenvalue $\mu(\epsilon)$ of $H^d(\lambda) + \epsilon M^d(\lambda)$, and the spectral data (eigenvectors, number of Jordan blocks, partial multiplicities, etc) are the same in both cases. The minimal reducing subspaces of a pencil and its dual are equal. Given a KCF (6) of $H(\lambda)$, the rows of P and the columns of Q corresponding to the "infinite" Jordan blocks are the rows and columns associated with the Jordan blocks of the zero eigenvalue in the KCF of the dual pencil.

If the zero eigenvalue $\mu_0 = 0$ of $H^d(\lambda)$ has spectral structure (14) in $H^d(\lambda)$, then we can define the matrices Φ_j^∞, $j = 1, \ldots, q+1$, for the infinite eigenvalue of $H(\lambda)$ as the matrices Φ_j corresponding to the zero eigenvalue in $H^d(\lambda)$. In addition, we can use the matrices P and Q of the KCF of $H(\lambda)$ to construct these matrices.

Therefore, to obtain the Puiseux expansions of the eigenvalues $\lambda(\epsilon)$ coming from infinity we just apply Theorem 2 above to the eigenvalues $\mu(\epsilon)$ of $H^d(\lambda) + \epsilon M^d(\lambda)$ with $\mu(0) = 0$, and compute the leading term of $\mu(\epsilon)^{-1}$. This leads to the following result.

Corollary 1. *Let $H(\lambda)$ be an $n \times n$ matrix pencil with Kronecker form* (6), *and $M(\lambda)$ another pencil with the same dimension. Let $\mu_0 = 0$ be an eigenvalue of $H^d(\lambda)$ with spectral structure given by* (14) *and* (15). *Let Φ_j^∞ and E_j, $j = 1, \ldots, q$, be the matrices defined in* (28) *and* (31), *and Φ_{q+1}^∞ be the matrix defined in* (29), *for the zero eigenvalue of the dual pencil $H^d(\lambda)$. If* $\det \Phi_{j+1}^\infty \neq 0$ *for some $j \in \{1, 2, \ldots, q\}$, let $\xi_1, \ldots, \xi_{r_j}$ be the r_j finite eigenvalues of the pencil $\Phi_j^\infty + \zeta E_j$, and $(\xi_r)_s^{1/n_j}$, $s = 1, \ldots, n_j$, be the n_j determinations of the n_jth root. Then, in a neighborhood of $\epsilon = 0$, the pencil $H(\lambda) + \epsilon M(\lambda)$ has $r_j n_j$ eigenvalues satisfying*

$$\lambda_j^{rs}(\epsilon) = (\xi_r)_s^{-1/n_j}\, \epsilon^{-1/n_j} + o(\epsilon^{-1/n_j}), \qquad r = 1, 2, \ldots, r_j,\ s = 1, 2, \ldots, n_j, \tag{38}$$

where ϵ^{1/n_j} is the principal determination of the n_jth root of ϵ. If, in addition, $\det\Phi_j^\infty \neq 0$, *then all ξ_r in* (38) *are nonzero, and* (38) *are all the expansions with leading exponent $-1/n_j$.*

5.4. Expansions for simple eigenvalues

The expansions in Theorem 2 and Corollary 1 depend on the matrices Φ_j defined in (28), and these matrices are constructed by using very specific vectors of the null spaces of $H(\lambda_0)$, easily obtained from the matrices P and Q transforming $H(\lambda)$ into its KCF (6). However *the matrices P and Q (or the blocks that we need) are very difficult to compute in the presence of multiple defective eigenvalues*. This is not the case for simple eigenvalues, because then we can use any bases of the left and right null spaces of $H(\lambda_0)$ to construct the corresponding matrices. This is shown in this section.

Theorem 3. *Let $H(\lambda) = A_0 + \lambda A_1$ be an arbitrary $n \times n$ matrix pencil (singular or not), $M(\lambda) = B_0 + \lambda B_1$ be another pencil with the same dimension, and λ_0 be a finite simple eigenvalue of $H(\lambda)$. Denote by W a matrix whose rows form any basis of the left null space of $H(\lambda_0)$ and by Z a matrix whose columns form any basis of the right null space of $H(\lambda_0)$. Then*

1. *The pencil $WM(\lambda_0)Z + \zeta WA_1Z$ is generically regular and has only one finite eigenvalue, i.e., this holds for all pencils $M(\lambda)$ except those in an algebraic manifold of positive codimension.*
2. *If the pencil $WM(\lambda_0)Z + \zeta WA_1Z$ is regular and has only one finite eigenvalue equal to ξ, then there is a unique eigenvalue of $H(\lambda) + \epsilon M(\lambda)$ such that*
   $\lambda(\epsilon) = \lambda_0 + \xi\epsilon + O(\epsilon^2)$,
   as ϵ tends to zero.
3. *In addition, if $H(\lambda)$ is regular then $WM(\lambda_0)Z + \zeta WA_1Z$ is 1×1, and it is regular with only one finite eigenvalue for all perturbations $M(\lambda)$. Therefore $\xi = -(WM(\lambda_0)Z)/WA_1Z$.*

Proof. The spectral properties, in particular the eigenvalues, of $WM(\lambda_0)Z + \zeta WA_1Z$ are the same for any pair of bases W and Z of the left and right null spaces of $H(\lambda_0)$, because changing bases simply transforms the pencil into a strictly equivalent pencil. Therefore, we can choose a pair of specific bases to prove the theorem. To this purpose, let $\mathscr{R}$ and $\mathscr{R}_T$ be, respectively, the minimal reducing and the row minimal reducing subspaces of $H(\lambda)$, and let $\mathscr{N}(H(\lambda_0))$ and $\mathscr{N}_T(H(\lambda_0))$ be the right and left null spaces of the matrix $H(\lambda_0)$. Let us denote by $W_{\mathscr{R}} \in \mathbb{C}^{d\times n}$ a matrix whose rows form any basis of $\mathscr{N}_T(H(\lambda_0)) \cap \mathscr{R}_T$, and by $Z_{\mathscr{R}} \in \mathbb{C}^{n\times d}$ a matrix whose columns form any basis of $\mathscr{N}(H(\lambda_0)) \cap \mathscr{R}$. Now, consider the KCF (6) of $H(\lambda)$ and the partitions (11) and (10) of P and Q, and notice that P_{λ_0} and Q_{λ_0} have, respectively, only one row and only one column because λ_0 is simple. From the KCF and Lemma 1, it is easy to see that the rows of $[P_{\lambda_0} \quad W_{\mathscr{R}}]$ form a basis of $\mathscr{N}_T(H(\lambda_0))$, and the columns of $[Q_{\lambda_0} \quad Z_{\mathscr{R}}]$ form a basis of $\mathscr{N}(H(\lambda_0))$. In addition, notice that the spectral structure (14) is simply $q = 1$, $n_1 = 1$, and $r_1 = 1$, and that, in this case, the matrices $\Phi_1, \ldots, \Phi_{q+1}$ defined in (28) and (29) are just two, more precisely

$$\Phi_1 = \begin{bmatrix} P_{\lambda_0} \\ W_{\mathscr{R}} \end{bmatrix} M(\lambda_0) \begin{bmatrix} Q_{\lambda_0} & Z_{\mathscr{R}} \end{bmatrix} \quad \text{and} \quad \Phi_2 = W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}}.$$

If the pencil is regular, then Φ_1 is 1×1 and Φ_2 does not exist.

Let us choose $W = [P_{\lambda_0} \quad W_{\mathscr{R}}]$, and $Z = [Q_{\lambda_0}\ Z_{\mathscr{R}}]$. Again from (6) and Lemma 1,

$$WA_1Z = \begin{bmatrix} 1 & 0 \\ 0 & 0_{d\times d} \end{bmatrix}. \tag{39}$$

Note that this matrix is E_1, according to (31). So,

$$WM(\lambda_0)Z + \zeta WA_1Z = \begin{bmatrix} P_{\lambda_0} M(\lambda_0) Q_{\lambda_0} + \zeta & P_{\lambda_0} M(\lambda_0) Z_{\mathscr{R}} \\ W_{\mathscr{R}} M(\lambda_0) Q_{\lambda_0} & W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}} \end{bmatrix}. \tag{40}$$

Laplace expansion across the first column yields

$$\det(WM(\lambda_0)Z + \zeta WA_1Z) = (P_{\lambda_0} M(\lambda_0) Q_{\lambda_0} + \zeta)\ \det(W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}}) + b,$$

where b is a constant independent of ζ. This equation shows that $WM(\lambda_0)Z + \zeta WA_1Z$ is regular and has only one finite eigenvalue if and only if $\det(W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}}) \neq 0$. Clearly, this condition is generic because $\det(W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}})$ is a multivariate polynomial in the entries of B_0 and B_1. This proves the first item of the theorem. In the regular case $WA_1Z = 1$, therefore the pencil is 1×1, regular, and has one finite eigenvalue for any $M(\lambda)$.

To prove the second item simply notice that the condition $\det\ \Phi_2 = \det(W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}}) \neq 0$ allows us to apply Theorem 2, and that (40) is $\Phi_1 + \zeta E_1$. The only point to discuss is that here we have $O(\epsilon^2)$ while in (32) we have $o(\epsilon)$. This is a simple consequence of Algebraic Function Theory: note that, by using Lemma 2, $\det\ \Phi_2 = \det(W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}}) \neq 0$ implies $\det(U_2(\lambda_0)\ M(\lambda_0)\ V_2(\lambda_0)) \neq 0$, so $\det(U_2(\lambda)\ M(\lambda)\ V_2(\lambda)) \not\equiv 0$ in (20). Hence λ_0 is a simple root of (20), and $\lambda(\epsilon)$ is analytic in ϵ and unique in a neighborhood of $\epsilon = 0$.

Finally, item 3 is a simple consequence of previous comments. □

Theorem 3 allows us to get the first order eigenvalue perturbation expansion, and to check its existence, by using arbitrary bases of left and right null spaces of the matrix $H(\lambda_0)$. To compute these bases is a basic linear algebra task. If particular bases are chosen, an explicit expression for ξ can be obtained. This is done in Corollary 2. However, the reader should notice that this expression requires to know the subspaces $\mathscr{N}_T(H(\lambda_0)) \cap \mathscr{R}_T$ and $\mathscr{N}(H(\lambda_0)) \cap \mathscr{R}$, something that is only possible with additional work.

Corollary 2. *Let $H(\lambda) = A_0 + \lambda A_1$ be an arbitrary $n \times n$ matrix pencil (singular or not), $M(\lambda) = B_0 + \lambda B_1$ be another pencil with the same dimension, and λ_0 be a finite simple eigenvalue of $H(\lambda)$. Let $\mathscr{R}$ and $\mathscr{R}_T$ be, respectively, the minimal reducing and the row minimal reducing subspaces of $H(\lambda)$, and let $\mathscr{N}(H(\lambda_0))$ and $\mathscr{N}_T(H(\lambda_0))$ be the right and left null spaces of the matrix $H(\lambda_0)$. Denote by $W_{\mathscr{R}} \in \mathbb{C}^{d \times n}$ a matrix whose rows form any basis of $\mathscr{N}_T(H(\lambda_0)) \cap \mathscr{R}_T$, and by $Z_{\mathscr{R}} \in \mathbb{C}^{n \times d}$ a matrix whose columns form any basis of $\mathscr{N}(H(\lambda_0)) \cap \mathscr{R}$, and construct from these matrices the matrices*

(i) $W = \begin{bmatrix} w \\ W_{\mathscr{R}} \end{bmatrix}$ *whose rows form a basis of $\mathscr{N}_T(H(\lambda_0))$, and*
(ii) $Z = [z\ Z_{\mathscr{R}}]$ *whose columns form a basis of $\mathscr{N}(H(\lambda_0))$.*

If $\det(W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}}) \neq 0$ then there is a unique eigenvalue of $H(\lambda) + \epsilon M(\lambda)$ such that

$$\lambda(\epsilon) = \lambda_0 - \frac{\det(W M(\lambda_0) Z)}{(w A_1 z) \cdot \det(W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}})}\, \epsilon + O(\epsilon^2),$$

as ϵ tends to zero.

Proof. Using the matrices appearing in the proof of Theorem 3, it is obvious that

$$\begin{bmatrix} w \\ W_{\mathscr{R}} \end{bmatrix} = \begin{bmatrix} s_{11} & S_{12} \\ 0 & I_d \end{bmatrix} \begin{bmatrix} P_{\lambda_0} \\ W_{\mathscr{R}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z & Z_{\mathscr{R}} \end{bmatrix} = \begin{bmatrix} Q_{\lambda_0} & Z_{\mathscr{R}} \end{bmatrix} \begin{bmatrix} t_{11} & 0 \\ T_{21} & I_d \end{bmatrix}.$$

Thus from (39)

$$\begin{bmatrix} w \\ W_{\mathscr{R}} \end{bmatrix} A_1 \begin{bmatrix} z & Z_{\mathscr{R}} \end{bmatrix} = \begin{bmatrix} w A_1 z & 0 \\ 0 & 0_{d \times d} \end{bmatrix}$$

and in this case

$$W M(\lambda_0) Z + \zeta W A_1 Z = \begin{bmatrix} w M(\lambda_0) z + \zeta (w A_1 z) & w M(\lambda_0) Z_{\mathscr{R}} \\ W_{\mathscr{R}} M(\lambda_0) z & W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}} \end{bmatrix}.$$

Finally,

$$(W M(\lambda_0) Z + \zeta W A_1 Z) \begin{bmatrix} 1 & 0 \\ -(W_{\mathscr{R}} M(\lambda_0) Z_{\mathscr{R}})^{-1}\, W_{\mathscr{R}} M(\lambda_0) z & I_d \end{bmatrix}$$

$$= \begin{bmatrix} wM(\lambda_0)z - wM(\lambda_0)Z_{\mathscr{R}}\,(W_{\mathscr{R}}M(\lambda_0)Z_{\mathscr{R}})^{-1}\,W_{\mathscr{R}}M(\lambda_0)z + \zeta(wA_1z) & wM(\lambda_0)Z_{\mathscr{R}} \\ 0 & W_{\mathscr{R}}M(\lambda_0)Z_{\mathscr{R}} \end{bmatrix}.$$

The result follows from equating the determinant to zero and noting that

$$\begin{aligned}\det(WM(\lambda_0)Z) &= (wM(\lambda_0)z - wM(\lambda_0)Z_{\mathscr{R}}\,(W_{\mathscr{R}}M(\lambda_0)Z_{\mathscr{R}})^{-1}W_{\mathscr{R}}M(\lambda_0)z)\\ &\quad\times \det(W_{\mathscr{R}}M(\lambda_0)Z_{\mathscr{R}}). \qquad \square\end{aligned}$$

6. Approximate eigenvectors of the perturbed pencil

We have commented that eigenvectors are not defined for singular pencils, even in the case of simple eigenvalues. Therefore, a perturbation theory for eigenvectors makes no sense. However, for $\epsilon \neq 0$, the perturbed pencil (4) is generically regular, has simple eigenvalues, and has well defined eigenvectors. For small ϵ, it is natural to expect that the eigenvectors corresponding to eigenvalues of (4) whose limits are the eigenvalues of $H(\lambda)$ are related to some properties of $H(\lambda)$. Given a finite eigenvalue λ_0 of $H(\lambda)$, in this section we will show that generically the eigenvectors of (4) corresponding to eigenvalues $\lambda(\epsilon)$ such that $\lambda(0) = \lambda_0$ satisfy three properties: (i) they can be expanded as Puiseux series $v(\epsilon)$ with $v(0) \neq 0$; (ii) $v(0)$ is in the null space of $H(\lambda_0)$; and (iii) inside this null space, $v(0)$ is completely determined by the perturbation $M(\lambda)$. In addition, we will show how to determine $v(0)$. Therefore, $v(0)$ is an approximate eigenvector of (4) for small $\epsilon \neq 0$, but it has no special meaning in $H(\lambda)$ except being in $\mathscr{N}(H(\lambda_0))$. Loosely speaking, it can be said that each perturbation $M(\lambda)$ selects a different direction in the null space of $H(\lambda_0)$ as an approximate eigenvector of $\lambda(\epsilon)$. For the sake of brevity, we focus on right eigenvectors. The reader can deduce similar results for left eigenvectors. As in the case of eigenvalues, the results when λ_0 is a simple eigenvalue of $H(\lambda)$ are easier and independent of any special normalization of bases.

The reader should notice that we are in a situation different from that in the expansions (32) for eigenvalues: in (32) the zero order term λ_0 was known and our task was to determine the next term, while in the case of eigenvectors we want to determine the zero order term. In fact, the results we present are meaningless for simple eigenvalues of regular pencils, since then the zero order term is obvious.

In the developments of this section we will assume that the generic condition $\det \Phi_{q+1} \neq 0$ holds. This condition can be relaxed at the cost of complicating the proof of Lemma 7, which shows the existence of expansions for eigenvectors.

Lemma 7. *Let us consider the same notation and assumptions as in Theorem 2 together with* $\det \Phi_{q+1} \neq 0$, $\det \Phi_j \ \det \Phi_{j+1} \neq 0$, *and that the r_j finite eigenvalues of the pencil $\Phi_j + \zeta E_j$, $\xi_1, \ldots, \xi_{r_j}$, are distinct. Then for each perturbed eigenvalue of the form* (32) *defined in a neighborhood of $\epsilon = 0$, there exists in the same neighborhood for $\epsilon \neq 0$ an associated right eigenvector of the regular pencil $H(\lambda) + \epsilon M(\lambda)$ which is of the form*

$$v_j^{rs}(\epsilon) = v_j^{rs} + \sum_{k=1}^{\infty} u_{jk}^{rs}\epsilon^{k/n_j}. \tag{41}$$

Proof. We simply sketch the proof. Note that the assumptions $\det \Phi_j \det \Phi_{j+1} \neq 0$ and that $\xi_1, \ldots, \xi_{r_j}$ are distinct imply that the eigenvalues in (32) are simple for $\epsilon \neq 0$ small enough. Let us consider without loss of generality that $\lambda_0 = 0$ as in the proof of Theorem 2. We proceed as in (33), and use the same notation. For $\epsilon \neq 0$, the eigenvalues and right eigenvectors of $\widehat{\Delta}(\lambda) + G(\lambda, \epsilon)$ are the same as the eigenvalues and eigenvectors of

$$F(\lambda, \epsilon) = \mathrm{diag}(I_g, I, (1/\epsilon)I_d)\ (\widehat{\Delta}(\lambda) + G(\lambda, \epsilon))$$

with

$$F(\lambda, 0) = \begin{bmatrix} D(\lambda) & & \\ & I & \\ G_{31}(\lambda) & G_{32}(\lambda) & G_{33}(\lambda) \end{bmatrix}$$

satisfying $\det F(\lambda, 0) \not\equiv 0$ by (34). Therefore, $F(\lambda, \epsilon)$ is an analytic matrix function that is regular at $\epsilon = 0$, so the variation with the small parameter ϵ of the eigenvalues of $F(\lambda, \epsilon)$ is a regular perturbation problem of an analytic matrix function. Taking into account that $\mathscr{P}(\lambda)$ and $\mathscr{Q}(\lambda)$ are nonsingular and analytic in a neighborhood of λ_0, the eigenvalues in a neighborhood of λ_0 of $H(\lambda) + \epsilon M(\lambda)$ and $F(\lambda, \epsilon)$ are the same for $\epsilon \neq 0$, in particular the expansions in (32) are eigenvalues of $F(\lambda, \epsilon)$. Lemma 2 in [12] can be applied to show that $F(\lambda, \epsilon)$ has corresponding right eigenvectors $w_j^{rs}(\epsilon)$ of the type (41). Finally, for $\epsilon \neq 0$ the right eigenvectors (41) of $H(\lambda) + \epsilon M(\lambda)$ corresponding to the eigenvalues (32) are $\mathscr{Q}(\lambda_j^{rs}(\epsilon))w_j^{rs}(\epsilon)$. □

Now we present the main result in this section, Theorem 4, that determines the zero order terms v_j^{rs} in the expansions (41). The reader should notice that this theorem in fact shows that v_j^{rs} does not depend on s, i.e., once ξ_r is fixed in (32) the n_j eigenvectors of the eigenvalues corresponding to the determinations of the n_jth roots $(\xi_r)_s^{1/n_j}$ have the same zero order term. Note also the big-O symbol in Eq. (42).

Theorem 4. *Let $H(\lambda)$ be an arbitrary $n \times n$ matrix pencil (singular or not) with Kronecker form (6), and $M(\lambda)$ another pencil with the same dimension. Let λ_0 be a finite eigenvalue of $H(\lambda)$ with spectral structure given by (14) and (15). Let Z_j, $j = 1, \ldots, q$, be the matrices defined in (17), and $Z_{\mathscr{R}} \in \mathbb{C}^{n\times d}$ a matrix whose columns form any basis of $\mathscr{N}(H(\lambda_0)) \cap \mathscr{R}$. Let Φ_j and E_j, $j = 1, \ldots, q$, be the matrices defined in (28) and (31), and Φ_{q+1} be the matrix defined in (29). If $\det \Phi_{q+1} \neq 0$, $\det \Phi_j\ \det \Phi_{j+1} \neq 0$ for some $j \in \{1, 2, \ldots, q\}$, and the r_j finite eigenvalues of the pencil $\Phi_j + \zeta E_j, \xi_1, \ldots, \xi_{r_j}$, are distinct and have eigenvectors $c_1, \ldots, c_{r_j}$, then, in a punctured neighborhood $0 < |\epsilon| < b$, the eigenvectors of the regular pencil $H(\lambda) + \epsilon M(\lambda)$ corresponding to its $r_j n_j$ eigenvalues (32) satisfy*

$$v_j^{rs}(\epsilon) = [Z_j \quad Z_{\mathscr{R}}]c_r + \mathrm{O}(\epsilon^{1/n_j}), \quad r = 1, 2, \ldots, r_j,\ s = 1, 2, \ldots, n_j. \tag{42}$$

Proof. For each eigenvalue $\lambda_j^{rs}(\epsilon)$ in (32), we consider for $\epsilon \neq 0$ the corresponding eigenvector $v_j^{rs}(\epsilon)$ given by (41). For brevity, we drop the superscripts and write λ_j and v_j instead of λ_j^{rs} and v_j^{rs}. Also, we take $\lambda_0 = 0$ as in the proof of Theorem 2. Again the proof is based on the local Smith form (24) in Lemma 5, which is well defined and analytic in a neighborhood of $\lambda_0 = 0$. To take advantage of this local Smith form we replace $v_j(\epsilon)$ with

$$w_j(\epsilon) = \mathscr{Q}(\lambda_j(\epsilon))^{-1}v_j(\epsilon), \tag{43}$$

which satisfies

$$[\Delta(\lambda_j(\epsilon)) + \epsilon\widetilde{M}(\lambda_j(\epsilon))]\, w_j(\epsilon) = 0, \tag{44}$$

where

$$\widetilde{M}(\lambda_j(\epsilon)) = \mathscr{P}(\lambda_j(\epsilon))\, M(\lambda_j(\epsilon))\, \mathscr{Q}(\lambda_j(\epsilon)).$$

Notice that one can easily recover $v_j = v_j(0)$ from $w_j(0)$, since $v_j(0) = \mathscr{Q}(0)w_j(0)$. We partition $\widetilde{M}(\lambda_j(\epsilon))$ as a 3×3 block matrix according to the three diagonal blocks of $\Delta(\lambda)$ specified in partition (25), and denote, as in the proof of Theorem 2, $[G_{ik}(\lambda_j(\epsilon))]_{i,k=1}^3 \equiv \widetilde{M}(\lambda_j(\epsilon))$. The vector $w_j(\epsilon)$ is partitioned accordingly, and (44) can be written as

$$\left(\begin{bmatrix} D(\lambda_j(\epsilon)) & & \\ & I & \\ & & 0_{d\times d} \end{bmatrix} + \epsilon \begin{bmatrix} G_{11}(\lambda_j(\epsilon)) & G_{12}(\lambda_j(\epsilon)) & G_{13}(\lambda_j(\epsilon)) \\ G_{21}(\lambda_j(\epsilon)) & G_{22}(\lambda_j(\epsilon)) & G_{23}(\lambda_j(\epsilon)) \\ G_{31}(\lambda_j(\epsilon)) & G_{32}(\lambda_j(\epsilon)) & G_{33}(\lambda_j(\epsilon)) \end{bmatrix}\right) \begin{bmatrix} w_j^{(1)}(\epsilon) \\ w_j^{(2)}(\epsilon) \\ w_j^{(3)}(\epsilon) \end{bmatrix} = 0. \tag{45}$$

For $\epsilon = 0$ this equation reduces to $w_j^{(2)}(0) = 0$. The rows corresponding to the first row of blocks are

$$\begin{aligned} &D(\lambda_j(\epsilon))\, w_j^{(1)}(\epsilon) + \epsilon(G_{11}(\lambda_j(\epsilon))\, w_j^{(1)}(\epsilon) \\ &\quad + G_{12}(\lambda_j(\epsilon))\, w_j^{(2)}(\epsilon)) + G_{13}(\lambda_j(\epsilon))\, w_j^{(3)}(\epsilon)) = 0. \end{aligned} \tag{46}$$

Notice that the terms of lower order in ϵ of the entries in $D(\lambda_j(\epsilon))$ are of the form $c\, \epsilon^{n_i/n_j}$, for $i = 1, \ldots, q$, with $c \neq 0$ because $\det \Phi_j \ \det \Phi_{j+1} \neq 0$. So taking into account (15), we can divide the first r_1 equations in (46) by ϵ^{n_1/n_j}, take the limit $\epsilon \to 0$, and prove that $w_{j,k}(0) = 0$ for $k \leqslant r_1$ (here $w_{j,k}(0)$ denotes the kth entry of $w_j(0)$). Dividing by ϵ^{n_2/n_j} the next r_2 equations in (46) and taking limits we prove $w_{j,k}(0) = 0$ for $k \leqslant r_1 + r_2$. This process continues by dividing successively by $\epsilon^{n_3/n_j}, \ldots, \epsilon^{n_{j-1}/n_j}$ to prove that $w_{j,k}(0) = 0$ for $k \leqslant r_1 + \cdots + r_{j-1}$.

Finally, denote by $\bar{w}_j(0)$ the vector obtained by removing from $w_j(0)$ the zero entries corresponding to $w_j^{(2)}(0) = 0$ and to $w_{j,k}(0) = 0$ for $k \leqslant r_1 + \cdots + r_{j-1}$. If we divide by ϵ the part of (45) corresponding to $\bar{w}_j$, set $\epsilon = 0$, and take into account (34), we get

$$\left(\xi_r E_j + \Phi_j\right) \bar{w}_j(0) = 0.$$

The result now follows from (26) and (43). □

6.1. *The case of simple eigenvalues*

We conclude by studying the case when λ_0 is a simple eigenvalue of $H(\lambda)$. The following result completes Theorem 3.

Corollary 3. *With the same notation and assumptions as in Theorem 3. If the pencil $WM(\lambda_0)Z + \zeta WA_1Z$ is regular and has only one finite eigenvalue equal to ξ with eigenvector c, then there is a unique eigenvalue of $H(\lambda) + \epsilon M(\lambda)$ such that $\lambda(\epsilon) = \lambda_0 + \xi\,\epsilon + \mathrm{O}(\epsilon^2)$, as ϵ tends to zero, and for $\epsilon \neq 0$ the corresponding eigenvector satisfies*

$$v(\epsilon) = Zc + \mathrm{O}(\epsilon).$$

Proof. The proof is a direct consequence of Theorem 4, (39) and (40) and an elementary change of bases. □

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