Polynomial Zigzag Matrices, Dual Minimal Bases, and the Realization of Completely Singular Polynomials

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Abstract

Minimal bases of rational vector spaces are a well-known and important tool in systems theory. If minimal bases for two subspaces of rational n-space are displayed as the rows of polynomial matrices $Z_1(\lambda)_{k\times n}$ and $Z_2(\lambda)_{m\times n}$, respectively, then Z_1 and Z_2 are said to be *dual* minimal bases if the subspaces have complementary dimension, i.e., $k+m=n$, and $Z_1(\lambda)Z_2^T(\lambda)=0$. In other words, each $Z_j(\lambda)$ provides a minimal basis for the nullspace of the other. It has long been known that for any dual minimal bases $Z_1(\lambda)$ and $Z_2(\lambda)$, the row degree sums of Z_1 and Z_2 are the same. In this paper we show that this is the only constraint on the row degrees, thus characterizing the possible row degrees of dual minimal bases. The proof is constructive, making extensive use of a new class of sparse, structured polynomial matrices that we have baptized *zigzag* matrices. Another application of these polynomial zigzag matrices is the constructive solution of the following inverse problem for minimal indices – given a list of left and right minimal indices and a desired degree d, does there exist a completely singular matrix polynomial (i.e., a matrix polynomial with no elementary divisors whatsoever) of degree d having exactly the prescribed minimal indices? We show that such a matrix polynomial exists if and only if d divides the sum of the minimal indices. The constructed realization is simple, and explicitly displays the desired minimal indices in a fashion analogous to the classical Kronecker canonical form of singular pencils.

Key words. zigzag matrices, singular matrix polynomials, minimal indices, dual minimal bases, inverse problem

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1 Introduction

The notion of a minimal basis, formed by vectors with polynomial entries, of a rational vector subspace was made popular by the books of Wolovich [20] and Kailath [12], and by the paper of Forney [8], although all three of them cite earlier work for the basic ideas of these so-called *minimal polynomial bases*. The main contribution of these authors is twofold: they provided computational schemes for constructing a minimal basis from an arbitrary polynomial basis, and they showed the importance of this notion for multivariable linear systems. These systems could be modeled by rational matrices, polynomial matrices, or linearized state-space models, and had tremendous potential for solving analysis and design problems in control theory as well as in coding theory.

One such classical design problem was to show the relations between left and right coprime factorizations of a rational matrix $R(\lambda)$ of size $m\times k$:

$$D_\ell(\lambda)^{-1}\,N_\ell(\lambda)=R(\lambda)=N_r(\lambda)\,D_r(\lambda)^{-1},$$

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where $D_\ell(\lambda), N_\ell(\lambda), N_r(\lambda), D_r(\lambda)$ are all polynomial matrices, and $D_\ell(\lambda)$, $D_r(\lambda)$ are square and invertible. The coprimeness condition amounts to saying that the $m \times (m+k)$ and $k \times (m+k)$ matrices

$$Z_\ell(\lambda) := [D_\ell(\lambda)\,,\, -N_\ell(\lambda)]\,, \quad \text{and} \quad Z_r(\lambda) := [N_r(\lambda)^T\,,\, D_r(\lambda)^T]$$

have full row rank for all $\lambda \in \mathbb{C}$. It is easy to see that

$$D_\ell(\lambda)^{-1}\,N_\ell(\lambda) = N_r(\lambda)\,D_r(\lambda)^{-1} \qquad \text{if and only if} \qquad Z_\ell(\lambda)Z_r(\lambda)^T = 0\,,$$

which implies that the row spaces of $Z_\ell(\lambda)$ and $Z_r(\lambda)$ over the field of rational functions are "dual" to each other in the sense of Forney [8, Section 6]. In order to better understand the structure of these rational row spaces, one could then look for polynomial bases that are "minimal" in the sense that the sum of the degrees of the vectors in the basis is minimal. In the literature mentioned in the preceding paragraph, it has been shown that this minimality condition makes the ordered list of degrees of the polynomial vectors in any of these minimal bases unique, although there exist infinitely many minimal bases for any given rational subspace. This is the reason why the degrees of the vectors in any minimal basis of a rational subspace are currently known as the "minimal indices" of that subspace; in [8], however, they are called "invariant dynamical indices". In this paper, following the classical reference [8], we often arrange the vectors of a minimal basis as the rows of a full row rank polynomial matrix, and refer to the matrix itself simply as a "minimal basis", for brevity. Since we are interested in dual rational subspaces, we also use the term "dual minimal bases" to denote any minimal bases of dual rational subspaces, although this terminology is not standard in the literature.

In the work by Forney [8, p.503, Corollary to Thm. 3], it was shown that dual rational subspaces have minimal indices that add up to the same sum. In other words, if $Z_\ell(\lambda)$ and $Z_r(\lambda)$ are $m \times (m+k)$ and $k \times (m+k)$ minimal bases such that

$$Z_\ell(\lambda)Z_r(\lambda)^T = 0,$$

then their respective row degrees η_i, for $i = 1, \ldots, m$, and ε_j, for $j = 1, \ldots, k$, satisfy

$$\sum_{i=1}^{m} \eta_i = \sum_{j=1}^{k} \varepsilon_j\,. \tag{1.1}$$

A proof of this result can be found in [8, Section 6]; see also [6, Lemma 3.6] for another proof based on techniques developed in [12, Chapter 6]. In this paper we study the associated inverse problem: given two lists of nonnegative integers $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ that have the same sum (1.1), *do there exist dual rational subspaces generated by some minimal (polynomial) bases having these numbers as their row degrees*? More specifically, we consider the constructive version of this question: *can we explicitly construct dual minimal bases having any lists of prescribed row degrees satisfying* (1.1)?

In order to answer this question, we introduce in Section 3 some minimal bases of a very special sparse form. When arranged as the rows of a matrix, we call these special minimal bases *zigzag polynomial matrices* because of their echelon-like form with alternating right and left turns, like a cab driving through Manhattan. This form will be crucial in showing that given any zigzag matrix Z_1, it is very easy to construct another zigzag matrix Z_2 such that Z_1 and Z_2 are dual minimal bases; this is proven in Section 4. In Section 5 we then solve, in a simple and explicit constructive way, the inverse row degree problem for dual zigzag matrices (Theorem 5.1); this result says that one can always construct a pair of dual zigzag minimal bases with any two prescribed lists of positive row degrees that satisfy (1.1) as long as $\sum_{i=1}^{\alpha} \eta_i \neq \sum_{j=1}^{\beta} \varepsilon_j$ whenever $(\alpha, \beta) \neq (m, k)$. Based on this inverse result for zigzag matrices, the inverse problem for general dual minimal bases is explicitly solved in Section 6, taking as necessary and sufficient condition only (1.1); more specifically, see Theorems 6.1 and 6.4 for this solution, which are the most important results in this paper. Finally, in Section 7 we show how zigzag matrices can be used to provide simple, explicit constructions of polynomial matrices with any prescribed degree d, any prescribed lists of left and right minimal indices, and no elementary divisors at all (neither finite nor infinite), subject to the single necessary and sufficient condition that d divides the sum of all the prescribed minimal indices. The results in Section 7 complement results recently presented in [6], where a much more general inverse problem for matrix polynomials has been solved, but via a rather complicated construction which does not explicitly display the realized complete eigenstructure. We begin with a preliminary Section 2, where we remind the reader of a number of basic results that are needed throughout this work.

Before proceeding, we emphasize that this paper is a new contribution to the active research area of inverse problems for polynomial matrices with fixed degree, a topic that has been considered in the literature since the 1970's, and has attracted considerable attention in recent years. See, for instance, [1, 10, 11, 13, 15, 16, 18] and the references therein.

2 Preliminaries

The results in this paper hold for an arbitrary field $\mathbb{F}$. The *algebraic closure* of $\mathbb{F}$ is denoted by $\overline{\mathbb{F}}$. By $\mathbb{F}[\lambda]$ we denote the ring of polynomials in the variable λ with coefficients in $\mathbb{F}$, and $\mathbb{F}(\lambda)$ denotes the field of fractions of $\mathbb{F}[\lambda]$, also known as the field of rational functions over $\mathbb{F}$. Vectors with entries in $\mathbb{F}[\lambda]$ will be termed *vector polynomials*, and the *degree* of a vector polynomial is the highest degree of all its entries. The set of $m \times n$ polynomial matrices with entries in $\mathbb{F}[\lambda]$ is denoted by $\mathbb{F}[\lambda]^{m\times n}$, and the set of $m \times n$ rational matrices is denoted by $\mathbb{F}(\lambda)^{m\times n}$. By I_n we denote the $n \times n$ identity matrix, by $0_{m\times n}$ the $m \times n$ null matrix, and square matrices of the form

$$\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

are referred to as reverse identity matrices. We use the terms "polynomial matrix" and "matrix polynomial" with exactly the same meaning.

For a polynomial matrix $P(\lambda) = \sum_{i=0}^{d} P_i\lambda^i$, where $P_i \in \mathbb{F}^{m\times n}$ and $P_d \neq 0$, we say that the degree of $P(\lambda)$ is d, denoted by $\deg(P) = d$. The rank of $P(\lambda)$ is defined in [9]; it is just the rank of $P(\lambda)$ considered as a matrix over the field $\mathbb{F}(\lambda)$, and is denoted by $\operatorname{rank}(P)$. Note also that $\operatorname{rank}(P)$ is equal to the number of invariant polynomials of $P(\lambda)$, which are also defined in [9]. The finite eigenvalues of $P(\lambda)$ are the roots of its invariant polynomials, and associated to each such eigenvalue are elementary divisors of $P(\lambda)$; see for instance [9] or [3, Section 2] for more details on this and other standard concepts used in this section. Polynomial matrices may also have infinity as an eigenvalue. Its definition is based on the so-called *reversal* polynomial matrix. The *reversal* polynomial matrix $\operatorname{rev}P(\lambda)$ of $P(\lambda)$ is

$$\operatorname{rev}P(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right) = P_d + P_{d-1}\,\lambda + \cdots + P_0\,\lambda^d\,. \tag{2.1}$$

We emphasize that in this paper the reversal is always taken with respect to the degree of the original polynomial. Note that other options are considered in [3, Definition 2.12]. We say that ∞ is an eigenvalue of $P(\lambda)$ if 0 is an eigenvalue of $\operatorname{rev}P(\lambda)$, and the elementary divisors for the eigenvalue 0 of $\operatorname{rev}P(\lambda)$ are the elementary divisors for ∞ of $P(\lambda)$. It is well known that $P(\lambda)$ is a polynomial matrix having no eigenvalue at ∞ if and only if its highest degree coefficient matrix P_d has rank equal to $\operatorname{rank}(P)$ [3, Remark 2.14].

This paper deals mainly with minimal bases and minimal indices of polynomial matrices. Therefore we introduce these concepts in some detail. An $m \times n$ polynomial matrix $P(\lambda)$ whose rank r is smaller than m and/or n has non-trivial left and/or right null-spaces, respectively, over the field $\mathbb{F}(\lambda)$:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T\right\}, \\ \mathcal{N}_r(P) &:= \left\{x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda)x(\lambda) \equiv 0\right\}. \end{aligned}$$

Polynomial matrices with non-trivial left and/or right null-spaces are called *singular* polynomial matrices.

It is well known that every rational vector subspace $\mathcal{V}$, i.e., every subspace $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$, has bases consisting entirely of vector polynomials. Among them some are minimal in the following sense [8].

Definition 2.1. *Let $\mathcal{V}$ be a subspace of $\mathbb{F}(\lambda)^n$. A* minimal basis *of $\mathcal{V}$ is a basis of $\mathcal{V}$ consisting of vector polynomials whose sum of degrees is minimal among all bases of $\mathcal{V}$ consisting of vector polynomials.*

It can be shown [8, 12, 14] that the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same. These degrees are then called the minimal indices of $\mathcal{V}$. This leads to the definition of the minimal indices of a polynomial matrix.

Definition 2.2. *Let $P(\lambda)$ be an $m \times n$ singular polynomial matrix with rank r over a field $\mathbb{F}$, and let the sets $\left\{y_1(\lambda)^T, \ldots, y_{m-r}(\lambda)^T\right\}$ and $\{x_1(\lambda), \ldots, x_{n-r}(\lambda)\}$ be minimal bases of $\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P)$, respectively, ordered so that $0 \le \deg(y_1) \le \cdots \le \deg(y_{m-r})$ and $0 \le \deg(x_1) \le \cdots \le \deg(x_{n-r})$. Let $\eta_i = \deg(y_i)$ for $i = 1, \ldots, m-r$ and $\varepsilon_j = \deg(x_j)$ for $j = 1, \ldots, n-r$. Then the scalars $\eta_1 \le \eta_2 \le \cdots \le \eta_{m-r}$ and $\varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_{n-r}$ are, respectively, the* left *and* right minimal indices *of $P(\lambda)$.*

In order to give a practical characterization of minimal bases, we introduce Definition 2.3. In the following, when referring to the column (resp., row) degrees $d_1, \ldots, d_n$ (resp., $d'_1, \ldots, d'_m$) of an $m \times n$ polynomial matrix $P(\lambda)$, we mean that d_j (resp., d'_j) is the degree of the jth column (resp., row) of $P(\lambda)$.

Definition 2.3. *Let $N(\lambda)$ be an $m \times n$ polynomial matrix with column degrees $d_1, \ldots, d_n$. The highest-column-degree coefficient matrix of $N(\lambda)$, denoted by N_{hc}, is the $m \times n$ constant matrix whose jth column is* *the vector coefficient of λ^{d_j} in the jth column of $N(\lambda)$. Then $N(\lambda)$ is said to be* column reduced *if N_{hc} has full column rank.*

Similarly, let $M(\lambda)$ be an $m \times n$ polynomial matrix with row degrees $d'_1, \ldots, d'_m$. The highest-row-degree coefficient matrix of $M(\lambda)$, denoted by M_{hr}, is the $m \times n$ constant matrix whose jth row is the vector coefficient of $\lambda^{d'_j}$ in the jth row of $M(\lambda)$. Then $M(\lambda)$ is said to be row reduced *if M_{hr} has full row rank.*

Theorem 2.4 now provides a characterization of those polynomial matrices whose columns or rows are minimal bases of the subspaces they span. Theorem 2.4 is a minor variation of [8, Main Theorem (2), p. 495] or [12, Theorem 6.5-10]; this minor variation was previously stated in [6, Theorem 2.14].

Theorem 2.4. *The columns (resp., rows) of a polynomial matrix $N(\lambda)$ over a field $\mathbb{F}$ are a minimal basis of the subspace they span if and only if $N(\lambda_0)$ has full column (resp., row) rank for all $\lambda_0 \in \overline{\mathbb{F}}$, and $N(\lambda)$ is column (resp., row) reduced.*

Remark 2.5. For the sake of brevity, we often refer to a $p \times q$ polynomial matrix $N(\lambda)$ itself as a *minimal basis*, if the columns (when $q < p$) or rows (when $p < q$) of $N(\lambda)$ are a minimal basis of the subspace they span. In addition, if $N(\lambda)$ is a minimal basis of $\mathcal{N}_r(P)$ (resp., $\mathcal{N}_\ell(P)$) for a given polynomial matrix $P(\lambda)$, then we refer to the matrix $N(\lambda)$ itself as a right (resp., left) minimal basis of $P(\lambda)$.

Theorem 2.4 allows us to easily prove two simple results that will be used in the next sections. They can also be proved from the results in [14, Section 6] via a completely different approach.

Lemma 2.6. *Let $M(\lambda)$ be a full row rank $m \times n$ polynomial matrix, and let*

$$\widetilde{M}(\lambda) := \begin{bmatrix} I_p & 0 \\ 0 & M(\lambda) \end{bmatrix} \quad \text{and} \quad \widehat{M}(\lambda) := \begin{bmatrix} M(\lambda) & 0 \\ 0 & I_p \end{bmatrix}.$$

Then $\widetilde{M}(\lambda)$ and $\widehat{M}(\lambda)$ both have full row rank, and both have right minimal indices equal to the right minimal indices of $M(\lambda)$.

Proof. First, we prove the result for $\widetilde{M}(\lambda)$. It is trivial to see that $\widetilde{M}(\lambda)$ has full row rank, and that $\dim \mathcal{N}_r(\widetilde{M}) = \dim \mathcal{N}_r(M) =: t$. Let the $n \times t$ polynomial matrix $N(\lambda)$ be a right minimal basis of $M(\lambda)$. Then $\widetilde{N}(\lambda) := \begin{bmatrix} 0_{p \times t} \\ N(\lambda) \end{bmatrix}$ is also a minimal basis by Theorem 2.4, and $\widetilde{M}(\lambda)\widetilde{N}(\lambda) = 0$. Since $\dim \mathcal{N}_r(\widetilde{M}) = t$, we conclude that $\widetilde{N}(\lambda)$ is in fact a right minimal basis of $\widetilde{M}(\lambda)$, and the result for the right minimal indices follows inmediately.

Since $\widehat{M}(\lambda)$ is obtained from $\widetilde{M}(\lambda)$ via column and row permutations, and such permutations change neither the rank nor the minimal indices, the result for $\widehat{M}(\lambda)$ follows from the one for $\widetilde{M}(\lambda)$. ☐

Remark 2.7. Since the left/right minimal indices of any polynomial matrix $P(\lambda)$ are equal to the right/left minimal indices of $P(\lambda)^T$, it follows immediately that an analogous version of Lemma 2.6 for the left minimal indices of full column rank polynomial matrices $M(\lambda)$ also holds.

The next lemma considers elementary divisors and minimal indices of direct sums of matrix polynomials.

Lemma 2.8. *Let $P_1(\lambda), \ldots, P_s(\lambda)$ be polynomial matrices with arbitrary sizes but all with the same degree, and let*

$$P(\lambda) := \begin{bmatrix} 0_{p_0 \times q_0} & & & \\ & P_1(\lambda) & & \\ & & \ddots & \\ & & & P_s(\lambda) \end{bmatrix}.$$

Then:

(a) *The list of elementary divisors of $P(\lambda)$ associated to its finite and infinite eigenvalues is the concatenation of the lists of elementary divisors of $P_i(\lambda)$ associated to its finite and infinite eigenvalues for $i = 1, \ldots, s$.*

(b) *The list of right minimal indices of $P(\lambda)$ is the concatenation of q_0 right minimal indices equal to 0 together with the lists of right minimal indices of $P_i(\lambda)$ for $i = 1, \ldots, s$.*

(c) *The list of left minimal indices of $P(\lambda)$ is the concatenation of p_0 left minimal indices equal to 0 together with the lists of left minimal indices of $P_i(\lambda)$ for $i = 1, \ldots, s$.*

As usual, in the case $p_0 \neq 0$ and $q_0 = 0$, $0_{p_0\times q_0}$ means that the first p_0 rows of $P(\lambda)$ are zero and no additional zero columns are placed in the first positions; in the case $p_0 = 0$ and $q_0 \neq 0$, $0_{p_0\times q_0}$ means that the first q_0 columns of $P(\lambda)$ are zero and no additional zero rows are placed in the first positions; in the case $p_0 = q_0 = 0$, $0_{p_0\times q_0}$ is just the empty matrix and $P_1(\lambda)$ is the top diagonal block of $P(\lambda)$.

Proof. (a) The result for the elementary divisors associated to finite eigenvalues follows from [9, Theorem 5, p. 142, Vol I], together with the fact that the zero polynomial matrix has no elementary divisors at all. For the elementary divisors associated to the infinite eigenvalue, observe that

$$\mathrm{rev}P(\lambda) = \begin{bmatrix} 0_{p_0\times q_0} & & & \\ & \mathrm{rev}P_1(\lambda) & & \\ & & \ddots & \\ & & & \mathrm{rev}P_s(\lambda) \end{bmatrix},$$

since all the polynomials $P_i(\lambda)$ have the same degree, and apply again [9, Theorem 5, p. 142, Vol I] to $\mathrm{rev}P(\lambda)$.
(b) Let $N_i(\lambda) \in \mathbb{F}[\lambda]^{q_i\times n_i}$ be a right minimal basis of $P_i(\lambda) \in \mathbb{F}[\lambda]^{p_i\times q_i}$ for $i = 1,\dots,s$, and consider the direct sum $N(\lambda) = I_{q_0} \oplus N_1(\lambda) \oplus \cdots \oplus N_s(\lambda)$. If some $\mathcal{N}_r(P_j) = \{0\}$, then we take $N_j(\lambda)$ to be a $q_j \times 0$ matrix, so that its effect on the direct sum $N(\lambda)$ is to add q_j zero rows and no columns. Next, observe that:

(1) the number of columns of $N(\lambda)$ is equal to $\dim \mathcal{N}_r(P) = q_0 + \dim \mathcal{N}_r(P_1) + \cdots + \dim \mathcal{N}_r(P_s)$,

(2) $P(\lambda)N(\lambda) = 0$, and

(3) $N(\lambda_0)$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and $N(\lambda)$ is column reduced, since for $i = 1,\dots,s$ the matrices $N_i(\lambda)$ satisfy these properties by Theorem 2.4, or are $q_i \times 0$ matrices.

Combining (1), (2), (3) and Theorem 2.4, we see that $N(\lambda)$ is a right minimal basis of $P(\lambda)$, and the result for the right minimal indices follows.
(c) It follows from applying (b) to $P(\lambda)^T$. □

Remark 2.9. Although it does not have any impact on the results presented in this paper, it is worth mentioning that the result stated in Lemma 2.8 for the elementary divisors associated to the infinite eigenvalue is no longer true if the polynomials $P_1(\lambda),\dots,P_s(\lambda)$ have different degrees. Let us illustrate this fact with a 2×2 block polynomial matrix. Assume that $P_1(\lambda)$ and $P_2(\lambda)$ have degrees 3 and 2, respectively. Then $P(\lambda) = P_1(\lambda) \oplus P_2(\lambda)$ has degree 3 and $\mathrm{rev}P(\lambda) = \mathrm{rev}P_1(\lambda) \oplus \lambda\, \mathrm{rev}P_2(\lambda)$, because the reversal of $P_2(\lambda)$ is taken with respect to $\deg P_2(\lambda) = 2$. Thus the elementary divisors of $P(\lambda)$ at infinity are those of $P_1(\lambda)$ at infinity concatenated with the elementary divisors at zero of $\lambda\, \mathrm{rev}P_2(\lambda)$. One option to avoid this complication is to define all the reversals with respect to a previously given *grade* larger than or equal to the maximum degree of all $P_1(\lambda),\dots,P_s(\lambda)$ in Lemma 2.8, although this strategy changes the degree of the elementary divisors at infinity of each $P_i(\lambda)$ with degree smaller than the grade by a uniform shift [3, 15].

The next concept introduced in this section is what we call *dual minimal bases*, and is one of the most important notions in this work. As far as we know, this exact name has not been used before in the literature, but it allows us to state and refer to certain fundamental results on rational vector subspaces included in [8, 12] in a concise way.

Definition 2.10. *Polynomial matrices $M(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ with full row ranks are said to be* dual minimal bases *if they are minimal bases satisfying $m + k = n$ and $M(\lambda)\,N(\lambda)^T = 0$.*

In the language of [8, Section 6], dual minimal bases span rational vector subspaces of $\mathbb{F}(\lambda)^n$ that are dual to each other. In the language we are using in this paper, we have that $M(\lambda)$ is a minimal basis of $\mathcal{N}_\ell(N(\lambda)^T)$ and that $N(\lambda)^T$ is a minimal basis of $\mathcal{N}_r(M(\lambda))$. Therefore the right minimal indices of $M(\lambda)$ are the row degrees of $N(\lambda)$ and the left minimal indices of $N(\lambda)^T$ are the row degrees of $M(\lambda)$. Note that we have defined dual minimal bases to have full row ranks because in the classical reference [8] minimal bases are always arranged as the rows of a matrix. Obviously, one could also use full column rank matrices to define dual minimal bases.

The discussion in the previous paragraph also allows us to establish the next fundamental (albeit easy to prove) result in the context of this paper.

Proposition 2.11. *For every minimal basis, there exists a minimal basis that is dual to it. In addition, every minimal basis is the minimal basis of some matrix polynomial.*

Proof. Let the rows of the polynomial matrix $M(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ with $m < n$ be a minimal basis of the rational subspace they span. The subspace $\mathcal{N}_r(M(\lambda))$ over the field $\mathbb{F}(\lambda)$ exists and has minimal bases. Then, the vectors of any of these minimal bases arranged as the rows of a matrix $N(\lambda)$ form a dual minimal basis for $M(\lambda)$. The relation $M(\lambda)N(\lambda)^T = 0$, together with the sizes of the matrices imposed by the duality, proves that $M(\lambda)$ is a left minimal basis of the matrix polynomial $N(\lambda)^T$. □

The next theorem is a key result on dual minimal bases. It was proved in [8, p. 503] by using minors of matrices. A different proof can be found in [6, Lemma 3.6], and a new proof is outlined in Remark 2.14.

Theorem 2.12. *Let $M(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ be dual minimal bases with row degrees $(\eta_1, \dots, \eta_m)$ and $(\varepsilon_1, \dots, \varepsilon_k)$, respectively. Then*

$$\sum_{i=1}^{m} \eta_i = \sum_{j=1}^{k} \varepsilon_j \,. \tag{2.2}$$

As explained in the introduction, the main result in this paper is to solve the inverse problem posed by Theorem 2.12, that is, to show that given any two lists of nonnegative integers $(\eta_1, \dots, \eta_m)$ and $(\varepsilon_1, \dots, \varepsilon_k)$ satisfying (2.2), there exists a pair of dual minimal bases $M(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+k)}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{k\times(m+k)}$ with precisely these row degrees, respectively. This is proved in Theorem 6.1 by using the properties of a new class of polynomial matrices, the zigzag polynomial matrices introduced in Section 3, which allow us to present a simple, explicit construction of these dual minimal bases.

We finish this section by recalling the Index Sum Theorem for polynomial matrices, and connecting this result with Theorem 2.12. The Index Sum Theorem is an important result presented first for real polynomials in [17], and extended to polynomials over any field in [3]. Recently, it has been shown in [6, Remark 3.2] that the Index Sum Theorem is an easy corollary of a more general result valid for arbitrary rational matrices proved in [19, Theorem 3], much earlier than in the previous references.

Theorem 2.13. (Index Sum Theorem). *Let $P(\lambda)$ be a polynomial matrix of degree d and rank r over an arbitrary field $\mathbb{F}$, having:*

- *elementary divisors associated to its finite eigenvalues with degrees $\alpha_1, \dots, \alpha_s$,*
- *elementary divisors associated to its infinite eigenvalue with degrees $\gamma_1, \dots, \gamma_t$,*
- *p right minimal indices $\varepsilon_1, \dots, \varepsilon_p$, and*
- *q left minimal indices $\eta_1, \dots, \eta_q$.*

Then

$$\sum_{j=1}^{s} \alpha_j + \sum_{j=1}^{t} \gamma_j + \sum_{j=1}^{p} \varepsilon_j + \sum_{j=1}^{q} \eta_j = dr. \tag{2.3}$$

Remark 2.14. The fundamental property of dual minimal bases expressed by Theorem 2.12 can be obtained as a corollary of the Index Sum Theorem applied to a minimal basis. To see this, consider any minimal basis $M(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$ with full row rank r. Let $w_i(\lambda)$ be the rows of $M(\lambda)$ with row degrees $\eta_i = \deg(w_i)$ for $i = 1, \dots, r$; without loss of generality we can assume that the rows are ordered so that $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_r$, hence $d := \deg M(\lambda) = \eta_1$. Let $\varepsilon_1, \dots, \varepsilon_k$ with $k + r = n$ be the right minimal indices of $M(\lambda)$; recall from the discussion following Definition 2.10 that $\varepsilon_1, \dots, \varepsilon_k$ are also the row degrees for any minimal basis that is dual to $M(\lambda)$. Clearly there are no left minimal indices for $M(\lambda)$ because of its full row rank. Now to apply the Index Sum Theorem to $M(\lambda)$, we must also find the degrees of all the elementary divisors of $M(\lambda)$. Since $M(\lambda)$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$ by Theorem 2.4, the Smith form of $M(\lambda)$ must be $\begin{bmatrix} I_r & 0_{r\times k} \end{bmatrix}$, i.e., $M(\lambda)$ has no finite eigenvalues at all. To find the elementary divisors at ∞, consider

$$\operatorname{rev} M(\lambda) = \begin{bmatrix} \lambda^{d-\eta_1} \operatorname{rev} w_1(\lambda) \\ \vdots \\ \lambda^{d-\eta_r} \operatorname{rev} w_r(\lambda) \end{bmatrix} = D(\lambda)R(\lambda)\,, \tag{2.4}$$

where $D(\lambda)_{r\times r} = \operatorname{diag}[\,\lambda^{d-\eta_1}, \dots, \lambda^{d-\eta_r}\,]$, and the jth row of $R(\lambda)_{r\times n}$ is $\operatorname{rev} w_j(\lambda)$. Now whenever a set $\{w_1(\lambda), \dots, w_r(\lambda)\}$ forms a minimal basis, it is known (see [2, Thm 3.2] or [15, Thm 7.5]) that $\{\operatorname{rev} w_1(\lambda), \dots, \operatorname{rev} w_r(\lambda)\}$ is also a minimal basis. And since $R(\lambda)$ is a minimal basis, it can be extended to an $n \times n$ unimodular matrix $\widetilde{R}(\lambda) = \begin{bmatrix} R(\lambda)_{r\times n} \\ W(\lambda)_{k\times n} \end{bmatrix}$, see [8, Thm 4]. Letting $S(\lambda) := \begin{bmatrix} D(\lambda) & 0_{r\times k} \end{bmatrix}$, we see that

$$S(\lambda)\widetilde{R}(\lambda) \;=\; \begin{bmatrix} D(\lambda) & 0 \end{bmatrix} \begin{bmatrix} R(\lambda) \\ W(\lambda) \end{bmatrix} \;=\; \operatorname{rev} M(\lambda)\,,$$

witnessing that $S(\lambda)$ is the Smith form of $\operatorname{rev} M(\lambda)$, and thus revealing the elementary divisors at ∞ for $M(\lambda)$. The Index Sum Theorem applied to $M(\lambda)$ then says

$$\sum_{i=1}^{r}(d-\eta_i) \,+\, \sum_{j=1}^{k}\varepsilon_j \;=\; d\,r\,,$$

from which (2.2) now immediately follows.

3 Zigzag Matrices: Definitions and Examples

We begin by defining the special class of polynomial matrices under consideration in this paper.

Definition 3.1 (Forward-zigzag polynomial matrices)**.** *An $m \times n$ polynomial matrix $Z(\lambda)$ with $m < n$ is said to be a* forward-zigzag polynomial matrix*, abbreviated to "forward-zigzag matrix", if*

(a) *each row of $Z(\lambda)$ is of the form*

$$\Big[\, \underbrace{0 \;\dots\; 0}_{Maybe\ none} \quad 1 \quad \lambda^{p_1} \quad \lambda^{p_2} \quad \dots \quad \lambda^{p_k} \quad \underbrace{0 \;\dots\; 0}_{Maybe\ none} \,\Big]\,, \tag{3.1}$$

with at least two *nonzero entries in each row: a leading* 1 *and at least one nontrivial power of λ. The nonzero entries in each row lie in consecutive adjacent columns, with the powers p_i in* strictly increasing order *going from* left to right*, i.e., $0 < p_1 < p_2 < \cdots < p_k$, with $k \geq 1$.*

(b) *$Z(\lambda)$ is in a special* double-echelon form*:*

> *For $i = 2, \dots, m$, the last nonzero entry of the $(i-1)$th row and the first nonzero entry of the ith row are in the* same *column.*

(c) *$Z(\lambda)$ has* no *zero columns.*

Remark 3.2. It is worth mentioning some concepts in the literature that are reminiscent of certain aspects of Definition 3.1. The notion of a matrix having a *zig-zag shape* in [7] is clearly related but not identical to the zero structure patterns of zigzag matrices in Definition 3.1. Even more closely related (but still not identical) is the zero structure of the *staircase matrices* in [5].

The reader should keep in mind from the outset the following fundamental property of forward-zigzag matrices.

Theorem 3.3. *The rows of any forward-zigzag matrix are a minimal basis of the rational subspace they span or, equivalently, any forward-zigzag matrix is a minimal basis.*

Proof. The double-echelon form of any forward-zigzag matrix $Z(\lambda)$ implies:
(a) $Z(\lambda_0)$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$ because of the position of the leading 1 in each row of $Z(\lambda)$, and
(b) $Z(\lambda)$ is row reduced because each row of $Z(\lambda)$ has a unique highest degree entry (the trailing one), and these highest degree entries are in distinct columns. Therefore, Theorem 2.4 guarantees that $Z(\lambda)$ is a minimal basis. □

In addition to being minimal bases, forward-zigzag matrices have a rich structure that is most easily described in terms of the definitions presented below. In particular, any column of a forward-zigzag matrix is of one of the following two types.

Definition 3.4 (Unit and non-unit columns)**.** *Any column of an $m \times n$ forward-zigzag matrix that contains the entry "1" is called a* unit column *of $Z(\lambda)$. Any column containing no entry "1" is called a* non-unit column *of $Z(\lambda)$. Using "U" to indicate a unit column and "N" for a non-unit column, we specify the location of the unit and non-unit columns of $Z(\lambda)$ by an n-symbol string*

$$\mathrm{S}_1\,,\, \mathrm{S}_2\,,\, \dots\,,\, \mathrm{S}_n$$

of U*'s and* N*'s, i.e., $\mathrm{S}_i \in \{\mathrm{U}, \mathrm{N}\}$ for $i = 1, \dots, n$. This string is the* unit column sequence *of $Z(\lambda)$.*
(*Observe that in a forward-zigzag matrix a unit column is the same as the usual notion of* pivot column.)

The following remarks on forward-zigzag matrices will be used later.

Remark 3.5. Note that the conditions in Definition 3.1 imply that the first column of a forward-zigzag matrix will always be a unit column, and the last column will never be a unit column, i.e., $\mathrm{S}_1 = \mathrm{U}$ and $\mathrm{S}_n = \mathrm{N}$.

Remark 3.6. The number of rows of a forward-zigzag matrix $Z(\lambda)$ is the same as the number of unit columns of $Z(\lambda)$, i.e., the number of U's in the unit column sequence of $Z(\lambda)$.

Example 3.7. Here is a simple example of a forward-zigzag matrix:

$$Z(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & \\ & & & 1 & \lambda^3 & & & & & & \\ & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & \\ & & & & & & & & 1 & \lambda^2 & \lambda^3 \end{bmatrix}. \tag{3.2}$$

The unit column sequence of $Z(\lambda)$ in (3.2) is U,N,N,U,U,N,N,N,U,N,N.

Example 3.8. Note that the canonical singular blocks for right (column) minimal indices that appear in the Kronecker canonical form for matrix pencils [9, Ch XII, Sec. 5], that is

$$\begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}_{\epsilon \times (\epsilon+1)},$$

constitute a familiar but very special type of forward-zigzag matrix, with only one non-unit entry in each row and only one non-unit column (the last column).

Remark 3.9. Observe that any column of a forward-zigzag matrix contains either *one or two* nonzero entries; one if it is a non-unit column or the first column, two if it is any unit column other than the first. There are no other possibilities. Also note that if two *adjacent* columns are non-unit, then their corresponding nonzero entries must lie in the *same row*, since when moving from left to right along nonzero entries in a forward-zigzag matrix, one can only switch rows at a unit column.

The following notion encodes relevant information on forward-zigzag matrices.

Definition 3.10 (Degree-gap sequence of a forward-zigzag matrix)**.** *For any two* adjacent *columns C_j and C_{j+1} in a forward-zigzag matrix $Z(\lambda)$, there is a unique row $\mathcal{R}_i$ having two nonzero entries in those columns. The positive degree difference of these two entries, i.e., $\delta_j := \deg Z_{i,j+1}(\lambda) - \deg Z_{i,j}(\lambda) \geq 1$, will be called the jth* degree gap *of $Z(\lambda)$. Note that if $Z(\lambda)$ is $m \times n$, then $Z(\lambda)$ has $n-1$ degree gaps. The ordered list*

$$\delta_1\,,\, \delta_2\,,\, \ldots\,,\, \delta_{n-1}$$

is the degree-gap sequence *of $Z(\lambda)$.*

Example 3.11. The degree-gap sequence of $Z(\lambda)$ in (3.2) is $2, 5, 1, 3, 1, 3, 4, 7, 2, 1$.

Definition 3.12 (Structure sequence of a forward-zigzag matrix)**.** *Let $Z(\lambda)$ be an $m \times n$ forward-zigzag matrix. Then the sequence of length $2n-1$ obtained by interleaving the unit column sequence and the degree-gap sequence of $Z(\lambda)$, i.e.,*

$$\mathcal{S} = \left[\ \mathrm{S}_1 \quad \delta_1 \quad \mathrm{S}_2 \quad \delta_2 \quad \ldots \quad \mathrm{S}_{n-1} \quad \delta_{n-1} \quad \mathrm{S}_n\ \right]. \tag{3.3}$$

is the structure sequence *of $Z(\lambda)$.*

Example 3.13. The structure sequence of $Z(\lambda)$ in (3.2) is

$$\left[\ \mathrm{U}\ \ 2\ \ \mathrm{N}\ \ 5\ \ \mathrm{N}\ \ 1\ \ \mathrm{U}\ \ 3\ \ \mathrm{U}\ \ 1\ \ \mathrm{N}\ \ 3\ \ \mathrm{N}\ \ 4\ \ \mathrm{N}\ \ 7\ \ \mathrm{U}\ \ 2\ \ \mathrm{N}\ \ 1\ \ \mathrm{N}\ \right]. \tag{3.4}$$

It is not hard to see that a forward-zigzag matrix is uniquely determined by its structure sequence, since it allows us to construct the matrix in a unique way starting from the 1 in the $(1,1)$ entry. We will consider this construction in more detail at the beginning of Section 4.1.

We now define another related type of zigzag matrix, using the reverse identity matrices R_k, defined by

$$R_k := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}_{k \times k}. \tag{3.5}$$

Note that the effect of multiplying an $m \times n$ matrix A by R_m on the left (by R_n on the right) is simply to reverse the order of the rows (columns) of A.

Definition 3.14 (Backward-zigzag polynomial matrices)**.** *An $m \times n$ polynomial matrix $\widehat{Z}(\lambda)$ with $m < n$ is said to be a* backward-zigzag polynomial matrix*, abbreviated to just "backward-zigzag matrix", if $R_m \widehat{Z}(\lambda) R_n$ is a forward-zigzag matrix.*

Clearly this relation between forward and backward-zigzag matrices is symmetric, since for any $m \times n$ forward-zigzag matrix $Z(\lambda)$, the matrix $R_m Z(\lambda) R_n$ is backward-zigzag.

From Definition 3.14 it is now easy to see that the structure of backward-zigzag matrices is very similar, but not identical, to that of forward-zigzag matrices. The main difference is that each row of a backward-zigzag matrix $\widehat{Z}(\lambda)$ is of the form

$$\begin{bmatrix} \underbrace{0 \ \dots \ 0}_{\text{Maybe none}} & \lambda^{q_\ell} & \dots & \lambda^{q_2} & \lambda^{q_1} & 1 & \underbrace{0 \ \dots \ 0}_{\text{Maybe none}} \end{bmatrix} \tag{3.6}$$

with at least two nonzero entries in each row, but now with a *trailing* unit entry 1 and at least one nontrivial power of λ. The nonzero entries in each row again lie in consecutive adjacent columns, but with the powers q_i in strictly increasing order going from *right to left* rather than from left to right, i.e., $q_\ell > \cdots > q_2 > q_1 > 0$, with $\ell \geq 1$. A backward-zigzag matrix $\widehat{Z}(\lambda)$ is still in *double-echelon form*, i.e., the last nonzero entry of the $(i-1)$th row and the first nonzero entry of the ith row are in the *same* column, and $\widehat{Z}(\lambda)$ has *no* zero columns. In addition, Definition 3.14 together with Remark 3.5 imply that the first column of a backward-zigzag matrix is never a unit column, while the last column is always a unit column, where the definitions of unit and non-unit columns, and unit column sequence are analogous to those for forward-zigzag matrices. Theorem 2.4 again guarantees that *any backward-zigzag matrix is a minimal basis.*

Example 3.15. Here is an example of a backward-zigzag matrix:

$$\widehat{Z}(\lambda) = \begin{bmatrix} \lambda^2 & 1 & & & & & & & & & \\ & \lambda^5 & 1 & & & & & & & & \\ & & \lambda^5 & \lambda^4 & \lambda & 1 & & & & & \\ & & & & & \lambda^3 & 1 & & & & \\ & & & & & & \lambda^4 & 1 & & & \\ & & & & & & & \lambda^9 & \lambda^2 & 1 & \\ & & & & & & & & & \lambda & 1 \end{bmatrix}. \tag{3.7}$$

The unit column sequence of $\widehat{Z}(\lambda)$ in (3.7) is N,U,U,N,N,U,U,U,N,U,U.

Remark 3.16. Just as for forward-zigzag matrices, any column of a backward-zigzag matrix contains either *one or two* nonzero entries; here, though, it is one if it is a non-unit column or the *last* column, two if it is any unit column other than the *last*. There are no other possibilities. This can be immediately seen from Remark 3.9 and Definition 3.14. It also follows from Remark 3.9 and Definition 3.14 that if two *adjacent* columns of a backward-zigzag matrix are non-unit, then their corresponding nonzero entries must once again lie in the *same row*.

Backward-zigzag matrices also have a natural degree-gap sequence, which can be defined by reducing to the forward-zigzag case.

Definition 3.17 (Degree-gap sequence of backward-zigzag matrices)**.** *Suppose the $m \times n$ matrix $\widehat{Z}(\lambda)$ is a backward-zigzag matrix, so that $Z(\lambda) := R_m \widehat{Z}(\lambda) R_n$ is forward-zigzag. Then the* degree-gap sequence *of $\widehat{Z}(\lambda)$ is the same as the* reverse *of the degree-gap sequence of $Z(\lambda)$.*

The degree-gap sequence of a backward-zigzag matrix could also be defined analogously to that for a forward-zigzag matrix, just taking into account that the degrees are now increasing from right to left instead of from left to right. The jth *degree gap* in a backward-zigzag matrix $\widehat{Z}(\lambda)$ is then the positive degree difference $\delta_j := \deg \widehat{Z}_{i,j}(\lambda) - \deg \widehat{Z}_{i,j+1}(\lambda) \geq 1$, where i is the unique row of $\widehat{Z}(\lambda)$ containing two nonzero entries in columns j and $j+1$. The *degree-gap sequence* of $\widehat{Z}(\lambda)$ is again the list of degree gaps, scanning through $\widehat{Z}(\lambda)$ from left to right.

Example 3.18. The degree-gap sequence of $\widehat{Z}(\lambda)$ in (3.7) is $(2, 5, 1, 3, 1, 3, 4, 7, 2, 1)$.

Definition 3.19 (Structure sequence of a backward-zigzag matrix)**.** *Let $\widehat{Z}(\lambda)$ be an $m \times n$ backward-zigzag matrix. Then the sequence of length $2n-1$ obtained by interleaving the unit column sequence and the degree-gap sequence of $\widehat{Z}(\lambda)$ is the* structure sequence *of $\widehat{Z}(\lambda)$. The structure sequences of $\widehat{Z}(\lambda)$ and $Z(\lambda) = R_m \widehat{Z}(\lambda) R_n$ are* reverses *of each other.*

Example 3.20. The structure sequence of $\widehat{Z}(\lambda)$ in (3.7) is

$$\left[\ \text{N}\ \ 2\ \ \text{U}\ \ 5\ \ \text{U}\ \ 1\ \ \text{N}\ \ 3\ \ \text{N}\ \ 1\ \ \text{U}\ \ 3\ \ \text{U}\ \ 4\ \ \text{U}\ \ 7\ \ \text{N}\ \ 2\ \ \text{U}\ \ 1\ \ \text{U}\ \right]. \tag{3.8}$$

Just as for forward-zigzag matrices, a backward-zigzag matrix is uniquely determined by its structure sequence, since it allows us to construct the matrix in a unique way starting from the 1 in the lower-right corner.

Definition 3.21 (Dual zigzag matrices)**.** *Suppose $Z(\lambda)$ is a forward-zigzag matrix and $\widehat{Z}(\lambda)$ is a backward-zigzag matrix with the same number of columns. Then $Z(\lambda)$ and $\widehat{Z}(\lambda)$ are said to be* dual zigzag matrices*, or to form a* dual zigzag pair*, if they have*

(a) *the same degree-gap sequence, but*

(b) *complementary unit column sequences, where* U *and* N *are each other's complement.*

Example 3.22. The matrices $Z(\lambda)$ in (3.2) and $\widehat{Z}(\lambda)$ in (3.7) form a dual zigzag pair.

The following result follows immediately from the complementarity property and Remark 3.6.

Corollary 3.23. *If $Z(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ and $\widehat{Z}(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ are dual zigzag matrices, then $m + k = n$.*

4 Properties of Zigzag Matrices

4.1 Basic Properties

It has been claimed that forward-zigzag and backward-zigzag matrices are uniquely determined by their structure sequences. Let us consider this in a little more detail, and describe a recursive procedure to reconstruct a zigzag matrix from its structure sequence.

We consider only the case of forward-zigzag matrices, constructing it from upper left to bottom right by reading the structure sequence from left to right. The construction of backward-zigzag matrices proceeds analogously, working instead from bottom right to upper left by reading the structure sequence from right to left.

The *size* of the forward-zigzag matrix $Z(\lambda)$ to be constructed is immediately determined by the given structure sequence

$$\mathcal{S} = \left[\ \text{S}_1\ \ \delta_1\ \ \text{S}_2\ \ \delta_2\ \ \ldots\ \ \text{S}_{n-1}\ \ \delta_{n-1}\ \ \text{S}_n\ \right],$$

where $\text{S}_i \in \{\text{N}, \text{U}\}$, for $i = 1, \ldots, n$. The number of rows is the same as the number of unit columns (which is the number of U's in the structure sequence), and the number of columns is n, where the length of the structure sequence is $2n - 1$.

The first row of $Z(\lambda)$ is determined by the initial subsequence $\mathcal{S}_{\text{init}}$ of the structure sequence between $\text{S}_1 = \text{U}$ and the *next* U in the structure sequence, i.e.,

$$\mathcal{S}_{\text{init}} = \left[\ \text{S}_1\ \ \delta_1\ \ \text{S}_2\ \ \delta_2\ \ \ldots\ \ \text{S}_{k-1}\ \ \delta_{k-1}\ \ \text{S}_k\ \right],$$

where $2 \le k < n$ (recall that S_n must be an N), $\text{S}_1 = \text{U} = \text{S}_k$ and $\text{S}_2 = \cdots = \text{S}_{k-1} = \text{N}$. If there is *no* second U in the structure sequence, then we take $k = n$ and $\mathcal{S}_{\text{init}} = \mathcal{S}$; the first row of $Z(\lambda)$ will in this case be the only row of $Z(\lambda)$. The first row then has k adjacent nonzero entries, beginning with a "1", and continuing by incrementally increasing the power of λ as you go from column to column according to the degree gaps δ_1 through δ_{k-1}; the row is then completed with zeroes to give

$$\text{Row}_1 Z(\lambda) = \left[\ 1\ \ \lambda^{\delta_1}\ \ \lambda^{\delta_1+\delta_2}\ \ \ldots\ \ \lambda^{\delta_1+\cdots+\delta_{k-1}}\ \ 0\ \ \ldots\ \ 0\ \right].$$

If $k < n$, then the rest of $Z(\lambda)$ is completed recursively as in the next step.

Observe that if $k < n$, then the remainder of the structure sequence $\mathcal{S} \setminus \mathcal{S}_{\text{init}}$, together with the initial entry $\text{S}_k = \text{U}$, that is,

$$\widetilde{\mathcal{S}} = \left[\ \text{S}_k\ \ \delta_k\ \ \text{S}_{k+1}\ \ \delta_{k+1}\ \ \ldots\ \ \text{S}_{n-1}\ \ \delta_{n-1}\ \ \text{S}_n\ \right],$$

is a structure sequence for a forward-zigzag matrix $\widetilde{Z}(\lambda)$ with one less row and $k - 1$ fewer columns than $Z(\lambda)$. Then $Z(\lambda)$ is recursively completed by attaching $\widetilde{Z}(\lambda)$ to $\text{Row}_1 Z(\lambda)$ with the first column of $\widetilde{Z}(\lambda)$ in the kth column of $Z(\lambda)$, as follows:

$$Z(\lambda) = \left[\begin{array}{cccc|ccc} 1 & \lambda^{\delta_1} & \ldots & \lambda^{\delta_1+\cdots+\delta_{k-2}} & \lambda^{\delta_1+\cdots+\delta_{k-1}} & 0\ \ldots & 0 \\ \hline & & 0 & & & \widetilde{Z}(\lambda) & \end{array}\right].$$

This reconstruction procedure thus proves the following proposition.

Proposition 4.1. *A forward-zigzag matrix is* uniquely determined *by its structure sequence. The same is true for backward-zigzag matrices.*

As an immediate corollary of Proposition 4.1 we see that every forward-zigzag matrix does indeed have a dual backward-zigzag matrix.

Corollary 4.2 (Existence of dual zigzag matrices)**.** *For every forward-zigzag matrix $Z(\lambda)$ there exists a unique backward-zigzag matrix that is dual to $Z(\lambda)$. Similarly, any backward-zigzag matrix has a unique forward-zigzag dual.*

Proof. A zigzag matrix (forward or backward) is uniquely defined by its structure sequence by Proposition 4.1. Then Definition 3.21 uniquely defines the structure sequence of its dual, from which the dual itself can be uniquely reconstructed, again by Proposition 4.1. □

Corollary 4.2 leads to the following definition.

Definition 4.3. *For any forward-zigzag matrix $Z(\lambda)$, the unique backward-zigzag matrix that is dual to $Z(\lambda)$ will be denoted by $Z^{\diamondsuit}(\lambda)$, and referred to as "Z dual". Similarly for any backward-zigzag matrix $\widehat{Z}(\lambda)$, the unique forward-zigzag matrix that is dual to $\widehat{Z}(\lambda)$ will be denoted by $\widehat{Z}^{\diamondsuit}(\lambda)$.*

Note that $(Z^{\diamondsuit})^{\diamondsuit} = Z$.

We next see how the information in the structure sequence of a forward-zigzag matrix $Z(\lambda)$ can be directly used, without first constructing $Z(\lambda)$ itself, to find not only the row degrees of $Z(\lambda)$ but also to deduce the row degrees of the dual $Z^{\diamondsuit}(\lambda)$.

Lemma 4.4 (Row degrees of a zigzag matrix and its dual)**.** *Suppose $Z(\lambda)$ is an $m \times n$ forward-zigzag matrix with structure sequence*

$$\mathcal{S} = \left[\ \text{s}_1 \quad \delta_1 \quad \text{s}_2 \quad \delta_2 \quad \ldots \quad \text{s}_{n-1} \quad \delta_{n-1} \quad \text{s}_n \ \right].$$

Then the row degrees of $Z(\lambda)$ and $Z^{\diamondsuit}(\lambda)$ can be found from $\mathcal{S}$ by the following (dual) rules.

(a) *$Z(\lambda)$ has row degrees equal to the partial sums of degree gaps between any two consecutive* U*'s and after the last* U*. This list of sums gives the row degrees of $Z(\lambda)$, ordered from top to bottom.*

(b) *$Z^{\diamondsuit}(\lambda)$ has row degrees equal to the partial sums of degree gaps before the first* N *and between any two consecutive* N*'s. This list of sums gives the row degrees of $Z^{\diamondsuit}(\lambda)$, ordered from top to bottom.*

Proof. The argument is based on the following two simple observations concerning zigzag matrices (forward or backward), both of which were used in the reconstruction procedure of Proposition 4.1:

(1) The nonzero entries in each row of a forward-zigzag matrix start at a unit column and continue up to the *next* unit column, or all the way to the last column if there is no next unit column. In terms of the structure sequence, this corresponds to the δ's between consecutive U's, or from the last U until the end.

For a backward-zigzag matrix, the nonzero entries of the first row start at the first column and continue up to the first unit column, while the nonzero entries of all remaining rows go between consecutive unit columns. In terms of the structure sequence of this backward-zigzag matrix, this corresponds to the δ's up until the first U, or between consecutive U's.

(2) The degree of any row in a zigzag matrix (forward or backward) is the sum of the degree gaps between the columns of the nonzero entries in that row.

□

Example 4.5. The structure sequence of $Z(\lambda)$ in (3.2) is

$$\left[\ \text{U} \quad 2 \quad \text{N} \quad 5 \quad \text{N} \quad 1 \quad \text{U} \quad 3 \quad \text{U} \quad 1 \quad \text{N} \quad 3 \quad \text{N} \quad 4 \quad \text{N} \quad 7 \quad \text{U} \quad 2 \quad \text{N} \quad 1 \quad \text{N} \ \right]. \tag{4.1}$$

It has the following partial sums between any two consecutive U's and after the last U

$$(2+5+1\,,\,3\,,\,1+3+4+7\,,\,2+1) \;=\; (8,3,15,3)\,.$$

These are exactly the row degrees of $Z(\lambda)$, as can be seen in (3.2). The partial sums corresponding to the "dual" sequence, i.e., before the first N and between any two consecutive N's, are

$$(2,5,1+3+1,3,4,7+2,1) \;=\; (2,5,5,3,4,9,1)\,,$$

which gives the row degrees of $Z^{\diamondsuit}(\lambda)$ in order from top to bottom, as can be seen in (3.7).

Corollary 4.6 (Row degree sums of dual zigzag matrices)**.** *Suppose* $Z(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *and* $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ *are dual zigzag matrices with row degrees* $(\eta_1, \eta_2, \ldots, \eta_m)$ *and* $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$*, respectively. Then:*

(a) $\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \varepsilon_i$*, that is, the sum of the row degrees of a zigzag matrix is equal to the sum of the row degrees of its dual.*

(b) $\sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \varepsilon_i$ *whenever* $(\alpha, \beta) \neq (m, k)$*,* $1 \leq \alpha \leq m$ *and* $1 \leq \beta \leq k$*; that is, a leading partial sum of row degrees of a zigzag matrix is never equal to a leading partial sum of row degrees of its dual.*

Proof. (a) From Lemma 4.4, the sum of the row degrees of any zigzag matrix (forward or backward) is equal to the sum of the degree gaps of the matrix. Since, by Definition 3.21, both $Z(\lambda)$ and $Z^{\diamondsuit}(\lambda)$ have the same degree-gap sequence, the result follows.

(b) Assume without loss of generality that $Z(\lambda)$ is a forward-zigzag matrix with structure sequence $\mathcal{S} = \begin{bmatrix} \mathrm{S}_1 & \delta_1 & \mathrm{S}_2 & \delta_2 & \cdots & \mathrm{S}_{n-1} & \delta_{n-1} & \mathrm{S}_n \end{bmatrix}$. If $\alpha < m$, then $\sum_{i=1}^{\alpha} \eta_i$ is the sum of all the degree gaps from δ_1 up through some δ_j occurring right before a U symbol by Lemma 4.4(a), while $\sum_{i=1}^{\beta} \varepsilon_i$ is the sum of all the degree gaps from δ_1 up through some δ_ℓ occurring right before an N symbol by Lemma 4.4(b). Since U and N symbols are always in different positions in the structure sequence of $Z(\lambda)$ and $\delta_j > 0$ for $j = 1, \ldots, n-1$, the two summations must be different. If $\alpha = m$, then note that $\sum_{i=1}^{\beta} \varepsilon_i < \sum_{i=1}^{k} \varepsilon_i = \sum_{i=1}^{m} \eta_i$. ☐

The final basic property of zigzag matrices in this section concerns a relationship between the rows of a forward-zigzag $Z(\lambda)$ and the rows of its dual $Z^{\diamondsuit}(\lambda)$. Roughly speaking, this next result shows that the rows of $Z(\lambda)$ and $Z^{\diamondsuit}(\lambda)$ mostly avoid each other, in the sense that their nonzero entries tend to be in different locations. But when they do overlap, then it is only in two adjacent entry locations. This result turns out to be key for proving the duality or "orthogonality" results of Section 4.2.

Lemma 4.7 (Overlap dichotomy lemma)**.** *Let* $Z(\lambda)$ *be a forward-zigzag matrix, with dual backward-zigzag matrix* $Z^{\diamondsuit}(\lambda)$*. Consider an arbitrary row* $\mathcal{R}_i$ *from* $Z(\lambda)$*, and an arbitrary row* $\widetilde{\mathcal{R}}_j$ *from* $Z^{\diamondsuit}(\lambda)$*. Then the nonzero entries of* $\mathcal{R}_i$ *and* $\widetilde{\mathcal{R}}_j$ *have either:*

- *no column locations in common, or*
- *exactly two adjacent column locations in common.*

These are the only two possibilities.

Proof. We consider the following four cases, each in turn.

(1a): $\mathcal{R}_i$ has exactly two nonzero entries and $\mathcal{R}_i$ is the last row of $Z(\lambda)$.

(1b): $\mathcal{R}_i$ has exactly two nonzero entries and $\mathcal{R}_i$ is not the last row of $Z(\lambda)$.

(2a): $\mathcal{R}_i$ has more than two nonzero entries and $\mathcal{R}_i$ is the last row of $Z(\lambda)$.

(2b): $\mathcal{R}_i$ has more than two nonzero entries and $\mathcal{R}_i$ is not the last row of $Z(\lambda)$.

In case (1a), the row $\mathcal{R}_i$ looks like

$$\begin{bmatrix} 0 & \ldots & 0 & 1 & * \end{bmatrix},$$

where $*$ denotes a nonzero entry (i.e., a positive power of λ), so that the last two columns of $Z(\lambda)$ are unit(U), then non-unit(N). Thus in the dual $Z^{\diamondsuit}(\lambda)$ the last two columns must be complementary, i.e., NU. This forces the last two columns of $Z^{\diamondsuit}(\lambda)$ to look like

$$\begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ * & 1 \end{bmatrix}.$$

Thus the overlap of the nonzero entries of $\mathcal{R}_i$ with any row $\widetilde{\mathcal{R}}_j$ from $Z^{\diamondsuit}(\lambda)$ has the dichotomy described in the statement of the lemma.

In case (1b), the row $\mathcal{R}_i$ looks like

$$\begin{bmatrix} 0 & \ldots & 0 & 1 & * & 0 & \ldots & 0 \end{bmatrix},$$

so the two nonzero columns of $\mathcal{R}_i$ must be UU in $Z(\lambda)$, i.e., both are unit columns. Hence the two corresponding columns of $Z^{\diamondsuit}(\lambda)$ are NN. By Remark 3.16 we can then conclude that these two columns look like

$$\begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ * & * \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

and once again we have the dichotomy described in the statement of the lemma.

Moving on to case (2a), the row $\mathcal{R}_i$ is the last row of $Z(\lambda)$ and looks like

$$\begin{bmatrix} 0 & \dots & 0 & 1 & * & \dots & * \end{bmatrix},$$

so the columns of $Z(\lambda)$ corresponding to these last nonzero entries are UNN...N. Thus the corresponding last columns of $Z^{\diamondsuit}(\lambda)$ are NUU...U. This implies that these last columns of $Z^{\diamondsuit}(\lambda)$ must look like

$$\begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & 0 \\ * & 1 & & & \\ & * & 1 & & \\ & & \ddots & \ddots & \\ & & & * & 1 \end{bmatrix}.$$

Comparison of $\mathcal{R}_i$ with each of these row fragments of $Z^{\diamondsuit}(\lambda)$ shows that the dichotomy of the lemma holds for this case.

Finally in case (2b), the row $\mathcal{R}_i$ looks like

$$\begin{bmatrix} 0 & \dots & 0 & 1 & * & \dots & * & 0 & \dots & 0 \end{bmatrix},$$

with the columns corresponding to the nonzero entries being UN...NU. Thus the corresponding columns of $Z^{\diamondsuit}(\lambda)$ must be NU...UN, which in turn implies that these columns of $Z^{\diamondsuit}(\lambda)$ must look like

$$\begin{bmatrix} 0 & 0 & & \dots & & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & \dots & & 0 \\ * & 1 & & & & \\ & * & 1 & & & \\ & & \ddots & \ddots & & \\ & & & * & 1 & \\ & & & & * & * \\ 0 & 0 & & \dots & & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & \dots & & 0 \end{bmatrix}.$$

Comparison of $\mathcal{R}_i$ with each of these row fragments of $Z^{\diamondsuit}(\lambda)$ shows that the dichotomy of the lemma also holds for this final case, which completes the proof. □

4.2 Dual Zigzag Matrices, Dual Minimal Bases, and Minimal Indices

Dual zigzag matrices allow us to construct explicit, simple examples of dual minimal bases (introduced in Definition 2.10). For this purpose we need to define the following auxiliary matrix.

Definition 4.8 (Alternating Signs Matrix)**.** *The* $n \times n$ diagonal alternating signs matrix Σ_n *is defined by*

$$\Sigma_n := \operatorname{diag}(1, -1, 1, -1, \dots, (-1)^{n-1}) \,. \tag{4.2}$$

As a consequence of Lemma 4.7, the truth of the "orthogonality" relation (4.3) in the next result becomes transparent. We can also immediately see why the alternating signs matrix Σ_n is *needed* in the story.

Lemma 4.9. *Suppose $Z(\lambda)$ is any $m \times n$ zigzag matrix, either forward or backward, and let $Z^{\diamondsuit}(\lambda)$ be its $(n-m) \times n$ dual zigzag matrix. Then*

$$Z(\lambda) \cdot \Sigma_n \cdot \left(Z^{\diamondsuit}(\lambda)\right)^T = 0_{m \times (n-m)}\,, \tag{4.3}$$

and $Z(\lambda)$ and $(Z^{\diamondsuit}(\lambda) \cdot \Sigma_n)$ are dual minimal bases.

Proof. Let us assume without loss of generality that $Z(\lambda)$ is a forward-zigzag matrix. As was proved in Lemma 4.7, the nonzero entries of a row of $Z(\lambda)$ and the nonzero entries of a row of $Z^{\diamondsuit}(\lambda)$ have exactly two adjacent column locations in common, or none at all. In the first case the degree *increase* in the nonzero entries in the row of $Z(\lambda)$ located in the adjacent columns is the same as the degree *decrease* in the nonzero entries in the same columns of the considered row of $Z^{\diamondsuit}(\lambda)$. Orthogonality then follows because of the sign matrix in the middle. In the second case, orthogonality is trivial. So, (4.3) is proved. Theorem 3.3 and the corresponding result for backward-zigzag matrices (see paragraph above Example 3.15) imply that $Z(\lambda)$ and $Z^{\diamondsuit}(\lambda)$ are both (full row rank) minimal bases, and from Theorem 2.4 we get that $(Z^{\diamondsuit}(\lambda) \cdot \Sigma_n)$ is also a minimal basis. Equation (4.3) yields $Z(\lambda) \cdot (Z^{\diamondsuit}(\lambda) \cdot \Sigma_n)^T = 0$, which combined with the sizes of $Z(\lambda)$ and $(Z^{\diamondsuit}(\lambda) \cdot \Sigma_n)$ proves that these two matrices are dual minimal bases according to Definition 2.10. □

As a direct corollary of the previous lemma, the comments in the paragraph just after Definition 2.10, and the developments in Remark 2.14, we obtain the complete eigenstructure of any zigzag matrix.

Corollary 4.10 (Eigenstructure and minimal bases of zigzag matrices)**.** *Let $Z(\lambda)$ be an $m \times n$ zigzag matrix, either forward or backward, with row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$, and let $d = \max_{i=1,\ldots,m} \eta_i$. Then:*

(a) *$Z(\lambda)$ has no finite eigenvalues.*

(b) *$Z(\lambda)$ has an eigenvalue at infinity if and only if not all row degrees $(\eta_1, \ldots, \eta_m)$ are equal. In this case, the degrees of the elementary divisors of $Z(\lambda)$ at ∞ are $\{d - \eta_i \,:\, d - \eta_i > 0,\ i = 1, \ldots, m\}$.*

(c) *$Z(\lambda)$ has no left minimal indices.*

(d) *The right minimal indices of $Z(\lambda)$ are the row degrees of its dual zigzag matrix $Z^{\diamondsuit}(\lambda)$, and $(Z^{\diamondsuit}(\lambda) \cdot \Sigma_n)^T$ is a right minimal basis of $Z(\lambda)$.*

We emphasize that as a consequence of Corollary 4.10 and Lemma 4.4, the complete eigenstructure of a zigzag matrix can be determined very easily, essentially by a simple inspection of the matrix entries.

5 The Inverse Row Degree Problem for Dual Zigzag Matrices

Corollary 4.6 establishes two properties that must be satisfied by the row degrees of any pair of dual zigzag matrices. In this section we solve the corresponding inverse problem, i.e., we prove that given any two lists of positive integers $(\eta_1, \ldots, \eta_m)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$ for which the equality in Corollary 4.6(a) and the inequalities in Corollary 4.6(b) hold, there exists a pair of dual zigzag matrices $Z(\lambda)$ and $Z^{\diamondsuit}(\lambda)$ with precisely these row degrees, and with the degrees in the given row order. In fact, this pair is unique once we decide which row degree list is attached to the forward-zigzag matrix and which list to the backward-zigzag matrix in the dual pair. In addition, we present a very simple procedure to construct this unique pair of dual zigzag matrices. In light of the result of Lemma 4.9, the solution of this problem also immediately solves an inverse row degree problem for dual minimal bases, but a very particular one with special additional conditions. In general some of the row degrees in a dual minimal basis pair may be zero, something that never happens for a zigzag matrix; furthermore, the row degrees in a dual minimal basis pair may not necessarily satisfy the inequalities in Corollary 4.6(b). The complete solution of the inverse row degree problem for general dual minimal bases is the subject of Section 6, and is based on the solution of the special inverse problem given here in Section 5.

The main result in this section is Theorem 5.1.

Theorem 5.1. *Let $(\eta_1, \ldots, \eta_m)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$ be any two lists of positive integers such that*

$$\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \varepsilon_i \quad \text{and} \quad \sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \varepsilon_i, \quad \text{whenever } (\alpha, \beta) \neq (m, k),\ 1 \leq \alpha \leq m \text{ and } 1 \leq \beta \leq k. \tag{5.1}$$

Then there exists a unique forward-zigzag matrix $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+k)}$ with row degrees $(\eta_1, \ldots, \eta_m)$ such that its dual backward-zigzag matrix $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{k \times (m+k)}$ has row degrees $(\varepsilon_1, \ldots, \varepsilon_k)$. In addition, the structure sequence of $Z(\lambda)$ is constructed via the following five-step procedure:

Step 1. *Define* $\ell_0 := 0$*, compute the partial sums* $\ell_\alpha := \sum_{i=1}^{\alpha} \eta_i$ *for* $\alpha = 1, \ldots, m-1$*, and the partial sums* $r_\beta := \sum_{i=1}^{\beta} \varepsilon_i$ *for* $\beta = 1, \ldots, k$*.*

Step 2. *Merge the lists of different integers* $\ell_0 < \ell_1 < \cdots < \ell_{m-1}$ *and* $r_1 < r_2 < \cdots < r_k$ *into a single ordered list of length* $n := m + k$

$$\ell_0 < \cdots < \ell_\alpha < \cdots < r_\beta < \cdots < \ell_\gamma < \cdots < r_\xi < \cdots < r_k \ . \tag{5.2}$$

Step 3. *The degree-gap sequence* $\delta_1, \ldots, \delta_{n-1}$ *of* $Z(\lambda)$ *is obtained by computing the* $n-1$ *differences between adjacent entries in the sequence* (5.2)*.*

Step 4. *The unit column sequence of* $Z(\lambda)$ *is obtained by replacing each* ℓ_α *in* (5.2) *by the symbol* U*, and each* r_β *in* (5.2) *by the symbol* N*.*

Step 5. *Interleave the unit column sequence from* Step 4 *with the degree-gap sequence from* Step 3 *to get the structure sequence of* $Z(\lambda)$*.*

Proof. The existence of a forward-zigzag matrix with the desired properties is established by showing that the five-step procedure in the statement always yields a structure sequence corresponding to a forward-zigzag matrix with these properties. Since Proposition 4.1 guarantees that a forward-zigzag matrix is determined by its structure sequence, this will certainly prove the existence part of the theorem. We use Lemma 4.4 to verify that the structure sequence generated by Steps 1-5 gives the desired row degrees for the corresponding $Z(\lambda)$ and $Z^\diamondsuit(\lambda)$. For the row degrees of $Z(\lambda)$, by Lemma 4.4(a) we have to look at the sum of the degree gaps between successive U's, and after the last U in the structure sequence. But successive U's in the structure sequence come from successive ℓ's in the merged list (5.2), let's say ℓ_{i-1} and ℓ_i. Extracting this sublist from (5.2) we have

$$\ell_{i-1} < r_j < r_{j+1} < \cdots < r_{j+s} < \ell_i \, . \tag{5.3}$$

The sum of the degree gaps corresponding to this sublist is the telescoping sum

$$(\ell_i - r_{j+s}) + (r_{j+s} - r_{j+s-1}) + \cdots + (r_{j+1} - r_j) + (r_j - \ell_{i-1}) \; = \; \ell_i - \ell_{i-1} \; = \; \eta_i \, .$$

In a similar way, the structure sequence after the last U comes from a sublist of (5.2) of the type $\ell_{m-1} < r_t < r_{t+1} < \cdots < r_k$ and the sum of the corresponding degree gaps is equal to $r_k - \ell_{m-1} = \eta_m$, where we have used the first equality in (5.1). Thus we see by Lemma 4.4(a) that we recover exactly the desired row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$ for $Z(\lambda)$.

For the row degrees of $Z^\diamondsuit(\lambda)$, by Lemma 4.4(b) we have to look at the sum of the degree gaps before the first N and between successive N's in the structure sequence. Two successive N's now correspond to a sublist of (5.2) of the form

$$r_{i-1} < \ell_j < \ell_{j+1} < \cdots < \ell_{j+s} < r_i \, , \tag{5.4}$$

whose associated sum of degree gaps is the telescoping sum

$$(r_i - \ell_{j+s}) + (\ell_{j+s} - \ell_{j+s-1}) + \cdots + (\ell_{j+1} - \ell_j) + (\ell_j - r_{i-1}) \; = \; r_i - r_{i-1} \; = \; \varepsilon_i \, ,$$

while the structure sequence before the first N comes from a sublist of (5.2) of the type $\ell_0 < \ell_1 < \ldots < \ell_s < r_1$, which gives a sum of degree gaps equal to $r_1 - \ell_0 = \varepsilon_1$. Thus by Lemma 4.4(b) we recover the desired row degrees $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ for $Z^\diamondsuit(\lambda)$ from the five-step procedure in the statement.

Now that existence has been proved, we establish the uniqueness of dual forward and backward-zigzag matrices $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+k)}$ and $Z^\diamondsuit(\lambda) \in \mathbb{F}[\lambda]^{k \times (m+k)}$ with prescribed row degrees $(\eta_1, \ldots, \eta_m)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$, respectively. To this end, we will show that these row degrees uniquely determine the structure sequence of $Z(\lambda)$ (and so also the one of $Z^\diamondsuit(\lambda)$). Let

$$\mathcal{S} = \begin{bmatrix} \text{U} & \delta_1 & \text{S}_2 & \delta_2 & \ldots & \text{S}_{n-1} & \delta_{n-1} & \text{N} \end{bmatrix}$$

be *any* forward-zigzag structure sequence of $Z(\lambda)$ compatible with the prescribed row degrees. The number of U's in $\mathcal{S}$ is m and the number of N's is k by Remark 3.6. As a consequence of Lemma 4.4(a), $\ell_\alpha = \sum_{i=1}^{\alpha} \eta_i$ is equal to the sum of the degree gaps from δ_1 up until the δ_i just before the $(\alpha+1)$th symbol U in $\mathcal{S}$ for $\alpha = 1, \ldots, m-1$, and, as a consequence of Lemma 4.4(b), $r_\beta = \sum_{i=1}^{\beta} \varepsilon_i$ is equal to the sum of the degree gaps from δ_1 up until the δ_j just before the βth symbol N in $\mathcal{S}$ for $\beta = 1, \ldots, k$. This and the fact that all the degree gaps are positive imply that the merged list (5.2) determines the order of all the U's and N's in $\mathcal{S}$ by replacing each ℓ_α by U and each r_β by N, and also that the differences between adjacent entries in the sequence (5.2) are precisely the degree gaps. Therefore $\mathcal{S}$ has been uniquely determined by the prescribed row degrees. □

Example 5.2. To illustrate the five-step procedure in Theorem 5.1, we use the lists $(\eta_1, \eta_2, \eta_3, \eta_4) = (8, 3, 15, 3)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_7) = (2, 5, 5, 3, 4, 9, 1)$ from Example 4.5 to reconstruct $Z(\lambda)$ in (3.2). For these two row degree lists, the partial sums lists in Step 1 are

$$\begin{bmatrix} \ell_0 & \ell_1 & \ell_2 & \ell_3 \\ 0 & 8 & 11 & 26 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 \\ 2 & 7 & 12 & 15 & 19 & 28 & 29 \end{bmatrix}.$$

The single merged list is then

$$\begin{bmatrix} \ell_0 & r_1 & r_2 & \ell_1 & \ell_2 & r_3 & r_4 & r_5 & \ell_3 & r_6 & r_7 \\ 0 & 2 & 7 & 8 & 11 & 12 & 15 & 19 & 26 & 28 & 29 \end{bmatrix}.$$

From this merged list we now read off the unit column sequence U,N,N,U,U,N,N,N,U,N,N and the degree gap sequence $2, 5, 1, 3, 1, 3, 4, 7, 2, 1$. Altogether, these two sequences define exactly the structure sequence of $Z(\lambda)$ in (3.2), as desired.

The five-step procedure in Theorem 5.1 builds the structure sequence of the unique forward-zigzag matrix $Z(\lambda)$ corresponding to the prescribed row degrees of $Z(\lambda)$ and $Z^\diamond(\lambda)$. In the particular case where all the elements in the list $(\eta_1, \ldots, \eta_m)$ are equal, except perhaps the last one which can be less than or equal to the others, the construction presented in Theorem 5.1 can be considerably simplified; indeed it is possible to give an explicit easy description of $Z(\lambda)$ directly in terms of its entries without first computing its structure sequence. This entrywise construction is presented in detail in Theorem 5.3, in a way that will be very convenient for Section 7. Observe that if $(\eta_1, \ldots, \eta_{m-1}, \eta_m) = (d, \ldots, d, \eta_m)$, where d is a positive integer and $\eta_m \leq d$, then $\sum_{i=1}^{\alpha} \eta_i = \alpha\, d$ *is a multiple of* d for $\alpha = 1, \ldots, m-1$; therefore the inequalities in (5.1) mean simply that $\sum_{i=1}^{\beta} \varepsilon_i$ *is not a multiple of* d for $\beta = 1, \ldots, k-1$, which is imposed as an assumption in Theorem 5.3. Since $\eta_m \leq d$ is not fixed, $\sum_{i=1}^{k} \varepsilon_i$ in Theorem 5.1 may or may not be a multiple of d, and we express this fact in Theorem 5.3 in a concise way as $\sum_{i=1}^{k} \varepsilon_i = dq_k + w_k$, with q_k and w_k nonnegative integers such that $0 < w_k \leq d$. So $\sum_{i=1}^{k} \varepsilon_i$ is a multiple of d if $w_k = d$, and otherwise is not. We emphasize the difference between $0 < w_k \leq d$ and the standard condition used in Euclidean integer division.

Theorem 5.3. *Let $(\varepsilon_1, \ldots, \varepsilon_k)$ be a list of positive integers, and d be another positive integer such that*

$$\sum_{i=1}^{\beta} \varepsilon_i = d\, q_\beta + w_\beta \qquad \text{with } \; 0 < w_\beta < d \; \text{ for } \; 1 \leq \beta \leq k-1 \; \text{ and } \; 0 < w_k \leq d, \tag{5.5}$$

for nonnegative integers $q_1, \ldots, q_k$ and positive integers $w_1, \ldots, w_k$. Then there exists a unique forward-zigzag matrix $Z(\lambda) \in \mathbb{F}[\lambda]^{(q_k+1)\times(q_k+1+k)}$ with row degrees $(d, \ldots, d, w_k)$, such that its dual backward-zigzag matrix $Z^\diamond(\lambda) \in \mathbb{F}[\lambda]^{k\times(q_k+1+k)}$ has row degrees $(\varepsilon_1, \ldots, \varepsilon_k)$. In addition:

(a) *The nonzero entries of $Z(\lambda)$ are*

$$\underbrace{1, \ldots, 1}_{q_k+1}, \underbrace{\lambda^d, \ldots, \lambda^d}_{q_k}, \text{ and } \lambda^{w_1}, \ldots, \lambda^{w_k}. \tag{5.6}$$

(b) *The $(q_k + 1)$ entries 1 in (5.6) are the leading 1's of the rows of $Z(\lambda)$ and are located in the positions $(1, 1)$ and*

$$(\, p\, ,\, p + \max\{\beta \, : \, q_\beta < (p-1)\}\,)\,, \quad \text{for } p = 2, \ldots, q_k + 1\,,$$

where if the set $\{\beta : q_\beta < (p-1)\}$ is empty, we take its maximum to be 0.

(c) *The q_k entries λ^d in (5.6) are the trailing nonzero entries in the rows $1, 2, \ldots, q_k$ of $Z(\lambda)$ and therefore are located in the columns corresponding to the leading 1's of the rows $2, 3, \ldots, q_k + 1$.*

(d) *Each entry λ^{w_i} in (5.6) is located in the position $(q_i + 1\,,\, q_i + 1 + i)$ of $Z(\lambda)$, for $i = 1, 2, \ldots, k$, where q_i is defined in (5.5). Observe that λ^{w_k} is the trailing entry of the last row of $Z(\lambda)$.*

Proof. If we take $(\eta_1, \eta_2, \ldots, \eta_{q_k+1}) = (d, \ldots, d, w_k)$, then $\sum_{i=1}^{q_k+1} \eta_i = dq_k + w_k$, $\sum_{i=1}^{\alpha} \eta_i = d\,\alpha$, for $\alpha = 1, \ldots, q_k$, and the assumptions (5.5) imply the assumptions (5.1) in Theorem 5.1 with $m = q_k + 1$. Therefore, Theorem 5.1 guarantees the existence and uniqueness of $Z(\lambda)$ with the properties of the statement. It remains to prove parts (a), (b), (c), and (d). For (a), note that $Z(\lambda)$ has $q_k + 1$ entries equal to 1 and q_k entries equal to λ^d, simply as a consequence of the definition of a forward-zigzag matrix and the row degrees that $Z(\lambda)$ has. The presence of the remaining nonzero entries will be established later.

Parts (b), (c), and (d) will be proved by using the five-step procedure in Theorem 5.1 applied to $(\eta_1, \eta_2, \ldots, \eta_{q_k+1}) = (d, \ldots, d, w_k)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$ for constructing the structure sequence of $Z(\lambda)$. In this situation, the partial sums in Step 1 are $\ell_0 = 0$, $\ell_\alpha = d\,\alpha$ for $\alpha = 1, \ldots, q_k$, and $r_\beta = dq_\beta + w_\beta$ for $\beta = 1, \ldots, k$. Next, recall that the position of the pth U column of $Z(\lambda)$ corresponds to the position of $\ell_{p-1} = (p-1)d$ in the list (5.2) (note that the first U column corresponds to ℓ_0) and observe that this position is

$$p + \#\{r_\beta \,:\, r_\beta < (p-1)d\},$$

where the summand p counts the terms $\ell_0, \ldots, \ell_{p-1}$. This proves (b), since the leading 1 of the pth row of $Z(\lambda)$ is located in the pth U column of $Z(\lambda)$. Taking into account which are the row degrees of $Z(\lambda)$ and the definition of a forward-zigzag matrix, we get also (c).

For proving (d) and the remaining part of (a), note that, according to Theorem 5.1, the N columns of $Z(\lambda)$ located between the pth and $(p+1)$th U columns ($p < q_k + 1$) correspond to those terms r_β's in (5.2) such that

$$(p-1)d = \ell_{p-1} < r_j < r_{j+1} < \cdots < r_{j+s} < \ell_p = pd\,. \tag{5.7}$$

Therefore, these terms are of the form

$$r_t = d(p-1) + w_t \quad \text{ with } \; 0 < w_t < d\,,$$

and in the list (5.2) they are in positions $p + t$ for $t = j, \ldots, j+s$, since in $\ell_0 < \cdots < \ell_{p-1} < \cdots < r_t$ there are p terms ℓ_α's and t terms r_β's. Moreover, note that the (unique) nonzero entries in these N columns are located in the positions $(p, p+t)$ for $t = j, \ldots, j+s$, since the 1 of the pth U column is in the pth row of $Z(\lambda)$. The degree gaps corresponding to the sublist (5.7) with ℓ_p removed are by Theorem 5.1

$$w_j,\, w_{j+1} - w_j,\, \ldots\, , w_{j+s} - w_{j+s-1},$$

and so the entries of $Z(\lambda)$ located in $(p, p+j), (p, p+j+1), \ldots, (p, p+j+s)$ are $\lambda^{w_j}, \lambda^{w_{j+1}}, \ldots, \lambda^{w_{j+s}}$, as a consequence of summing the degree gaps. This proves (d) for all w_i such that $r_i = dq_i + w_i$ corresponds in (5.2) to an N column located between two U columns. For those N columns of $Z(\lambda)$ located after its last U column, we proceed as follows. Since the last U column is the $(q_k + 1)$th U column, the N columns after it correspond to those terms r_β's in (5.2) such that

$$q_k d = \ell_{q_k} < r_j < r_{j+1} < \cdots < r_{k-1} < r_k\,. \tag{5.8}$$

Therefore they are of the form

$$r_t = dq_k + w_t \quad \text{with } \; 0 < w_t < d \; \text{ if } \; t < k\,, \; \text{and } \; 0 < w_k \le d\,,$$

i.e., they all have $q_t = q_k$, and in the list (5.2) they are in positions $q_k + 1 + t$ for $t = j, \ldots, k$. Moreover, note that the nonzero entries in these N columns are located in the positions $(q_k + 1, q_k + 1 + t)$ for $t = j, \ldots, k$, since the 1 of the $(q_k + 1)$th U column is in the $(q_k + 1)$th row. The rest of the argument is the same as the previous one except that in the computation of the needed degree gaps the last term in the sublist (5.8) is not removed. So the proof of (d) is complete. Observe that in the proof of (d) we have scanned all N columns and all the nonzero entries in those columns. Therefore the proof of (a) is also complete. □

Example 5.4. To illustrate the construction presented in Theorem 5.3, we consider the list $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = (2, 3, 5, 1, 5)$, with $k = 5$, and take $d = 4$. Then the partial sums in (5.5) are

$$\sum_{i=1}^{1} \varepsilon_i = 2 = 4 \cdot 0 + 2\,, \quad \sum_{i=1}^{2} \varepsilon_i = 5 = 4 \cdot 1 + 1\,, \quad \sum_{i=1}^{3} \varepsilon_i = 10 = 4 \cdot 2 + 2\,,$$
$$\sum_{i=1}^{4} \varepsilon_i = 11 = 4 \cdot 2 + 3\,, \quad \sum_{i=1}^{5} \varepsilon_i = 16 = 4 \cdot 3 + 4\,,$$

which implies $q_5 = 3$ and so $Z(\lambda)$ has size 4×9. In addition, $w_1 = 2, w_2 = 1, w_3 = 2, w_4 = 3, w_5 = 4$, which means in particular that all the row degrees of $Z(\lambda)$ are in this case equal to 4. Theorem 5.3(b) provides the following positions for the leading 1's of the four rows of $Z(\lambda)$:

$$(1,1),\; (2, 2+1) = (2,3),\; (3, 3+2) = (3,5),\; (4, 4+4) = (4,8),$$

and Theorem 5.3(d) provides the following positions for $\lambda^{w_1} = \lambda^2, \lambda^{w_2} = \lambda, \lambda^{w_3} = \lambda^2, \lambda^{w_4} = \lambda^3, \lambda^{w_5} = \lambda^4$:

$$\lambda^2 \text{ in } (1, 1+1) = (1,2), \quad \lambda \text{ in } (2, 2+2) = (2,4), \quad \lambda^2 \text{ in } (3, 3+3) = (3,6),$$
$$\lambda^3 \text{ in } (3, 3+4) = (3,7), \quad \lambda^4 \text{ in } (4, 4+5) = (4,9),$$

respectively. Therefore

$$Z(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^4 & & & & & & \\ & & 1 & \lambda & \lambda^4 & & & & \\ & & & & 1 & \lambda^2 & \lambda^3 & \lambda^4 & \\ & & & & & & & 1 & \lambda^4 \end{bmatrix}. \tag{5.9}$$

Applying Lemma 4.4(b) to $Z(\lambda)$, we easily check that $Z^{\diamond}(\lambda)$ has row degrees $(2, 3, 5, 1, 5)$, i.e., $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$ as predicted by Theorem 5.3.

The structure sequence of $Z(\lambda)$ can also be deduced from the five-step procedure in Theorem 5.1 by taking $(\eta_1, \eta_2, \eta_3, \eta_4) = (4, 4, 4, 4)$. For this purpose note that the partial sums in Step 1 are

$$\begin{bmatrix} \ell_0 & \ell_1 & \ell_2 & \ell_3 \\ 0 & 4 & 8 & 12 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 \\ 2 & 5 & 10 & 11 & 16 \end{bmatrix}.$$

The single merged list is then

$$\begin{bmatrix} \ell_0 & r_1 & \ell_1 & r_2 & \ell_2 & r_3 & r_4 & \ell_3 & r_5 \\ 0 & 2 & 4 & 5 & 8 & 10 & 11 & 12 & 16 \end{bmatrix}.$$

From this merged list we now read off the unit column sequence U,N,U,N,U,N,N,U,N and the degree-gap sequence $2, 2, 1, 3, 2, 1, 1, 4$. Altogether, these two sequences define the structure sequence of $Z(\lambda)$ in (5.9).

6 The Inverse Row Degree Problem for Dual Minimal Bases

In this section, we use the results in Section 5 to show how to concretely construct dual minimal bases (see Definition 2.10) for any two lists of prescribed row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ satisfying only the necessary condition (2.2). This result is stated in Theorem 6.1, which is one of the main contributions of this paper. Observe that in Theorem 6.1, we use as the $(1, 1)$ blocks inside the block diagonal matrices $M(\lambda)$ and $N(\lambda)$ the matrices

$$\begin{bmatrix} I_{m_0} & 0_{m_0 \times k_0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0_{k_0 \times m_0} & I_{k_0} \end{bmatrix}, \tag{6.1}$$

whose meaning is the standard one if $m_0 \neq 0$ and $k_0 \neq 0$. In the case $m_0 = k_0 = 0$, the matrices in (6.1) should be understood as empty matrices, i.e., they are not present in $M(\lambda)$ and $N(\lambda)$. In the cases $m_0 = 0, k_0 \neq 0$ and $m_0 \neq 0, k_0 = 0$, the meanings of the matrices in (6.1) are

$$\begin{bmatrix} I_0 & 0_{0 \times k_0} \end{bmatrix} = 0_{0 \times k_0}, \begin{bmatrix} 0_{k_0 \times 0} & I_{k_0} \end{bmatrix} = I_{k_0} \quad \text{and} \quad \begin{bmatrix} I_{m_0} & 0_{m_0 \times 0} \end{bmatrix} = I_{m_0}, \begin{bmatrix} 0_{0 \times m_0} & I_0 \end{bmatrix} = 0_{0 \times m_0},$$

where the matrices $0_{0 \times k_0}$ and $0_{0 \times m_0}$ on the right-hand sides contribute k_0 and m_0 initial zero columns to $M(\lambda)$ and $N(\lambda)$, respectively, with no additional rows.

Theorem 6.1. *Let $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ be any two lists of nonnegative integers such that*

$$\sum_{i=1}^{m} \eta_i = \sum_{j=1}^{k} \varepsilon_j . \tag{6.2}$$

Then there exist two matrix polynomials $M(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+k)}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{k \times (m+k)}$ that are dual minimal bases, and whose row degrees are $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, respectively.

Moreover, there are infinitely many pairs of dual minimal bases with row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, and one of these pairs can be constructed, up to a row permutation, as follows: Let the lists $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ be ordered so that

$$0 = \eta_1 = \cdots = \eta_{m_0}, \;\; 0 < \eta_i \;\text{ if } m_0 < i \qquad \text{and} \qquad 0 = \varepsilon_1 = \cdots = \varepsilon_{k_0}, \;\; 0 < \varepsilon_j \;\text{ if } k_0 < j ,$$

and define the set

$$\{(m_1, k_1), (m_2, k_2), \ldots, (m_t, k_t)\} = \left\{ (\gamma, \rho) : \sum_{i=m_0+1}^{\gamma} \eta_i = \sum_{j=k_0+1}^{\rho} \varepsilon_j, \; m_0 + 1 \le \gamma \le m, \; k_0 + 1 \le \rho \le k \right\},$$

where $m_1 < \cdots < m_t = m$ and $k_1 < \cdots < k_t = k$. Then:

(a) *For each $i = 1, \ldots, t$, there exists a unique forward-zigzag matrix $Z_i(\lambda)$ with row degrees $(\eta_{m_{i-1}+1}, \ldots, \eta_{m_i})$ such that its dual backward-zigzag matrix $Z_i^{\diamond}(\lambda)$ has row degrees $(\varepsilon_{k_{i-1}+1}, \ldots, \varepsilon_{k_i})$.*

(b) *The matrices*

$$M(\lambda) := \left[\begin{array}{cc|ccc} I_{m_0} & 0_{m_0\times k_0} & & & \\ \hline & & Z_1(\lambda) & & \\ & & & \ddots & \\ & & & & Z_t(\lambda) \end{array}\right] \quad \text{and}$$

$$N(\lambda) := \left[\begin{array}{cc|ccc} 0_{k_0\times m_0} & I_{k_0} & & & \\ \hline & & Z_1^{\diamondsuit}(\lambda)\cdot\Sigma^{(1)} & & \\ & & & \ddots & \\ & & & & Z_t^{\diamondsuit}(\lambda)\cdot\Sigma^{(t)} \end{array}\right],$$

where $\Sigma^{(1)}, \ldots, \Sigma^{(t)}$ are alternating signs matrices (as in Definition 4.8) of appropriate sizes, are dual minimal bases with row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, respectively, and sizes $m \times (m+k)$ and $k \times (m+k)$, respectively.

Proof. For brevity, in the proof we set $n := m+k$. First, observe that if $M(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ are dual minimal bases with row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, respectively, and Q is any $n\times n$ nonsingular constant matrix, then $\widetilde{M}(\lambda) = M(\lambda)Q$ and $\widetilde{N}(\lambda) = N(\lambda)Q^{-T}$ are dual minimal bases with the same row degrees and the same sizes as $M(\lambda)$ and $N(\lambda)$. It is obvious that $\widetilde{M}(\lambda)$ has the same row degrees and size as $M(\lambda)$, and that $\widetilde{N}(\lambda)$ has the same row degrees and size as $N(\lambda)$. Also $\widetilde{M}(\lambda)\widetilde{N}(\lambda)^T = M(\lambda)QQ^{-1}N(\lambda)^T = M(\lambda)N(\lambda)^T = 0$. To see that $\widetilde{M}(\lambda)$ is a minimal basis, we use Theorem 2.4 and the facts that $\widetilde{M}(\lambda_0) = M(\lambda_0)Q$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$, since $M(\lambda_0)$ has full row rank, and that $\widetilde{M}_{hr} = M_{hr}Q$ has also full row rank since M_{hr} has (recall that M_{hr} is the highest-row-degree coefficient matrix of $M(\lambda)$). A similar argument proves that $\widetilde{N}(\lambda)$ is a minimal basis. Therefore, if we find one pair of dual minimal bases with the degrees prescribed in the statement, we can construct infinitely many of them by choosing infinitely many nonsingular matrices Q.

Next, we prove (a). The definition of $\{(m_1, k_1), (m_2, k_2), \ldots, (m_t, k_t)\}$ implies

$$\sum_{j=m_{i-1}+1}^{m_i} \eta_j = \sum_{j=k_{i-1}+1}^{k_i} \varepsilon_j\,, \tag{6.3}$$

which is obvious for $i = 1$ and, for $2 \le i \le t$, follows by subtracting $\sum_{j=m_0+1}^{m_{i-1}} \eta_j = \sum_{j=k_0+1}^{k_{i-1}} \varepsilon_j$ from $\sum_{j=m_0+1}^{m_i} \eta_j = \sum_{j=k_0+1}^{k_i} \varepsilon_j$. Also

$$\sum_{j=m_{i-1}+1}^{\alpha} \eta_j \neq \sum_{j=k_{i-1}+1}^{\beta} \varepsilon_j, \quad \text{whenever } (\alpha,\beta) \neq (m_i,k_i),\ m_{i-1}+1 \le \alpha \le m_i \text{ and } k_{i-1}+1 \le \beta \le k_i, \tag{6.4}$$

since otherwise the set $\{(m_1, k_1), (m_2, k_2), \ldots, (m_t, k_t)\}$ would have additional elements $(\widetilde{m}, \widetilde{k})$ such that $m_{i-1}+1 \le \widetilde{m} < m_i$ and $k_{i-1}+1 \le \widetilde{k} < k_i$. Observe that (6.3) and (6.4) are precisely the assumptions (5.1) of Theorem 5.1 for the lists of positive integers $(\eta_{m_{i-1}+1}, \ldots, \eta_{m_i})$ and $(\varepsilon_{k_{i-1}+1}, \ldots, \varepsilon_{k_i})$. Therefore, (a) follows from Theorem 5.1.

Now we proceed to prove (b). The row degrees of $M(\lambda)$ and $N(\lambda)$ are $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, respectively, as a consequence of (a). The size of each $Z_i(\lambda)$ in (a) is $(m_i - m_{i-1}) \times (m_i - m_{i-1} + k_i - k_{i-1})$ and the one of $Z_i^{\diamondsuit}(\lambda)$ is $(k_i - k_{i-1}) \times (m_i - m_{i-1} + k_i - k_{i-1})$ by Corollary 3.23. So, by adding these sizes and the ones of the first diagonal blocks of $M(\lambda)$ and $N(\lambda)$, we get that $M(\lambda)$ has size $m \times (m+k)$ and $N(\lambda)$ has size $k \times (m+k)$. Part (a) and Lemma 4.9 now immediately imply that $M(\lambda)N(\lambda)^T = 0$. Finally, we prove that $M(\lambda)$ and $N(\lambda)$ are both minimal bases. For this purpose, note that $M(\lambda_0)$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$, because each $Z_i(\lambda_0)$ has full row rank for $i = 1, \ldots, t$, and that $M(\lambda)$ is row reduced, because each $Z_i(\lambda)$ is row reduced for $i = 1, \ldots, t$. Therefore, Theorem 2.4 guarantees that $M(\lambda)$ is a minimal basis. A similar argument shows that $N(\lambda)$ is also a minimal basis. This completes the proof of Theorem 6.1. □

There are several important points related to Theorem 6.1 that are worth highlighting.

- The first is the *complete straightforwardness* of the construction of $M(\lambda)$ and $N(\lambda)$ via the five-step procedure in Theorem 5.1, applied to each pair of lists $(\eta_{m_{i-1}+1}, \ldots, \eta_{m_i})$ and $(\varepsilon_{k_{i-1}+1}, \ldots, \varepsilon_{k_i})$, together with the construction described at the beginning of Section 4.1.

- The second is the *non-uniqueness* of dual minimal bases with prescribed row degrees satisfying the equal sum constraint (6.2). We have already seen one source of non-uniqueness in the proof of Theorem 6.1, but we emphasize that there are other sources of non-uniqueness. For instance, note that any of the diagonal blocks of $M(\lambda)$ could be replaced by a backward-zigzag matrix, together with an appropriate adjustment in the corresponding diagonal block of $N(\lambda)$. Re-ordering the lists of prescribed row degrees provides yet another source of non-uniqueness. Note that different orders of the nonzero row degrees may produce different sets of pairs of indices for matching sums $\{(m_1, k_1), (m_2, k_2), \ldots, (m_t, k_t)\}$, even yielding sets with different cardinalities and different numbers of dual zigzag matrices in the blocks of $M(\lambda)$ and $N(\lambda)$. Consider for instance $(\eta_1, \eta_2, \eta_3) = (1, 2, 3)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (2, 3, 1)$ with no matching partial sums and the reordering $(\eta'_1, \eta'_2, \eta'_3) = (1, 2, 3)$ and $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3) = (1, 2, 3)$, for which all partial sums match.

Example 6.2. Let us illustrate the construction in Theorem 6.1(a)-(b) with the lists $(\eta_1, \eta_2, \eta_3, \eta_4) = (0, 2, 4, 3)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1, 1, 2, 5)$. For these lists $m_0 = 1, k_0 = 0$,

$$\sum_{i=2}^{2} \eta_i = 2, \quad \sum_{i=2}^{3} \eta_i = 6, \quad \sum_{i=2}^{4} \eta_i = 9, \quad \text{and}$$
$$\sum_{i=1}^{1} \varepsilon_i = 1, \quad \sum_{i=1}^{2} \varepsilon_i = 2, \quad \sum_{i=1}^{3} \varepsilon_i = 4, \quad \sum_{i=1}^{4} \varepsilon_i = 9.$$

So, the set of pairs of indices for matching sums is $\{(m_1, k_1), (m_2, k_2)\} = \{(2, 2), (4, 4)\}$. Therefore, we need to apply the five-step procedure in Theorem 5.1 (in these cases it is also possible to use Theorem 5.3) to the following sublists of row degrees:

1. $(\eta_2) = (2), (\varepsilon_1, \varepsilon_2) = (1, 1)$ for building the dual zigzag matrices

$$Z_1(\lambda) = \begin{bmatrix} 1 & \lambda & \lambda^2 \end{bmatrix} \quad \text{and} \quad Z_1^{\diamond}(\lambda) = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \end{bmatrix},$$

   where the simple details of the application of Theorem 5.1 have been omitted for brevity, and

2. $(\eta_3, \eta_4) = (4, 3), (\varepsilon_3, \varepsilon_4) = (2, 5)$ for building the dual zigzag matrices

$$Z_2(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^4 & \\ & & 1 & \lambda^3 \end{bmatrix} \quad \text{and} \quad Z_2^{\diamond}(\lambda) = \begin{bmatrix} \lambda^2 & 1 & & \\ & \lambda^5 & \lambda^3 & 1 \end{bmatrix},$$

   where the details of the application of Theorem 5.1 have again been omitted.

With these zigzag matrices in hand, we finally construct the dual minimal bases in Theorem 6.1(b):

$$M(\lambda) = \left[\begin{array}{c|ccc|cccc} 1 & & & & & & & \\ \hline & 1 & \lambda & \lambda^2 & & & & \\ \hline & & & & 1 & \lambda^2 & \lambda^4 & \\ & & & & & & 1 & \lambda^3 \end{array}\right], \quad N(\lambda) = \left[\begin{array}{c|ccc|cccc} 0 & \lambda & -1 & & & & & \\ & & -\lambda & 1 & & & & \\ \hline & & & & \lambda^2 & -1 & & \\ & & & & & -\lambda^5 & \lambda^3 & -1 \end{array}\right],$$

which realize the prescribed row degrees $(\eta_1, \eta_2, \eta_3, \eta_4) = (0, 2, 4, 3)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1, 1, 2, 5)$, respectively.

Remark 6.3. An alternative method for doing many of the computations described in Sections 4, 5, and 6 makes use of the notion of a number zigzag tableau. Although this method requires some additional definitions and results, it has the advantage of providing a simple and unified solution of the inverse row degree problems for dual zigzag matrices and for dual minimal bases, i.e., for Theorems 5.1 and 6.1. A development of this tableau method, together with illustrative examples, can be found in Appendix A of the preprint [4].

We conclude this section by combining the classical result of Forney stated in Theorem 2.12 with the existence part of Theorem 6.1, to obtain the following definitive characterization theorem on row degrees of dual minimal bases.

Theorem 6.4. *There exists a pair* $M(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+k)}$ *and* $N(\lambda) \in \mathbb{F}[\lambda]^{k \times (m+k)}$ *of dual minimal bases with row degrees* $(\eta_1, \ldots, \eta_m)$ *and* $(\varepsilon_1, \ldots, \varepsilon_k)$*, respectively, if and only if* $\sum_{i=1}^{m} \eta_i = \sum_{j=1}^{k} \varepsilon_j$.

7 Explicit Realization of Completely Singular Polynomials

The recent paper [6] has established very simple necessary and sufficient conditions for the existence of a polynomial matrix when its degree, finite and infinite elementary divisors, and left and right minimal indices are prescribed. The proof of these necessary and sufficient conditions (see Theorem 3.3 or the equivalent formulation Theorem 3.12 in [6]) combines the Index Sum Theorem (stated in Theorem 2.13) with the construction of a polynomial matrix $P(\lambda)$ which realizes the prescribed structure and degree. However, the construction in [6] is rather complicated, and the prescribed elementary divisors and minimal indices are not apparent "by simple inspection" of $P(\lambda)$. In this section, we present a first step towards constructing a polynomial $P(\lambda)$ which displays "by simple inspection" the prescribed structure. To this end, we consider the more restricted problem of realizing prescribed lists of only left and right minimal indices by a *completely singular polynomial matrix* of arbitrary prescribed degree d, i.e., by a singular polynomial matrix of degree d *with no elementary divisors at all*, associated to either finite or infinite eigenvalues. In this case, the necessary and sufficient conditions established in [6] become even simpler, since given a degree d, a list $(\eta_1, \eta_2, \ldots, \eta_m)$ of left minimal indices, and a list $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ of right minimal indices, there exists a completely singular polynomial matrix $Q(\lambda)$ of degree d having these left and right minimal indices if and only if

$$d \text{ is a divisor of } \mu := \left(\sum_{i=1}^{m} \eta_i + \sum_{j=1}^{k} \varepsilon_j \right). \tag{7.1}$$

Observe that the necessary and sufficient condition (7.1) amounts to saying that d, $(\eta_1, \eta_2, \ldots, \eta_m)$, and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ satisfy the Index Sum Theorem. We will see that zigzag matrices and their properties enable us to give an elegant solution to this completely singular realization problem; based on them we will show how to explicitly construct a realization $Q(\lambda)$ that is very simple, and has the additional property that its left and right minimal indices are immediately apparent "by inspection", in the same sense as the minimal indices of a pencil in Kronecker canonical form can be read off essentially by inspection.

Our final realization result is Theorem 7.8, whose proof relies on the solution of two simpler realization problems in Lemmas 7.1 and 7.5. These simpler results provide the building blocks of the definitive realization result, but are also of independent interest. Our first result, Lemma 7.1, considers the realization of a list of positive right minimal indices which is not decomposable (in the given order) into shorter realizable lists. It is a direct corollary of Theorem 5.3 and Corollary 4.10.

Lemma 7.1. *Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ be a list of positive integers and d another positive integer such that*

$$\sum_{i=1}^{k} \varepsilon_i = d\, r \quad \text{and} \quad \sum_{i=1}^{\beta} \varepsilon_i \text{ is not a multiple of } d \text{ for } \beta = 1, \ldots, k-1, \tag{7.2}$$

where r is an integer. Then there exists a unique forward-zigzag matrix $Z(\lambda) \in \mathbb{F}[\lambda]^{r\times(r+k)}$ with row degrees all equal to d such that its dual backward-zigzag matrix $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{k\times(r+k)}$ has row degrees $(\varepsilon_1, \ldots, \varepsilon_k)$. This $Z(\lambda)$ is a completely singular polynomial matrix of degree d with right minimal indices $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ and no left minimal indices.

Proof. The assumptions (7.2) are precisely the assumptions (5.5) in Theorem 5.3 for $w_k = d$ and $q_k = r - 1$. Therefore, Theorem 5.3 implies the existence and uniqueness of $Z(\lambda) \in \mathbb{F}[\lambda]^{r\times(r+k)}$ and $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{k\times(r+k)}$ with the row degrees of the statement. It is obvious that $Z(\lambda)$ is a polynomial matrix with degree d, and Corollary 4.10 guarantees that $Z(\lambda)$ has no elementary divisors at all, has right minimal indices $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, and has no left minimal indices. ☐

Remark 7.2. Since $Z(\lambda)^T$ in Lemma 7.1 is a completely singular polynomial matrix of degree d which realizes the list of positive *left minimal indices* $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, we see that Lemma 7.1 also allows us to realize prescribed lists of left minimal indices with prescribed degree.

Remark 7.3. Observe that given a list of prescribed positive right minimal indices $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ satisfying (7.2), parts (a), (b), (c), and (d) in Theorem 5.3 allow us to very easily construct the completely singular polynomial matrix $Z(\lambda)$ mentioned in Lemma 7.1, which has degree d and realizes these minimal indices. In addition, since $Z(\lambda)$ is a forward-zigzag matrix, it has a very simple structure, and its right minimal indices can be read off essentially by inspection of $Z(\lambda)$ via Lemma 4.4(b), i.e., as the row degrees of its dual.

Example 7.4. Given the list $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = (2, 3, 5, 1, 5)$ of prescribed right minimal indices and the prescribed degree $d = 4$, Example 5.4 shows how to construct the completely singular polynomial matrix $Z(\lambda)$ in (5.9) with the prescribed degree 4 and the prescribed right minimal indices.

Note that if the condition $\sum_{i=1}^{k} \varepsilon_i = dr$ in (7.2) holds, but $\sum_{i=1}^{\beta} \varepsilon_i$ is a multiple of d for some $\beta = 1, \ldots, k-1$, then $\sum_{i=\beta+1}^{k} \varepsilon_i$ is also a multiple of d, and we can separately realize with two completely singular polynomials $Z_1(\lambda)$ and $Z_2(\lambda)$ of degree d the two lists of right minimal indices $(\varepsilon_1, \ldots, \varepsilon_\beta)$ and $(\varepsilon_{\beta+1}, \ldots, \varepsilon_k)$, respectively; in fact, if both conditions in (7.2) hold for the corresponding sublists, then $Z_1(\lambda)$ and $Z_2(\lambda)$ can be chosen to be forward-zigzag matrices by Lemma 7.1. Then, according to Lemma 2.8, $Z_1(\lambda) \oplus Z_2(\lambda)$ is a completely singular polynomial matrix of degree d which realizes the whole list $(\varepsilon_1, \ldots, \varepsilon_k)$ of right minimal indices. In short, we see that if the second condition in (7.2) is not satisfied, then we can decompose the realization problem into two smaller realization subproblems for the same degree. However, *we emphasize* that even when (7.2) is satisfied, it may be possible to decompose the realization problem into smaller ones with the same degree by a suitable re-ordering of the given list of minimal indices. Consider for instance the list $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = (2, 3, 5, 1, 5)$ of right minimal indices in Example 7.4 and its sublists $(\varepsilon_1, \varepsilon_4, \varepsilon_5) = (2, 1, 5)$ and $(\varepsilon_2, \varepsilon_3) = (3, 5)$, which each satisfy both conditions in (7.2) for $d = 4$, and so can be realized independently by two completely singular polynomials

$$Z_1(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^3 & \lambda^4 & \\ & & & 1 & \lambda^4 \end{bmatrix} \quad \text{and} \quad Z_2(\lambda) = \begin{bmatrix} 1 & \lambda^3 & \lambda^4 & \\ & & 1 & \lambda^4 \end{bmatrix}$$

of degree 4. Therefore, $Z_1(\lambda) \oplus Z_2(\lambda)$ is another completely singular polynomial of degree 4 with right minimal indices $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = (2, 3, 5, 1, 5)$.

Lemma 7.5 considers the joint realization of two lists of positive left and right minimal indices which are not decomposable (in the given order) into shorter realizable lists.

Lemma 7.5. *Let $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ be two lists of positive integers and d another positive integer such that*

(i) $\displaystyle\sum_{i=1}^{m} \eta_i + \sum_{j=1}^{k} \varepsilon_j = d\, r$, *for an integer r,*

(ii) $\displaystyle\sum_{i=1}^{\alpha} \eta_i$ *is not a multiple of d for $\alpha = 1, \ldots, m$, and* $\displaystyle\sum_{i=1}^{\beta} \varepsilon_i$ *is not a multiple of d for $\beta = 1, \ldots, k$.*

Let

$$\sum_{i=1}^{m} \eta_i = d\, \widetilde{q}_m + \widetilde{w}_m \ \text{ with } 0 < \widetilde{w}_m < d \qquad \text{and} \qquad \sum_{i=1}^{k} \varepsilon_i = d\, q_k + w_k, \ \text{ with } 0 < w_k < d, \tag{7.3}$$

for nonnegative integers $\widetilde{q}_m, q_k$ and positive integers $\widetilde{w}_m, w_k$. Then:

(a) *There exists a unique forward-zigzag matrix $\widetilde{Z}(\lambda) \in \mathbb{F}[\lambda]^{(\widetilde{q}_m+1)\times(\widetilde{q}_m+1+m)}$ with row degrees $(d, \ldots, d, \widetilde{w}_m)$, such that its dual backward-zigzag matrix $\widetilde{Z}^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{m\times(\widetilde{q}_m+1+m)}$ has row degrees $(\eta_1, \ldots, \eta_m)$.*

(b) *There exists a unique forward-zigzag matrix $Z(\lambda) \in \mathbb{F}[\lambda]^{(q_k+1)\times(q_k+1+k)}$ with row degrees $(d, \ldots, d, w_k)$, such that its dual backward-zigzag matrix $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{k\times(q_k+1+k)}$ has row degrees $(\varepsilon_1, \ldots, \varepsilon_k)$.*

(c) *The polynomial matrix*

$$P(\lambda) := \left[\begin{array}{c|c} I_{q_k} & \\ \hline & R \cdot \widetilde{Z}(\lambda)^T \cdot R' \end{array}\right] \left[\begin{array}{c|c} Z(\lambda) & \\ \hline & I_{\widetilde{q}_m} \end{array}\right], \tag{7.4}$$

where R and R' are reverse identity matrices, is a completely singular polynomial matrix of degree d, rank r, size $(r+m) \times (r+k)$, with left minimal indices $(\eta_1, \eta_2, \ldots, \eta_m)$, and with right minimal indices $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$.

Proof. (a) The first assumption in Lemma 7.5(ii) and the first equality in (7.3) are a particular case of the assumptions (5.5) in Theorem 5.3 with $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ replaced by $(\eta_1, \eta_2, \ldots, \eta_m)$. Therefore, Theorem 5.3 implies the existence and uniqueness of $\widetilde{Z}(\lambda)$ and $\widetilde{Z}^{\diamondsuit}(\lambda)$ with the sizes and row degrees stated in part (a).

(b) The proof is the same as that of (a), using the second assumption in Lemma 7.5(ii) and the second equality in (7.3).

(c) Observe first that, from (7.3) and the assumption Lemma 7.5(i), we have $d(\widetilde{q}_m + q_k) + (\widetilde{w}_m + w_k) = dr$, which implies $(\widetilde{w}_m + w_k) = d(r - \widetilde{q}_m - q_k)$, and so that $\widetilde{w}_m + w_k$ is a multiple of d. In fact, since $0 < \widetilde{w}_m < d$ and $0 < w_k < d$,

$$\widetilde{w}_m + w_k = d \qquad \text{and} \qquad \widetilde{q}_m + q_k + 1 = r \tag{7.5}$$

must hold. Therefore, the factors defining $P(\lambda)$ in (7.4) have the sizes

$$A(\lambda) := \left[\begin{array}{c|c} I_{q_k} & \\ \hline & R \cdot \widetilde{Z}(\lambda)^T \cdot R' \end{array}\right] \in \mathbb{F}[\lambda]^{(r+m)\times r}, \qquad B(\lambda) := \left[\begin{array}{c|c} Z(\lambda) & \\ \hline & I_{\widetilde{q}_m} \end{array}\right] \in \mathbb{F}[\lambda]^{r\times(r+k)}, \tag{7.6}$$

hence $P(\lambda)$ has size $(r+m)\times(r+k)$. Since any forward-zigzag matrix has full row rank, we see that $A(\lambda)$ has full column rank equal to r and $B(\lambda)$ has full row rank equal to r; consequently, the rank of $P(\lambda)$ is also r. Moreover, these rank properties imply that the null spaces of $P(\lambda)$ over $\mathbb{F}(\lambda)$ satisfy

$$\mathcal{N}_\ell(P) = \mathcal{N}_\ell(A) \quad \text{and} \quad \mathcal{N}_r(P) = \mathcal{N}_r(B)\,,$$

which in turn imply that the left minimal indices of $P(\lambda)$ and $A(\lambda)$ are equal, and that the right minimal indices of $P(\lambda)$ and $B(\lambda)$ are equal. By using Lemma 2.6, we see that the right minimal indices of $B(\lambda)$ are those of $Z(\lambda)$, which are $(\varepsilon_1, \ldots, \varepsilon_k)$ by Corollary 4.10. The left minimal indices of $A(\lambda)$ are the right minimal indices of $A(\lambda)^T$, which are the same as those of $R' \cdot \widetilde{Z}(\lambda) \cdot R$, by Lemma 2.6; these in turn are the same as those of $\widetilde{Z}(\lambda)$, since R' and R are constant nonsingular matrices, that is, they are $(\eta_1, \eta_2, \ldots, \eta_m)$, again by Corollary 4.10. This establishes that $P(\lambda)$ has left minimal indices $(\eta_1, \eta_2, \ldots, \eta_m)$ and right minimal indices $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$.

Next, we prove that the degree of $P(\lambda)$ is d. For this purpose we partition $R \cdot \widetilde{Z}(\lambda)^T \cdot R'$ into its first column and its remaining $\widetilde{q}_m$ columns, and $Z(\lambda)$ into its first q_k rows and its last row. More precisely

$$R \cdot \widetilde{Z}(\lambda)^T \cdot R' = \left[\begin{array}{c|c} X_{11}(\lambda) & X_{12}(\lambda) \\ \hline 0 & X_{22}(\lambda) \end{array}\right] \qquad \text{and} \qquad Z(\lambda) = \left[\begin{array}{c|c} Z_{11}(\lambda) & Z_{12}(\lambda) \\ \hline 0 & Z_{22}(\lambda) \end{array}\right], \tag{7.7}$$

where $X_{11}(\lambda) = \begin{bmatrix} \lambda^{\widetilde{w}_m} & \cdots & 1 \end{bmatrix}^T$ and $Z_{22}(\lambda) = \begin{bmatrix} 1 & \cdots & \lambda^{w_k} \end{bmatrix}$. Although it is not important in our argument, note that the zigzag structures of $Z(\lambda)$ and $\widetilde{Z}(\lambda)$ imply that the only nonzero entry in $X_{12}(\lambda)$ and the only nonzero entry in $Z_{12}(\lambda)$ is, in both cases, λ^d placed in the lower-left corner. Inserting (7.7) into (7.4), we get partitions of the factors defining $P(\lambda)$ which are conformable for matrix multiplication:

$$P(\lambda) = \left[\begin{array}{c|c|c} I_{q_k} & 0 & 0 \\ \hline 0 & X_{11}(\lambda) & X_{12}(\lambda) \\ \hline 0 & 0 & X_{22}(\lambda) \end{array}\right]\left[\begin{array}{c|c|c} Z_{11}(\lambda) & Z_{12}(\lambda) & 0 \\ \hline 0 & Z_{22}(\lambda) & 0 \\ \hline 0 & 0 & I_{\widetilde{q}_m} \end{array}\right] = \left[\begin{array}{c|c|c} Z_{11}(\lambda) & Z_{12}(\lambda) & 0 \\ \hline 0 & X_{11}(\lambda)Z_{22}(\lambda) & X_{12}(\lambda) \\ \hline 0 & 0 & X_{22}(\lambda) \end{array}\right]. \tag{7.8}$$

Since $\begin{bmatrix} Z_{11}(\lambda) & Z_{12}(\lambda) \end{bmatrix}$ and $\begin{bmatrix} X_{12}(\lambda)^T & X_{22}(\lambda)^T \end{bmatrix}^T$ both have degree d (if they are not empty), and

$$X_{11}(\lambda)Z_{22}(\lambda) = \begin{bmatrix} \lambda^{\widetilde{w}_m} \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & \lambda^{w_k} \end{bmatrix} = \begin{bmatrix} \lambda^{\widetilde{w}_m} & \cdots & \lambda^{\widetilde{w}_m + w_k} \\ \vdots & & \vdots \\ 1 & \cdots & \lambda^{w_k} \end{bmatrix}$$

also has degree d by (7.5), we see that $P(\lambda)$ does indeed have degree d.

It only remains to prove that the polynomial matrix $P(\lambda)$ is completely singular. This follows immediately from assumption Lemma 7.5(i) and the Index Sum Theorem (Theorem 2.13), since we have already proved that $P(\lambda)$ has degree d, rank r, left minimal indices $(\eta_1, \eta_2, \ldots, \eta_m)$, and right minimal indices $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$. □

Observe that, as we commented in Remark 7.3, given the lists $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ of prescribed positive left and right minimal indices satisfying the assumptions of Lemma 7.5, parts (a), (b), (c), and (d) in Theorem 5.3 allow us to very easily construct the forward-zigzag matrices $\widetilde{Z}(\lambda)$ and $Z(\lambda)$ and, therefore, the completely singular polynomial matrix $P(\lambda)$ in (7.4), which has degree d and realizes these left and right minimal indices. $P(\lambda)$ inherits a very simple factored structure from $\widetilde{Z}(\lambda)$ and $Z(\lambda)$, and we can read off the left and right minimal indices of $P(\lambda)$ essentially by inspection of its factors via Lemma 4.4(b) applied to $\widetilde{Z}(\lambda)$ and $Z(\lambda)$, i.e., as the row degrees of the dual backward-zigzag matrices $\widetilde{Z}^\diamond(\lambda)$ and $Z^\diamond(\lambda)$. Even more, the proof of Lemma 7.5 shows us that the multiplied-out form of $P(\lambda)$ in (7.8) is also very simple and can be directly and explicitly constructed via Theorem 5.3. In addition, we can also read off the left and right minimal indices of $P(\lambda)$ by inspection of (7.8) since $\widetilde{Z}(\lambda)$ and $Z(\lambda)$ are totally visible in (7.8); the nonzero entries of the first column of $R \cdot \widetilde{Z}(\lambda)^T \cdot R'$ form the first column of $X_{11}(\lambda)Z_{22}(\lambda)$ and the nonzero entries of the last row of $Z(\lambda)$ form the last row of $X_{11}(\lambda)Z_{22}(\lambda)$. Therefore, it is worth highlighting the explicit multiplied-out form (7.8) of $P(\lambda)$ in the next corollary of Lemma 7.5.

Corollary 7.6. *Consider the same assumptions and notation as in Lemma 7.5, and partition the matrices $R \cdot \widetilde{Z}(\lambda)^T \cdot R'$ and $Z(\lambda)$ as follows:*

$$R \cdot \widetilde{Z}(\lambda)^T \cdot R' = \left[\begin{array}{c|c} X_{11}(\lambda) & X_{12}(\lambda) \\ \hline 0 & X_{22}(\lambda) \end{array}\right] \qquad \text{and} \qquad Z(\lambda) = \left[\begin{array}{c|c} Z_{11}(\lambda) & Z_{12}(\lambda) \\ \hline 0 & Z_{22}(\lambda) \end{array}\right],$$

where $X_{11}(\lambda)$ has only one column and $Z_{22}(\lambda)$ has only one row, with the structures $X_{11}(\lambda) = \begin{bmatrix} \lambda^{\widetilde{w}_m} & \cdots & 1 \end{bmatrix}^T$ and $Z_{22}(\lambda) = \begin{bmatrix} 1 & \cdots & \lambda^{w_k} \end{bmatrix}$. Then the polynomial matrix $P(\lambda)$ in (7.4) can be written as

$$P(\lambda) = \left[\begin{array}{c|c|c} Z_{11}(\lambda) & Z_{12}(\lambda) & 0 \\ \hline 0 & X_{11}(\lambda)Z_{22}(\lambda) & X_{12}(\lambda) \\ \hline 0 & 0 & X_{22}(\lambda) \end{array}\right], \quad \text{where} \quad X_{11}(\lambda)Z_{22}(\lambda) = \begin{bmatrix} \lambda^{\widetilde{w}_m} & \cdots & \lambda^{\widetilde{w}_m + w_k} \\ \vdots & & \vdots \\ 1 & \cdots & \lambda^{w_k} \end{bmatrix}.$$

Example 7.7. To illustrate the construction of $P(\lambda)$ in Lemma 7.5 and Corollary 7.6, we consider the lists $(\eta_1, \eta_2) = (3, 4)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1, 2, 3, 2)$, take $d = 5$, and observe that for this example $m = 2$ and $k = 4$. Note that

$$\sum_{i=1}^{1} \eta_i = 3 = 5 \cdot 0 + 3, \quad \sum_{i=1}^{2} \eta_i = 7 = 5 \cdot 1 + 2,$$
$$\sum_{i=1}^{1} \varepsilon_i = 1 = 5 \cdot 0 + 1, \quad \sum_{i=1}^{2} \varepsilon_i = 3 = 5 \cdot 0 + 3, \quad \sum_{i=1}^{3} \varepsilon_i = 6 = 5 \cdot 1 + 1, \quad \sum_{i=1}^{4} \varepsilon_i = 8 = 5 \cdot 1 + 3,$$

and $\sum_{i=1}^{2} \eta_i + \sum_{i=1}^{4} \varepsilon_i = 15$. Therefore, these lists satisfy the assumptions in Lemma 7.5, and we have the following values for the parameters appearing in Lemma 7.5: $r = 3$, $\widetilde{q}_2 = 1$, and $q_4 = 1$, so $P(\lambda)$ has size $(r + m) \times (r + k) = 5 \times 7$.

Theorem 5.3 applied to (η_1, η_2) allows us to construct the forward-zigzag matrix $\widetilde{Z}(\lambda) \in \mathbb{F}[\lambda]^{2 \times 4}$: the two leading 1's are in positions $(1, 1)$ and $(2, 2 + 1) = (2, 3)$; λ^3 is in position $(1, 1 + 1) = (1, 2)$; and λ^2 is in position $(2, 2 + 2) = (2, 4)$. With this information, we get

$$\widetilde{Z}(\lambda) = \begin{bmatrix} 1 & \lambda^3 & \lambda^5 & \\ & & 1 & \lambda^2 \end{bmatrix} \quad \text{and} \quad R \cdot \widetilde{Z}(\lambda)^T \cdot R' = \left[\begin{array}{c|c} \lambda^2 & \\ 1 & \lambda^5 \\ \hline & \lambda^3 \\ & 1 \end{array}\right],$$

where we have indicated in $R \cdot \widetilde{Z}(\lambda)^T \cdot R'$ the partition used in Corollary 7.6.

Theorem 5.3 applied to $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ allows us to construct the forward-zigzag matrix $Z(\lambda) \in \mathbb{F}[\lambda]^{2 \times 6}$: the two leading 1's are in positions $(1, 1)$ and $(2, 2 + 2) = (2, 4)$; λ is in position $(1, 1 + 1) = (1, 2)$; λ^3 is in position $(1, 1 + 2) = (1, 3)$; λ is in position $(2, 2 + 3) = (2, 5)$; and λ^3 is in position $(2, 2 + 4) = (2, 6)$. With this information, we get

$$Z(\lambda) = \left[\begin{array}{ccc|ccc} 1 & \lambda & \lambda^3 & \lambda^5 & & \\ \hline & & & 1 & \lambda & \lambda^3 \end{array}\right],$$

where we have indicated in $Z(\lambda)$ the partition used in Corollary 7.6.

The factored form of $P(\lambda)$ in (7.4) follows immediately from $R \cdot \widetilde{Z}(\lambda)^T \cdot R'$ and $Z(\lambda)$, as does the multiplied-out form in Corollary 7.6, which is

$$P(\lambda) = \left[\begin{array}{ccc|ccc|c} 1 & \lambda & \lambda^3 & \lambda^5 & & & \\ \hline & & & \lambda^2 & \lambda^3 & \lambda^5 & \\ & & & 1 & \lambda & \lambda^3 & \lambda^5 \\ \hline & & & & & & \lambda^3 \\ & & & & & & 1 \end{array}\right].$$

Now we are in position to state and prove our final realizability result for completely singular polynomial matrices with prescribed degree, which is Theorem 7.8. In this result, we assume without loss of generality that zero terms, if any, in the prescribed lists of minimal indices are placed in the initial positions. Also, we assume implicitly that there is at least one positive minimal index, since otherwise the lists of minimal indices can be trivially realized with constant matrices of infinitely many different sizes, and cannot be realized with polynomials of any degree larger than or equal to 1. Except for this, the only assumption in Theorem 7.8 is that the prescribed minimal indices satisfy the Index Sum Theorem.

Theorem 7.8. *Let* $(\eta_1, \eta_2, \dots, \eta_m)$ *and* $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ *be two lists of nonnegative integers, and* d *another positive integer such that*

$$\sum_{i=1}^{m} \eta_i + \sum_{j=1}^{k} \varepsilon_j = d\, r, \tag{7.9}$$

for a positive integer r. *Let the lists* $(\eta_1, \eta_2, \dots, \eta_m)$ *and* $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ *be ordered so that*

$$0 = \eta_1 = \cdots = \eta_{m_0}, \;\; 0 < \eta_i \;\; \textit{if } m_0 < i \qquad \textit{and} \qquad 0 = \varepsilon_1 = \cdots = \varepsilon_{k_0}, \;\; 0 < \varepsilon_j \;\; \textit{if } k_0 < j\,.$$

Define the sets

$$\{m_1, \dots, m_s\} = \left\{ \alpha \, : \, m_0 < \alpha \leq m \quad \textit{and} \quad d \textit{ is a divisor of } \sum_{i=1}^{\alpha} \eta_i \right\},$$

$$\{k_1, \dots, k_t\} = \left\{ \beta \, : \, k_0 < \beta \leq k \quad \textit{and} \quad d \textit{ is a divisor of } \sum_{i=1}^{\beta} \varepsilon_i \right\},$$

where $m_1 < \cdots < m_s$ *and* $k_1 < \cdots < k_t$, *and define the sequences of integers*

$$\widetilde{r}_j = \frac{\eta_{m_{j-1}+1} + \cdots + \eta_{m_j}}{d}, \;\; \textit{for } j = 1, \dots, s, \qquad \textit{and} \qquad r_j = \frac{\varepsilon_{k_{j-1}+1} + \cdots + \varepsilon_{k_j}}{d}, \;\; \textit{for } j = 1, \dots, t\,.$$

Then:

(a) *For each* $j = 1, \dots, s$, *there exists a unique forward-zigzag matrix* $\widetilde{Z}_j(\lambda) \in \mathbb{F}[\lambda]^{\widetilde{r}_j \times (\widetilde{r}_j + m_j - m_{j-1})}$ *with row degrees all equal to* d, *such that its dual backward-zigzag matrix has row degrees* $(\eta_{m_{j-1}+1}, \dots, \eta_{m_j})$.

(b) *For each* $j = 1, \dots, t$, *there exists a unique forward-zigzag matrix* $Z_j(\lambda) \in \mathbb{F}[\lambda]^{r_j \times (r_j + k_j - k_{j-1})}$ *with row degrees all equal to* d, *such that its dual backward-zigzag matrix has row degrees* $(\varepsilon_{k_{j-1}+1}, \dots, \varepsilon_{k_j})$.

(c) *If* $m_s \neq m$ *(or equivalently if* $k_t \neq k$*), then there exists a completely singular polynomial matrix* $Q(\lambda)$ *of degree* d, *with left minimal indices* $(\eta_{m_s+1}, \dots, \eta_m)$, *and with right minimal indices* $(\varepsilon_{k_t+1}, \dots, \varepsilon_k)$. *In addition,* $Q(\lambda)$ *can be constructed by applying Lemma 7.5 to the lists* $(\eta_{m_s+1}, \dots, \eta_m)$ *and* $(\varepsilon_{k_t+1}, \dots, \varepsilon_k)$.

(d) *The polynomial matrix*

$$P(\lambda) = \begin{bmatrix} 0_{m_0 \times k_0} & & & & & & & \\ & Z_1(\lambda) & & & & & & \\ & & \ddots & & & & & \\ & & & Z_t(\lambda) & & & & \\ & & & & Q(\lambda) & & & \\ & & & & & R^{(1)} \cdot \widetilde{Z}_1(\lambda)^T \cdot R'^{(1)} & & \\ & & & & & & \ddots & \\ & & & & & & & R^{(s)} \cdot \widetilde{Z}_s(\lambda)^T \cdot R'^{(s)} \end{bmatrix},$$

where $R^{(i)}$ *and* $R'^{(i)}$ *are reverse identity matrices for* $i = 1, \dots, s$, *is a completely singular polynomial matrix of degree* d, *rank* r, *size* $(r+m) \times (r+k)$, *with left minimal indices* $(\eta_1, \eta_2, \dots, \eta_m)$, *and right minimal indices* $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$.

Proof. (a) Since $\sum_{i=1}^{m_j} \eta_i$ and $\sum_{i=1}^{m_{j-1}} \eta_i$ are multiples of d, we have that $\eta_{m_{j-1}+1} + \cdots + \eta_{m_j} = \sum_{i=1}^{m_j} \eta_i - \sum_{i=1}^{m_{j-1}} \eta_i$ is also a multiple of d. In addition, observe that $\eta_{m_{j-1}+1} + \cdots + \eta_\alpha$ is not a multiple of d for any $\alpha = m_{j-1}+1, \dots, m_j - 1$, because otherwise the set $\{m_1, \dots, m_s\}$ would have additional elements. Therefore the list $(\eta_{m_{j-1}+1}, \dots, \eta_{m_j})$ satisfies (7.2), so Lemma 7.1 guarantees the existence and uniqueness of $\widetilde{Z}_j(\lambda)$.

(b) The proof is completely analogous to that of (a).

(c) $m_s \neq m$ if and only if $k_t \neq k$, because from the definitions of m_s and k_t and (7.9) we get that $m_s = m$ implies that $\sum_{j=1}^{k} \varepsilon_j$ is a multiple of d, and so $k_t = k$; similarly $k_t = k$ implies that $\sum_{i=1}^{m} \eta_i$ is a multiple of d, and so $m_s = m$. Therefore if $m_s \neq m$, we get from (7.9) that

$$\sum_{i=m_s+1}^{m} \eta_i + \sum_{j=k_t+1}^{k} \varepsilon_j = d\, r_{mix}$$

for some positive integer r_{mix}. In addition, $\sum_{i=m_s+1}^{\alpha} \eta_i$ and $\sum_{j=k_t+1}^{\beta} \varepsilon_j$ are not multiples of d for $\alpha = m_s+1, \ldots, m$ and $\beta = k_t+1, \ldots, k$, because otherwise the sets $\{m_1, \ldots, m_s\}$ and $\{k_1, \ldots, k_t\}$ would have additional elements. Therefore the lists $(\eta_{m_s+1}, \ldots, \eta_m)$ and $(\varepsilon_{k_t+1}, \ldots, \varepsilon_k)$ satisfy the assumptions in Lemma 7.5(i)-(ii), so Lemma 7.5 provides the construction of the desired $Q(\lambda)$.

(d) Lemma 2.8, Corollary 4.10, and the properties of $Z_1(\lambda), \ldots, Z_t(\lambda), Q(\lambda), \widetilde{Z}_1(\lambda)^T, \ldots, \widetilde{Z}_s(\lambda)^T$ imply that $P(\lambda)$ has degree d, is completely singular, has left minimal indices $(\eta_1, \eta_2, \ldots, \eta_m)$, and right minimal indices $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$. Observe that $\widetilde{Z}_i(\lambda)^T$ and $R^{(i)} \cdot \widetilde{Z}_i(\lambda)^T \cdot R'^{(i)}$ have the same minimal indices (and elementary divisors, i.e., none) for $i = 1, \ldots, s$. The fact that the rank of $P(\lambda)$ is r follows from the Index Sum Theorem (Theorem 2.13) and (7.9), and its size is $(r+m) \times (r+k)$ as a consequence of the rank-nullity theorem applied to the dimensions of the left and right null spaces of $P(\lambda)$. □

Remark 7.9. Observe that the blocks $R^{(i)} \cdot \widetilde{Z}_i(\lambda)^T \cdot R'^{(i)}$ in the polynomial matrix $P(\lambda)$ in Theorem 7.8(d) can be replaced by $\widetilde{Z}_i(\lambda)^T$ for $i = 1, \ldots, s$, without changing any of the properties proved for $P(\lambda)$. This second option is commonly used in the literature in the description of the Kronecker canonical form of pencils [9]. However, the use of $R^{(i)} \cdot \widetilde{Z}_i(\lambda)^T \cdot R'^{(i)}$ might have advantages in proving structured versions of Theorem 7.8 for structured matrix polynomials and, moreover, it is coherent with the factored form of the block $Q(\lambda)$, in the middle of $P(\lambda)$, presented in Lemma 7.5(c). Note that in Lemma 7.5 the use of reverse identity matrices is needed in order to construct a polynomial with the right degree.

Example 7.10. We illustrate Theorem 7.8 by realizing the prescribed lists $(\eta_1, \eta_2, \eta_3, \eta_4) = (0, 5, 3, 4)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) = (6, 4, 1, 2, 3, 2)$ of left and right minimal indices with a completely singular polynomial of degree $d = 5$. Observe that

$$\sum_{i=1}^{4} \eta_i + \sum_{j=1}^{6} \varepsilon_j = 30\,,$$

so $r = 6$, and the realizing polynomial $P(\lambda)$ will have size $(r+m) \times (r+k) = 10 \times 12$. The key parameters in Theorem 7.8 are $m_0 = 1$, $k_0 = 0$, $s = 1$ and $m_1 = 2$, and $t = 1$ and $k_1 = 2$. Therefore, we have to realize independently the lists $(\eta_2) = (5)$ of one left minimal index, $(\varepsilon_1, \varepsilon_2) = (6, 4)$ of two right minimal indices, and jointly the lists $(\eta_3, \eta_4) = (3, 4)$ and $(\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) = (1, 2, 3, 2)$. Observe that (η_3, η_4) and $(\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6)$ were already realized in Example 7.7.

The list $(\eta_2) = (5)$ is immediately realized by $\widetilde{Z}_1(\lambda) = \begin{bmatrix} 1 & \lambda^5 \end{bmatrix}$, although the reader is invited to check that the procedure in Theorem 5.3 also produces this forward-zigzag matrix.

For the list $(\varepsilon_1, \varepsilon_2) = (6, 4)$, we have

$$\sum_{i=1}^{1} \varepsilon_i = 6 = 5 \cdot 1 + 1, \qquad \sum_{i=1}^{2} \varepsilon_i = 10 = 5 \cdot 1 + 5\,.$$

So in Theorem 5.3, $q_1 = 1$, $w_1 = 1$, $q_2 = 1$, $w_2 = 5$, which implies that $Z(\lambda)$ has size 2×4. The positions of the leading 1s are $(1,1)$ and $(2, 2+0) = (2,2)$, the position of λ is $(2,3)$, and the position of λ^5 is $(2,4)$. With this information, we get

$$Z_1(\lambda) = \begin{bmatrix} 1 & \lambda^5 & & \\ & 1 & \lambda & \lambda^5 \end{bmatrix}.$$

Finally we gather all these matrices together in the desired 10×12 $P(\lambda)$:

$$P(\lambda) = \left[\begin{array}{cccc|ccccccc|c}
0 & & & & & & & & & & & \\ \hline
1 & \lambda^5 & & & & & & & & & & \\
 & 1 & \lambda & \lambda^5 & & & & & & & & \\ \hline
 & & & & 1 & \lambda & \lambda^3 & \lambda^5 & & & & \\
 & & & & & & & \lambda^2 & \lambda^3 & \lambda^5 & & \\
 & & & & & & & 1 & \lambda & \lambda^3 & \lambda^5 & \\
 & & & & & & & & & & \lambda^3 & \\
 & & & & & & & & & & 1 & \\ \hline
 & & & & & & & & & & & \lambda^5 \\
 & & & & & & & & & & & 1
\end{array}\right].$$

Remark 7.11. The polynomial $P(\lambda)$ in Theorem 7.8(d) can be factored as a product of a full column rank polynomial matrix times a full row rank polynomial matrix of sizes $(r+m) \times r$ and $r \times (r+k)$, respectively, and such that the left minimal indices of the first factor are the left minimal indices of $P(\lambda)$ and the right minimal indices of the second factor are the right minimal indices of $P(\lambda)$. Such a factorization follows from factoring the polynomial $Q(\lambda)$ in the middle of the block diagonal of $P(\lambda)$ according to (7.4) in Lemma 7.5. If $\widetilde{Z}(\lambda)$ and $Z(\lambda)$ are the forward-zigzag matrices involved in the factorization of $Q(\lambda)$, and we define $B(\lambda) = Z_1(\lambda) \oplus \cdots \oplus Z_t(\lambda) \oplus Z(\lambda)$ and $A(\lambda) = \big(R' \cdot \widetilde{Z}(\lambda) \cdot R\big) \oplus \big(R'^{(1)} \cdot \widetilde{Z}_1(\lambda) \cdot R^{(1)}\big) \oplus \cdots \oplus \big(R'^{(s)} \cdot \widetilde{Z}_s(\lambda) \cdot R^{(s)}\big)$, then

$$P(\lambda) = \left[\begin{array}{cc} \multicolumn{2}{c}{0_{m_0 \times r}} \\ \hline I & 0 \\ 0 & A(\lambda)^T \end{array}\right] \left[\begin{array}{c|cc} 0_{r \times k_0} & B(\lambda) & 0 \\ & 0 & I \end{array}\right],$$

where the column index of the first column of $A(\lambda)^T$ is the same as the row index of the last row of $B(\lambda)$.

8 Conclusions

We have introduced the new class of polynomial zigzag matrices, and employed them to solve the inverse row degree problem for dual minimal bases in its most general form; equivalently, we have solved the inverse minimal index problem for dual rational subspaces. To the best of our knowledge, this problem is settled for the first time in this paper, with a solution that is both constructive and simple. In addition, we have shown how to use zigzag matrices to explicitly and easily construct completely singular polynomial matrices with any desired degree d and arbitrarily prescribed left and right minimal indices, provided only that d divides the sum of all of the prescribed minimal indices. A key feature of this approach is that the minimal indices are immediately apparent by inspection from the constructed polynomial matrix. This result therefore complements the solution of the more general inverse problem given recently in [6], which is based on a rather complicated construction that does not display the realized structure at all. It also opens up the possibility that it may be feasible to realize a prescribed degree, minimal index list, *and* elementary divisor collection with a polynomial matrix that immediately displays the complete eigenstructure, or at least a significant part of it; this remains an open question. We believe that zigzag matrices will continue to be a useful tool in many other problems involving polynomial matrices; indeed, we are presently working on several research problems in which they play a key role.

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