SIAM J. MATRIX ANAL. APPL.
Vol. 30, No. 2, pp. 538–547

LOW RANK PERTURBATION OF WEIERSTRASS STRUCTURE*

FERNANDO DE TERÁN†, FROILÁN M. DOPICO†, AND JULIO MORO†

Abstract. Let $A_0 + \lambda A_1$ be a regular matrix pencil, and let λ_0 be one of its finite eigenvalues having g elementary Jordan blocks in the Weierstrass canonical form. We show that for most matrices B_0 and B_1 with $\operatorname{rank}(B_0+\lambda_0 B_1) < g$ there are $g - \operatorname{rank}(B_0+\lambda_0 B_1)$ Jordan blocks corresponding to the eigenvalue λ_0 in the Weierstrass form of the perturbed pencil $A_0+B_0+\lambda(A_1+B_1)$. If $\operatorname{rank}(B_0+\lambda_0 B_1)+\operatorname{rank}(B_1)$ does not exceed the number of λ_0-Jordan blocks in $A_0+\lambda A_1$ of dimension greater than one, then the λ_0-Jordan blocks of the perturbed pencil are the $g-\operatorname{rank}(B_0+\lambda_0 B_1)-\operatorname{rank}(B_1)$ smallest λ_0-Jordan blocks of $A_0+\lambda A_1$, together with $\operatorname{rank}(B_1)$ blocks of dimension one. Otherwise, all $g-\operatorname{rank}(B_0+\lambda_0 B_1)$ λ_0-Jordan blocks of the perturbed pencil are of dimension one. This happens for any pair of matrices B_0 and B_1 except those in a proper algebraic submanifold in the set of matrix pairs. If $A_0 + \lambda A_1$ has an infinite eigenvalue, then the corresponding result follows from considering the zero eigenvalue of the dual pencils $A_1 + \lambda A_0$ and $A_1 + B_1 + \lambda(A_0 + B_0)$.

Key words. regular matrix pencils, Weierstrass canonical form, low rank perturbations, matrix spectral perturbation theory

AMS subject classifications. 15A22, 15A18, 15A21

DOI. 10.1137/050633020

1. Introduction.

The change of the Jordan structure of a matrix A under perturbations B of low rank has been recently studied by several authors [5, 7, 8, 9, 10]. It is known that if λ_0 is one of the eigenvalues of A having g elementary Jordan blocks in the Jordan canonical form of A, then *for most matrices* B satisfying $\operatorname{rank}(B) < g$, the Jordan blocks of $A + B$ with eigenvalue λ_0 are just the $g - \operatorname{rank}(B)$ smallest Jordan blocks of A with eigenvalue λ_0. As far as we know, this generic behavior was first proved in [5] and again in [7] and [8, 9, 10]. The proof in [7] uses only elementary linear algebra results, and allows us to explicitly characterize the set of perturbation matrices B for which this generic behavior does not happen. This is done through a scalar determinantal equation involving B and some of the λ_0-eigenvectors of A. Thus, this behavior can be properly termed as *generic*, since it happens for any perturbation matrix B except those belonging to a proper algebraic submanifold in the set of $n\times n$ matrices of given rank. It is interesting to note that the result in [5] remains valid for infinite dimensional compact linear operators in Banach spaces.

The purpose of this paper is to study which is the generic change of the Weierstrass canonical form [4] of a regular $n \times n$ pencil of matrices $A_0 + \lambda A_1$ under a low rank perturbation $B_0 + \lambda B_1$. We will see that this change is rather different from the change described above for matrices. The regular matrix pencil $A_0+\lambda A_1$ may have an *infinite* eigenvalue, whose Jordan blocks in the Weiertrass canonical form are precisely the Jordan blocks associated with the *zero* eigenvalue in the Weierstrass form of the *dual* pencil $A_1 + \lambda A_0$. Therefore, we may focus on finite eigenvalues of $A_0 + \lambda A_1$. The perturbation results for the infinite eigenvalue follow from results for the zero eigenvalue of the dual pencil.

*Received by the editors June 3, 2005; accepted for publication (in revised form) by B. T. Kågström January 3, 2008; published electronically May 16, 2008. This research was partially supported by the Ministerio de Educación y Ciencia of Spain through grant BFM 2003-00223.
http://www.siam.org/journals/simax/30-2/63302.html

†Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911-Leganés, Madrid, Spain (fteran@math.uc3m.es, dopico@math.uc3m.es, jmoro@math.uc3m.es).

Let λ_0 be a finite eigenvalue with geometric multiplicity g of the regular $n \times n$ matrix pencil $A_0 + \lambda A_1$. Recall that a pencil is regular if the polynomial $\det(A_0 + \lambda A_1)$ in λ is not identically zero, and that the geometric multiplicity of λ_0 is $g = \dim \ker(A_0 + \lambda_0 A_1)$, where ker denotes the null space. The elementary inequalities $\operatorname{rank}(C + D) \leq \operatorname{rank}(C) + \operatorname{rank}(D)$ and $\operatorname{rank}(C) \leq \operatorname{rank}(C + D) + \operatorname{rank}(D)$, valid for any pair of matrices C and D, lead to

$$\operatorname{rank}(A_0 + \lambda_0 A_1 + B_0 + \lambda_0 B_1) \leq \operatorname{rank}(A_0 + \lambda_0 A_1) + \operatorname{rank}(B_0 + \lambda_0 B_1),$$
$$\operatorname{rank}(A_0 + \lambda_0 A_1) \leq \operatorname{rank}(A_0 + \lambda_0 A_1 + B_0 + \lambda_0 B_1) + \operatorname{rank}(B_0 + \lambda_0 B_1).$$

Combining both inequalities, one gets

$$g - \operatorname{rank}(B_0 + \lambda_0 B_1) \leq \dim \ker(A_0 + \lambda_0 A_1 + B_0 + \lambda_0 B_1) \leq g + \operatorname{rank}(B_0 + \lambda_0 B_1). \tag{1.1}$$

Therefore, whenever

$$\operatorname{rank}(B_0 + \lambda_0 B_1) < g, \tag{1.2}$$

the eigenvalue λ_0 of $A_0 + \lambda A_1$ stays as an eigenvalue of the perturbed pencil

$$A_0 + B_0 + \lambda(A_1 + B_1). \tag{1.3}$$

As a consequence, by "low" rank perturbation we will mean in what follows that B_0 and B_1 satisfy (1.2), a condition which depends on the particular eigenvalue λ_0 we are considering. It is well known that for a regular pencil $L_0 + \lambda L_1$ the number of Jordan blocks associated with λ_0 in its Weierstrass canonical form is equal to $\dim \ker(L_0 + \lambda_0 L_1)$. Therefore, assuming that (1.3) is still regular, (1.1) implies that the perturbation $B_0 + \lambda B_1$ can destroy at most $\operatorname{rank}(B_0 + \lambda_0 B_1)$ Jordan blocks of $A_0 + \lambda A_1$, and can create at most $\operatorname{rank}(B_0 + \lambda_0 B_1)$ new Jordan blocks associated with the finite eigenvalue λ_0 of $A_0 + \lambda A_1$. This allows many different choices for the number and dimensions of the Jordan blocks appearing in the Weierstrass form of $A_0 + B_0 + \lambda(A_1 + B_1)$. The goal of this work is to find out which is the generic behavior in this respect.[1]

The result we present depends on two quantities *for each eigenvalue* λ_0, namely

$$\rho_0 = \operatorname{rank}(B_0 + \lambda_0 B_1) \quad \text{and} \quad \rho_1 = \operatorname{rank}(B_1).$$

Assuming that condition (1.2) holds, we will prove that *for generic matrices* B_0 *and* B_1 there are precisely $g - \rho_0$ Jordan blocks associated with λ_0 in the Weierstrass canonical form of the perturbed pencil (1.3). Moreover, if we denote by d_0 the number of Jordan blocks in $A_0 + \lambda A_1$ with eigenvalue λ_0 of dimension greater than one, we will prove that whenever $\rho_0 + \rho_1 \leq d_0$, the largest ρ_0 Jordan blocks of $A_0 + \lambda A_1$ associated with λ_0 disappear, and the second-largest ρ_1 blocks of λ_0 turn into 1×1 blocks, while the rest of the Jordan blocks of λ_0 in $A_0 + \lambda A_1$ remain as Jordan blocks in the perturbed pencil (1.3). If $\rho_0 + \rho_1 > d_0$, then there will be only 1×1 blocks corresponding to λ_0 in the Weierstrass form of (1.3). This generic behavior coincides with the one previously described for low rank perturbations of the Jordan canonical form of matrices in the case $B_1 = 0$, while it is rather different when $B_1 \neq 0$. *Describing this behavior and proving that it is generic is our major contribution.*

[1] The assumption that $A_0 + B_0 + \lambda(A_1 + B_1)$ is a regular pencil holds except for very particular choices of B_0 and B_1.

Inequality (1.1) makes clear that $B_0 + \lambda_0 B_1$ is bound to play a relevant role in the perturbation of the Weierstrass structure, since it determines the geometric multiplicity of λ_0 in (1.3). To understand why B_1 plays a separate role on its own, recall that a Jordan chain of $A_0 + \lambda A_1$ of length s associated with λ_0 satisfies the equations $(A_0 + \lambda_0 A_1)v_1 = 0$ and $(A_0 + \lambda_0 A_1)v_k = A_1 v_{k-1}$ for $2 \leq k \leq s$. Therefore, it is expected that perturbing A_1 affects to the length of the Jordan chains. In plain words, the generic behavior described above corresponds to a cooperation between $B_0 + \lambda_0 B_1$ and B_1 to destroy some of the blocks, and to decrease the dimension of as many of the largest Jordan blocks as possible, while still fulfilling the constraint (1.1) on the geometric multiplicity.

The results obtained in the present paper, as those in [7], are valid for perturbations of *any size* satisfying the low rank condition (1.2), i.e., they are not first-order perturbation results. Notice also that we are not paying attention to the perturbation of the eigenvalues corresponding to the destroyed Jordan blocks. First order perturbation results for this problem are enumerated in [6] for general matrix polynomials, and, more recently, in [2]. In [13] first order multiparametric perturbations have been considered for multiple semisimple eigenvalues. Several perturbation bounds, valid for perturbations of finite size, appear in [11], but they do not apply to multiple defective eigenvalues, except in the case of some Gerschgorin-like inclusion regions.

Now, we summarize the *Weierstrass canonical form* of a regular pencil [4], and introduce some notation to be used throughout the paper. For any regular $n \times n$ complex matrix pencil $A_0 + \lambda A_1$ having λ_0 as one of its eigenvalues, there exist nonsingular $n \times n$ matrices P and Q, independent of λ, such that

$$Q(A_0 + \lambda A_1)P = \operatorname{diag}(J_{n_1}(-\lambda_0), \ldots, J_{n_g}(-\lambda_0), \widetilde{J}, I_\infty) + \lambda \operatorname{diag}(I_1, I_2, N), \tag{1.4}$$

where $\operatorname{diag}(C, E)$ denotes a block diagonal matrix with square diagonal blocks C and E; $J_{n_i}(-\lambda_0)$ stands for a Jordan block of dimension n_i with $-\lambda_0$ on the main diagonal; $\widetilde{J}$ is a matrix in Jordan canonical form corresponding to the other finite eigenvalues of the pencil; and N is a matrix in Jordan canonical form whose eigenvalues are all equal to zero. N contains the spectral structure of the infinite eigenvalue of the pencil. Finally, I_1, I_2 and I_∞ are identity matrices of matching dimensions to those of $\operatorname{diag}(J_{n_1}(-\lambda_0), \ldots, J_{n_g}(-\lambda_0))$, $\widetilde{J}$ and N, respectively. The right-hand side of (1.4) is the Weierstrass canonical form of the pencil $A_0 + \lambda A_1$, and it is unique up to permutation of the diagonal Jordan blocks. The Weierstrass canonical form displays all of the spectral information of the regular pencil $A_0 + \lambda A_1$. From (1.4), one can easily see that the *geometric multiplicity* of λ_0 is g and its *algebraic multiplicity* is

$$a_{A_0 + \lambda A_1}(\lambda_0) = n_1 + \cdots + n_g. \tag{1.5}$$

Without loss of generality, we assume the dimensions n_i to be ordered decreasingly, i.e.,

$$n_1 \geq n_2 \geq \cdots \geq n_g. \tag{1.6}$$

The paper is organized as follows: Section 2 contains one of the main results (Theorem 2.2) concerning the change of the Weierstrass structure. It gives a lower bound on the algebraic multiplicity associated with each eigenvalue in the perturbed pencil, and suggests that the generic behavior for the Jordan blocks of the perturbed pencil is the one happening when this lower bound is attained. In section 3, we prove that this behavior is indeed generic by showing that it holds for all perturbations except those in a proper algebraic submanifold in the set of matrix pencils. Finally, Theorem 3.3 summarizes the results obtained throughout this paper.

2. Lower bounds on the algebraic multiplicities and the dimensions of Jordan blocks in the perturbed pencil. Throughout this section we follow a notation consistent with (1.5), and denote by

$$a_{R(\lambda)}(\lambda_0) \tag{2.1}$$

the algebraic multiplicity of the eigenvalue λ_0 in the regular matrix pencil $R(\lambda)$. Our aim is to determine the generic Weierstrass structure of λ_0 in a perturbed matrix pencil $A_0 + B_0 + \lambda(A_1 + B_1)$, starting from the structure of this eigenvalue in the unperturbed pencil $A_0 + \lambda A_1$. For this, we need to know $a_{A_0+B_0+\lambda(A_1+B_1)}(\lambda_0)$, as well as how this algebraic multiplicity is distributed among the Jordan blocks of λ_0. At least two approaches are possible to solve this problem. First, one can start with Jordan chains of $A_0 + \lambda A_1$ associated with λ_0, and then explicitly build new Jordan chains for λ_0 in $A_0 + B_0 + \lambda(A_1 + B_1)$, exhausting the algebraic multiplicity $a_{A_0+B_0+\lambda(A_1+B_1)}(\lambda_0)$. This approach was used in [7, 8, 9] for the standard eigenvalue problem $A - \lambda I$. It has the advantage of providing the new Jordan chains, and the drawback of being rather intricate in the case of pencils. In this paper we use a simpler approach: First, we determine lower bounds on the number and the dimensions of the Jordan blocks associated with λ_0 in the perturbed pencil. This method, based on a result by Thompson [12], involves the use of the invariant factors of the pencils. Then, we prove that the generic behavior corresponds to the case when these lower bounds are attained.

We begin by recalling that the rank of an arbitrary matrix pencil, regular or singular, $T(\lambda) = T_0 + \lambda T_1$ is r if all of the minors of $T(\lambda)$ of dimension greater than r are identically equal to zero, but $T(\lambda)$ has minors of dimension r which are polynomials in λ not identically equal to zero. As a consequence, the rank of a regular $n \times n$ matrix pencil is equal to n.

The next auxiliary lemma is a consequence of [12, Theorem 1]. It establishes lower bounds on the number and dimensions of the Jordan blocks in the Weierstrass form of a regular matrix pencil $(R+T)(\lambda)$, where $R(\lambda)$ is a regular matrix pencil and $T(\lambda)$ is any pencil of rank r.

LEMMA 2.1. *Let $R(\lambda) = R_0 + \lambda R_1$ be a complex regular square pencil, and $T(\lambda) = T_0 + \lambda T_1$ be another complex pencil of the same dimension with rank at most r. Let λ_0 be an eigenvalue of $R(\lambda)$ with g associated Jordan blocks of dimensions $d_1 \geq \cdots \geq d_g$ in the Weierstrass form of $R(\lambda)$. If $(R + T)(\lambda)$ is also a regular pencil and $r \leq g$, then in the Weierstrass form of $(R + T)(\lambda)$ there are at least $g - r$ Jordan blocks associated with λ_0 of dimensions $\beta_{r+1} \geq \cdots \geq \beta_g$ such that $\beta_i \geq d_i$ for $r + 1 \leq i \leq g$.*

Proof. First, let us assume that the rank of $T(\lambda)$ is exactly r. We begin by proving that any pencil $T(\lambda)$ of rank r is the sum of r singular pencils of rank 1. This can be seen by using the Kronecker canonical form of singular pencils [4, Chapter XII]. Let $K_0 + \lambda K_1$ be the Kronecker canonical form of $T(\lambda) = T_0 + \lambda T_1$, and write $K_0 + \lambda K_1$ as the sum of the following matrices:

1. For any singular block L_k of dimension $k \times (k + 1)$ appearing in $K_0 + \lambda K_1$ [4, p. 39], we have that $L_k = L_k^{(1)} + \cdots + L_k^{(k)}$, where the jth row of $L_k^{(j)}$ is equal to the jth row of L_k, and the rest of the rows of $L_k^{(j)}$ are zero. Therefore L_k is the sum of k singular pencils with rank 1.

2. An analogous expression holds for any singular block L_p^T of dimension $(p + 1) \times p$ appearing in $K_0 + \lambda K_1$.

3. Finally, for the $m \times m$ regular part $F(\lambda) = F_0 + \lambda F_1$ of $K_0 + \lambda K_1$, we have again that $F(\lambda) = F^{(1)} + \cdots + F^{(m)}$, where $F^{(j)}$ has the jth row equal to the jth row of $F(\lambda)$ and the rest of the rows of $F^{(j)}$ are zero. Therefore $F(\lambda)$ is the sum of m singular pencils with rank 1.

The pencil $K_0 + \lambda K_1$ can be expressed as the sum of r singular pencils of rank 1 just by combining the previous expansions of its singular blocks and of its regular part. The same holds for $T(\lambda)$ because it is strictly equivalent to $K_0 + \lambda K_1$. Let this decomposition be

$$T(\lambda) = T_1(\lambda) + \cdots + T_r(\lambda),$$

where $\operatorname{rank} T_i(\lambda) = 1$ for $1 \le i \le r$.

For any $n \times n$ regular pencil $P(\lambda) = P_0 + \lambda P_1$, we denote by

$$h_n(P) | h_{n-1}(P) | \cdots | h_1(P)$$

its invariant polynomials [3, Chapter VI], also called invariant factors. As usual, $h_n(P)|h_{n-1}(P)$ means that $h_n(P)$ divides $h_{n-1}(P)$. Notice also that $h_1(P) \neq 0$ because the pencil is regular.

Let

$$(\lambda - \lambda_0)^{d_g} | (\lambda - \lambda_0)^{d_{g-1}} | \cdots | (\lambda - \lambda_0)^{d_1}$$

be the elementary divisors [3, Chapter VI] of $R(\lambda)$ associated with λ_0. Each elementary divisor $(\lambda - \lambda_0)^{d_i}$ corresponds to a Jordan block of λ_0 of dimension d_i in the Weierstrass form of $R(\lambda)$. It is well known that

$$\begin{array}{ccc} (\lambda - \lambda_0)^{d_1} & | & h_1(R) \\ (\lambda - \lambda_0)^{d_2} & | & h_2(R) \\ \vdots & \vdots & \vdots \\ (\lambda - \lambda_0)^{d_g} & | & h_g(R). \end{array}$$

Now, consider the sequence of pencils $R(\lambda), R(\lambda)+T_1(\lambda), R(\lambda)+T_1(\lambda)+T_2(\lambda), \ldots,$ $R(\lambda) + T(\lambda)$, and note that each of them is a rank 1 perturbation of the preceding one. Applying [12, Theorem 1][2] to this sequence leads to

$$\begin{array}{l} (\lambda - \lambda_0)^{d_{r+1}} | h_{r+1}(R) | h_r(R + T_1) | \ldots | h_1(R + T), \\ (\lambda - \lambda_0)^{d_{r+2}} | h_{r+2}(R) | h_{r+1}(R + T_1) | \ldots | h_2(R + T), \\ \qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \;\; \vdots \\ (\lambda - \lambda_0)^{d_g} | h_g(R) | h_{g-1}(R + T_1) | \ldots | h_{g-r}(R + T), \end{array}$$

where $h_1(R+T) \neq 0$ because the pencil $(R+T)(\lambda)$ is regular. These divisibility chains mean that the pencil $(R+T)(\lambda)$ has *at least* $g-r$ elementary divisors associated with λ_0:

$$(\lambda - \lambda_0)^{\beta_g} | (\lambda - \lambda_0)^{\beta_{g-1}} | \cdots | (\lambda - \lambda_0)^{\beta_{r+1}},$$

[2]Theorem 1 in [12] states that if $P(\lambda)$ and $Q(\lambda)$ are $n \times n$ matrix polynomials with invariant polynomials $h_n(P)|h_{n-1}(P)|\cdots|h_1(P)$ and $h_n(Q)|h_{n-1}(Q)|\cdots|h_1(Q)$, respectively, and if the rank of $P(\lambda) - Q(\lambda)$ is equal to one, then $h_n(P)|h_{n-1}(Q)|h_{n-2}(P)|h_{n-3}(Q)|\cdots$ and $h_n(Q)|h_{n-1}(P)|h_{n-2}(Q)|h_{n-3}(P)|\cdots$.

with $d_i \leq \beta_i$ for $r+1 \leq i \leq g$. Each of these elementary divisors corresponds to a $\beta_i \times \beta_i$ Jordan block associated with λ_0 in the Weierstrass form of $(R+T)(\lambda)$.

If the rank of $T(\lambda)$ is $r_1 < r$, then the result we have just proved can be applied to show that the Weierstrass form of $(R+T)(\lambda)$ has at least $g - r_1 > g - r$ Jordan blocks associated with λ_0 of dimensions $\beta_i \geq d_i$, $i = r_1 + 1, \dots, g$, and the result follows. ∎

The previous lemma allows us to obtain the main result in the first part of the present paper.

THEOREM 2.2. *Let λ_0 be an eigenvalue of the complex regular matrix pencil $A_0 + \lambda A_1$, and $n_1 \geq \cdots \geq n_g$ be the dimensions of the Jordan blocks associated with λ_0 in its Weierstrass canonical form. Let $B_0 + \lambda B_1$ be any complex pencil such that the perturbed pencil $A_0 + B_0 + \lambda(A_1 + B_1)$ is also regular. Assume that $g \geq \operatorname{rank}(B_0 + \lambda_0 B_1)$. Set $\rho = \operatorname{rank}(B_0 + \lambda_0 B_1) + \operatorname{rank} B_1$ and $n_m = 1$ for any $m = g+1, \dots, \rho$. Then the algebraic multiplicities of λ_0 in the perturbed and unperturbed pencils satisfy*

$$a_{A_0+B_0+\lambda(A_1+B_1)}(\lambda_0) \geq a_{A_0+\lambda A_1}(\lambda_0) + \operatorname{rank} B_1 - n_1 - \cdots - n_\rho, \tag{2.2}$$

using the notation in (2.1). *Moreover, if the equality in this inequality holds, then the dimensions of the Jordan blocks for λ_0 in the Weierstrass canonical form of $A_0 + B_0 + \lambda(A_1 + B_1)$ are obtained by removing the first ρ members in the list $n_1, \dots, n_g, \underbrace{1, \dots, 1}_{\operatorname{rank} B_1}$.*

Proof. Notice that

$$\operatorname{rank}(B_0 + \lambda B_1) = \operatorname{rank}(B_0 + \lambda_0 B_1 + (\lambda - \lambda_0) B_1) \leq \operatorname{rank}(B_0 + \lambda_0 B_1) + \operatorname{rank}(B_1) = \rho.$$

So, in the case $\rho < g$, Lemma 2.1 guarantees the existence of $g - \rho$ Jordan blocks associated with λ_0 of dimensions $\beta_{\rho+1} \geq n_{\rho+1}, \dots, \beta_g \geq n_g$ in the Weierstrass canonical form of the perturbed pencil $A_0 + B_0 + \lambda(A_1 + B_1)$. Moreover, the left side in the inequality (1.1) implies that there are at least $\rho_1 = \operatorname{rank} B_1$ additional Jordan blocks of sizes $\alpha_1 \geq 1, \dots, \alpha_{\rho_1} \geq 1$ associated with λ_0. Thus,

$$a_{A_0+B_0+\lambda(A_1+B_1)}(\lambda_0) \geq \beta_{\rho+1} + \cdots + \beta_g + \alpha_1 + \cdots + \alpha_{\rho_1} \geq n_{\rho+1} + \cdots + n_g + \rho_1.$$

Obviously, this inequality is equivalent to (2.2). If $g \leq \rho$, then inequality (2.2) becomes

$$a_{A_0+B_0+\lambda(A_1+B_1)}(\lambda_0) \geq g - \operatorname{rank}(B_0 + \lambda_0 B_1).$$

This fact is trivial because of inequality (1.1) and the evident spectral inequality

$$a_{A_0+B_0+\lambda(A_1+B_1)}(\lambda_0) \geq \dim \ker (A_0 + \lambda_0 A_1 + B_0 + \lambda_0 B_1).$$

Finally, notice that the previous inequalities become equalities if and only if the number and dimensions of the Jordan blocks associated with λ_0 in the Weierstrass form of $A_0 + B_0 + \lambda(A_1 + B_1)$ are those appearing in the statement of Theorem 2.2. ∎

Remark 1. Notice that the unnatural definition $n_m = 1$ for $m = g+1, \dots, \rho$ allows us to express inequality (2.2) in a unified way for both cases $\rho < g$ and $\rho \geq g$. The reader is invited to check that the number and dimensions of the Jordan blocks of the perturbed pencil associated with λ_0 in the case of equality in (2.2) are precisely those appearing in the generic behavior described in the abstract and the introduction.

Theorem 2.2 gives us all the sizes of the Jordan blocks associated with λ_0 in the perturbed pencil $A_0+B_0+\lambda(A_1+B_1)$ when the inequality in (2.2) is an equality. As we will see in the following section, this is the case for most perturbations $B_0+\lambda B_1$.

3. The generic behavior. The quantity

$$\widetilde{a} = a_{A_0+\lambda A_1}(\lambda_0) + \text{rank } B_1 - n_1 - \cdots - n_\rho$$

in (2.2), where $\rho = \text{rank}\,(B_0+\lambda_0 B_1) + \text{rank } B_1$ as in the statement of Theorem 2.2, is a lower bound on the algebraic multiplicity of λ_0 as an eigenvalue of the perturbed pencil $A_0+B_0+\lambda(A_1+B_1)$. This means that for each perturbation $B_0+\lambda B_1$ of $A_0+\lambda A_1$ such that $g \geq \text{rank}(B_0+\lambda_0 B_1)$,

$$\det\left(A_0+B_0+\lambda(A_1+B_1)\right) = (\lambda-\lambda_0)^{\widetilde{a}} q(\lambda-\lambda_0) \tag{3.1}$$

for some polynomial $q(\lambda-\lambda_0)$. Therefore, if the perturbed pencil is regular the algebraic multiplicity of λ_0 in the perturbed pencil is exactly $\widetilde{a}$ if and only if the coefficient $q(0)$ of $(\lambda-\lambda_0)^{\widetilde{a}}$ in (3.1) is not equal to zero. Clearly, once A_0 and A_1 are fixed, this coefficient is a multivariate polynomial in the entries of B_0 and B_1. Therefore, if this coefficient is not identically zero for all B_0 and B_1 such that $\text{rank}(B_0+\lambda_0 B_1)=\rho_0 \leq g$ and $\text{rank}(B_1)=\rho_1$, for fixed integers ρ_0 and ρ_1, the equation $q(0)=0$ defines an algebraic submanifold in the set of pairs (B_0,B_1) with $\text{rank}(B_0+\lambda_0 B_1)=\rho_0\leq g$ and $\text{rank}(B_1)=\rho_1$ that characterizes the set of perturbation pencils for which the generic behavior described in the introduction does not happen. The only goal of this section is to show that this algebraic submanifold is proper or, in other words, that the coefficient $q(0)$ is not zero for all perturbations $B_0+\lambda B_1$ such that $\text{rank}(B_0+\lambda_0 B_1)=\rho_0\leq g$ and $\text{rank}(B_1)=\rho_1$. This is done in Lemma 3.2. This will allow us to say that the change in the dimensions of the Jordan blocks described in Theorem 2.2, when the equality in (2.2) holds, is *generic*. The reader is referred to [1] for a detailed description of the algebraic submanifold $q(0)=0$ in terms of a determinantal equation involving the entries of B_0 and B_1.

The simple Lemma 3.1 studies some specific perturbations of the blocks appearing in the Weierstrass canonical form (1.4). It will be used in the proof of Lemma 3.2.

LEMMA 3.1. *Let $J_k(\alpha)$ be a $k\times k$ Jordan block with α on the main diagonal, $E_k(\beta)$ be a $k\times k$ matrix that is everywhere zero except for β in the $(k,1)$ entry, and $D_k(\lambda)=(\lambda-\lambda_0)\,\text{diag}(s_1,\ldots,s_k)$, where $s_i=0$ or 1 for all i. Note that $D_k(\lambda)$ may be the zero matrix. Then,*

1. *λ_0 is an eigenvalue of $\lambda I + J_k(-\lambda_0)+E_k(\lambda-\lambda_0)+D_k(\lambda)$ with algebraic multiplicity 1,*
2. *λ_0 is not an eigenvalue of $\lambda I + J_k(-\lambda_0)+E_k(1)+D_k(\lambda)$,*
3. *λ_0 is not an eigenvalue of $\lambda I + J_k(-\lambda_1)+D_k(\lambda)$ if $\lambda_1\neq\lambda_0$,*
4. *λ_0 is not an eigenvalue of $\lambda J_k(0)+I+D_k(\lambda)$.*

Proof. Check that

1.

$$\begin{aligned}\det(\lambda I + J_k(-\lambda_0)+E_k(\lambda-\lambda_0)+D_k(\lambda))&\\ = (\lambda-\lambda_0)\left[(\lambda-\lambda_0)^{k-1}\left(\prod_{i=1}^{k}(1+s_i)\right)+(-1)^{k+1}\right]&,\end{aligned}$$

2.

$$\det(\lambda I + J_k(-\lambda_0) + E_k(1) + D_k(\lambda)) = (\lambda - \lambda_0)^k \left(\prod_{i=1}^{k}(1+s_i)\right) + (-1)^{k+1},$$

3.

$$\det(\lambda I + J_k(-\lambda_1) + D_k(\lambda)) = \prod_{i=1}^{k}\left[(\lambda-\lambda_1) + s_i(\lambda-\lambda_0)\right],$$

4.

$$\det(\lambda J_k(0) + I + D_k(\lambda)) = \prod_{i=1}^{k}\left[1 + s_i(\lambda-\lambda_0)\right]. \qquad \square$$

LEMMA 3.2. *Let λ_0 be an eigenvalue of the complex $n \times n$ regular matrix pencil $A_0 + \lambda A_1$, and $n_1 \geq \cdots \geq n_g$ be the dimensions of the Jordan blocks associated with λ_0 in its Weierstrass canonical form. Let ρ_0 and ρ_1 be two nonnegative integers, with $\rho_0 \leq g$ and $\rho_1 \leq n$. Then, there exists a complex matrix pencil $B_0 + \lambda B_1$ such that*

$$\rho_0 = \operatorname{rank}(B_0 + \lambda_0 B_1), \qquad \rho_1 = \operatorname{rank}(B_1),$$

and the algebraic multiplicity of λ_0 in the perturbed pencil $A_0 + B_0 + \lambda(A_1 + B_1)$ is exactly $a_{A_0+\lambda A_1}(\lambda_0) + \rho_1 - n_1 - \cdots - n_\rho$, where $\rho := \rho_0 + \rho_1$ and $n_m = 1$ for $m = g+1, \ldots, \rho$.

Proof. It suffices to prove the result when $A_0+\lambda A_1$ is in Weierstrass canonical form because, otherwise, we can consider the strict equivalence (1.4), apply the result to the matrix pencil in Weierstrass canonical form in the right-hand side (with $B_0+\lambda B_1$ as the perturbation pencil), and take $Q^{-1}(B_0 + \lambda B_1)P^{-1}$.

So, assume that $A_0+\lambda A_1$ is in Weierstrass canonical form given by the right-hand side of (1.4). We consider separately the following two cases.

(i) Case $\rho < g$. Define the matrices

$$B_0 = \operatorname{diag}(E_{n_1}(1), \ldots, E_{n_{\rho_0}}(1), E_{n_{\rho_0+1}}(-\lambda_0), \ldots, E_{n_{\rho_0+\rho_1}}(-\lambda_0), 0, \ldots, 0)$$

and

$$B_1 = \operatorname{diag}(\overbrace{0, \ldots, 0}^{\rho_0 \ blocks}, E_{n_{\rho_0+1}}(1), \ldots, E_{n_{\rho_0+\rho_1}}(1), 0, \ldots, 0),$$

where zeros denote matrices, and the partition in diagonal blocks is conformal to the one of the Weierstrass form (1.4). It can be checked that the pencil $B_0 + \lambda B_1$ verifies the conditions mentioned in the statement, by using the first two items in Lemma 3.1 with $D_k(\lambda) = 0$.

(ii) Case $\rho \geq g$. Now, we define

$$\widehat{B}_0 = \operatorname{diag}(E_{n_1}(1), \ldots, E_{n_{\rho_0}}(1), E_{n_{\rho_0+1}}(-\lambda_0), \ldots, E_g(-\lambda_0), 0, \ldots, 0)$$

and

$$\widehat{B}_1 = \operatorname{diag}(\overbrace{0, \ldots, 0}^{\rho_0 \ blocks}, E_{n_{\rho_0+1}}(1), \ldots, E_{n_g}(1), 0, \ldots, 0),$$

where the partition is again conformal to the one in the Weierstrass canonical form (1.4). Notice that $\operatorname{rank}(\widehat{B}_1) = g - \rho_0 \leq \rho_1$, and that by appropriate choices of $\{s_1, \dots, s_n\}$, $s_i = 0$ or 1 for all i, $\operatorname{rank}(\widehat{B}_1 + \operatorname{diag}(s_1, \dots, s_n))$ may take any value between $g - \rho_0$ and n. Let $\{\tilde{s}_1, \dots, \tilde{s}_n\}$ be such that $\operatorname{rank}(\widehat{B}_1 + \operatorname{diag}(\tilde{s}_1, \dots, \tilde{s}_n)) = \rho_1$, and define the pencil $D(\lambda) = D_0 + \lambda D_1 = (\lambda - \lambda_0) \operatorname{diag}(\tilde{s}_1, \dots, \tilde{s}_n)$. Then the pencil $B_0 + \lambda B_1 \equiv \widehat{B}_0 + \lambda \widehat{B}_1 + D(\lambda)$ verifies the conditions mentioned in the statement, because $\operatorname{rank}(B_1) = \rho_1$,

$$\operatorname{rank}(B_0 + \lambda_0 B_1) = \operatorname{rank}(\widehat{B}_0 + \lambda_0 \widehat{B}_1) = \rho_0,$$

and Lemma 3.1 implies that the algebraic multiplicity of λ_0 in $A_0 + B_0 + \lambda(A_1 + B_1)$ is $g - \rho_0$, which is exactly $a_{A_0+\lambda A_1}(\lambda_0) + \rho_1 - n_1 - \cdots - n_\rho$. ☐

Theorem 2.2 and Lemma 3.2 allow us to give a complete answer to the problem originally posed in the introduction: Given a regular pencil $A_0 + \lambda A_1$ with eigenvalue λ_0, perturbed by a pencil $B_0 + \lambda B_1$, determine the generic Weierstrass structure associated with λ_0 as an eigenvalue of the perturbed pencil (1.3) when the low rank condition (1.2) holds for the perturbation. Notice that if $B_0 + \lambda B_1$ is in the set consisting of perturbation pencils for which $q(0) \neq 0$ (with $q(\lambda)$ as in (3.1)), then the perturbed pencil $A_0 + B_0 + \lambda(A_1 + B_1)$ is regular. With this observation in mind we can state the main theorem of this paper in the following way.

THEOREM 3.3. *Let λ_0 be an eigenvalue of the complex regular $n \times n$ matrix pencil $A_0 + \lambda A_1$ with Weierstrass canonical form* (1.4)*, and let g be the geometric multiplicity of λ_0 in $A_0 + \lambda A_1$. Let $B_0 + \lambda B_1$ be any $n \times n$ pencil, and set*

$$\rho_0 := \operatorname{rank}(B_0 + \lambda_0 B_1), \qquad \rho_1 := \operatorname{rank}(B_1), \qquad \rho := \rho_0 + \rho_1.$$

If $\rho_0 < g$, then λ_0 is an eigenvalue of the perturbed pencil

$$A_0 + B_0 + \lambda(A_1 + B_1), \tag{3.2}$$

and, generically, the pencil $A_0 + B_0 + \lambda(A_1 + B_1)$ is regular and the dimensions of the Jordan blocks for λ_0 in the Weierstrass canonical form of $A_0 + B_0 + \lambda(A_1 + B_1)$ are obtained by removing the first ρ members in the sequence $n_1, \dots, n_g, \underbrace{1, \dots, 1}_{\rho_1}$.

Remark 2. 1. An analogous result holds for the infinite eigenvalue of $A_0 + \lambda A_1$, by applying the previous theorem to the zero eigenvalue of the dual pencils $A_1 + \lambda A_0$ and $A_1 + B_1 + \lambda(A_0 + B_0)$.

2. Theorem 3.3 describes in a concise way the generic behavior presented in the introduction of this paper.

Acknowledgments. The third author thanks Prof. Sergey Savchenko for bringing to his attention Theorem 1 in [12] and the reference [5]. The authors are also indebted to an anonymous referee for valuable comments which very much improved the original manuscript.

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