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Eigenvalue condition numbers and Pseudospectra of Fiedler matrices

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Abstract The aim of the present paper is to analyze the behavior of Fiedler companion matrices in the polynomial root-finding problem from the point of view of conditioning of eigenvalues. More precisely, we compare: (a) the condition number of a given root λ of a monic polynomial $p(z)$ with the condition number of λ as an eigenvalue of any Fiedler matrix of $p(z)$, (b) the condition number of λ as an eigenvalue of an arbitrary Fiedler matrix with the condition number of λ as an eigenvalue of the classical Frobenius companion matrices, and (c) the pseudozero sets of $p(z)$ and the pseudospectra of any Fiedler matrix of $p(z)$. We prove that, if the coefficients of the polynomial $p(z)$ are not too large and not all close to zero, then the conditioning of any root λ of $p(z)$ is similar to the conditioning of λ as an eigenvalue of any Fiedler matrix of $p(z)$. On the contrary, when $p(z)$ has some large coefficients, or they are all close to zero, the conditioning of λ as an eigenvalue of any Fiedler matrix can be arbitrarily much larger than its conditioning as a root of $p(z)$ and, moreover, when $p(z)$ has some large coefficients there can be two different Fiedler matrices such that the ratio between the condition numbers of λ as an eigenvalue of these two matrices can be arbitrarily large. Finally, we relate asymptotically the pseudozero sets of $p(z)$ with the pseudospectra of any given Fiedler matrix of $p(z)$, and the pseudospectra of any two Fiedler matrices of $p(z)$.

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1 Introduction

The computation of roots of scalar polynomials, that we term as *the polynomial root-finding problem*, is one of the oldest mathematical problems [7, 34]. In addition, computing roots of polynomials has important recent applications in many areas. Low-degree polynomial equations appear in physics, chemistry, engineering, business management and statistics and they are solved very often using tools of numerical linear algebra [34]. These tools are sufficiently effective in most cases. Polynomial equations with large degree (typically 100 and sometimes of order of several thousands) arise in computer algebra, computational algebraic geometry, control theory and signal processing (see [27–29, 33, 34] and the bibliography therein), and, until the 1980's, they were a challenge for the available software.

Nowadays we are aware of many algorithms for computing roots of polynomials, together with their numerical analysis and applications. These algorithms can be classified into two categories. The first category comprises algorithms that are implemented using the standard machine floating point arithmetic. Some of these algorithms are the Madsen–Reid [25], Jenkins–Traub [21], Aberth-Ehrlich [1, 15], Durand-Kerner [23], Laguerre [37], QR-algorithm applied to companion matrices, and QZ-algorithm applied to companion pencils (we refer to [30, 31] for thorough surveys on numerical methods for computing the roots of polynomials, and to [19] for the QR and QZ-algorithms). In the second category we have algorithms that work using different levels of working machine precision with an increasing number of bits. In this second category we have the algorithm MPSolve [5, 6], which can compute roots of polynomials with any number of digits of precision.

Among the algorithms that work with a fixed floating point arithmetic precision, the most widely used are the ones that compute the roots of polynomials as the eigenvalues of companion matrices. The main advantages of this approach are: (a) the methods are easy to implement (assuming available any algorithm to compute the eigenvalues of a matrix), (b) there are standard backward stable eigenvalue algorithms, like the QR-algorithm, that can be used to compute the eigenvalues of the companion matrix, and (c) the convergence is guaranteed in practice. This is the approach followed by the MATLAB command `roots`, which uses the QR-algorithm on the balanced Frobenius companion matrix (2.4) to compute its eigenvalues, though this method requires order n^2 storage and order n^3 floating point operations [32], and is not backward stable in the polynomial sense [13, 14]. In the last decades, much effort has been devoted to devise a fast, memory efficient, and backward-stable algorithm for the polynomial root-finding problem using eigenvalue solvers. In the recent work [2], that contains many references on this issue, it is presented an algorithm with order n storage and order n^2 floating point operations that is backward stable in the matrix sense, that is, the computed roots are the exact eigenvalues of a small perturbation of the Frobenius companion matrix. However, it is known that small perturbations of the companion matrix might not correspond to small perturbations of the polynomial (see, for example, [13, 14]), so the problem of devising a fast, memory efficient, and backward-stable (in the polynomial sense) root-finding method via the computation of eigenvalues of companion matrices remains open. However, it is worth to emphasize that backward stability in the polynomial sense can be achieved via the computation of eigenvalues of companion pencils [42], though it considerably increases the computational cost.

On a different but related line of research, a new family of companion matrices was introduced in 2003 [16]. This family of matrices is now known as the *Fiedler companion matrices* [9], or Fiedler matrices for brevity, and it includes the classical Frobenius companion matrices. Also, among Fiedler matrices, we find pentadiagonal matrices, which have potential numerical advantages since their small bandwidth could be exploited by some eigenvalue solvers to devise fast algorithms for computing roots of polynomials [3]. For this reason, it makes sense to analyze the numerical performance of this family, in order to conclude whether or not there are other Fiedler matrices than the Frobenius ones that can be used in the polynomial root-finding problem with better (or similar) reliability.

When computing a root λ of a polynomial $p(z)$ using a backward stable algorithm on any companion matrix A, the relative (forward) error of λ can be bounded as

$$\frac{|\lambda - \widehat{\lambda}|}{|\lambda|} \leq \kappa(\lambda, A) \cdot O(u), \tag{1.1}$$

where $\widehat{\lambda}$ is the computed (approximate) eigenvalue (root), u is the unit roundoff and $\kappa(\lambda, A)$ is the relative condition number of λ as an eigenvalue of A with respect to perturbations $A+E$ of A such that $\|E\|_2 = O(u)\|A\|_2$. However, it is natural to look for a bound in terms of the original polynomial root-finding problem, instead of the eigenvalue problem. Then, in order to get a bound for the forward error in terms of the relative condition number, $\kappa(\lambda, p)$, of λ as a root of $p(z)$, one can rewrite (1.1) as

$$\frac{|\lambda - \widehat{\lambda}|}{|\lambda|} \leq \frac{\kappa(\lambda, A)}{\kappa(\lambda, p)} \cdot \kappa(\lambda, p) \cdot O(u). \tag{1.2}$$

Hence, the ratio $\kappa(\lambda, A)/\kappa(\lambda, p)$ measures how far is the error obtained after solving the polynomial root-finding problem as an eigenvalue problem from the ideal error expected when considering it just as a polynomial root-finding problem, that is, from the ideal error corresponding to the sensitivity of the original input $p(z)$.

As mentioned above, the standard approach to solve the polynomial root-finding problem as an eigenvalue problem uses the Frobenius companion matrices (2.4) or some variants of them. The behavior of these matrices in the polynomial root-finding problem, from the point of view of conditioning, was studied in [39]. In that reference the authors compare the conditioning of the polynomial root-finding problem with the conditioning of the eigenvalue problem by two means. The first one is by measuring the ratio $\kappa(\lambda, C)/\kappa(\lambda, p)$, with C being a Frobenius companion matrix. The second one is by comparing the pseudozero sets of the polynomial with the pseudospectra of C. The numerical experiments carried out in that paper show that, once the companion matrix is balanced, the conditioning of these two problems tend to be equivalent. However, no proof of this fact is provided.

Following the approach in [39], and motivated also by (1.2), in this paper we compare: (a) the condition number of a given root of a monic polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with the condition number of this root as an eigenvalue of any Fiedler matrix, and (b) the pseudozero sets of $p(z)$ with the pseudospectra of the associated Fiedler matrices.

Regarding (a), we present explicit expressions for $\kappa(\lambda, M_\sigma)$, with M_σ being any Fiedler matrix, and then we analyze the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, as well as the ratios $\kappa(\lambda, C)/\kappa(\lambda, M_\sigma)$ and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$, with C being a Frobenius companion matrix. The conclusions of this analysis can be roughly summarized as follows:

1. There is a quantity $\alpha(p)$, depending on the coefficients of $p(z)$, satisfying

$$\frac{1}{n} \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \leq n^{5/2}\alpha(p),$$

   and such that (see Theorem 6.1 for complete details):
   (i) if $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is not much larger or much smaller than 1 (that is, of order $\Theta(1)$), then $\alpha(p)$ is not much larger than 1, and
   (ii) if $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is much larger or much smaller than 1, then $\alpha(p)$ can be much larger than 1.
2. If C is a Frobenius companion matrix, then

$$1 \leq \max\left\{\frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)}, \frac{\kappa(\lambda, C)}{\kappa(\lambda, M_\sigma)}\right\} \leq n^{5/2}\|p\|_2$$

   (see Theorem 6.2), where $\|p\|_2 = \sqrt{\sum_{k=0}^{n} |a_k|^2}$.

From 1 and 2 we get the following conclusions:

(C1) When $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is of order $\Theta(1)$, then the eigenvalues of any Fiedler matrix are as well conditioned as the roots of the monic polynomial $p(z)$. In this case, from the point of view of condition numbers, any Fiedler matrix is a good tool for solving the root-finding problem for $p(z)$.

(C2) When $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is much larger than 1 or close to zero, the eigenvalues of any Fiedler companion matrix may be potentially much more ill conditioned than the roots of the monic polynomial $p(z)$.

(C3) From the point of view of condition numbers, when $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is not large, any Fiedler matrix can be used for solving the root-finding problem for $p(z)$ with the same reliability as Frobenius companion matrices.

(C4) The ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ may be potentially large (or small) for monic polynomials with large coefficients. In this case, for some polynomials there can be other Fiedler matrices than the Frobenius ones which are more convenient from the point of view of conditioning, and viceversa.

Despite (C4), we will show in Theorem 6.3 that, from the point of view of condition numbers, Frobenius companion matrices are better suited than the rest of Fiedler matrices in the polynomial root-finding problem only for monic polynomials for which it is not recommended to compute their roots as eigenvalues of any Fiedler matrix (including the Frobenius ones). Finally, we show that there are monic polynomials for which one should avoid computing their roots as the eigenvalues of Frobenius companion matrices and to use, instead, another Fiedler matrix. However, the problem of identifying these polynomials in advance and knowing which is the most convenient Fiedler matrix one might use are still ongoing works.

Notice that there is a difference in the conditions of (C1)–(C2) and the ones in (C3)–(C4) regarding the size of the coefficients of $p(z)$. In particular, it may happen that $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is not large, so that $\max\left\{\frac{\kappa(\lambda,M_\sigma)}{\kappa(\lambda,C)}, \frac{\kappa(\lambda,C)}{\kappa(\lambda,M_\sigma)}\right\}$ is not large, but $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is close to zero, so that $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ can be large.

This work is related to the recent paper [13], where the authors have analyzed the backward error of the polynomial root-finding problem solved via Fiedler companion matrices and a backward stable eigenvalue algorithm. It is shown there that only for polynomials having coefficients with absolute values much larger than 1 the solution of the polynomial root-finding problem using Fiedler companion matrices is not normwise backward stable. However, to guarantee coefficient-wise backward stability we need to impose that the coefficients do not have small absolute values as well. As a consequence, we can say that, when the coefficients of the polynomial are of order $\Theta(1)$, all Fiedler matrices have a similar (good) behavior in the polynomial root-finding problem, from the point of view of both conditioning and backward error.

Regarding pseudozero sets and pseudospectra, we first present an asymptotic formula for the norm of the resolvent of any Fiedler matrix M_σ. This formula provides a fast method to approximate the level curves of the pseudospectra of M_σ. Then, we present asymptotic relationships between: (i) the pseudozero sets of $p(z)$ and the pseudospectra of M_σ, and (ii) the pseudospectra of any two Fiedler matrices. Regarding (i), we conclude that the ϵ-pseudozero set of $p(z)$ coincides asymptotically with the ϵ'-pseudospectrum of M_σ, with $\epsilon' = \epsilon\kappa(\lambda, p)/\kappa(\lambda, M_\sigma)$. As for (ii), we conclude that, for a given couple of Fiedler matrices M_{σ_1} and M_{σ_2}, the ϵ-pseudospectrum of M_{σ_1} coincides asymptotically with the ϵ'-pseudospectrum of M_{σ_2}, with $\epsilon' = \epsilon\kappa(\lambda, M_{\sigma_1})/\kappa(\lambda, M_{\sigma_2})$. This means that the ϵ-pseudozero sets of $p(z)$ and the ϵ-pseudospectra of M_σ tend to coincide, but for different values of ϵ, related through the ratio $\kappa(\lambda, p)/\kappa(\lambda, M_\sigma)$. The same happens with the ϵ-pseudospectra of two different Fiedler matrices $M_{\sigma_1}, M_{\sigma_2}$ and the ratio $\kappa(\lambda, M_{\sigma_1})/\kappa(\lambda, M_{\sigma_2})$.

The paper is organized as follows. In Section 2 we introduce the basic notation and definitions, and we review the basic facts about Fiedler matrices that are used along the paper. In Section 3 we present expressions for the condition numbers of the roots of a monic polynomial $p(z)$. In Section 4 we give new explicit formulas for the right and left eigenvectors of Fiedler matrices, which are simpler than the ones already present in the literature. These formulas are used in Section 5 to get expressions for the eigenvalue condition numbers of Fiedler matrices associated with $p(z)$. In Section 6 we compare the eigenvalue condition numbers of Fiedler matrices with the condition numbers of the roots of $p(z)$, and we also compare the eigenvalue condition numbers of the Frobenius companion matrices with the eigenvalue condition numbers of Fiedler matrices other than Frobenius ones. Section 7 is devoted to the study of pseudospectra of Fiedler matrices, and to compare them with the pseudozero sets of $p(z)$. Finally, in Section 8 we present numerical experiments to illustrate our theoretical results, and to study the effect of balancing Fiedler matrices from the point of view of eigenvalue condition numbers and pseudospectra. We summarize the main contributions of the paper in Section 9.

The contents of this paper are part of the PhD. Thesis of the third author, and the reader is referred to this work for further details [36].

2 Basic notation and definitions

The symbol I_n stands for the $n \times n$ identity matrix. For a given variable x, by $O(x)$ we mean a quantity bounded, in absolute value, by $\alpha|x|$ as x approaches 0, with α being a nonzero constant.

In this work, the right and left eigenvectors of a matrix $A \in \mathbb{C}^{n\times n}$ associated with the eigenvalue λ are two nonzero vectors $x, y \in \mathbb{C}^n$, respectively, satisfying $Ax = \lambda x$ and $y^T A = \lambda y^T$. The definition of left eigenvector differs from the usual definition, which is a nonzero vector $y \in \mathbb{C}^n$ such that $y^* A = \lambda y^*$. However, it will be more convenient for our developments.

In order to better express our results, we need to distinguish between norms on the vector space of polynomials of degree less than or equal to n and norms on the vector space of nontrivial coefficients of monic polynomials (i. e., excluding the leading coefficient $a_n = 1$) of degree equal to n. In particular, for a polynomial

$$p(z) = \sum_{k=0}^{n} a_k z^k, \qquad a_k \in \mathbb{C}, \tag{2.1}$$

non necessarily monic, $\|p\|_2$ is the norm on the vector space of polynomials of degree less than or equal to n, defined as

$$\|p\|_2 := \sqrt{\sum_{k=0}^{n} |a_k|^2}.$$

In addition, for a monic polynomial of degree n,

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \tag{2.2}$$

we define $|||p|||_2$ as

$$|||p|||_2 := \sqrt{\sum_{k=0}^{n-1} |a_k|^2}.$$

Notice that $|||p|||_2$ is not a norm on the vector space of polynomials of degree less than or equal to n. Note also that when $p(z)$ is monic, then $\|p\|_2 \geq 1$.

By $\|A\|_F$ we denote the *Frobenius norm* of the matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, that is:

$$\|A\|_F := \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}.$$

Similarly, $\|A\|_2$ denotes the 2-norm of A, namely, the largest singular value of A.

Given $p(z)$ as in (2.1) and $d = 0, 1, \ldots, n$, the *degree d Horner shift of* $p(z)$ is the polynomial

$$p_d(z) := a_n z^d + a_{n-1} z^{d-1} + \cdots + a_{n-d+1} z + a_{n-d}.$$

The Horner shifts of $p(z)$ satisfy the following recurrence relation:

$$\begin{cases} p_0(z) = a_n\,, & \text{and} \\ p_d(z) = zp_{d-1}(z) + a_{n-d}\,, & \text{for } \ d = 1, 2, \ldots, n. \end{cases} \tag{2.3}$$

For a general polynomial $p(z)$ as in (2.1), the *reversal polynomial of* $p(z)$ is obtained by reversing the order of its coefficients, that is, $p^{\text{rev}}(z) := \sum_{k=0}^{n} a_k z^{n-k}$.

2.1 A brief introduction to Fiedler companion matrices

Fiedler matrices first appeared in the context of *companion matrices of monic polynomials* in [16]. We present in Definition 2.1 what we mean by a companion matrix of a monic polynomial as in (2.2). This definition is a particular case of the more general definition presented in [11] of companion forms of grade ℓ for matrix polynomials.

Definition 2.1 Given a monic polynomial $p(z)$ of degree $n \geq 2$, a companion matrix of $p(z)$ is a matrix $A \in \mathbb{C}^{n \times n}$ satisfying the following two properties:

(i) Each entry of the matrix A is either a constant $\alpha \in \mathbb{C}$, or a constant times one of the coefficients of $p(z)$, i.e., βa_j for some $\beta \in \mathbb{C}$ and $0 \leq j \leq n-1$, and
(ii) $\det(zI - A) = p(z)$, so the eigenvalues of A, counting algebraic multiplicities, are equal to the roots of $p(z)$, counting multiplicities.

The best well-known examples of companion matrices of a monic polynomial (2.2) are the matrices

$$C_1 := \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C_2 := \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.4}$$

known as the *first* and *second Frobenius companion matrices* of $p(z)$, respectively. Notice that C_1 and C_2 can be constructed directly from the coefficients of the polynomial $p(z)$. The use of companion matrices goes back at least to Frobenius (1879) in his "rational canonical form" of a matrix [17]. Other similar Frobenius companion matrices that appear in the literature are obtained by transposition and/or by reversing the order of rows and columns of C_1 or C_2 (see, for instance, [20, pp. 194–200] and [24, p. 105]).

Frobenius companion matrices are important in theory, in numerical computations, and in applications. For instance, MATLAB command `roots` computes all the roots of a polynomial by applying the Francis' implicitly-shifted QR-algorithm, or QR-algorithm for short, to a balanced Frobenius companion matrix [26]. Frobenius companion matrices are also widely used in control theory and signal processing, for example, in the observable and the controllable canonical forms (see [22] and [24, Section 10.4] and the references therein).

In 2003, Fiedler expanded significantly the family of companion matrices associated with a monic polynomial [16]. These matrices were named *Fiedler matrices* in [9]. The family of Fiedler matrices includes C_1 and C_2 and, provided that $n \geq 3$, it contains some other different matrices and, in fact, many others when n is large.

Next, we are going to recall the main notions and notation regarding Fiedler matrices, where we essentially follow [13].

For the monic polynomial $p(z)$ in (2.2), we define the $n \times n$ matrices

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix} \quad \text{and} \quad M_k := \begin{bmatrix} I_{n-k-1} & & & \\ & -a_k & 1 & \\ & 1 & 0 & \\ & & & I_{k-1} \end{bmatrix}, \quad k = 1, \dots, n-1. \tag{2.5}$$

The matrices (2.5) are the basic factors used to build all Fiedler matrices. To be precise, let $\sigma = (i_1, i_2, \dots, i_n)$ be a permutation of the n-tuple $(0, 1, \dots, n-1)$. Then, the *Fiedler matrix of* $p(z)$ *associated with* σ is defined as the product

$$M_\sigma = M_{i_1} M_{i_2} \cdots M_{i_n}. \tag{2.6}$$

Note that the Frobenius companion matrices are particular Fiedler matrices. More precisely: $C_1 = M_{n-1} \cdots M_1 M_0$ and $C_2 = M_0 M_1 \cdots M_{n-1}$.

It is shown in [16] that given a monic polynomial $p(z)$, all Fiedler matrices M_σ associated with $p(z)$ are similar to each other. Since Frobenius companion matrices are particular examples of Fiedler matrices and the characteristic polynomial of a matrix is invariant under similarity, we conclude that the characteristic polynomial of all Fiedler matrices of $p(z)$ is the same $p(z)$.

The identity (2.6) allows us to get all nonzero entries of any Fiedler matrix M_σ. These are: (a) $(n-1)$ entries identically equal to 1, and (b) n entries equal to a_i, for $i = 0, \dots, n-1$, with exactly one copy of each. According to Definition 2.1, this justifies that Fiedler matrices are also called Fiedler companion matrices. Moreover, this implies the following identity:

$$\|M_\sigma\|_F = \sqrt{(n-1) + |a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2}. \tag{2.7}$$

The positions of the $(2n-1)$ nonzero entries of M_σ depend on the particular permutation σ. However, Fiedler matrices are constructible from the polynomial $p(z)$, without performing any arithmetic operation, by means of uniform templates valid for all polynomials [10, Algorithm 1].

Due to the identities

$$M_i M_j = M_j M_i\,, \qquad \text{for} \qquad |i-j| \neq 1,$$

some Fiedler matrices associated with different permutations σ are always equal, regardless of the values of the coefficients of $p(z)$. This fact motivates us to introduce some key notions. In the following, and according to the notation in [9,10, 12,13], we will also see the permutation σ as a bijection $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$, such that $\sigma(j)$ denotes the position of j in the list $(i_1, i_2, \dots, i_n)$.

Definition 2.2 Let $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$ be a bijection.

(a) For $i = 0, 1 \dots, n-2$, we say that σ has a consecution at i if $\sigma(i) < \sigma(i+1)$ and that σ has an inversion at i if $\sigma(i) > \sigma(i+1)$.

(b) The positional consecution-inversion sequence of σ, denoted by PCIS(σ), is the $(n-1)$-tuple $(v_0, \dots, v_{n-2})$ such that $v_j = 1$ if σ has a consecution at j and $v_j = 0$ otherwise.

Note that σ has a consecution at i, that is $v_i = 1$, if and only if M_i is to the left of M_{i+1} in the product (2.6), while σ has an inversion at i, that is $v_i = 0$, if and only if M_i is to the right of M_{i+1} in (2.6).

The relevance of the PCIS relies on the following fact: two Fiedler matrices M_{σ_1} and M_{σ_2} are symbolically equal (that is, they are equal for any polynomial $p(z)$ as in (2.1), regardless of the values of its coefficients) if and only if PCIS(σ_1)=PCIS(σ_2) [36, Prop. 2.7].

We will also use the following notions.

Definition 2.3 Let $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$ be a bijection with PCIS(σ) $= (v_0, v_1, \dots, v_{n-2})$. Then:

(a) The extended positional consecution-inversion sequence of σ, denoted by EPCIS(σ), is the n-tuple $(v_0, v_1, \dots, v_{n-1})$, where $v_{n-1} = v_{n-2}$.
(b) For $0 \le i \le j \le n-2$, we set

$$\mathfrak{i}_\sigma(i:j) := \sum_{k=i}^{j}(1 - v_k) \qquad \text{and} \qquad \mathfrak{c}_\sigma(i:j) := \sum_{k=i}^{j} v_k$$

for, respectively, the number of inversions and consecutions of σ from i to j. We also set $\mathfrak{i}_\sigma(i:j) := \mathfrak{c}_\sigma(i:j) := 0$ for $i > j$.

3 Condition numbers of roots of monic polynomials

The *condition number* $\kappa(\lambda, p)$ of a nonzero simple root λ of a monic polynomial $p(z)$ is defined as

$$\kappa(\lambda, p) := \lim_{\epsilon \to 0} \sup \left\{ \tfrac{|\widetilde{\lambda} - \lambda|}{\epsilon|\lambda|} : \widetilde{\lambda} \text{ is the closest root to } \lambda \text{ of } \widetilde{p}(z) = z^n + \textstyle\sum_{k=0}^{n-1} \widetilde{a}_k z^k \text{ with } |||\widetilde{p} - p|||_2 \le \epsilon |||p|||_2 \right\}. \tag{3.1}$$

The condition number $\kappa(\lambda, p)$ measures the relative sensitivity of the simple root λ with respect to relative normwise perturbations of $p(z)$ that preserve the monic character of the polynomial. It is also possible to introduce an appropriate definition of *coefficientwise condition number* to measure the relative sensitivity of the simple root λ with respect to relative coefficientwise perturbations of $p(z)$. For more information on this we refer the reader to [36, Ch. 8].

Note that the condition number $\kappa(\lambda, p)$ in (3.1) is defined only for perturbations of the coefficients $a_{n-1}, \dots, a_1, a_0$, in order to preserve the monic character of the polynomial $p(z)$. The reason for doing this is twofold. In the first place, we follow the same approach as Toh and Trefethen in [39], since our aim is to extend their results from Frobenius companion matrices to arbitrary Fiedler companion matrices. In the second place, it is natural to keep the monic property of $p(z)$ because we are considering the polynomial root-finding problem as an standard eigenvalue problem using companion matrices, and companion matrices are associated to monic polynomials. Allowing perturbations of the leading coefficient would lead to a generalized eigenvalue problem instead. However, it is possible to carry out a similar analysis to the one developed in further sections, by introducing the appropriate condition number, if we allow for perturbations of the leading coefficient as well.

In Theorem 3.1 we present a practical formula for the condition number $\kappa(\lambda, p)$. This formula can be found in [39, Prop. 2.2] and also in [38, Th. 5], after taking $\|E_j\|_2 = |||p|||_2$, for all $j = 0, 1, \ldots, n-1$.

Theorem 3.1 *Let* $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ *be a monic polynomial of degree* n*, let* λ *be a simple nonzero root of* $p(z)$*, let* $\Lambda(z)$ *be the vector*

$$\Lambda(z) = \left[z^{n-1} \cdots z \; 1\right]^T, \tag{3.2}$$

and let $\kappa(\lambda, p)$ *be the condition number defined in* (3.1)*. Then*

$$\kappa(\lambda, p) = \frac{|||p|||_2 \|\Lambda(\lambda)\|_2}{|\lambda| \cdot |p'(\lambda)|}, \tag{3.3}$$

where $\|\Lambda(\lambda)\|_2$ *denotes the usual Euclidean vector norm of* $\Lambda(\lambda)$*.*

4 Explicit formulas for the eigenvectors of Fiedler matrices

In Theorem 4.1, we present explicit expressions, in terms of λ and the coefficients of $p(z)$, of the right and left eigenvectors of any Fiedler matrix associated with an eigenvalue λ. Formulas for the left and right eigenvectors of Fiedler matrices are already known (see [8, Th. 3.1] or [9, Lemma 5.3], for example), but the ones we present here are simpler. We obtain them with a new technique that makes use of the expressions for $\operatorname{adj}(zI - M_\sigma)$ in [13, Th. 3.3]. Note that, since any Fiedler matrix M_σ is a *non-derogatory matrix*[1], the right and left eigenvectors of M_σ associated with an eigenvalue λ are unique, up to a multiplicative factor.

Theorem 4.1 *Let* $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ *be a monic polynomial of degree* $n \geq 2$*, let* $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ *be a bijection with* EPCIS$(\sigma)=(v_0, v_1, \ldots, v_{n-1})$*, and let* M_σ *be the Fiedler matrix of* $p(z)$ *associated with* σ*. Let* $x_\sigma(z)$*,* $y_\sigma(z)$*,* $v_\sigma(z)$*, and* $w_\sigma(z)$ *be the rational vectors whose* k*th entry is:*

$$x_\sigma^{(k)}(z) = \begin{cases} z^{\mathfrak{i}_\sigma(0:n-k-1)} p_{k-1}(z) & \text{if } v_{n-k} = 1, \\ z^{\mathfrak{i}_\sigma(0:n-k-1)} & \text{if } v_{n-k} = 0, \end{cases},$$

$$y_\sigma^{(k)}(z) = \begin{cases} z^{\mathfrak{c}_\sigma(0:n-k-1)} p_{k-1}(z) & \text{if } v_{n-k} = 0, \\ z^{\mathfrak{c}_\sigma(0:n-k-1)} & \text{if } v_{n-k} = 1, \end{cases},$$

$$v_\sigma^{(k)}(z) = \begin{cases} -z^{\mathfrak{i}_\sigma(0:n-k-1)-1} p_{n-k}^{\text{rev}}(z^{-1}) & \text{if } v_{n-k} = 1, \\ z^{\mathfrak{i}_\sigma(0:n-k-1)} & \text{if } v_{n-k} = 0, \end{cases},$$

$$w_\sigma^{(k)}(z) = \begin{cases} -z^{\mathfrak{c}_\sigma(0:n-k-1)-1} p_{n-k}^{\text{rev}}(z^{-1}) & \text{if } v_{n-k} = 0, \\ z^{\mathfrak{c}_\sigma(0:n-k-1)} & \text{if } v_{n-k} = 1, \end{cases},$$

[1] The first and second Frobenius companion matrices, C_1 and C_2, of a monic polynomial $p(z)$ are non-derogatory matrices [20], that is, the geometric multiplicity of each eigenvalue is equal to 1. Since Fiedler matrices of $p(z)$ are similar to each other, and since the geometric multiplicities of eigenvalues do not change under similarity, all Fiedler matrices are non-derogatory matrices.

for $k = 1, 2, \dots, n$, and where $p_j(z), p_j^{\text{rev}}(z)$ denote the degree j Horner shifts of $p(z)$ and $p^{\text{rev}}(z)$, respectively. If λ is a root of $p(z)$, then $x_\sigma(\lambda)$ and $y_\sigma(\lambda)$ are right and left eigenvectors of M_σ, respectively, associated with λ. Moreover, if λ is nonzero, then $x_\sigma(\lambda) = v_\sigma(\lambda)$ and $y_\sigma(\lambda) = w_\sigma(\lambda)$.

Proof From $\text{adj}(zI_n - M_\sigma)(zI_n - M_\sigma) = (zI_n - M_\sigma)\text{adj}(zI_n - M_\sigma) = p(z)I_n$ (see [18, Ch. 4, §4]), it follows that any nonzero row of $\text{adj}(\lambda I_n - M_\sigma)$ is the transpose of a left eigenvector of M_σ associated with the eigenvalue λ, and any nonzero column of $\text{adj}(\lambda I_n - M_\sigma)$ is a right eigenvector of M_σ associated with the eigenvalue λ. Moreover, using $p(\lambda) = 0$ and [13, Th. 3.3] we get that $\text{adj}(\lambda I_n - M_\sigma) = x_\sigma(\lambda) y_\sigma^T(\lambda)$.

We are now going to prove that both $x_\sigma(\lambda)$ and $y_\sigma(\lambda)$ have an entry identically equal to 1, and, therefore, the matrix $\text{adj}(\lambda I_n - M_\sigma)$ will have a nonzero column equal to $x_\sigma(\lambda)$ and a nonzero row equal to $y_\sigma^T(\lambda)$.

To prove that both $x_\sigma(\lambda)$ and $y_\sigma(\lambda)$ have an entry identically equal to 1 we distinguish two cases: when $M_\sigma = C_1, C_2$ and when $M_\sigma \neq C_1, C_2$. The Frobenius companion matrices C_1 and C_2 are Fiedler matrices associated with bijections σ_1 and σ_2, respectively, such that $\text{EPCIS}(\sigma_1) = (0, 0, ..., 0) \in \mathbb{R}^n$ and $\text{EPCIS}(\sigma_2) = (1, 1, ..., 1) \in \mathbb{R}^n$. Therefore, $x_{\sigma_1}(z) = y_{\sigma_2}(z) = \begin{bmatrix} z^{n-1} & \dots & z & 1 \end{bmatrix}^T$ and $y_{\sigma_1}(z) = x_{\sigma_2}(z) = \begin{bmatrix} p_0(z) & p_1(z) & \dots & p_{n-1}(z) \end{bmatrix}^T$. The result follows by inspection of the entries of these two vectors (notice that $p_0(z) = 1$).

If $M_\sigma \neq C_1, C_2$, suppose that $v_0 = 1$, that is, σ has a consecution at 0 (the case $v_0 = 0$ is similar and we omit it), and let $t \in \{0, 1, ..., n-3\}$ be such that $v_{t+1} = 0$ and $v_j = 1$ for any $j \leq t$ (note that a number t satisfying those conditions always exists when $M_\sigma \neq C_1, C_2$). Then, from the expression of the kth entry of $x_\sigma(z)$ and $y_\sigma(z)$ in the statement, we get $x_\sigma^{(n-t-1)}(z) = z^{\mathfrak{i}_\sigma(0:t)} = 1$ and $y_\sigma^{(n)}(z) = z^{\mathfrak{c}_\sigma(0:-1)} = 1$.

Finally, the fact that $x_\sigma(\lambda)$ and $y_\sigma(\lambda)$ are equal to $v_\sigma(\lambda)$ and $w_\sigma(\lambda)$, respectively, when $\lambda \neq 0$, follows from the expressions for the entries of the vectors $x_\sigma(z)$, $y_\sigma(z)$, $v_\sigma(z)$, and $w_\sigma(z)$ in the statement, together with the following relation between the Horner shifts of $p(z)$ and the Horner shifts of the reversal polynomial $p^{\text{rev}}(z)$ evaluated at λ:

$$p_{k-1}(\lambda) = -\lambda^{-1} p_{n-k}^{\text{rev}}(\lambda^{-1}),$$

for $k = 1, 2, \dots, n$, which may be easily obtained from the identity $p(\lambda) = 0$. □

Theorem 4.1 provides us two different couples of rational vectors, namely $(x_\sigma(z), y_\sigma(z))$ and $(v_\sigma(z), w_\sigma(z))$, that, when evaluated at an eigenvalue λ of M_σ provide right and left eigenvectors associated with λ. The expressions $x_\sigma(\lambda)$ and $y_\sigma(\lambda)$ will be used thoroughly along the manuscript for these eigenvectors. By contrast, $v_\sigma(z)$ and $w_\sigma(z)$ will not be used until Section 7.2, where they are employed to get asymptotic expressions for the resolvent $(zI - M_\sigma)^{-1}$ for $|z| > 1$.

Remark 4.1 For the sake of simplicity, we will frequently use the notation $x_\sigma := x_\sigma(\lambda), y_\sigma := y_\sigma(\lambda)$, with $x_\sigma(z), y_\sigma(z)$ as in Theorem 4.1.

Theorem 4.1 allows us to easily get explicit expressions for the right and left eigenvectors of any Fiedler matrix of $p(z)$ associated with an eigenvalue λ. These expressions depend on the eigenvalue λ and the Horner shifts of $p(z)$ evaluated at λ. To illustrate Theorem 4.1 we provide the following examples.

Example 4.1 Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \geq 2$, let C_1 and C_2 be the first and second Frobenius companion matrices of $p(z)$, respectively, and let λ be a root of $p(z)$. Recall that C_1 is a Fiedler matrix M_{σ_1} associated with any bijection σ_1 such that $\mathrm{PCIS}(\sigma_1) = (0, 0, \ldots, 0)$ and that C_2 is a Fiedler matrix M_{σ_2} associated with any bijection σ_2 such that $\mathrm{PCIS}(\sigma_2) = (1, 1, \ldots, 1)$. Then, from Theorem 4.1 we get the following expressions for the right and left eigenvectors of C_1 and C_2 associated with λ:

$$x_{\sigma_1} = y_{\sigma_2} = \begin{bmatrix}\lambda^{n-1} \cdots \lambda \ 1\end{bmatrix}^T \quad \text{and} \quad y_{\sigma_1} = x_{\sigma_2} = \begin{bmatrix}p_0(\lambda) \ p_1(\lambda) \cdots p_{n-1}(\lambda)\end{bmatrix}^T .$$

Remark 4.2 Note that the right and left eigenvectors of C_1 (resp. the left and right eigenvectors of C_2) associated with an eigenvalue λ are equal to the vectors $\Lambda(z)$ in (3.2) and

$$\Pi(z) := \begin{bmatrix} p_0(z) \ p_1(z) \ \ldots \ p_{n-1}(z)\end{bmatrix}^T , \tag{4.1}$$

evaluated at $z = \lambda$.

Example 4.2 Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n > 2$, let λ be a root of $p(z)$, and let M_σ be the Fiedler matrix of $p(z)$ associated with any bijection σ such that $\mathrm{PCIS}(\sigma) = (0, 1, \ldots, 1)$, namely:

$$M_\sigma = \begin{bmatrix} -a_{n-1} & 1 & & & \\ -a_{n-2} & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ -a_1 & & & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} .$$

Then, from Theorem 4.1 we get the following expressions for the right and left eigenvectors of M_σ associated with λ:

$$x_\sigma = \begin{bmatrix}\lambda p_0(\lambda) \ \lambda p_1(\lambda) \cdots \lambda p_{n-2}(\lambda) \ 1\end{bmatrix}^T \quad \text{and} \quad y_\sigma = \begin{bmatrix}\lambda^{n-2} \cdots \lambda \ 1 \ p_{n-1}(\lambda)\end{bmatrix}^T .$$

Example 4.3 Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n > 2$, let λ be a root of $p(z)$, and let M_σ be the pentadiagonal Fiedler matrix of $p(z)$ associated with any bijection σ such that $\mathrm{PCIS}(\sigma) = (1, 0, 1, 0, 1, 0, \ldots)$. Then, from Theorem 4.1 we get the following expressions for the right and left eigenvectors of M_σ associated with λ:

$$\begin{aligned} x_\sigma &= \left[\lambda^{\frac{n-2}{2}} p_0(\lambda), \lambda^{\frac{n-2}{2}} p_1(\lambda), \lambda^{\frac{n-4}{2}}, \lambda^{\frac{n-4}{2}} p_3(\lambda), \cdots \ 1, p_{n-1}(\lambda)\right]^T \quad \text{and} \\ y_\sigma &= \left[\lambda^{\frac{n}{2}}, \lambda^{\frac{n-2}{2}}, \lambda^{\frac{n-2}{2}} p_2(\lambda), \cdots \ \lambda, \lambda p_{n-2}(\lambda), 1\right]^T , \end{aligned}$$

if n is even, and

$$\begin{aligned} x_\sigma &= \left[\lambda^{\frac{n-1}{2}}, \lambda^{\frac{n-3}{2}}, \lambda^{\frac{n-3}{2}} p_2(\lambda), \lambda^{\frac{n-5}{2}}, \lambda^{\frac{n-5}{2}} p_4(\lambda), \cdots \ 1, p_{n-1}(\lambda)\right]^T \quad \text{and} \\ y_\sigma &= \left[\lambda^{\frac{n-1}{2}}, \lambda^{\frac{n-1}{2}} p_1(\lambda), \lambda^{\frac{n-3}{2}}, \lambda^{\frac{n-3}{2}} p_3(\lambda), \cdots \ \lambda, \lambda p_{n-2}(\lambda), 1\right]^T , \end{aligned}$$

if n is odd.

5 Eigenvalue condition numbers of Fiedler matrices

The condition number of a simple nonzero eigenvalue λ of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$\kappa(\lambda, A) := \lim_{\epsilon\to 0} \sup \left\{ \tfrac{|\widetilde{\lambda}-\lambda|}{\epsilon|\lambda|} : \widetilde{\lambda} \text{ is the eigenvalue of } A+E \text{ closest to } \lambda, \right. \\ \left. \text{with } \|E\|_2 \le \epsilon\|A\|_2 \right\}. \tag{5.1}$$

This condition number, which measures the relative sensitivity of the simple eigenvalue λ with respect to relative normwise perturbations of A, was introduced in [4] and it is a slight modification of the Wilkinson condition number [44], which measures the absolute sensitivity of a simple eigenvalue with respect to absolute normwise perturbations of the matrix.

The condition number $\kappa(\lambda, A)$ can be computed explicitly through the formula (see [19])

$$\kappa(\lambda, A) = \frac{\|x\|_2\|y\|_2}{|y^T x|}\frac{\|A\|_2}{|\lambda|}, \tag{5.2}$$

where $x, y \in \mathbb{C}^n$ are the right and left eigenvectors of A, respectively, associated with the simple eigenvalue λ.

As a direct consequence of (5.2) and Theorem 4.1 we get, for any Fiedler matrix M_σ, an explicit formula for the condition number $\kappa(\lambda, M_\sigma)$.

Corollary 5.1 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \ge 2$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, and let M_σ be the Fiedler companion matrix of $p(z)$ associated to the bijection σ. If λ is a simple nonzero eigenvalue of M_σ, then*

$$\kappa(\lambda, M_\sigma) = \frac{\|x_\sigma(\lambda)\|_2\ \|y_\sigma(\lambda)\|_2}{|p'(\lambda)|}\frac{\|M_\sigma\|_2}{|\lambda|}, \tag{5.3}$$

where the vectors $x_\sigma(\lambda), y_\sigma(\lambda)$ are defined in Theorem 4.1.

Proof From (5.2) and Theorem 4.1 we get

$$\kappa(\lambda, M_\sigma) = \frac{\|x_\sigma\|_2\ \|y_\sigma\|_2}{|y_\sigma^T x_\sigma|}\frac{\|M_\sigma\|_2}{|\lambda|},$$

where we follow Remark 4.1 for the notation. Hence, we just need to check that $x_\sigma^T y_\sigma = p'(\lambda)$. Indeed, from Theorem 4.1 we get

$$\begin{aligned} y_\sigma^T x_\sigma &= \sum_{k=1}^{n} y_\sigma^{(k)} x_\sigma^{(k)} = \sum_{k=1}^{n} \lambda^{\mathfrak{i}_\sigma(0:n-k-1)+\mathfrak{c}_\sigma(0:n-k-1)} p_{k-1}(\lambda) \\ &= \sum_{k=1}^{n} \lambda^{n-k} p_{k-1}(\lambda) = \sum_{k=1}^{n} k a_k \lambda^{k-1} = p'(\lambda), \end{aligned}$$

where $a_n = 1$, and where we have used that $\mathfrak{i}_\sigma(0 : n-k-1)+\mathfrak{c}_\sigma(0 : n-k-1) = n-k$. □

Remark 5.1 There are no explicit known formulas for the 2-norm $\|M_\sigma\|_2$, except for the case $M_\sigma = C_1, C_2$. Nevertheless, in [12] we have obtained explicit expressions for the 1-,∞-, and Frobenius norms of any Fiedler matrix M_σ, and, since the 2-norm is equivalent to any of those norms, there exist constants c_n and $\widehat{c}_n$ (that only depend on n) such that $c_n\|M_\sigma\|_i \le \|M_\sigma\|_2 \le \widehat{c}_n\|M_\sigma\|_i$, for $i = 1, \infty, F$ [20, p. 314].

As a particular case of Corollary 5.1, if C denotes the first or the second Frobenius companion matrix of the polynomial (2.2), we recover the expression for $\kappa(\lambda, C)$ given in [39]:

$$\kappa(\lambda, C) = \frac{\|C\|_2}{|\lambda|} \frac{\|\Lambda(\lambda)\|_2 \|\Pi(\lambda)\|_2}{|p'(\lambda)|}, \tag{5.4}$$

where the vectors $\Lambda(z)$ and $\Pi(z)$ are defined in (3.2) and (4.1), respectively.

6 Comparing condition numbers

Ideally, given a monic polynomial $p(z)$, for solving the root-finding problem for $p(z)$ by using a backward stable eigenvalue algorithm on a Fiedler companion matrix of $p(z)$, one would like the eigenvalues of the Fiedler matrix to be as well conditioned as the roots of the original polynomial. Since relative forward errors of computed eigenvalues are bounded by (1.2), in order for the roots of $p(z)$ to be computed with the forward errors expected from the sensitivity of the original data, i.e., from $p(z)$, one would like

$$\frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \le K,$$

with K a moderate constant, and where, from (5.3) and (3.3), this ratio is equal to

$$\frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} = \frac{\|M_\sigma\|_2}{|||p|||_2} \frac{\|x_\sigma\|_2 \|y_\sigma\|_2}{\|\Lambda(\lambda)\|_2}, \tag{6.1}$$

with x_σ, y_σ being the right and left eigenvectors of M_σ associated with the eigenvalue λ (see Theorem 4.1). In particular, if C denotes the first or the second Frobenius companion matrix of $p(z)$, one gets, according to Remark 4.2,

$$\frac{\kappa(\lambda, C)}{\kappa(\lambda, p)} = \frac{\|C\|_2}{|||p|||_2} \|\Pi(\lambda)\|_2, \tag{6.2}$$

where the vector $\Pi(z)$ is defined in (4.1).

The ratio of condition numbers (6.1) is a function of λ and of the coefficients of the polynomial $p(z)$. Theorem 6.1 provides simple upper and lower bounds for this ratio in terms of the absolute values of the coefficients of $p(z)$. To prove that the bounds in Theorem 6.1 hold we need Lemmas 6.1 and 6.2. Lemma 6.1 gives a simple upper bound for the absolute value of any Horner shift of a monic polynomial $p(z)$ evaluated at a root of $p(z)$.

Lemma 6.1 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree n, let $\lambda \in \mathbb{C}$ be a root of $p(z)$, and let $\{p_k(z)\}_{k=0}^{n-1}$ be the Horner shifts of $p(z)$. Then,*

$$|p_k(\lambda)| \leq \sqrt{n}\|p\|_2,$$

for $k = 0, 1, \ldots, n-1$.

Proof First, suppose that $|\lambda| \leq 1$. Then, from $|p_k(\lambda)| = |\lambda^k + a_{n-1}\lambda^{k-1} + \cdots + a_{n-k+1}\lambda + a_{n-k}| \leq 1 + |a_{n-1}| + \cdots + |a_0| \leq \sqrt{n}\|p\|_2$, we get the result. Second, suppose that $|\lambda| \geq 1$. Recall that the Horner shifts of $p(z)$ satisfy $p_k(z) = \lambda p_{k-1}(z) + a_{n-k}$, for $k = 1, 2, \ldots, n-1$, where $p_0(\lambda) = 1$. Since $p(\lambda) = \lambda p_{n-1}(\lambda) + a_0 = 0$, we have that $p_{n-1}(\lambda) = -a_0/\lambda$. With the previous equation, the recurrence relation $p_{k-1}(\lambda) = p_k(\lambda)/\lambda - a_{n-k}/\lambda$, for $k = 1, 2, \ldots, n-1$, implies that $p_k(\lambda) = -a_0/\lambda^{n-k} - a_1/\lambda^{n-k-1} - \cdots - a_{n-k-1}/\lambda$. Then, from $|p_k(\lambda)| = |a_0/\lambda^{n-k} + a_1/\lambda^{n-k-1} + \cdots + a_{n-k-1}/\lambda| \leq 1 + |a_{n-1}| + \cdots + |a_0| \leq \sqrt{n}\|p\|_2$, we get the result. □

Lemma 6.2 shows that Fiedler matrices associated with a polynomial $p(z)$ have, up to dimensional constants, a norm equal to the norm of the polynomial $p(z)$, and it also gives lower and upper bounds for the ratios $\|C\|_2/|||p|||_2$ and $\|M_\sigma\|_2/|||p|||_2$ in terms of the coefficients of the monic polynomial $p(z)$.

Lemma 6.2 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n > 2$, let C be the first or the second Frobenius companion matrix of $p(z)$, let M_σ be a Fiedler matrix of $p(z)$ other than the Frobenius ones, and let $\rho(p)$ be defined as*

$$\rho(p) := \sqrt{1 + \frac{1}{\max\limits_{0 \leq k \leq n-1} |a_k|^2}}. \tag{6.3}$$

Then,

$$\frac{1}{\sqrt{2}} \leq \frac{\|C\|_2}{|||p|||_2} \leq \rho(p), \qquad \frac{1}{\sqrt{n}} \leq \frac{\|M_\sigma\|_2}{|||p|||_2} \leq \sqrt{n}\rho(p),$$

and

$$\frac{1}{\sqrt{2}} \leq \frac{\|C\|_2}{\|p\|_2} \leq 1, \qquad \frac{1}{\sqrt{n}} \leq \frac{\|M_\sigma\|_2}{\|p\|_2} \leq \sqrt{n}.$$

Proof The lower and upper bounds for $\|C\|_2/|||p|||_2$ and $\|C\|_2/\|p\|_2$ are immediate consequences of the formula for $\|C\|_2$ in [24, Section 10.4]. Also, using that, for any matrix $A \in \mathbb{C}^{n\times n}$, $\|A\|_F \geq \|A\|_2 \geq n^{-1/2}\|A\|_F$ [20, pp. 314], the lower and upper bounds for $\|M_\sigma\|_2/|||p|||_2$ and $\|M_\sigma\|_2/\|p\|_2$ follow from (2.7). □

Remark 6.1 Notice that to bound $\|M_\sigma\|_2/|||p|||_2$ and $\|M_\sigma\|_2/\|p\|_2$ we have used the Frobenius norm $\|M_\sigma\|_F$ instead of $\|M_\sigma\|_2$ because explicit expressions for the 2-norm of Fiedler matrices are not known when $M_\sigma \neq C_1, C_2$.

Remark 6.2 Notice that $\rho(p)$ in (6.3) is always greater than 1, and that there are polynomials for which $\rho(p)$, and therefore the upper bounds for $\|C\|_2/|||p|||_2$ and $\|M_\sigma\|_2/|||p|||_2$ in Lemma 6.2, may be as large as desired.

Theorem 6.1 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n > 2$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, and let M_σ be the Fiedler matrix of $p(z)$ associated with σ. If λ is a simple nonzero root of $p(z)$, then*

$$\frac{1}{\sqrt{2}} \le \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \le n\rho(p)\|p\|_2 \tag{6.4}$$

if $M_\sigma = C_1, C_2$, and

$$\frac{1}{n} \le \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \le n^{5/2}\rho(p)\|p\|_2^2 \tag{6.5}$$

if $M_\sigma \neq C_1, C_2$, where $\rho(p)$ is defined in (6.3)*.*

Proof We prove first (6.4), that is, we have to bound (6.2). Since $\kappa(\lambda, C_1) = \kappa(\lambda, C_2)$ we will focus only on C_1. From Example 4.1 and Lemma 6.1, we have that the modulus of all the entries of $y_\sigma = \Pi(\lambda)$ are bounded by $\sqrt{n}\|p\|_2$, and, therefore, $\|y_\sigma\|_2 = \|\Pi(\lambda)\|_2 \le n\|p\|_2$. With the previous inequality, the upper bound in (6.4) follows from Lemma 6.2. Finally, the lower bound in (6.4) follows from Lemma 6.2 and from $\|\Pi(\lambda)\|_2 \ge 1$, since $p_0(\lambda) = 1$.

Next, we prove (6.5), that is, we have to bound (6.1) when $M_\sigma \neq C_1, C_2$. From Lemma 6.2 we have that $\|M_\sigma\|_2/\|\|p\|\|_2 \le \sqrt{n}\rho(p)$. So, to prove that the upper bound in (6.5) holds, we need to show that $\|x_\sigma\|_2\|y_\sigma\|_2/\|\Lambda(\lambda)\|_2 \le n^2\|p\|_2^2$. In order to do this, we have to distinguish two cases: $|\lambda| \le 1$ and $|\lambda| > 1$.

When $|\lambda| \le 1$, from Theorem 4.1 it follows that, for $k = 1, 2, \ldots, n$, the modulus of the kth entry of x_σ and of y_σ is bounded by $\max\{1, |p_{k-1}(\lambda)|\}$, so, using Lemma 6.1 we get that the modulus of these entries are bounded by $\sqrt{n}\|p\|_2$, and, therefore, $\|x_\sigma\|_2\|y_\sigma\|_2 \le n^2\|p\|_2^2$. With the previous inequality and using that $\|\Lambda(\lambda)\|_2 \ge 1$, the result follows.

When $|\lambda| > 1$, using $\|\Lambda(\lambda)\|_2 \ge |\lambda^{n-1}|$ and $n-1 = \mathfrak{i}_\sigma(0:n-2) + \mathfrak{c}_\sigma(0:n-2)$, we get

$$\frac{\|x_\sigma\|_2\|y_\sigma\|_2}{\|\Lambda(\lambda)\|_2} \le \frac{\|x_\sigma\|_2\|y_\sigma\|_2}{|\lambda^{n-1}|} = \frac{\|x_\sigma\|_2}{|\lambda^{\mathfrak{i}_\sigma(0:n-2)}|}\frac{\|y_\sigma\|_2}{|\lambda^{\mathfrak{c}_\sigma(0:n-2)}|}.$$

Hence, we need to bound the modulus of the entries of $x_\sigma/\lambda^{\mathfrak{i}_\sigma(0:n-2)}$ and $y_\sigma/\lambda^{\mathfrak{c}_\sigma(0:n-2)}$. Since $|\lambda| > 1$, and, for $k = 1, 2, \ldots, n$, $\mathfrak{i}_\sigma(0 : n-2) \ge \mathfrak{i}_\sigma(0 : n-k-1)$ and $\mathfrak{c}_\sigma(0 : n-2) \ge \mathfrak{c}_\sigma(0 : n-k-1)$, from Theorem 4.1 it follows that the modulus of the kth entry of $x_\sigma/\lambda^{\mathfrak{i}_\sigma(0:n-2)}$ and of $y_\sigma/\lambda^{\mathfrak{c}_\sigma(0:n-2)}$ is upper bounded by $\max\{1, |p_{k-1}(\lambda)|\}$, so, using Lemma 6.1 we get that the modulus of these entries is bounded by $\sqrt{n}\|p\|_2$, and, therefore, $\|x_\sigma\|_2\|y_\sigma\|_2/\|\Lambda(\lambda)\|_2 \le n^2\|p\|_2^2$. With the last inequality, the result follows.

Finally, we prove that the lower bound in (6.5) holds. From Lemma 6.2, we have that $\|M_\sigma\|_2/\|\|p\|\|_2 \ge 1/\sqrt{n}$, so, we only need to show that $\|x_\sigma\|_2\|y_\sigma\|_2/\|\Lambda(\lambda)\|_2 \ge 1/\sqrt{n}$. In order to prove the previous inequality, first notice that $\|x_\sigma\|_2\|y_\sigma\|_2 \ge \max\{1, |x_\sigma^{(1)} y_\sigma^{(1)}|\} = \max\{1, |\lambda|^{n-1}\}$, where we have used that the vectors x_σ and y_σ have, at least, an entry equal to 1 (see the proof of Theorem 4.1), and that $x_\sigma^{(1)} y_\sigma^{(1)} = \lambda^{n-1}$ (see Theorem 4.1). Also notice that $\|\Lambda(\lambda)\|_2 = \sqrt{\sum_{i=0}^{n-1} |\lambda|^{2i}} \le \sqrt{n}\max\{1, |\lambda|^{n-1}\}$. Then, from the two precedent observations, it follows that $\|x_\sigma\|_2\|y_\sigma\|_2/\|\Lambda(\lambda)\|_2 \ge 1/\sqrt{n}$. □

Remark 6.3 Probably, if explicit expressions for the 2-norm of Fiedler matrices other than the Frobenius ones were available, upper and lower bounds sharper than the ones in (6.5) could be found. Although, in Example 6.1, we will show that the presence of $\|p\|_2^2$ is necessary in the upper bound in (6.5).

The presence of $\|p\|_2$ and $\rho(p)$ in the upper bounds in (6.4) and (6.5) shows that these bounds are large if and only if $\max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}$ is either large or close to zero.

It is evident that there exist polynomials for which the upper bounds in (6.4) and (6.5) can be as large as desired, but this does not necessarily mean that the ratio (6.1) is large for these polynomials. Nevertheless, Example 6.1 shows that there exist polynomials for which a large upper bound in (6.4) or in (6.5) implies that the ratio (6.1) is also large.

Example 6.1 Let $a \geq 2$ be a constant and consider the monic polynomial $p(z) = z^n + (a-1)z^{n-1} - a$. It may be checked that $\lambda = 1$ is a root of $p(z)$, and that the Horner shifts of $p(z)$, evaluated at $z = 1$, satisfy $p_0(1) = 1$, and, for $k = 1, 2, \dots, n-1$, $p_k(1) = a$. Now, given a Fiedler matrix M_σ of $p(z)$, we are going to prove that, for $\lambda = 1$, the ratio between (6.1) and the upper bounds in Theorem 6.1 is larger than a quantity that depends only on n.

First, suppose that $M_\sigma = C_1, C_2$. From (6.2) and Lemma 6.2 we get

$$\frac{\kappa(1, M_\sigma)}{\kappa(1,p)} = \frac{\|M_\sigma\|_2}{|||p|||_2} \| \begin{bmatrix} 1 & a & a & \cdots & a \end{bmatrix}^T \|_2 \geq \sqrt{\frac{n-1}{2}} a.$$

Also, the upper bound in (6.4) is equal to $n(1+1/a^2)^{1/2} \| \begin{bmatrix} 1 & a-1 & 0 & \cdots & 0 & -a \end{bmatrix}^T \|_2$ wich is less than or equal to $\sqrt{6}na$. From this, we get that the ratio between (6.2) and the upper bound in (6.4) is larger than or equal to $(n-1)^{1/2}/(2\sqrt{3}n)$. So, taking A large enough and n not too large, we have a polynomial $p(z)$ for which a large upper bound in (6.4) implies that the ratio (6.2) is large.

Second, suppose that $M_\sigma \neq C_1, C_2$. Observe that, since M_σ is not one of the Frobenius companion matrices, if $\text{PCIS}(\sigma) = (v_0, v_1, \dots, v_{n-2})$, then there exist $i, j \in \{2, 3, \dots, n\}$ such that $v_{n-i} \neq v_{n-j}$. Suppose that $v_{n-i} = 1$ and $v_{n-j} = 0$ (when $v_{n-i} = 0$ and $v_{n-j} = 1$ the argument is similar, so we omit it). If $v_{n-i} = 1$ and $v_{n-j} = 0$, and if x_σ and y_σ are the right and left eigenvectors of M_σ associated with $\lambda = 1$, respectively, then Theorem 4.1 implies $\|x_\sigma\|_2 \|y_\sigma\|_2 \geq |x_\sigma^{(i)}| \cdot |y_\sigma^{(j)}| = |p_{i-1}(1)| \cdot |p_{j-1}(1)| = a^2$, and, therefore,

$$\frac{\kappa(1, M_\sigma)}{\kappa(1,p)} = \frac{\|M_\sigma\|_2}{|||p|||_2} \frac{\|x_\sigma\|_2 \|y_\sigma\|_2}{\|\Lambda(1)\|_2} \geq \frac{a^2}{n},$$

where we have used Lemma 6.2 and $\|\Lambda(1)\|_2 = n^{1/2}$. Also, the upper bound in (6.5) is equal to $n^{5/2}(1+1/a^2)^{1/2} \| \begin{bmatrix} 1 & a-1 & 0 & \cdots & 0 & -a \end{bmatrix}^T \|_2^2$ which is less than or equal to $3\sqrt{2}n^{5/2}a^2$. From this, we get that the ratio between (6.1) and the upper bound in (6.5) is larger than or equal to $1/(3\sqrt{2}n^{7/2})$. So, again, taking a large enough and n not too large, we have a polynomial $p(z)$ for which a large upper bound in (6.5) implies that the ratio (6.1) is large.

Notice that the upper bound in (6.5) is larger than the upper bound in (6.4). This suggests that the eigenvalues of Fiedler companion matrices other than the Frobenius ones may be potentially more ill conditioned than the eigenvalues of the Frobenius companion matrices. For solving the polynomial root-finding problem using Fiedler companion matrices is important to know whether or not the eigenvalue condition numbers of Fiedler companion matrices other than the Frobenius ones are much larger (or smaller) than the eigenvalue condition numbers of Frobenius companion matrices. For this, we study the ratio

$$\frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)} = \frac{\|M_\sigma\|_2}{\|C\|_2} \frac{\|x_\sigma\|_2 \|y_\sigma\|_2}{\|\Lambda(\lambda)\|_2 \|\Pi(\lambda)\|_2}, \tag{6.6}$$

where $C = C_1, C_2$ and $M_\sigma \neq C_1, C_2$, that we obtain from (5.3) and (5.4). In Theorem 6.2 we give upper bounds for (6.6) in terms of the norm of the vector $\Pi(\lambda)$ and, also, in terms of the absolute values of the coefficients of $p(z)$. Lemma 6.3 will be useful in establishing these bounds.

Lemma 6.3 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \geq 2$, let C be the first or the second Frobenius companion matrix, let $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$ be a bijection, and let M_σ be the Fiedler matrix of $p(z)$ associated with σ. Then, $n^{-1/2}\|M_\sigma\|_2 \leq \|C\|_2 \leq n^{1/2}\|M_\sigma\|_2$.*

Proof The result follows from $n^{-1/2}\|A\|_F \leq \|A\|_2 \leq \|A\|_F$, valid for any matrix $A \in \mathbb{C}^{n\times n}$, [20, pp. 314] and the identity $\|M_\sigma\|_F = \|C\|_F$ (see (2.7)). □

Theorem 6.2 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n > 2$, let C denote the first or the second Frobenius companion matrix of $p(z)$, let $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$ be a bijection, and let M_σ be the Fiedler matrix of $p(z)$ associated with σ. Assume that $M_\sigma \neq C$. If λ is a simple nonzero root of $p(z)$, then,*

$$1 \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)} \leq n^{3/2}\|\Pi(\lambda)\|_2 \leq n^{5/2}\|p\|_2, \tag{6.7}$$

if $\kappa(\lambda, M_\sigma) \geq \kappa(\lambda, C)$, and

$$1 \leq \frac{\kappa(\lambda, C)}{\kappa(\lambda, M_\sigma)} \leq n\|\Pi(\lambda)\|_2 \leq n^2\|p\|_2, \tag{6.8}$$

if $\kappa(\lambda, C) \geq \kappa(\lambda, M_\sigma)$, where $\Pi(\lambda)$ is the vector defined in (4.1).

Proof First, we prove (6.7). From (6.6) and Lemma 6.3, we get

$$1 \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)} \leq n^{1/2} \frac{\|x_\sigma\|_2 \|y_\sigma\|_2}{\|\Lambda(\lambda)\|_2 \|\Pi(\lambda)\|_2}.$$

Therefore, to get the second inequality in (6.7) we need to check that

$$\|x_\sigma\|_2 \|y_\sigma\|_2 / (\|\Lambda(\lambda)\|_2 \|\Pi(\lambda)\|_2) \leq n\|\Pi(\lambda)\|_2$$

holds. In order to do this we distinguish two cases: $|\lambda| \leq 1$ and $|\lambda| > 1$.

If $|\lambda| \leq 1$, using $\|\Lambda(\lambda)\|_2 \geq 1$, we get

$$\frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)} \leq n^{1/2} \frac{\|x_\sigma\|_2 \|y_\sigma\|_2}{\|\Pi(\lambda)\|_2}.$$

So, we need to bound the norm of the vectors $x_\sigma/\|\Pi(\lambda)\|_2$ and y_σ. For $k = 1, 2, \ldots, n$, Theorem 4.1 implies

$$\frac{|x_\sigma^{(k)}|}{\|\Pi(\lambda)\|_2} = \frac{|\lambda^{\mathfrak{i}_\sigma(0:n-k-1)} p_{k-1}(\lambda)|}{\|\Pi(\lambda)\|_2} \leq \frac{|p_{k-1}(\lambda)|}{\|\Pi(\lambda)\|_2} \leq 1,$$

if $v_{n-k} = 1$, or

$$\frac{|x_\sigma^{(k)}|}{\|\Pi(\lambda)\|_2} = \frac{|\lambda^{\mathfrak{i}_\sigma(0:n-k-1)}|}{\|\Pi(\lambda)\|_2} \leq \frac{1}{\|\Pi(\lambda)\|_2} \leq 1,$$

if $v_{n-k} = 0$. Also

$$|y_\sigma^{(k)}| = |\lambda^{\mathfrak{c}_\sigma(0:n-k-1)} p_{k-1}(\lambda)| \leq |p_{k-1}(\lambda)| \leq \|\Pi(\lambda)\|_2,$$

if $v_{n-k} = 0$, or

$$|y_\sigma^{(k)}| = |\lambda^{\mathfrak{c}_\sigma(0:n-k-1)}| \leq 1 \leq \|\Pi(\lambda)\|_2,$$

if $v_{n-k} = 1$. We have used that $\|\Pi(\lambda)\|_2 \geq |p_{k-1}(\lambda)|$, for $k = 1, 2, \ldots, n$, (in particular, for $k = 1$, we have that $\|\Pi(\lambda)\|_2 \geq 1$, since $p_0(\lambda) = 1$). From the previous bounds, we get $\|x_\sigma\|_2/\|\Pi(\lambda)\|_2 \leq n^{1/2}$ and $\|y_\sigma\|_2 \leq n^{1/2}\|\Pi(\lambda)\|_2$, and from this, the second inequality in (6.7) follows. The rightmost bound in (6.7) follows from Lemma 6.1.

If $|\lambda| > 1$, then

$$\frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)} \leq n^{1/2} \frac{\|x_\sigma\|_2 \|y_\sigma\|_2}{|\lambda^{n-1}| \|\Pi(\lambda)\|_2} \leq n^{1/2} \frac{\|x_\sigma\|_2}{\|\Pi(\lambda)\|_2 |\lambda^{\mathfrak{i}_\sigma(0:n-2)}|} \frac{\|y_\sigma\|_2}{|\lambda^{\mathfrak{c}_\sigma(0:n-2)}|},$$

where we have used Lemma 6.3 together with the fact that $\|\Lambda(\lambda)\|_2 \geq |\lambda^{n-1}|$, and also that $n-1 = \mathfrak{i}_\sigma(0:n-2) + \mathfrak{c}_\sigma(0:n-2)$. So we need to bound the norm of the vectors $x_\sigma/(\|\Pi(\lambda)\|_2 |\lambda^{\mathfrak{i}_\sigma(0:n-2)}|)$ and $y_\sigma/|\lambda^{\mathfrak{c}_\sigma(0:n-2)}|$. For $k = 1, 2, \ldots, n$, Theorem 4.1 implies

$$\frac{|x_\sigma^{(k)}|}{\|\Pi(\lambda)\|_2 |\lambda^{\mathfrak{i}_\sigma(0:n-2)}|} = \frac{|\lambda^{\mathfrak{i}_\sigma(0:n-k-1)} p_{k-1}(\lambda)|}{\|\Pi(\lambda)\|_2 |\lambda^{\mathfrak{i}_\sigma(0:n-2)}|} \leq \frac{|p_{k-1}(\lambda)|}{\|\Pi(\lambda)\|_2} \leq 1,$$

if $v_{n-k} = 1$, or

$$\frac{|x_\sigma^{(k)}|}{\|\Pi(\lambda)\|_2 |\lambda^{\mathfrak{i}_\sigma(0:n-2)}|} = \frac{|\lambda^{\mathfrak{i}_\sigma(0:n-k-1)}|}{\|\Pi(\lambda)\|_2 |\lambda^{\mathfrak{i}_\sigma(0:n-2)}|} \leq \frac{1}{\|\Pi(\lambda)\|_2} \leq 1,$$

if $v_{n-k} = 0$, and

$$\frac{|y_\sigma^{(k)}|}{|\lambda^{\mathfrak{c}_\sigma(0:n-2)}|} = \frac{|\lambda^{\mathfrak{c}_\sigma(0:n-k-1)} p_{k-1}(\lambda)|}{|\lambda^{\mathfrak{c}_\sigma(0:n-2)}|} \leq |p_{k-1}(\lambda)| \leq \|\Pi(\lambda)\|_2$$

if $v_{n-k} = 0$, or

$$\frac{|y_\sigma^{(k)}|}{|\lambda^{\mathfrak{c}_\sigma(0:n-2)}|} = \frac{|\lambda^{\mathfrak{c}_\sigma(0:n-k-1)}|}{|\lambda^{\mathfrak{c}_\sigma(0:n-2)}|} \leq 1 \leq \|\Pi(\lambda)\|_2,$$

if $v_{n-k} = 1$, where we have used $\|\Pi(\lambda)\|_2 \geq |p_{k-1}(\lambda)|$, and $\mathfrak{i}_\sigma(0:n-2) \geq \mathfrak{i}_\sigma(0:n-k-1)$ and $\mathfrak{c}_\sigma(0:n-2) \geq \mathfrak{c}_\sigma(0:n-k-1)$, for $k = 1, 2, \ldots, n$. From the previous bounds, we have $\|x_\sigma\|_2/(\|\Pi(\lambda)\|_2|\lambda^{\mathfrak{i}_\sigma(0:n-2)}|) \leq n^{1/2}$ and $\|y_\sigma\|_2/|\lambda^{\mathfrak{c}_\sigma(0:n-2)}| \leq n^{1/2}\|\Pi(\lambda)\|_2$, and from this, the second inequality in (6.7) follows. The rightmost bound follows from Lemma 6.1.

Next, we prove (6.8). From (6.6) and Lemma 6.3, we get

$$\frac{\kappa(\lambda, C)}{\kappa(\lambda, M_\sigma)} = \frac{\|C\|_2}{\|M_\sigma\|_2} \frac{\|\Lambda(\lambda)\|_2\|\Pi(\lambda)\|_2}{\|x_\sigma\|_2\|y_\sigma\|_2} \leq n^{1/2} \frac{\|\Lambda(\lambda)\|_2\|\Pi(\lambda)\|_2}{\|x_\sigma\|_2\|y_\sigma\|_2}.$$

Therefore, to get the second inequality in (6.8) we need to check that

$$\|\Lambda(\lambda)\|_2\|\Pi(\lambda)\|_2/(\|x_\sigma\|_2\|y_\sigma\|_2) \leq n^{1/2}\|\Pi(\lambda)\|_2$$

holds. In order to do this, again, we distinguish the cases: $|\lambda| \leq 1$ and $|\lambda| > 1$.

If $|\lambda| \leq 1$ then, using $\|x_\sigma\|_2, \|y_\sigma\|_2 \geq 1$ (since both x_σ and y_σ have at least one entry equal to 1, see the proof of Theorem 4.1), and $\|\Lambda(\lambda)\|_2 \leq n^{1/2}$ when $|\lambda| \leq 1$, we get

$$\frac{\kappa(\lambda, C)}{\kappa(\lambda, M_\sigma)} \leq n\|\Pi(\lambda)\|_2.$$

If $|\lambda| > 1$, then, using $\|x_\sigma\|_2\|y_\sigma\|_2 \geq |x_\sigma^{(1)} y_\sigma^{(1)}| = |\lambda^{n-1}|$, and $\|\Lambda(\lambda)\|/|\lambda^{n-1}| \leq n^{1/2}$ when $|\lambda| \geq 1$, we get

$$\frac{\kappa(\lambda, C)}{\kappa(\lambda, M_\sigma)} \leq n^{1/2} \frac{\|\Lambda(\lambda)\|_2}{|\lambda^{n-1}|}\|\Pi(\lambda)\|_2 \leq n\|\Pi(\lambda)\|_2.$$

The rightmost bound in (6.8) follows from Lemma 6.1. □

The presence of $\|p\|_2$ in the rightmost upper bounds in (6.7) and (6.8) shows that these bounds are large if and only if $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is large.

It is evident that there exist polynomials for which the rightmost upper bounds in (6.7) and (6.8) can be as large as desired, but this does not imply necessarily that the ratio of eigenvalue condition numbers is also large for these polynomials. In Example 6.2, given any Fiedler matrix $M_\sigma \neq C_1, C_2$, we show that there exist polynomials such that a large upper bound in Theorem 6.2 implies a large ratio between the eigenvalue condition numbers of M_σ and C_1 or C_2, or viceversa. In fact, we show that for these polynomials, up to a constant that depends only on the size of the problem, the rightmost bounds in Theorem 6.2 correctly predict the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ or $\kappa(\lambda, C)/\kappa(\lambda, M_\sigma)$.

Example 6.2 We first focus on the rightmost bound in (6.7). Given $a > 2$, consider the polynomial $p(z) = z^n + (a-1)z^{n-1} - a$. Recall from Example 6.1 that $\lambda = 1$ is a root of $p(z)$, and that the Horner shifts of $p(z)$, evaluated at $z = 1$, satisfy $p_0(1) = 1$, and, for $k = 1, 2, \ldots, n-1$, $p_k(1) = a$. If C denotes one of the Frobenius companion matrices of $p(z)$ and M_σ denotes a Fiedler matrix of $p(z)$ other than the Frobenius ones, we are going to show that the ratio between $\kappa(1, M_\sigma)/\kappa(1, C)$ and the rightmost bound in (6.7) are both larger than a quantity that depends only on n. Therefore, if n is fixed and a is very large, both are very large.

First, we need to get a lower bound on $\kappa(1, M_\sigma)/\kappa(1, C)$. This may be done using the following inequalities. From Lemma 6.3 we get $\|M_\sigma\|_2/\|C\|_2 \geq n^{-1/2}$, also, from Example 6.1, recall that if x_σ and y_σ are the right and left eigenvectors of M_σ, respectively, associated with $\lambda = 1$, then $\|x_\sigma\|_2\|y_\sigma\|_2 \geq a^2$, and, finally,

$\|\Lambda(1)\|_2\|\Pi(1)\|_2 = n^{1/2}\| \begin{bmatrix} 1 & a & \cdots & a \end{bmatrix}^T \|_2 \leq na$. From these three inequalities we get

$$\frac{\kappa(1,M_\sigma)}{\kappa(1,C)} = \frac{\|M_\sigma\|_2}{\|C\|_2} \frac{\|x_\sigma\|_2\|y_\sigma\|_2}{\|\Lambda(1)\|_2\|\Pi(1)\|_2} \geq n^{-3/2}a.$$

Also, the rightmost bound in (6.7) is equal to $n^{5/2}\| \begin{bmatrix} 1 & a-1 & 0 & \cdots & 0 & -a \end{bmatrix}^T \|_2$, which is larger than or equal to $n^{5/2}a$ and smaller than or equal to $\sqrt{3}n^{5/2}a$. Hence, the ratio between $\kappa(1,M_\sigma)/\kappa(1,C)$ and the rightmost bound in (6.7) is larger than $n^{-4}/\sqrt{3}$. Therefore, taking a large enough and n not too large compared to a, for the polynomial $p(z)$ a large rightmost bound in (6.7) implies that the ratio $\kappa(1,M_\sigma)/\kappa(1,C)$ is also large.

Next, we focus on the rightmost bound in (6.8). Given a bijection $\sigma : \{0,1,\ldots, n-1\} \to \{1,\ldots,n\}$ with $\mathrm{PCIS}(\sigma) = (v_0, v_1, \ldots, v_{n-2}) \neq (0,\ldots,0),(1,\ldots,1)$, then, it may be checked that there are $i,j \in \{2,3,\ldots,n\}$ with $i > j$ such that $v_{n-i} = 1$ and $v_{n-j} = 0$ or $v_{n-i} = 0$ and $v_{n-j} = 1$. Suppose that $v_{n-i} = 1$ and $v_{n-j} = 0$ (if $v_{n-i} = 0$ and $v_{n-j} = 1$ the argument is completely similar, so we omit it). Let $a > 1$ and $\epsilon < 1$ such that $\epsilon a < 1$, and consider the monic polynomial $q(z) = z^n + (a-\epsilon)z^{n-1} - \epsilon a z^{n-2}$ if $j = 2$, or $q(z) = z^n - \epsilon z^{n-1} + a z^{n-j+1} - \epsilon a z^{n-j}$ if $j > 2$. It may be easily checked that $\lambda = \epsilon$ is a root of $q(z)$ and that the Horner shifts of $q(z)$ satisfy $q_0(\epsilon) = 1$, $q_{j-1}(\epsilon) = a$, and $q_{k-1}(\epsilon) = 0$ when $k \neq 0, j$.

Now, let C denote the first or the second Frobenius companion matrix of $q(z)$, and let M_σ be the Fiedler matrix of $q(z)$ associated with the bijection σ. If x_σ and y_σ denote the right and left eigenvector of M_σ, respectively, associated with $\lambda = \epsilon$, from the values of the Horner shifts of $q(z)$ in ϵ, Theorem 4.1, and $\epsilon a < 1$ it can be proved that $\|x_\sigma\|_2\|y_\sigma\|_2 \leq n$. Hence

$$\frac{\kappa(\epsilon,C)}{\kappa(\epsilon,M_\sigma)} = \frac{\|C\|_2}{\|M_\sigma\|_2} \frac{\|\Lambda(\epsilon)\|_2\|\Pi(\epsilon)\|_2}{\|x_\sigma\|_2\|y_\sigma\|_2} \geq \frac{a}{n^{3/2}}.$$

Also the rightmost upper bound in (6.8) is less than or equal to $2n^2a$. So, the ratio between $\kappa(\epsilon,C)/\kappa(\epsilon,M_\sigma)$ and the rightmost bound in (6.8) is larger than $n^{-7/2}/2$. Therefore, taking a large enough and ϵ small enough, for the polynomial $q(z)$ a large rightmost bound in (6.8) implies a large ratio $\kappa(\epsilon,C)/\kappa(\epsilon,M_\sigma)$ when n is not too large.

The bound (6.7) in Theorem 6.2 and Example 6.2 suggest that for some polynomials and some roots one should avoid using a Fiedler matrix M_σ other than the Frobenius ones, and to use, instead, the Frobenius companion matrices. However, Theorem 6.3 shows that this could only happen for a polynomial $p(z)$ whose roots are very ill-conditioned both as eigenvalues of the Frobenius companion matrices and as eigenvalues of the Fiedler matrix M_σ compared to the conditioning of the roots of $p(z)$. In other words:

$$\kappa(\lambda,M_\sigma) \gg \kappa(\lambda,C) \quad \text{implies} \quad \text{“}\kappa(\lambda,M_\sigma) \gg \kappa(\lambda,p) \quad \text{and} \quad \kappa(\lambda,C) \gg \kappa(\lambda,p)\text{”}.$$

By contrast, if the roots of a polynomial $p(z)$ are much more ill-conditioned as eigenvalues of one of the Frobenius companion matrices than they are as eigenvalues of another Fiedler matrix M_σ (that is, the situation corresponding to bound (6.8) and the second part of Example 6.2), then, Theorem 6.3 implies that the roots of $p(z)$ are very ill-conditioned as eigenvalues of the Frobenius matrices compared to their conditioning as a roots of $p(z)$. In other words:

$$\kappa(\lambda, C) \gg \kappa(\lambda, M_\sigma) \quad \text{implies} \quad \kappa(\lambda, C) \gg \kappa(\lambda, p).$$

But $\kappa(\lambda, C) \gg \kappa(\lambda, M_\sigma)$ does not imply $\kappa(\lambda, M_\sigma) \gg \kappa(\lambda, p)$.

Theorem 6.3 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial, let C denote the first or the second Frobenius companion matrix of $p(z)$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection such that $\mathrm{PCIS}(\sigma) \neq (0, \ldots, 0), (1, \ldots, 1)$, and let M_σ be the Fiedler matrix of $p(z)$ associated with σ. If λ is a simple nonzero root of $p(z)$, then the following results hold.*

(a) *If $\kappa(\lambda, M_\sigma) \geq \kappa(\lambda, C)$, then*

$$\frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \geq \frac{1}{\sqrt{2}} \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)} \quad \text{and} \quad \frac{\kappa(\lambda, C)}{\kappa(\lambda, p)} \geq \frac{1}{n^{3/2}\sqrt{2}} \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)}. \tag{6.9}$$

(b) *If $\kappa(\lambda, C) \geq \kappa(\lambda, M_\sigma)$, then*

$$\frac{\kappa(\lambda, C)}{\kappa(\lambda, p)} \geq \frac{1}{n\sqrt{2}} \frac{\kappa(\lambda, C)}{\kappa(\lambda, M_\sigma)}. \tag{6.10}$$

Proof First, we prove part (a). From Lemma 6.2 and using $\|\Pi(\lambda)\|_2 \geq 1$, we have

$$\begin{aligned} \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)} &= \frac{\|M_\sigma\|_2}{\|C\|_2} \frac{\|x_\sigma\|_2 \|y_\sigma\|_2}{\|\Lambda(\lambda)\|_2 \|\Pi(\lambda)\|_2} \leq \frac{\|M_\sigma\|_2}{\|C\|_2} \frac{\|x_\sigma\|_2 \|y_\sigma\|_2}{\|\Lambda(\lambda)\|_2} \\ &= \frac{\|M_\sigma\|_2}{|\lambda|} \frac{\|x_\sigma\|_2 \|y_\sigma\|_2}{|p'(\lambda)|} \frac{|\lambda|}{|||p|||_2} \frac{|p'(\lambda)|}{\|\Lambda(\lambda)\|_2} \frac{|||p|||_2}{\|C\|_2} \leq \sqrt{2}\, \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)}. \end{aligned}$$

Also, from Lemma 6.2 and using the upper bound in (6.7), we have

$$\begin{aligned} \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)} &\leq n^{3/2} \|\Pi(\lambda)\|_2 = n^{3/2} \frac{\|C\|_2}{|\lambda|} \frac{\|\Pi(\lambda)\|_2 \|\Lambda(\lambda)\|_2}{|p'(\lambda)|} \frac{|\lambda|}{|||p|||_2} \frac{|p'(\lambda)|}{\|\Lambda(\lambda)\|_2} \frac{|||p|||_2}{\|C\|_2} \\ &\leq n^{3/2}\sqrt{2}\, \frac{\kappa(\lambda, C)}{\kappa(\lambda, p)}. \end{aligned}$$

Next, we prove part (b). From Lemma 6.2 and using the upper bound in (6.8), we have

$$\begin{aligned} \frac{\kappa(\lambda, C)}{\kappa(\lambda, M_\sigma)} &\leq n \|\Pi(\lambda)\|_2 = n \frac{\|C\|_2}{|\lambda|} \frac{\|\Pi(\lambda)\|_2 \|\Lambda(\lambda)\|_2}{|p'(\lambda)|} \frac{|\lambda|}{|||p|||_2} \frac{|p'(\lambda)|}{\|\Lambda(\lambda)\|_2} \frac{|||p|||_2}{\|C\|_2} \\ &\leq n\sqrt{2}\, \frac{\kappa(\lambda, C)}{\kappa(\lambda, p)}. \end{aligned}$$

□

From Theorem 6.2, Example 6.2, and Theorem 6.3 we conclude the following:

- From the point of view of condition numbers, there are polynomials for which Frobenius companion matrices may be better suited than the rest of Fiedler matrices in the problem of computing their roots, but only in situations where it is not recommended to compute them as eigenvalues of any Fiedler matrix.

– From the point of view of condition numbers, there are polynomials for which one should avoid computing their roots as the eigenvalues of Frobenius companion matrices and to use, instead, another Fiedler matrix. However, neither Theorem 6.2 nor Theorem 6.3 show how to identify these polynomials and how to know in advance which Fiedler matrix might be used.

The difference between the statements of parts (a) and (b) in Theorem 6.3 is striking, but the next example shows that if the ratio $\kappa(\lambda, C)/\kappa(\lambda, M_\sigma)$ is large, then the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ is not necessarily large.

Example 6.3 Consider the following cubic polynomial: $p(z) = z^3 - z^2(\epsilon + 1/\epsilon) + z$, with $\epsilon \ll 1$, whose roots are $\epsilon, \epsilon^{-1}, 0$. Let C denote the first or the second Frobenius companion matrix of $p(z)$, and let M_σ be the Fiedler matrix of $p(z)$ associated with a bijection σ such that $\mathrm{PCIS}(\sigma) = (0, 1)$. It may be checked that

$$\frac{\kappa(\epsilon, C)}{\kappa(\epsilon, M_\sigma)} = \frac{\|C\|_2}{\|M_\sigma\|_2} \frac{\| \begin{bmatrix} \epsilon^2 \ \epsilon \ 1 \end{bmatrix}^T \|_2 \| \begin{bmatrix} 1 \ -\epsilon^{-1} \ 0 \end{bmatrix}^T \|_2}{\| \begin{bmatrix} \epsilon \ 1 \ 0 \end{bmatrix}^T \|_2 \| \begin{bmatrix} \epsilon \ -1 \ 1 \end{bmatrix}^T \|_2} \geq \frac{1}{3\epsilon},$$

and

$$\frac{\kappa(\epsilon, M_\sigma)}{\kappa(\epsilon, p)} = \frac{\|M_\sigma\|_2}{|||p|||_2} \frac{\| \begin{bmatrix} \epsilon \ 1 \ 0 \end{bmatrix}^T \|_2 \| \begin{bmatrix} \epsilon \ -1 \ 1 \end{bmatrix}^T \|_2}{\| \begin{bmatrix} \epsilon^2 \ \epsilon \ 1 \end{bmatrix}^T \|_2} \leq 3,$$

where to get the first inequality we have used Lemma 6.3, and to get the last inequality we have used that $\|M_\sigma\|_2/|||p|||_2 \leq \|M_\sigma\|_F/|||p|||_2 \leq \sqrt{3}$. So, taking ϵ small enough, $\kappa(\lambda, C)/\kappa(\lambda, M_\sigma)$ may be as large as desired, but $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ is bounded by a constant that is independent of ϵ.

7 Pseudospectra of Fiedler matrices

In this section we establish several mathematical relationships between the pseudozero sets of a monic polynomial $p(z)$ and the pseudospectra of the associated Fiedler matrices, and we also show how to estimate accurately pseudospectra of Fiedler matrices in a fast way. These results are generalizations of the results in [39], valid only for the Frobenius companion matrices, to all Fiedler matrices. The information obtained from pseudospectra and pseudozero sets expands the one coming from condition numbers to perturbations of finite magnitudes.

Given a monic polynomial $p(z)$ as in (2.2), the *ϵ-pseudozero set of* $p(z)$, denoted by $Z_\epsilon(p)$, is the set

$$Z_\epsilon(p) = \left\{ z \in \mathbb{C} : z \text{ is a root of } \widetilde{p}(z) = z^n + \sum_{k=0}^{n-1} \widetilde{a}_k z^k \quad \text{with} \quad |||\widetilde{p} - p|||_2 \leq \epsilon |||p|||_2 \right\}.$$

The pseudozero set $Z_\epsilon(p)$ can be characterized in terms of the level curves of a certain function [39, Proposition 2.1]:

$$Z_\epsilon(p) = \left\{ z \in \mathbb{C} : \quad \psi(z) = \frac{|p(z)|}{|||p|||_2 \|\Lambda(z)\|_2} \leq \epsilon \right\}, \tag{7.1}$$

where the vector $\Lambda(z)$ is defined in (3.2).

For a given Fiedler companion matrix M_σ of $p(z)$, the *ϵ-pseudospectrum of M_σ*, denoted by $\Lambda_\epsilon(M_\sigma)$, is the set

$$\Lambda_\epsilon(M_\sigma) := \{z \in \mathbb{C} : z \text{ is an eigenvalue of } M_\sigma + E \text{ for some } E \text{ with } \|E\|_2 \leq \epsilon\|M_\sigma\|_2\}.$$

The ϵ-pseudospectrum $\Lambda_\epsilon(M_\sigma)$ can be characterized in terms of the level curves of the norm of the resolvent [41, Theorem 2.2] as:

$$\Lambda_\epsilon(M_\sigma) = \left\{ z : \|(zI - M_\sigma)^{-1}\|_2 \geq (\epsilon\|M_\sigma\|_2)^{-1} \right\}, \tag{7.2}$$

where by convention $\|(zI - M_\sigma)^{-1}\|_2$ takes the value ∞ in the spectrum of M_σ. Based on (7.2) and the explicit expression of $(zI - M_\sigma)^{-1}$ given in [36, Th. 8.21] for any Fiedler matrix, we show how to compute $\Lambda_\epsilon(M_\sigma)$ in a fast way in Section 7.1.

7.1 Fast computation of pseudospectra of Fiedler matrices

Since $\|(zI - M_\sigma)^{-1}\|_2$ is equal to the inverse of the minimum singular value of $zI - M_\sigma$, one obvious way to determine $\Lambda_\epsilon(M_\sigma)$ numerically is to compute the minimum singular value of $zI - M_\sigma$, via the SVD, on a grid in the complex plane and, then, generate a contour plot from this data. The problem with this approach is that computing the whole SVD of an $n \times n$ matrix on a $m \times m$ grid requires a number of floating point operations (flops) of order $m^2 n^3$, which is highly expensive. Different techniques have been introduced to make the computation of the norm of the resolvent matrix as efficient as possible. With these techniques the overall complexity can be reduced to a number of order $n^3 + n^2 m^2$ [40]. Nevertheless, we will show that $\Lambda_\epsilon(M_\sigma)$ can be accurately estimated on a $m \times m$ grid in only a number of flops of order $m^2 n$. This result relies on Theorem 7.1, which shows that the ϵ-pseudospectrum of a Fiedler matrix M_σ is the region bounded by the ϵ-level curve of a certain function (denoted by $\phi_\sigma(z)$), that is easy and fast to compute, defined over the complex plane when ϵ is sufficiently small. This result was proved in [39, Proposition 6.2] only when M_σ is one of the Frobenius companion matrices and it is extended here to any Fiedler matrix via the result in [36, Thm. 8.21], which relies on the expression for the adjugate matrix of $zI - M_\sigma$ in [13, Th. 3.3]. Hence, though the proof of Theorem 7.1 we provide here is rather short, we emphasize that it is based on strong technical results on Fiedler matrices presented in [13,36].

Theorem 7.1 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \geq 2$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, let M_σ be the Fiedler companion matrix of $p(z)$ associated with σ, and let $x_\sigma(z)$, $y_\sigma(z)$, $v_\sigma(z)$ and $w_\sigma(z)$ be the vectors defined in Theorem 4.1. Then,*

$$\frac{\|(zI - M_\sigma)^{-1}\|_2}{\phi_\sigma(z)} = 1 + O\left(\frac{1}{\phi_\sigma(z)}\right) \quad \text{as} \quad \phi_\sigma(z) \longrightarrow \infty, \tag{7.3}$$

where

$$\phi_\sigma(z) = \begin{cases} \|x_\sigma(z)\|_2 \|y_\sigma(z)\|_2 / |p(z)| & \text{if } |z| \leq 1, \quad \text{and} \\ \|v_\sigma(z)\|_2 \|w_\sigma(z)\|_2 / |p(z)| & \text{if } |z| > 1. \end{cases} \tag{7.4}$$

Proof From [36, Thm. 8.21] we get

$$(zI - M_\sigma)^{-1} = \frac{1}{p(z)} x_\sigma(z) y_\sigma(z)^T - A_\sigma(z) = \frac{1}{p(z)} v_\sigma w_\sigma^T + B_\sigma(z).$$

where $A_\sigma(z), B_\sigma(z)$ are the matrices introduced in [36, Th. 5.3] (see also [13, Th. 3.3] for $A_\sigma(z)$). Therefore:

$$\|(zI - M_\sigma)^{-1}\|_2 = \left\| \frac{x_\sigma(z) y_\sigma^T(z)}{p(z)} - A_\sigma(z) \right\|_2 = \left\| \frac{v_\sigma(z) w_\sigma^T(z)}{p(z)} + B_\sigma(z) \right\|_2. \quad (7.5)$$

From [13, Th. 3.3] we know that the entries of $A_\sigma(z)$ are polynomials in z. In a similar way, it can be proved that the entries of $B_\sigma(z)$ are polynomials in z^{-1} (see [36, Lemma 5.8]). Therefore, there exist finite constants $0 \le m_1, m_2 < \infty$, such that

$$\|A_\sigma(z)\|_2 \le m_1 \quad \text{if} \quad |z| \le 1 \quad \text{and} \quad \|B_\sigma(z)\|_2 \le m_2 \quad \text{if} \quad |z| \ge 1. \quad (7.6)$$

Finally, to prove the result, we have to distinguish two cases: when $|z| \le 1$ and when $|z| > 1$.

(a) If $|z| \le 1$, from (7.5) and (7.6) we get

$$\left| \|(zI - M_\sigma)^{-1}\|_2 - \frac{\|x_\sigma(z)\|_2 \|y_\sigma(z)\|_2}{|p(z)|} \right| \le \|A_\sigma(z)\|_2 \le m_1.$$

(b) If $|z| > 1$, from (7.5) and (7.6) we get

$$\left| \|(zI - M_\sigma)^{-1}\|_2 - \frac{\|v_\sigma(z)\|_2 \|w_\sigma(z)\|_2}{|p(z)|} \right| \le \|B_\sigma(z)\|_2 \le m_2.$$

Therefore,

$$\left| \frac{\|(zI - M_\sigma)^{-1}\|_2}{\phi_\sigma(z)} - 1 \right| \le \frac{\max\{m_1, m_2\}}{\phi_\sigma(z)},$$

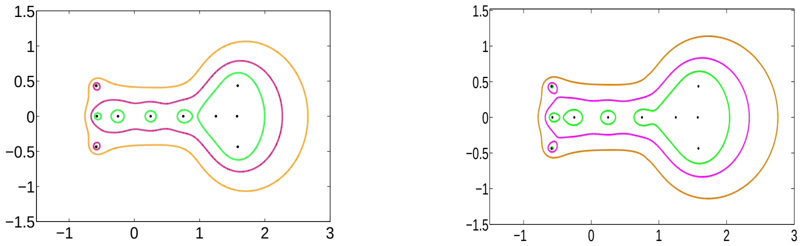
which implies (7.3). □

Note that the only information that we need in the proof of Theorem 7.1 about the matrices $A_\sigma(z)$ and $B_\sigma(z)$ is that their entries are, respectively, polynomials in z and z^{-1}.

Remark 7.1 The entries of the vectors $x_\sigma(z)$, $y_\sigma(z)$, $v_\sigma(z)$, and $w_\sigma(z)$ are either powers of z or powers of z times a Horner shift of $p(z)$ or $p^{\text{rev}}(z^{-1})$. This observation, together with the recurrence relation (2.3) to compute the Horner shifts of a polynomial and with the Horner's rule for evaluating polynomials, implies that $\phi_\sigma(z)$ can be computed in a number of flops of order n.

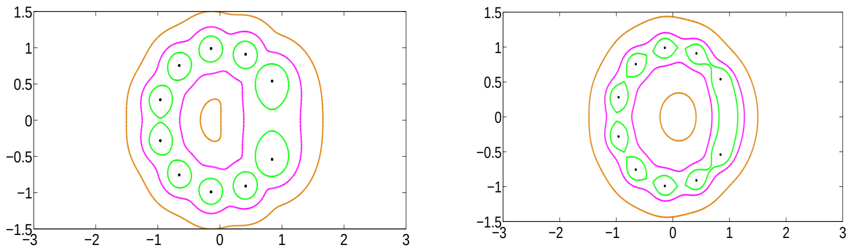
Notice that in the neighborhood of a root of $p(z)$ we have $\phi_\sigma(z) \gg 1$. In this setting, Theorem 7.1 shows that $\phi_\sigma(z)$ provides an accurate estimate of $\|(zI - M_\sigma)^{-1}\|_2$. Therefore, in the limit $\epsilon \to 0$, the pseudospectrum $\Lambda_\epsilon(M_\sigma)$ agrees with the region bounded by the $(\epsilon\|M_\sigma\|_2)^{-1}$−level curve of $\phi_\sigma(z)$. This result has practical applications since pseudospectra, as we said in the paragraph before Theorem 7.1, are expensive to compute. As it is stated in Remark 7.1, only order n flops are needed to calculate $\phi_\sigma(z)$, as compared with the order n^3 flops needed to calculate $\|(zI - M_\sigma)^{-1}\|_2$ by the SVD. Therefore, the function $\phi_\sigma(z)$ can be evaluated on a $m \times m$ grid in a number of flops of order only $m^2 n$.

We illustrate how the ϵ-pseudospectrum of a Fiedler matrix is accurately estimated by the $(\epsilon\|M_\sigma\|_2)^{-1}$−level curve of the function $\phi_\sigma(z)$ with three examples. In Figure 7.1, we plot, for $\epsilon = 10^{-2.5}, 10^{-3}, 10^{-3.5}$, in (a) the ϵ−pseudospectra and in (b) the $(\epsilon\|M_\sigma\|_2)^{-1}$−level curves of the function $\phi_\sigma(z)$ of the Fiedler matrix $M_\sigma = C_2$ of the Bernoulli polynomial of degree 10: $z^{10} - 5z^9 + (15/2)z^8 - 7z^6 + 5z^4 - (3/2)z^2 + 5/66$. In Figure 7.2, we plot, for $\epsilon = 10^{-1.25}, 10^{-1}, 10^{-0.75}$, in (a) the ϵ−pseudospectra and in (b) the $(\epsilon\|M_\sigma\|_2)^{-1}$−level curves of the function $\phi_\sigma(z)$, for the Fiedler matrix M_σ from Example 4.3 of the polynomial $z^{10} + z^9 + \cdots + z + 1$. In Figure 7.3, we plot, for $\epsilon = 10^{-16}, 10^{-15}, 10^{-14}$, in (a) the ϵ−pseudospectra and in (b) the $(\epsilon\|M_\sigma\|_2)^{-1}$−level curves of the function $\phi_\sigma(z)$, for the Fiedler matrix M_σ fom Example 4.2 of the monic polynomial with zeros in $1, 2, \ldots, 10$.



(a) (b)

Fig. 7.1 For the Bernoulli polynomial of degree 10, $z^{10} - 5z^9 + (15/2)z^8 - 7z^6 + 5z^4 - (3/2)z^2 + 5/66$, and for $\epsilon = 10^{-3.5}, 10^{-3}, 10^{-2.5}$, we plot in (a) the ϵ−pseudospectra of C_2 and in (b) the $(\epsilon\|C_2\|_2)^{-1}$ − level curves of the function $\phi_\sigma(z)$ defined in (7.4), in green, magenta and brown, respectively.



(a) (b)

Fig. 7.2 For the monic polynomial $z^{10} + z^9 + \cdots + z + 1$, and for $\epsilon = 10^{-1.25}, 10^{-1}, 10^{-0.75}$, we plot in (a) the ϵ−pseudospectra of M_σ with PCIS(σ) = $(1, 0, 1, 0, \ldots)$ and in (b) the $(\epsilon\|M_\sigma\|_2)^{-1}$ − level curves of the function $\phi_\sigma(z)$ defined in (7.4), in green, magenta and brown, respectively.

Notice that Figures 7.1-(a) and 7.1-(b), and 7.3-(a) and 7.3-(b) are almost indistinguishable. Note also that, by contrast with Figures 7.1 and 7.3, there are

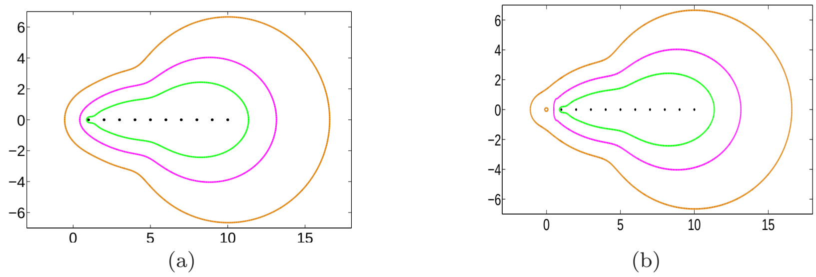


Fig. 7.3 For the monic polynomial with zeros in $1, 2, \ldots, 10$, and for $\epsilon = 10^{-16}, 10^{-15}, 10^{-14}$, we plot in (a) the $\epsilon-$pseudospectra of M_σ with $\text{PCIS}(\sigma) = (0, 1, \ldots, 1)$ and in (b) the $(\epsilon \|M_\sigma\|_2)^{-1}-$ level curves of the function $\phi_\sigma(z)$ defined in (7.4), in green, magenta and brown, respectively.

perceptible differences between Figures 7.2-(a) and 7.2-(b). The main reason for these differences is that in Figure 7.2 we are computing pseudospectra for rather large values of ϵ and, in addition, close to the region $|z| = 1$ where the 2-norm of the matrices $A_\sigma(z)$ and $B_\sigma(z)$ in the proof of Theorem 7.1 might be large and, therefore, $\|(zI - M_\sigma)^{-1}\|_2 \approx \phi_\sigma(z)$ might not hold.

7.2 Asymptotic relations between pseudozero sets of monic polynomials and pseudospectra of Fiedler matrices

Theorem 7.1 is also the key tool to prove the main results in this section, that is, Corollaries 7.1 and 7.2. These two corollaries give several asymptotic relations between the ϵ-pseudozero set of a monic polynomial $p(z)$ and the pseudospectra of the Fiedler matrices of $p(z)$ in a neighborhood of a simple nonzero root λ of $p(z)$.

Corollary 7.1 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \geq 2$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, and let M_σ be the Fiedler companion matrix of $p(z)$ associated with σ. If λ is a simple nonzero root of $p(z)$, then*

$$\lim_{z \to \lambda} \frac{\|(zI - M_\sigma)^{-1}\|_2 \|M_\sigma\|_2}{1/\psi(z)} = \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)},$$

where $\psi(z) = |p(z)|/(\|\Lambda(z)\|_2 \| p \|_2)$ is the function in (7.1), and where $\Lambda(z)$ is defined in (3.2).

Proof From Theorem 4.1 together with (5.3) and (7.4), we have

$$\lim_{z \to \lambda} \frac{\|M_\sigma\|_2}{|z|} \frac{|p(z)|\phi_\sigma(z)}{|p'(z)|} = \kappa(\lambda, M_\sigma). \tag{7.7}$$

Therefore,

$$\frac{\|(zI - M_\sigma)^{-1}\|_2 \|M_\sigma\|_2}{1/\psi(z)} = \frac{\|(zI - M_\sigma)^{-1}\|_2}{\phi_\sigma(z)} \frac{|p'(z)| \cdot |z|}{\|\Lambda(z)\|_2 \| p \|_2} \frac{|p(z)|\phi_\sigma(z)\|M_\sigma\|_2}{|z| \cdot |p'(z)|} \xrightarrow[z \to \lambda]{} \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)},$$

where we have used (3.3), (7.7), and Theorem 7.1. □

In words, Corollary 7.1 says that, if M_σ is a Fiedler matrix of $p(z)$, then, in the limit $\epsilon \to 0$, the components of $\Lambda_\epsilon(M_\sigma)$ and $Z_{\epsilon'}(p)$ containing λ, where $\epsilon' = \epsilon\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, agree with each other (see (7.1) and (7.2)).

Corollary 7.1, together with Theorem 6.1, suggest that the ϵ-pseudospectrum of a Fiedler matrix of a monic polynomial $p(z)$ may be potentially much larger than the ϵ-pseudozero set of that polynomial when $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is large or close to zero. Nevertheless, Corollary 7.1 reveals that when $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is moderate and not close to zero, that is, of order $\Theta(1)$, the pseudozero sets of a monic polynomial and the pseudospectra of the associated Fiedler matrices will be quite close to each other for the same values of ϵ, for ϵ small enough.

Corollary 7.2 *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \geq 2$, let $\sigma_1, \sigma_2 : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be two bijections, and let M_{σ_1} and M_{σ_2} be the Fiedler companion matrices of $p(z)$ associated with σ_1 and σ_2, respectively. Then, if λ is a nonzero simple root of $p(z)$,*

$$\lim_{z \to \lambda} \frac{\|(zI - M_{\sigma_1})^{-1}\|_2 \|M_{\sigma_1}\|_2}{\|(zI - M_{\sigma_2})^{-1}\|_2 \|M_{\sigma_2}\|_2} = \frac{\kappa(\lambda, M_{\sigma_1})}{\kappa(\lambda, M_{\sigma_2})}. \tag{7.8}$$

Proof From (7.7), with $\sigma = \sigma_1, \sigma_2$, and Theorem 7.1, we have

$$\begin{aligned}
&\frac{\|(zI - M_{\sigma_1})^{-1}\|_2 \|M_{\sigma_1}\|_2}{\|(zI - M_{\sigma_2})^{-1}\|_2 \|M_{\sigma_2}\|_2} \\
&= \frac{\phi_{\sigma_2}(z)\|(zI - M_{\sigma_1})^{-1}\|_2}{\phi_{\sigma_1}(z)\|(zI - M_{\sigma_2})^{-1}\|_2} \frac{\phi_{\sigma_1}(z)|p(z)|}{|p'(z)|} \frac{\|M_{\sigma_1}\|_2}{|z|} \frac{|p'(z)|}{\phi_{\sigma_2}(z)|p(z)|} \frac{|z|}{\|M_{\sigma_2}\|_2} \\
&\xrightarrow[z \to \lambda]{} \frac{\kappa(\lambda, M_{\sigma_1})}{\kappa(\lambda, M_{\sigma_2})}.
\end{aligned}$$

□

In words, Corollary 7.2 says that, if M_{σ_1} and M_{σ_2} are two different Fiedler matrices of $p(z)$, then, in the limit $\epsilon \to 0$, the components of $\Lambda_\epsilon(M_{\sigma_1})$ and $\Lambda_{\epsilon'}(M_{\sigma_2})$, where $\epsilon' = \epsilon\kappa(\lambda, M_{\sigma_1})/\kappa(\lambda, M_{\sigma_2})$, containing λ agree with each other.

Corollary 7.2 together with Theorem 6.2 suggest that the ϵ-pseudospectrum of a Fiedler matrix $M_\sigma \neq C_1, C_2$ may be potentially much larger (or much smaller) than the ϵ-pseudospectrum of the Frobenius companion matrices for polynomials that have large coefficients, since, in this case, the ratios $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ or $\kappa(\lambda, C)/\kappa(\lambda, M_\sigma)$ can be large (with $C = C_1, C_2$). Nevertheless, if $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is moderate, Corollary 7.2 and Theorem 6.2 guarantee that the pseudospectra of Fiedler matrices other than the Frobenius ones are quite close to the pseudospectra of the Frobenius companion matrices for the same values of ϵ, for ϵ small enough.

8 Numerical experiments

In this section we provide numerical experiments that support our theoretical results. In addition, we investigate numerically the effect of balancing Fiedler matrices on eigenvalue condition numbers and pseudospectra. We emphasize that the main obstacle for studying theoretically the effect of balancing is that the balancing diagonal matrix is a very complicated function of the coefficients of $p(z)$.

The specific goals of this section are: (i) to show whether or not the bounds in Theorem 6.1 correctly predict the dependence on the coefficients of $p(z)$ of the largest ratios $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ that may be obtained; (ii) to show whether or not the bounds in Theorem 6.2 correctly predict the dependence on the coefficients of $p(z)$ of the largest and smallest ratios $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ that may be obtained, where C denotes one of the Frobenius companion matrices; (iii) to study the ratios $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ when the coefficients of $p(z)$ are bounded in absolute value by a moderate constant; and (iv) to investigate, from the point of view of condition numbers and pseudospectra, the effect of balancing Fiedler matrices in practice.

Given a monic polynomial $p(z)$ of degree n, the second Frobenius companion matrix C_2 of $p(z)$, and a Fiedler matrix M_σ other than the Frobenius ones associated with $p(z)$, we are interested in the following quantities:

– $\max_\lambda \kappa(\lambda, C_2)/\kappa(\lambda, p)$ and $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$,
– $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$, and
– $\min_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$,

where λ runs over all the nonzero simple roots of $p(z)$, and where the ratios of condition numbers $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, $\kappa(\lambda, C_2)/\kappa(\lambda, p)$, and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ are, respectively, the ratios in (6.1), (6.2) and (6.6).

In the numerical experiments, we consider monic polynomials of degree 10 and the following Fiedler companion matrices associated with degree-10 polynomials:

(a) the second Frobenius companion matrix $C_2 = M_{\sigma_1}$ with $\mathrm{PCIS}(\sigma_1) = (1, 1, 1, 1, 1, 1, 1, 1, 1)$,
(b) the pentadiagonal Fiedler matrix $P_1 = M_{\sigma_2}$ with $\mathrm{PCIS}(\sigma_2) = (1, 0, 1, 0, 1, 0, 1, 0, 1)$,
(c) the Fiedler matrix $F = M_{\sigma_3}$ with $\mathrm{PCIS}(\sigma_3) = (0, 1, 1, 1, 1, 1, 1, 1, 1)$, and
(d) the Fiedler matrix M_{σ_4} with $\mathrm{PCIS}(\sigma_4) = (0, 0, 1, 1, 1, 0, 1, 0, 0)$.

Recall that the matrices M_{σ_2} and M_{σ_3} are the Fiedler matrices considered in Examples 4.3 and 4.2, respectively.

To compute the ratios $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ we proceed as follows. First, we compute the roots of $p(z)$ as the eigenvalues of the second Frobenius companion matrix C_2 with 64 digits of accuracy in MATLAB using the function `vpa` (variable precision arithmetic) followed by the command `eig`. Then, we compute $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ using (6.1)-(6.2) and (6.6), respectively.

8.1 Numerical experiments that show the dependence of $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ on the coefficients of $p(z)$

In this subsection, we perform numerical experiments to determine whether or not the ratio of condition numbers $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ behave like the bounds in Theorems 6.1 and 6.2 predict. In particular, we provide two sets of numerical experiments to study the dependence of $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ on $\|p\|_2$ and on the function $\rho(p)$ (defined in (6.3)), and a set of numerical experiments to study the dependence of $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ on $\|p\|_2$.

In the first set of numerical experiments we study the dependence of the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ on $\|p\|_2$. For this purpose, we consider a random sample of two hundred degree-10 monic polynomials $p(z) = z^{10} + \sum_{i=0}^{9} a_i z^i$ with coefficients of the form $a_i = c_i \times 10^{e_i}$, for $i = 0, 1, \ldots, 9$, where c_i and e_i are drawn from the uniform distributions on the intervals $[-1, 1]$ and $[0, 5]$, respectively. All the generated polynomials where chosen to satisfy $\max\{|a_0|, |a_1|, \ldots, |a_9|\} \geq 1$, and, so, the function $\rho(p)$ satisfies $\rho(p) \leq \sqrt{2}$ for these polynomials. Then, Theorem 6.1 predicts that

$$\frac{1}{\sqrt{2}} \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \leq 10\sqrt{2}\|p\|_2 \tag{8.1}$$

if $M_\sigma = C_1, C_2$, and

$$\frac{1}{10} \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \leq 100\sqrt{20}\|p\|_2^2 \tag{8.2}$$

if $M_\sigma \neq C_1, C_2$.

In Figures 8.1-(a), 8.1-(b), 8.1-(c), and 8.1-(d) we plot for each of the 200 random polynomials the quantity $\max_\lambda \kappa(\lambda, M_{\sigma_i})/\kappa(\lambda, p)$, for $i = 1, 2, 3, 4$, against the norm $\|p\|_2$. As may be seen in those figures, the largest ratios obtained in these numerical experiments are bounded by a function that grows like $\|p\|_2$ in Figure 8.1-(a), and like $\|p\|_2^2$ in Figures 8.1-(b), 8.1-(c), and 8.1-(d). These results are consistent with the bounds in (8.1) and (8.2).

In the second set of numerical experiments we study the dependence of the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ on the function $\rho(p)$. For this purpose, we consider a random sample of two hundred degree-10 monic polynomials with coefficients close to zero. To generate those polynomials we proceed as follows. For $k = 1, 2, \ldots, 10$, we generate twenty degree-10 monic polynomials $p(z) = z^{10} + \sum_{i=0}^{9} a_i z^i$ with coefficients of the form $a_i = c_i \times 10^{e_i}$, for $i = 0, 1, \ldots, 9$, where c_i and e_i are drawn from the uniform distribution on the intervals $[-1, 1]$ and $[-k/2, -k/2 + 0.5]$, respectively. All the generated polynomials satisfy $\max\{|a_0|, |a_1|, \ldots, |a_9|\} \leq 1$, and, therefore, their norms satisfy $\|p\|_2 \leq \sqrt{11}$. Then, Theorem 6.1 predicts that

$$\frac{1}{\sqrt{2}} \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \leq 11^{3/2}\rho(p), \tag{8.3}$$

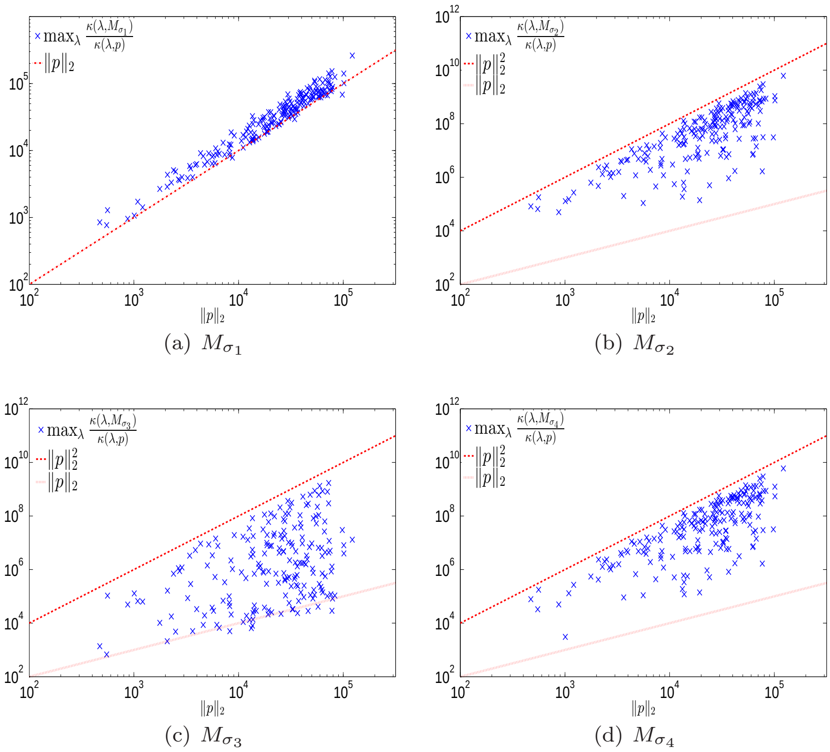
if $M_\sigma = C_1, C_2$, and

$$\frac{1}{10} \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \leq 11^{7/2}\rho(p), \tag{8.4}$$

if $M_\sigma \neq C_1, C_2$.

In Figures 8.2-(a), 8.2-(b), 8.2-(c), and 8.2-(d) we plot, for each of the 200 random polynomials, the quantity $\max_\lambda \kappa(\lambda, M_{\sigma_i})/\kappa(\lambda, p)$, for $i = 1, 2, 3, 4$, against the function $\rho(p)$. As may be seen in those figures, the ratios obtained in these numerical experiments grow like the function $\rho(p)$. These results are consistent with the bounds in (8.3) and (8.4). Also notice that the four plots in Figure 8.2 are almost indistinguishable. This result is in accordance with Theorem 6.2, which predicts, from the point of view of conditioning, that for polynomials with moderate coefficients all Fiedler matrices behave like the Frobenius ones.

In the third set of numerical experiments we study the dependence of the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ on $\|p\|_2$. For this purpose, we consider again a random



(a) M_{σ_1} (b) M_{σ_2}

(c) M_{σ_3} (d) M_{σ_4}

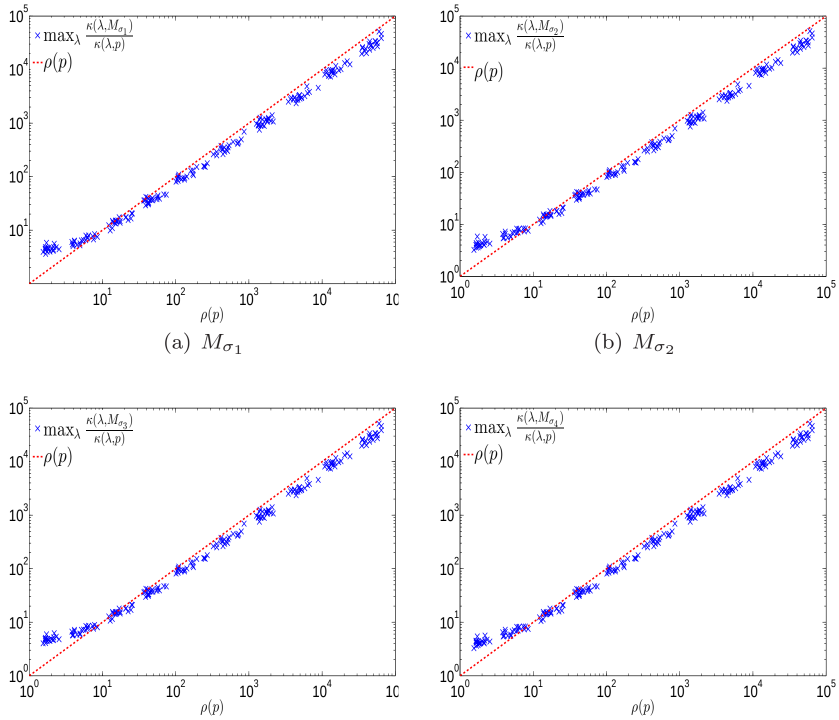
Fig. 8.1 Maximum ratio $\max_\lambda \kappa(\lambda, M_{\sigma_i})/\kappa(\lambda, p)$, for $i = 1, 2, 3, 4$, for each of the 200 random degree-10 monic polynomials with coefficients of the form $a_i = c_i \times 10^{e_i}$, for $i = 0, 1, \ldots, 9$, where c_i and e_i are drawn, respectively, from the uniform distributions on the intervals $[-1, 1]$ and $[0, 5]$.

sample of two hundred degree-10 monic polynomials $p(z) = z^{10} + \sum_{i=0}^{9} a_i z^i$ with coefficients of the form $a_i = c_i \times 10^{e_i}$, for $i = 0, 1, \ldots, 9$, where c_i and e_i are drawn from the uniform distributions on the intervals $[-1, 1]$ and $[0, 5]$.

In Figures 8.3-(a), 8.3-(b), and 8.3-(c) we plot for each of the 200 random polynomials the quantities $\max_\lambda \kappa(\lambda, M_{\sigma_i})/\kappa(\lambda, C_2)$ and $\min_\lambda \kappa(\lambda, M_{\sigma_i})/\kappa(\lambda, C_2)$, for $i = 2, 3, 4$, against the norm $\|p\|_2$. As may be seen in those figures, the largest ratios obtained in these numerical experiments are upper bounded by a function that grows like $\|p\|_2$, and the smallest ratios are lower bounded by a function that decreases like $\|p\|_2^{-1}$. These results are consistent with the bounds in Theorem 6.2.

8.2 Numerical experiments with polynomials having coefficients not much larger than 1

In this subsection we study the ratios $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ when the coefficients of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ are bounded in absolute value by a moderate number. In particular, we provide numerical evidence to show that, as predicted by the bounds in Theorems 6.1 and 6.2: (i) both ratios are moderate



(a) M_{σ_1} (b) M_{σ_2}

(c) M_{σ_3} (d) M_{σ_4}

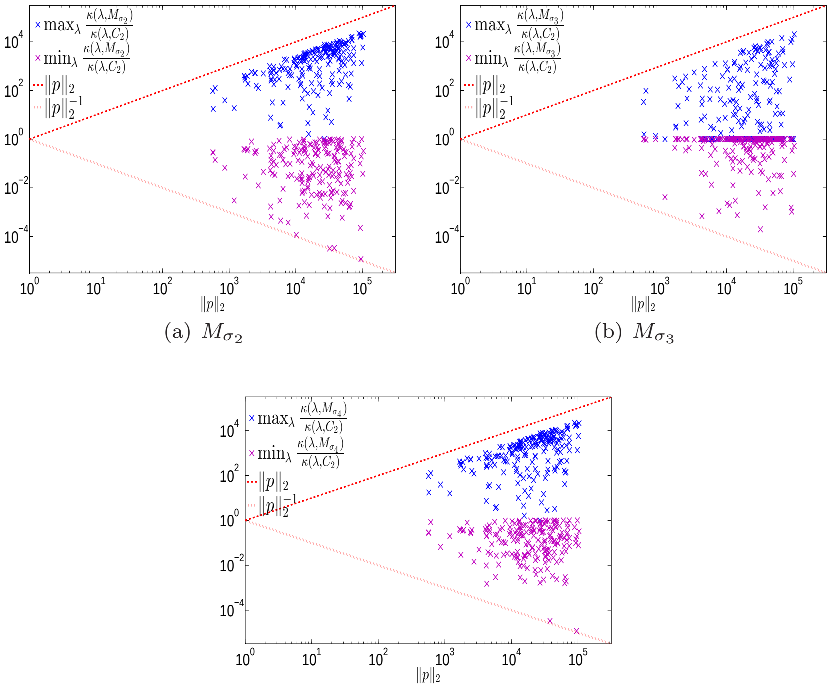
Fig. 8.2 Maximum ratio $\max_\lambda \kappa(\lambda, M_{\sigma_i})/\kappa(\lambda, p)$, for $i = 1, 2, 3, 4$, for each of the 20 random degree-10 monic polynomials of each of the 10 samples of random polynomials with coefficients of the form $a_i = c_i \times 10^{e_i}$, for $i = 0, 1, \ldots, 9$, where c_i and e_i are drawn, respectively, from the uniform distributions on the intervals $[-1, 1]$ and $[-k/2, -k/2 + 0.5]$, for $k = 1, 2, \ldots, 10$.

when all the coefficients or $p(z)$ are moderate and not close to zero; and (ii) the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ is moderate when $\max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$ is moderate, but, in this situation, the ratios $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ or $\kappa(\lambda, C_2)/\kappa(\lambda, p)$ may be large when $\max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$ is close to zero.

In the first set of numerical experiments, we consider a random sample of 1000 degree-10 monic polynomials with coefficients drawn from the uniform distribution on the interval $[-10, 10]$, so that all the coefficients of every polynomial in the sample are moderate and not close to zero. In Table 8.1, we give the mean and the maximum of the decimal logarithms (Log-Mean and Log-Maximum, respectively) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, and the maximum of the decimal logarithms (Log-Maximum) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ and $\max_\lambda \kappa(\lambda, C_2)/\kappa(\lambda, M_\sigma)$, where λ runs over all nonzero simple roots of $p(z)$, obtained for the Fiedler matrices $M_{\sigma_1}, M_{\sigma_2}, M_{\sigma_3}, M_{\sigma_4}$.

As may be seen from the data in Table 8.1, both ratios $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$, obtained for the polynomials in the first sample, are moderate.

In the second set of numerical experiments, we consider a random sample of 1000 degree-10 monic polynomials with coefficients of the form $a_i = c \cdot 10^e$ where c and e are drawn from the uniform distributions on the intervals $[-1, 1]$



(a) M_{σ_2} (b) M_{σ_3}

(c) M_{σ_4}

Fig. 8.3 Maximum and minimum ratio $\kappa(\lambda, M_{\sigma_i})/\kappa(\lambda, C_2)$, in blue and purple, respectively, for $i = 2, 3, 4$, for each of the 200 random degree-10 monic polynomials with coefficients of the form $a_i = c_i \times 10^{e_i}$, for $i = 0, 1, \ldots, 9$, where c_i and e_i are drawn, respectively, from the uniform distributions on the intervals $[-1, 1]$ and $[0, 5]$.

	M_{σ_1}	M_{σ_2}	M_{σ_3}	M_{σ_4}
Log-Mean $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$	1.5	2.0	1.9	2.1
Log-Maximum $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$	1.8	2.6	2.4	2.6
Log-Maximum $\max_\lambda \kappa(\lambda, C_2)/\kappa(\lambda, M_\sigma)$	0.0	1.1	0.8	0.9
Log-Maximum $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	0.0	0.8	0.8	0.8

Table 8.1 Mean and maximum of the decimal logarithms (Log-Mean and Log-Maximum, respectively) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, and maximum of the decimal logarithms (Log-Maximum) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ and $\max_\lambda \kappa(\lambda, C_2)/\kappa(\lambda, M_\sigma)$, where λ runs over all nonzero simple roots of $p(z)$, obtained for 1000 random degree-10 polynomials, with coefficients drawn from the uniform distribution on the interval $[-10, 10]$.

and $[-10, -8]$, respectively, so that all polynomials in this sample satisfy that $\max\{|a_9|, \ldots, |a_1|, |a_0|\}$ is close to zero. The reason for this choice of random polynomials is to show that $\max\{|a_9|, \ldots, |a_1|, |a_0|\} = M$, where M is not much larger than 1, is not enough to guarantee that $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ is moderate. In Table 8.2, we display the mean and the maximum of the decimal logarithms (Log-

Mean and Log-Maximum, respectively) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, and the maximum of the decimal logarithms (Log-Maximum) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ and $\max_\lambda \kappa(\lambda, C_2)/\kappa(\lambda, M_\sigma)$, where λ runs over all nonzero simple roots of $p(z)$, obtained for the Fiedler matrices $M_{\sigma_1}, M_{\sigma_2}, M_{\sigma_3}, M_{\sigma_4}$.

	M_{σ_1}	M_{σ_2}	M_{σ_3}	M_{σ_4}
Log-Mean $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$	8.3	8.3	8.3	8.3
Log-Maximum $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$	9.2	9.2	9.2	9.2
Log-Maximum $\max_\lambda \kappa(\lambda, C_2)/\kappa(\lambda, M_\sigma)$	0.0	0.0	0.0	0.0
Log-Maximum $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	0.0	0.0	0.0	0.0

Table 8.2 Mean and maximum of the decimal logarithms (Log-Mean and Log-Maximum, respectively) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, and maximum of the decimal logarithms (Log-Maximum) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ and $\max_\lambda \kappa(\lambda, C_2)/\kappa(\lambda, M_\sigma)$, where λ runs over all nonzero simple roots of $p(z)$, obtained for 1000 random degree-10 polynomials, with coefficients of the form $a_i = c \cdot 10^e$ where c and e are drawn from the uniform distributions on the intervals $[-1, 1]$ and $[-10, -8]$, respectively.

As may be seen in Table 8.2, the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$, obtained for the polynomials in the second sample of random polynomials, are moderate, but, as may be seen from the data in Log-Mean $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ is large very often. Notice, in addition, that the data in the four columns are identical. This is in accordance with Theorem 6.2, which predicts, from the point of view of conditioning, that for polynomials with moderate coefficients all Fiedler matrices behave like the Frobenius ones.

8.3 Numerical experiments balancing Fiedler matrices

Numerical algorithms that compute the eigenvalues of a nonsymmetric matrix A are typically affected by backward roundoff errors of size roughly $\alpha_n u\|A\|_F$ [19], where u is the machine epsilon and α_n is a low degree polynomial on the size of the matrix. Balancing, an idea introduced in [35], is a standard technique in computing the eigenvalues of a given matrix A, which leads, very often, to more accurate results, especially when the entries of A have very different magnitudes (although there are situations where balancing has the opposite effect [43]). Actually, balancing is implemented by default as an initial step in the MATLAB command `eig` for computing the eigenvalues of arbitrary matrices. Balancing consists of performing a diagonal similarity DAD^{-1} (i.e., with D diagonal) in order to reduce the norm of A by equilibrating as much as possible the ∞-norm of all rows and columns. In addition, very frequently balancing reduces the eigenvalue condition numbers [19, §7.2.2]. Balancing is performed with matrices D whose entries are powers of two, so as not to introduce any roundoff errors.

In the context of polynomial root-finding, given a polynomial $p(z)$ and an associated Fiedler companion matrix M_σ, it would be desirable to find a similarity transformation that makes the eigenvalue problem no worse conditioned than the polynomial root-finding problem. If D is a nonsingular diagonal matrix, then the condition number $\kappa(\lambda, DM_\sigma D^{-1})$ of a nonzero simple eigenvalue λ of M_σ is given by

$$\kappa(\lambda, DM_\sigma D^{-1}) = \frac{\|DM_\sigma D^{-1}\|_2}{|\lambda|} \frac{\|Dx_\sigma\|_2 \|D^{-1}y_\sigma\|_2}{|p'(\lambda)|}, \tag{8.5}$$

where the vectors x_σ and y_σ are defined in Theorem 4.1. In this subsection we perform numerical experiments to study, from the point of view of condition numbers, the effect of balancing Fiedler matrices.

In the first set of numerical experiments, we consider a random sample of 1000 degree-10 monic polynomials $p(z)$ as in (2.2) with coefficients of the form $a_k = b_1 \times 10^{e_1} + \mathrm{i}\, b_2 \times 10^{e_2}$, with i being the imaginary unit, and where, for $i = 1, 2$, b_i and e_i are drawn from the uniform distributions on the intervals $[-1, 1]$ and $[-10, 10]$, respectively. Our goal is to study the ratios $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ and $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ when the Fiedler matrices are not or are balanced. To compute the diagonal matrix D that balance a Fiedler matrix M_σ we use the command `balance` in MATLAB.

In Table 8.3, we give the mean and the maximum of the decimal logarithms (Log-Mean and Log-Maximum, respectively) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$, and the mean and the minimum of the decimal logarithms (Log-Mean and Log-Minimum, respectively) of $\min_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$, for the Fiedler matrices $M_\sigma = M_{\sigma_2}, M_{\sigma_3}$, M_{σ_4}, with and without balancing them (and with the Frobenius matrix C_2 balanced).

(a) The Fiedler matrices are not balanced.

	M_{σ_2}	M_{σ_3}	M_{σ_4}
Log-Mean $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	6.3	3.7	6.1
Log-Maximum $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	9.4	9.4	9.4
Log-Mean $\min_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	-4.8	-2.7	-5.4
Log-Minimum $\min_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	-9.5	-8.4	-9.3

(b) The Fiedler matrices are balanced.

	M_{σ_2}	M_{σ_3}	M_{σ_4}
Log-Mean $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	0.1	0.1	0.1
Log-Maximum $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	0.9	1.2	1.3
Log-Mean $\min_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	-0.7	-0.1	-0.6
Log-Minimum $\min_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$	-2.6	-0.9	-2.3

Table 8.3 Mean and maximum of the decimal logarithms (Log-Mean and Log-Maximum, respectively) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$, and mean and minimum of the decimal logarithms (Log-Mean and Log-Minimum, respectively) of $\min_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$, where λ runs over all nonzero simple roots of $p(z)$, obtained for a sample of 1000 random degree-10 monic polynomials, with coefficients of the form $b_1 \times 10^{e_1} + \mathrm{i}\, b_2 \times 10^{e_2}$, where, for $i = 1, 2$, b_i and e_i are drawn from the uniform distributions on the intervals $[-1, 1]$ and $[-10, 10]$, respectively, for the Fiedler matrices $M_\sigma = M_{\sigma_2}, M_{\sigma_3}, M_{\sigma_4}$, without balancing and with balancing.

Several observations may be drawn from the data in Tables 8.3-(a) and 8.3-(b). First note, from the data in Table 8.3-(a), that if the Fiedler matrices are not balanced, the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ may be large or small, as it is predicted by the bounds in Theorem 6.2. Also note that the largest and smallest of these ratios are consistent with the bounds in (6.7) and (6.8), since the largest value of $\|p\|_2$ is approximately 10^{10}. Second, note, from the data in Table 8.3-(b), that the process of balancing the Fiedler matrices makes the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C_2)$ moderate, so that, from the point of view of conditioning, balanced Fiedler Matrices can be used with the same reliability as balanced Frobenius companion matrices even with polynomials with large norms.

In Table 8.4, we give the mean and the maximum of the decimal logarithms (Log-Mean and Log-Maximum, respectively) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, for the Fiedler matrices $M_{\sigma_1}, M_{\sigma_2}, M_{\sigma_3}, M_{\sigma_4}$, with and without balancing them.

(a) The Fiedler matrices are not balanced.

	M_{σ_1}	M_{σ_2}	M_{σ_3}	M_{σ_4}
Log-Mean $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$	8.6	14.6	11.8	14.4
Log-Maximum $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$	10.4	19.4	19.4	19.3

(b) The Fiedler matrices are balanced.

	M_{σ_1}	M_{σ_2}	M_{σ_3}	M_{σ_4}
Log-Mean $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$	2.7	2.4	2.8	2.5
Log-Maximum $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$	7.9	7.5	8.2	7.7

Table 8.4 Mean and maximum of the decimal logarithms (Log-Mean and Log-Maximum, respectively) of $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, where λ runs over all nonzero simple roots of $p(z)$, obtained for a sample of 1000 random degree-10 monic polynomials, with coefficients of the form $b_1 \times 10^{e_1} + ib_2 \times 10^{e_2}$, where, for $i = 1, 2$, b_i and e_i are drawn from the uniform distributions on the intervals $[-1, 1]$ and $[-10, 10]$, respectively, for the Fiedler matrices $M_\sigma = M_{\sigma_1}, M_{\sigma_2}, M_{\sigma_3}, M_{\sigma_4}$, without balancing and with balancing.

Again, several conclusions may be drawn from the data in Tables 8.4-(a) and 8.4-(b). First note, from the point of view of conditioning, that if the Fiedler matrices are not balanced, the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ may be large, as it is predicted in Theorem 6.1 when there are coefficients of $p(z)$ with large absolute values. Note also that the largest of these ratios is consistent with the upper bound in (6.4) for the Frobenius companion matrix, and with the upper bound in (6.5) for Fiedler matrices other than the Frobenius ones, since the largest value of $\|p\|_2$ is approximately 10^{10}. Second, note, comparing the data in Table 8.4-(a) with the data in Table 8.4-(b), that the process of balancing Fiedler matrices may reduce considerably the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$, i. e., it may reduce considerably $\kappa(\lambda, M_\sigma)$. In particular, from the data in Log-Mean $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ in Table 8.4-(b), we can see that, *usually*, balancing makes the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ moderate, although, from the data in Log-Maximum $\max_\lambda \kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$ in Table 8.4-(b), we can see that balancing is not always enough to guarantee a moderate ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, p)$.

8.4 Numerical experiments with polynomials having clustered roots

Clustered polynomial roots and clustered eigenvalues are ill-conditioned, that is, they have huge condition numbers. Despite the ill-conditioning of the clustered roots, the eigenvalue condition numbers of Fiedler matrices still behave as our results in Theorem 6.1 predict: if λ is a root in a cluster of a monic polynomial $p(\lambda)$ with $\rho(p) \approx 1$, then $\kappa(\lambda, M_\sigma)$ behaves approximately as $\|p\|_2\, \kappa(\lambda, p)$, if M_σ is one of the Frobenius companion matrices, or as $\|p\|_2^2\, \kappa(\lambda, p)$ if M_σ is a Fidler matrix other than the Frobenius ones. The goal of this section is to provide numerical experiments illustrating this behavior. Additionally, we study the effect of balancing Fiedler matrices with clustered eigenvalues.

In a first set of numerical experiments, we consider a random sample of one hundred degree-10 monic polynomials with large norms and with a cluster of five roots near $z = 1$. These polynomials are generated as follows. First, we generate the cluster of five roots as

$$\lambda_\ell = 1 + \texttt{eps} \cos\left(\frac{2\pi(\ell-1)}{5}\right) + \mathfrak{i}\, \texttt{eps} \sin\left(\frac{2\pi(\ell-1)}{5}\right), \quad \text{for } \ell = 1, \ldots, 5, \quad (8.6)$$

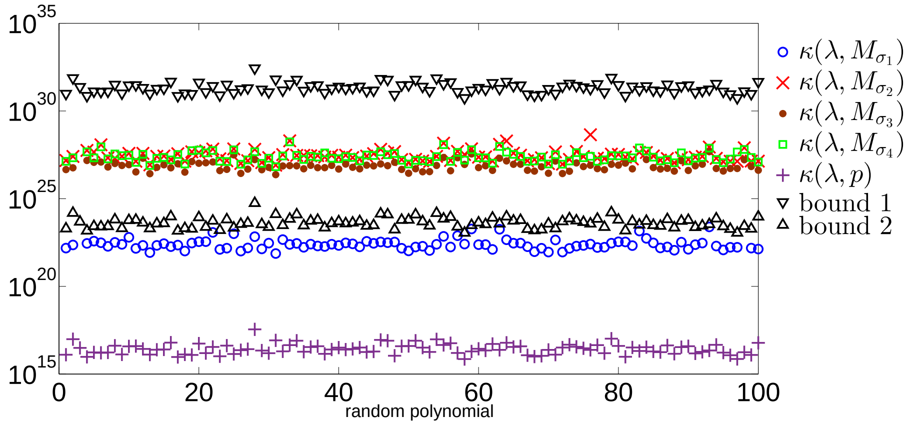
where the MATLAB function `eps` returns the unit roundoff in double precision. Then, we randomly generate the other five roots as

$$\lambda_\ell = 10 \cos\theta_\ell + \mathfrak{i}\, 10 \sin\theta_\ell, \quad \text{for } \ell = 6, \ldots, 10,$$

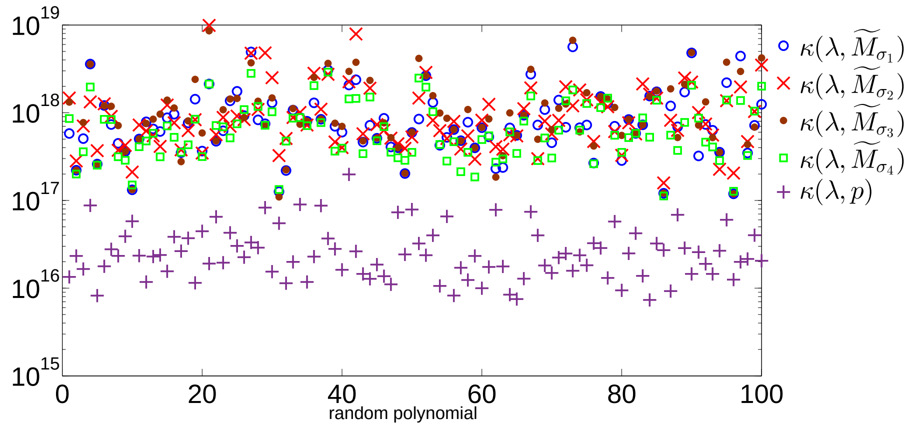
where θ_ℓ is drawn from the uniform distribution on the interval $[0, 2\pi)$. Finally, we compute the coefficients of the monic polynomial with those generated roots in quadruple precision (32 decimal digits of accuracy) using the function `vpa` followed by the command `charpoly` applied on a diagonal matrix whose diagonal entries are $\{\lambda_1, \lambda_2, \ldots, \lambda_{10}\}$. We discard the polynomial if its norm does not satisfy $10^6 \leq \|p\|_2 \leq 2 \cdot 10^6$, and generate a new one.

In Figure 8.4-(a) we plot $\kappa(\lambda_1, p)$ and $\kappa(\lambda_1, M_{\sigma_j})$, for $j = 1, 2, 3, 4$, for each of the one hundred degree-10 polynomials. Additionally, we also plot our theoretical upper bounds for $\kappa(\lambda_1, M_{\sigma_j})$ in Theorem 6.1, that is, we plot the quantities `bound 1` $= n^{5/2}\rho(p)\|p\|_2^2\, \kappa(\lambda_1, p)$ and `bound 2` $= n\rho(p)\|p\|_2\, \kappa(\lambda_1, p)$. We, then, run a similar experiment, but this time balancing the Fiedler matrices, and plot the results in Figure 8.4-(b), where $\widetilde{M}_{\sigma_j}$ denotes the balanced Fiedler matrix corresponding to M_{σ_j}.

The results in Figure 8.4-(a) show that despite the ill-conditioning of the root λ_1 in the cluster near $z = 1$, our theoretical results in Theorem 6.1 correctly explain the behavior of the eigenvalue condition numbers of λ_1 as eigenvalue of the Fiedler matrices: just notice that $\rho(p) \approx 1$ in this case and $\kappa(\lambda_1, M_{\sigma_1}) \approx \|p\|_2\, \kappa(\lambda_1, p) \approx 10^6 \kappa(\lambda_1, p)$, and that $\kappa(\lambda_1, M_{\sigma_j}) \approx \|p\|_2^2\, \kappa(\lambda_1, p) \approx 10^{12} \kappa(\lambda_1, p)$, for $j = 2, 3, 4$. Also, we want to remark that the discrepancy between the eigenvalue condition numbers $\kappa(\lambda_1, M_{\sigma_j})$ and `bound 1` $= n^{5/2}\rho(p)\|p\|_2^2\, \kappa(\lambda_1, p)$ is mainly due to the pessimistic dimensional factor $n^{5/2}$, which most of the times makes the upper bound to overestimate the eigenvalue condition numbers. The results in Figure 8.4-(b) show the drastic effect of balancing the Fiedler matrices even for roots in a cluster: notice that the eigenvalue condition numbers $\kappa(\lambda_1, \widetilde{M}_{\sigma_j})$ are just a couple of orders of magnitude larger than $\kappa(\lambda_1, p)$ (compare this with the 6 and 12 orders of magnitude displayed in Figure 8.4-(a)).



(a) The Fiedler matrices are not balanced.



(b) The Fiedler matrices are balanced.

Fig. 8.4 Condition numbers $\kappa(\lambda_1, p)$, $\kappa(\lambda_1, M_{\sigma_j})$ and $\kappa(\lambda_1, \widetilde{M}_{\sigma_j})$, where $\widetilde{M}_{\sigma_j}$ denotes the balanced Fiedler matrix M_{σ_j}, for $j = 1, 2, 3, 4$, for each of the one hundred degree-10 random polynomials with a cluster of five roots $\lambda_\ell = 1 + 2^{-52}\cos(2\pi(\ell-1)/5) + \mathrm{i}\, 2^{-52}\sin(2\pi(\ell-1)/5)$, for $\ell = 1, \ldots, 5$, near $z = 1$, and with the other five roots of the form $\lambda_\ell = 10\cos\theta_j + \mathrm{i}10\sin\theta_\ell$, where θ_ℓ is drawn from the uniform distribution on the interval $[0, 2\pi)$, for $\ell = 6, \ldots, 10$, and the upper bounds for $\kappa(\lambda_1, M_{\sigma_j})$ in Theorem 6.1, that is $\texttt{bound 1} = n^{5/2}\rho(p)\|p\|_2^2\,\kappa(\lambda_1, p)$ and $\texttt{bound 2} = n\rho(p)\|p\|_2\,\kappa(\lambda_1, p)$.

In a second set of numerical experiments, we consider a random sample of one hundred degree-10 monic polynomials with coefficients not much larger than 1. These polynomials are generated as follows. As in the previous set of random polynomials, we generate a cluster of five roots as in (8.6) and, then, we randomly generate the other five roots as

$$\lambda_\ell = \frac{1}{2}\cos\theta_\ell + \frac{\mathrm{i}}{2}\sin\theta_\ell, \quad \text{for } \ell = 6, \ldots, 10,$$

where θ_ℓ is drawn from the uniform distribution on the interval $[0, 2\pi)$. Finally, we compute the coefficients of the monic polynomial with those generated roots in quadruple precision (32 decimal digits of accuracy) using the function `vpa` followed by the command `charpoly` applied on a diagonal matrix whose diagonal entries are $\{\lambda_1, \lambda_2, \ldots, \lambda_{10}\}$. We discard the polynomial if its norm does not satisfy $1 \leq \|p\|_2 \leq 50$, and generate a new one.

In Figure 8.5 we plot $\kappa(\lambda_1, p)$ and $\kappa(\lambda_1, M_{\sigma_j})$, for $j = 1, 2, 3, 4$, for each of the one hundred degree-10 polynomials, together with our theoretical upper bounds for $\kappa(\lambda_1, M_{\sigma_j})$ in Theorem 6.1, that is, we also plot the quantities $\texttt{bound 1} = n^{5/2}\rho(p)\|p\|_2^2\,\kappa(\lambda_1, p)$ and $\texttt{bound 2} = n\rho(p)\|p\|_2\,\kappa(\lambda_1, p)$.

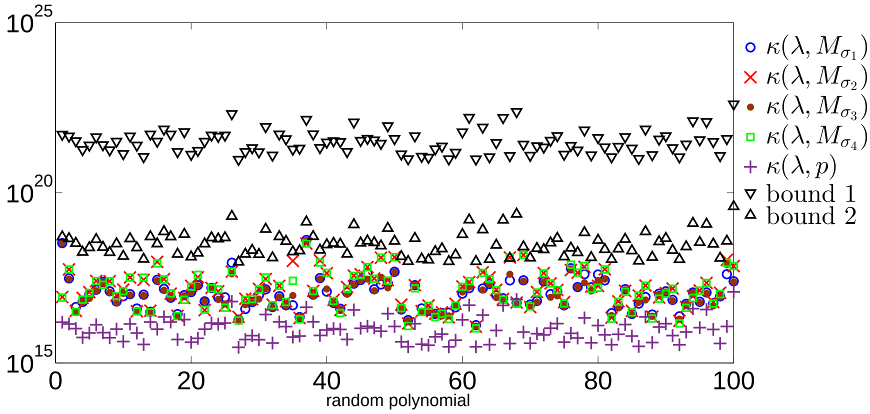


Fig. 8.5 Condition numbers $\kappa(\lambda_1, p)$ and $\kappa(\lambda_1, M_{\sigma_j})$, for $j = 1, 2, 3, 4$, for each of the one hundred degree-10 random polynomials with a cluster of five roots $\lambda_j = 1 + 2^{-52}\cos(2\pi(j-1)/5) + \mathrm{i}\,2^{-52}\sin(2\pi(j-1)/5)$, for $j = 1, \ldots, 5$, near $z = 1$, and with the other five roots of the form $\lambda_j = (\cos\theta_j + \mathrm{i}\sin\theta_j)/2$, where θ_j is drawn from the uniform distribution on the interval $[0, 2\pi)$, for $j = 6, \ldots, 10$, and the upper bounds for $\kappa(\lambda_1, M_{\sigma_j})$ in Theorem 6.1, that is, $\texttt{bound 1} = n^{5/2}\rho(p)\|p\|_2^2\,\kappa(\lambda_1, p)$ and $\texttt{bound 2} = n\rho(p)\|p\|_2\,\kappa(\lambda_1, p)$.

The results in Figure 8.5 show that despite the ill-conditioning of the root λ_1, the condition numbers ratio $\kappa(\lambda_1, M_{\sigma_j})/\kappa(\lambda_1, p)$ is moderate, for $j = 1, 2, 3, 4$, as predicted by the bounds in Theorem 6.1. We want to mention, finally, that the factor $n^{5/2}$ in `bound 1` is making, again, this upper bound to overestimate the condition numbers by 2-3 orders of magnitude.

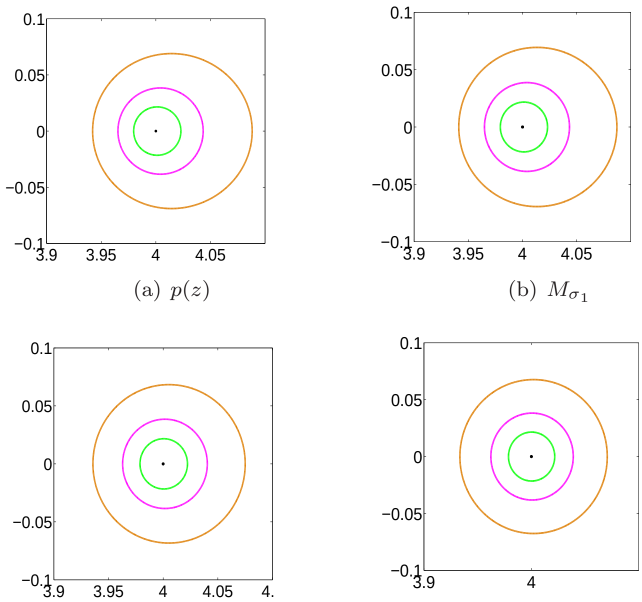
8.5 Numerical experiments for studying pseudospectra of Fiedler matrices

In this section we perform some numerical experiments to illustrate the theoretical results in Section 7 concerning pseudospectra of Fiedler matrices.

In the first numerical experiment, we illustrate Corollary 7.1, that is, in a neighborhood of a nonzero simple root λ of a monic polynomial $p(z)$, the component of the pseudospectrum $\Lambda_{\epsilon'}(M_\sigma)$ and the pseudozero set $Z_\epsilon(p)$ containing λ, where $\epsilon' = \epsilon\kappa(\lambda, p)/\kappa(\lambda, M_\sigma)$, agree with each other.

In Figure 8.6 we plot, in the neoghborhood of 4, for $\epsilon = 10^{-10.75}, 10^{-10.5}, 10^{-10.25}$, the ϵ-pseudozero sets $Z_\epsilon(p)$, and, for $i = 1, 2, 3$, the ϵ_i-pseudospectra $\Lambda_{\epsilon_i}(M_{\sigma_i})$, where $p(z)$ is the monic polynomial $p(z) = \prod_{j=1}^{10}(z - j)$, and, for

$i = 1,2,3$, $\epsilon_i = \epsilon\kappa(4,p)/\kappa(4,M_{\sigma_i})$. The ratios $\kappa(4,M_{\sigma_1})/\kappa(4,p) = 3.97\cdot 10^6$, $\kappa(4,M_{\sigma_2})/\kappa(4,p) = 1.18\cdot 10^9$ and $\kappa(4,M_{\sigma_3})/\kappa(4,p) = 5.13\cdot 10^7$ are computed using (6.1) and (6.2) in MATLAB. As can be seen in those figures, the pseudozero sets and the pseudospectra of the three Fiedler matrices are almost identical for these values of $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3$.



(a) $p(z)$ (b) M_{σ_1} (c) M_{σ_2} (d) M_{σ_3}

Fig. 8.6 For $p(z) = \prod_{j=1}^{10}(z-j)$ and for $\epsilon = 10^{-10.75}, 10^{-10.5}, 10^{-10.25}$ we plot, in green, magenta and brown, respectively, the ϵ-pseudozero set $Z_\epsilon(p)$ and, for $i = 1,2,3$, the ϵ_i-pseudospectra $\Lambda(M_{\sigma_i})$, where $\epsilon_i = \epsilon\kappa(4,p)/\kappa(4,M_{\sigma_i})$, in the neighborhood of 4.

In the second numerical experiment, we present a graphical comparison between the pseudozero sets of a monic polynomial $p(z)$ with a large norm $\|p\|_2$, and the pseudospectra of the Fiedler matrices $M_{\sigma_1}, M_{\sigma_2}, M_{\sigma_3}$ associated with $p(z)$, to show that when $\|p\|_2$ is large there may be relevant differences between pseudozero sets and pseudospectra of Fiedler matrices.

In Figures 8.7-(a), 8.7-(b), 8.7-(c), and 8.7-(d) we plot, for $\epsilon = 10^{-13}, 10^{-12}, 10^{-11}$, the ϵ-pseudozero sets $Z_\epsilon(p)$ of the monic polynomial $p(z) = \prod_{j=1}^{10}(z-j)$, and the ϵ-pseudospectra $\Lambda_\epsilon(M_{\sigma_1}), \Lambda_\epsilon(M_{\sigma_2}), \Lambda_\epsilon(M_{\sigma_3})$ of the Fiedler matrices $M_{\sigma_1}, M_{\sigma_2}, M_{\sigma_3}$ associated with $p(z)$. As may be seen in those figures, the pseudozero sets and the pseudospectra of the Fiedler matrices are very different. Notice also that there are relevant differences between the pseudospectra $\Lambda_\epsilon(M_{\sigma_1})$, $\Lambda_\epsilon(M_{\sigma_2})$ and $\Lambda_\epsilon(M_{\sigma_3})$, although, for a fixed ϵ, the three pseudospectra have, approximately, the same area.

For the interested reader, we refer to [36, Ch. 8] for further numerical experiments comparing the pseudozero sets of a polynomial with the pseudospectra of balanced Fiedler matrices.

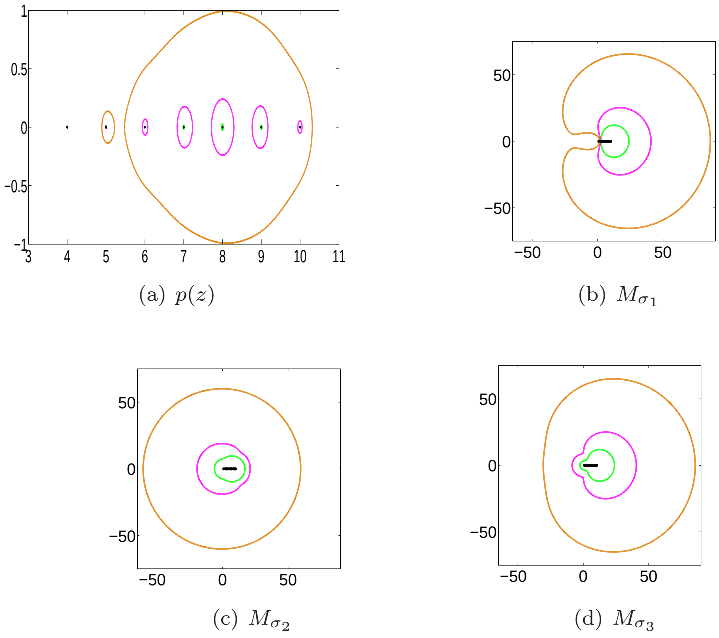


Fig. 8.7 For $p(z) = \prod_{j=1}^{10}(z-j)$ and for $\epsilon = 10^{-13}, 10^{-12}, 10^{-11}$ we plot, in green, magenta and brown, respectively, the ϵ-pseudozero set $Z_\epsilon(p)$ and, for $i = 1, 2, 3$, the ϵ-pseudospectra $\Lambda(M_{\sigma_i})$.

9 Conclusions

We have carried out a study of the eigenvalue condition numbers of Fiedler companion matrices of a monic polynomial $p(z)$. Ideally, in the polynomial root-finding problem using Fiedler companion matrices, one would like the eigenvalues of the Fiedler matrix to be as well conditioned as the roots of the original polynomial, that is,

$$\frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} = \Theta(1),$$

where $\kappa(\lambda, M_\sigma)$ and $\kappa(\lambda, p)$ denote, respectively, the condition number of λ as an eigenvalue of M_σ and the condition number of λ as a root of $p(z)$, and where $\Theta(1)$ denotes a quantity not much larger than 1 and not too close to zero. However, we have proved that

$$\frac{1}{\sqrt{2}} \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \leq n\rho(p)\|p\|_2$$

if $M_\sigma = C_1, C_2$, and

$$\frac{1}{n} \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)} \leq n^{5/2}\rho(p)\|p\|_2^2,$$

if $M_\sigma \neq C_1, C_2$, where

$$\|p\|_2 = \sqrt{1 + \sum_{k=0}^{n-1} |a_k|^2} \quad \text{and} \quad \rho(p) = \sqrt{1 + \frac{1}{\max_{0 \leq k \leq n-1} |a_k|^2}}.$$

These bounds have led us to conclude that, from the point of view of eigenvalue condition numbers, any Fiedler matrix can be used for solving the root-finding problem for $p(z)$ when the absolute values of the coefficients of $p(z)$ are moderate and not all are close to zero. By contrast, when $\max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}$ is large or close to zero, the eigenvalues of any Fiedler companion matrix may be potentially much more ill conditioned than the roots of $p(z)$, a fact that happens very often, as has been checked in many numerical experiments.

We have also studied the ratio between the eigenvalue condition numbers of Fiedler matrices other that the Frobenius ones and the eigenvalue condition number of Frobenius companion matrices, that is, the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$, where C denotes the first or the second Frobenius companion matrices. We have proved that

$$(n^2\|p\|_2)^{-1} \leq \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, C)} \leq n^{5/2}\|p\|_2,$$

which allows us to conclude that, from the point of view of eigenvalue condition numbers, when $\max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}$ is moderate, any Fiedler matrix can be used for solving the root-finding problem for $p(z)$ with the same reliability as Frobenius companion matrices. On the other hand, when $\max\{|a_0|, |a_1|, \dots, |a_{n-1}|\}$ is large, we have seen in many numerical tests that the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ may be arbitrarily large or arbitrarily small. In addition to this, we have proved that if this ratio is very large, then λ is very ill conditioned as an eigenvalue of M_σ and as an eigenvalue of the Frobenius companion matrices compared to $\kappa(\lambda, p)$. We have also proved that the opposite is not true. There exist polynomials for which the ratio $\kappa(\lambda, M_\sigma)/\kappa(\lambda, C)$ is arbitrarily small, but this only implies that λ is very ill-conditioned as an eigenvalue of the Frobenius companion matrices compared with $\kappa(\lambda, p)$. From the point of view of eigenvalue condition numbers, this allows to conclude that there are polynomials for which one should avoid computing their roots as the eigenvalues of Frobenius companion matrices and to use, instead, another Fiedler matrix. Although how to identify these polynomials and how to know which Fiedler matrix one might use instead of the Frobenius ones is an interesting open problem in this area.

We have also studied numerically the effect of balancing Fiedler companion matrices on the eigenvalue condition numbers. We have provided numerical experiments that show that the eigenvalues of Fiedler matrices that have been previously balanced and the roots of the corresponding monic polynomials are essentially equally conditioned in practice.

We have also analyzed several questions on pseudospectra of Fiedler matrices. In particular, we have seen how to estimate accurately these pseudospectra in an $m \times m$ grid using only order nm^2 flops. This is a much lower cost compared with the order $n^3 + n^2m^2$ flops needed for general matrices. We have also established an asymptotic relationship between the pseudozero sets of a monic polynomial $p(z)$ and the pseudospectra of the associated Fiedler matrices. This relationship (Corollary 7.1) depends on the radio $\frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)}$, so, again, if this ratio is large, the ϵ-pseudozero set of $p(z)$ and the ϵ-pseudospectra of Fiedler matrices may differ substantially being the latter much larger. However, after scaling with the parameter $\epsilon_\sigma = \epsilon \frac{\kappa(\lambda, M_\sigma)}{\kappa(\lambda, p)}$, the ϵ_σ-pseudospectra of M_σ are almost identical to the ϵ-pseudozero sets of $p(z)$, for sufficiently small ϵ.

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