

# Generic skew-symmetric matrix polynomials with fixed rank and fixed odd grade<sup>☆</sup>

Andrii Dmytryshyn<sup>a</sup>, Froilán M. Dopico<sup>b</sup>

<sup>a</sup>*Department of Computing Science, Umeå University, SE-901 87 Umeå, Sweden.*

<sup>b</sup>*Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911, Leganés, Spain.*

---

## Abstract

We show that the set of  $m \times m$  complex skew-symmetric matrix polynomials of odd grade  $d$ , i.e., of degree at most  $d$ , and (normal) rank at most  $2r$  is the closure of the single set of matrix polynomials with the certain, explicitly described, complete eigenstructure. This complete eigenstructure corresponds to the most generic  $m \times m$  complex skew-symmetric matrix polynomials of odd grade  $d$  and rank at most  $2r$ . In particular, this result includes the case of skew-symmetric matrix pencils ( $d = 1$ ).

*Keywords:* complete eigenstructure, genericity, matrix polynomials, skew-symmetry, normal rank, orbits, pencils

*2000 MSC:* 15A18, 15A21

---

## 1. Introduction

The structure of the sets of matrix pencils and of matrix polynomials with fixed grade and fixed rank is not trivial and, as a consequence, only recently has been investigated in the literature [9, 12, 20], with the main purpose of providing reasonably simple descriptions of these sets as the closures of certain “generic sets” of pencils and matrix polynomials that can be easily described in terms of their eigenstructures or in terms of certain parameterizations. One motivation for this type of research comes from its applications in the study of the effect of low rank perturbations on the spectral information of pencils and matrix polynomials, since these problems have received considerable attention in the last years. See for instance the references [2, 3, 4, 8, 11, 17, 43] for different low rank perturbation problems related to matrix pencils, and the particular study in [10] on certain

---

<sup>☆</sup>Preprint Report UMINF 17.07, Department of Computing Science, Umeå University  
*Email addresses:* `andrii@cs.umu.se` (Andrii Dmytryshyn), `dopico@math.uc3m.es` (Froilán M. Dopico)

low rank perturbations of matrix polynomials. Moreover, low rank perturbations of matrix pencils have been applied recently to some classical problems as the eigenvalue placement problem [30] or the estimation of the distance of a regular pencil to the nearest singular pencil [42]. Finally, from a more theoretical perspective, the study of the sets of matrix pencils and of matrix polynomials with fixed grade and fixed rank generalizes classical studies [48] on the algebraic structure of the set of  $n \times n$  singular pencils, i.e., those whose rank is at most  $n - 1$ .

As can be observed in the titles of several of the references mentioned in the paragraph above, some of the problems concerning low rank perturbations of matrix pencils involve pencils with particular structures (symmetric, Hermitian, palindromic, skew-symmetric, alternating, etc), since the pencils and matrix polynomials arising in applications have often such particular structures [37]. In this scenario, it is natural to consider the extension of the results in [9, 12, 20] to the structured setting, with the aim of providing simple descriptions of sets of structured matrix pencils and of structured matrix polynomials with fixed grade and fixed rank in the case of the structures appearing in applications. The results in this paper can be seen as the first step available in the literature towards the solution of this ambitious and wide problem. In particular, we prove that the set of  $m \times m$  complex skew-symmetric matrix polynomials of odd grade  $d$ , i.e., of degree at most  $d$ , and (normal) rank at most  $2r$  is the closure of the single set of matrix polynomials with the certain, explicitly described, complete eigenstructure, which is termed as the *unique generic eigenstructure* of this set of skew-symmetric matrix polynomials of odd grade  $d$  and rank at most  $2r$ . By taking  $d = 1$ , we obtain that such uniqueness also holds for skew-symmetric pencils. This result greatly simplifies the description of the set of skew-symmetric polynomials of fixed odd grade and fixed rank and illustrates how strong can be the effect of imposing a structure on this type of problems, since the corresponding unstructured problem has many more generic complete eigenstructures. More precisely, it has been proved in [9, 20] that there are  $rd + 1$  generic complete eigenstructures in the set of  $m \times m$  complex *general (unstructured)* matrix polynomials of odd grade  $d$  and rank at most  $r$ .

There are two reasons for restricting our attention to skew-symmetric matrix polynomials of odd grade in this paper. The first one is that the general (that is, without imposing any rank constraint) sets of skew-symmetric matrix pencils and of skew-symmetric matrix polynomials of fixed odd grade  $d$  have been deeply studied in the literature and their stratification hierarchies are well understood [19, 22], while similar results for other classes of structured pencils and matrix polynomials are not yet known. The second reason is that skew-symmetric pencils and matrix polynomials appear in a number

of interesting applications, e.g., analysis of passive velocity field controllers [36], bi-Hamiltonian systems [44], multisymplectic PDEs [7], investigation of a product of two skew-symmetric matrices and particular symmetric factorizations of skew-symmetric rational matrices [40].

The paper is organized as follows. Section 2 includes some basic results on general and skew-symmetric pencils, which allow us to describe in Section 3 the set of skew-symmetric matrix pencils with rank at most  $2w$  as the closure of the matrix pencils that have a single and simple skew-symmetric Kronecker canonical form. Section 4 presents a number of preliminary results on general and skew-symmetric matrix polynomials that combined with the results for pencils of Section 3 allow us to prove our main result for skew-symmetric matrix polynomials of fixed odd grade and fixed rank, i.e., Theorem 5.2 in Section 5. Finally, the conclusions, some lines of future research, and some difficulties for extending the results of this paper are discussed in Section 6.

The reader should bear in mind throughout this paper that all the matrices that we consider have complex entries, and that the rank of any pencil or any matrix polynomial is defined as the largest size of the minors that are not identically zero scalar polynomials [29], which is sometimes called the normal rank [26]. However, for brevity, we do not use the word “normal” in this paper.

## 2. Preliminaries on skew-symmetric matrix pencils

We start by recalling the Kronecker canonical form (KCF) of general matrix pencils and the canonical form of skew-symmetric matrix pencils under congruence. Define  $\overline{\mathbb{C}} := \mathbb{C} \cup \infty$ . For each  $k = 1, 2, \dots$ , define the  $k \times k$  matrices

$$J_k(\mu) := \begin{bmatrix} \mu & 1 & & \\ & \mu & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}, \quad I_k := \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

and for each  $k = 0, 1, \dots$ , define the  $k \times (k + 1)$  matrices

$$F_k := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \quad G_k := \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}.$$

All non-specified entries of  $J_k(\mu)$ ,  $I_k$ ,  $F_k$ , and  $G_k$  are zeros.

An  $m \times n$  matrix pencil  $\lambda A - B$  is called *strictly equivalent* to  $\lambda C - D$  if and only if there are non-singular matrices  $Q$  and  $R$  such that  $Q^{-1}AR = C$  and  $Q^{-1}BR = D$ .

**Theorem 2.1.** [29, Ch. XII, Sect. 4] *Each  $m \times n$  matrix pencil  $\lambda A - B$  is strictly equivalent to a direct sum, uniquely determined up to permutation of summands, of pencils of the form*

$$\begin{aligned} \mathcal{E}_k(\mu) &:= \lambda I_k - J_k(\mu), \text{ in which } \mu \in \mathbb{C}, \quad \mathcal{E}_k(\infty) := \lambda J_k(0) - I_k, \\ \mathcal{L}_k &:= \lambda G_k - F_k, \quad \text{and} \quad \mathcal{L}_k^T := \lambda G_k^T - F_k^T. \end{aligned}$$

*This direct sum is called the KCF of  $\lambda A - B$ .*

The regular part of  $\lambda A - B$  consists of the blocks  $\mathcal{E}_k(\mu)$  and  $\mathcal{E}_k(\infty)$  corresponding to the finite and infinite eigenvalues, respectively. The singular part of  $\lambda A - B$  consists of the blocks  $\mathcal{L}_k$  and  $\mathcal{L}_k^T$  corresponding to the column and row minimal indices, respectively. The number of blocks  $\mathcal{L}_k$  (respectively,  $\mathcal{L}_k^T$ ) in the KCF of  $\lambda A - B$  is equal to the dimension of the right (respectively, left) rational null-space of  $\lambda A - B$ .

Define an *orbit* of  $\lambda A - B$  under the action of the group  $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$  on the space of all matrix pencils by strict equivalence as follows:

$$O^e(\lambda A - B) = \{Q^{-1}(\lambda A - B)R : Q \in GL_m(\mathbb{C}), R \in GL_n(\mathbb{C})\}. \quad (1)$$

The orbit of  $\lambda A - B$  is a manifold in the complex  $2mn$  dimensional space.

The space of all  $m \times n$  matrix pencils is denoted by  $\text{PENCIL}_{m \times n}$ . A distance in  $\text{PENCIL}_{m \times n}$  can be defined with the Frobenius norm of complex matrices [32] as  $d(\lambda A - B, \lambda C - D) := \sqrt{\|A - C\|_F^2 + \|B - D\|_F^2}$ , which makes  $\text{PENCIL}_{m \times n}$  into a metric space. This metric allows us to consider closures of subsets of  $\text{PENCIL}_{m \times n}$ , in particular, closures of orbits by strict equivalence, denoted by  $\overline{O^e}(\lambda A - B)$ . By using these concepts, the following result from [5], see also [27], describes all the possible changes in the KCF of a matrix pencil under arbitrarily small perturbations. If in the KCF of a matrix pencil the blocks  $\mathcal{X}$  are changed to the blocks  $\mathcal{Y}$  ( $\mathcal{X}$  and  $\mathcal{Y}$  of the same size) we write  $\mathcal{X} \rightsquigarrow \mathcal{Y}$ .

**Theorem 2.2.** [5] *Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two matrix pencils in KCF. Then,  $\overline{O^e}(\mathcal{P}_1) \supset O^e(\mathcal{P}_2)$  if and only if  $\mathcal{P}_1$  can be obtained from  $\mathcal{P}_2$  changing canonical blocks of  $\mathcal{P}_2$  by applying a sequence of rules and each rule is one of the six types below:*

1.  $\mathcal{L}_{j-1} \oplus \mathcal{L}_{k+1} \rightsquigarrow \mathcal{L}_j \oplus \mathcal{L}_k, 1 \leq j \leq k;$
2.  $\mathcal{L}_{j-1}^T \oplus \mathcal{L}_{k+1}^T \rightsquigarrow \mathcal{L}_j^T \oplus \mathcal{L}_k^T, 1 \leq j \leq k;$
3.  $\mathcal{L}_j \oplus \mathcal{E}_{k+1}(\mu) \rightsquigarrow \mathcal{L}_{j+1} \oplus \mathcal{E}_k(\mu), j, k = 0, 1, 2, \dots \text{ and } \mu \in \overline{\mathbb{C}};$
4.  $\mathcal{L}_j^T \oplus \mathcal{E}_{k+1}(\mu) \rightsquigarrow \mathcal{L}_{j+1}^T \oplus \mathcal{E}_k(\mu), j, k = 0, 1, 2, \dots \text{ and } \mu \in \overline{\mathbb{C}};$
5.  $\mathcal{E}_j(\mu) \oplus \mathcal{E}_k(\mu) \rightsquigarrow \mathcal{E}_{j-1}(\mu) \oplus \mathcal{E}_{k+1}(\mu), 1 \leq j \leq k \text{ and } \mu \in \overline{\mathbb{C}};$

6.  $\mathcal{L}_p \oplus \mathcal{L}_q^T \rightsquigarrow \bigoplus_{i=1}^t \mathcal{E}_{k_i}(\mu_i)$ , if  $p+q+1 = \sum_{i=1}^t k_i$  and  $\mu_i \neq \mu_{i'}$  for  $i \neq i'$ ,  $\mu_i \in \overline{\mathbb{C}}$ .

Observe that in the rules above any block  $\mathcal{E}_0(\mu)$  should be understood as the empty matrix.

An  $n \times n$  matrix pencil  $\lambda A - B$  is called *congruent* to  $\lambda C - D$  if there is a non-singular matrix  $S$  such that  $S^T A S = C$  and  $S^T B S = D$ . In the following theorem we recall the canonical form under congruence of skew-symmetric matrix pencils, i.e., those satisfying  $(\lambda A - B)^T = -(\lambda A - B)$ .

**Theorem 2.3.** [45] *Each skew-symmetric  $n \times n$  matrix pencil  $\lambda A - B$  is congruent to a direct sum, determined uniquely up to permutation of summands, of pencils of the form*

$$\begin{aligned} \mathcal{H}_h(\mu) &:= \lambda \begin{bmatrix} 0 & I_h \\ -I_h & 0 \end{bmatrix} - \begin{bmatrix} 0 & J_h(\mu) \\ -J_h(\mu)^T & 0 \end{bmatrix}, \quad \mu \in \mathbb{C}, \\ \mathcal{K}_k &:= \lambda \begin{bmatrix} 0 & J_k(0) \\ -J_k(0)^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}, \\ \mathcal{M}_m &:= \lambda \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix}. \end{aligned}$$

This direct sum is called the *skew-symmetric KCF* of  $\lambda A - B$ . Observe that the block  $\mathcal{M}_0$  is just the  $1 \times 1$  zero matrix.

Similarly to KCF, the regular part of  $\lambda A - B$  consists of the blocks  $\mathcal{H}_h(\mu)$  and  $\mathcal{K}_k$  corresponding to the finite and infinite eigenvalues, respectively. The singular part of  $\lambda A - B$  consists of the blocks  $\mathcal{M}_m$  corresponding to the right (column) and left (row) minimal indices, which are equal in the case of skew-symmetric matrix pencils. Define an *orbit* of  $\lambda A - B$  under the action of the group  $GL_n(\mathbb{C})$  on the space of all skew-symmetric matrix pencils by congruence as follows:

$$\text{O}^c(\lambda A - B) = \{S^T(\lambda A - B)S : S \in GL_n(\mathbb{C})\}. \quad (2)$$

The orbit of  $\lambda A - B$  is a manifold in the complex  $n^2 - n$  dimensional space of skew-symmetric pencils.

The fact that two skew-symmetric matrix pencils are congruent if and only if they have the same skew-symmetric KCF follows immediately from Theorem 2.3, as well as the well-known property that skew-symmetric matrix pencils have always even rank.

### 3. Generic skew-symmetric matrix pencils with bounded rank

In this section we state and prove our main results about skew-symmetric matrix pencils, in particular, we will find the most generic congruence orbit of skew-symmetric matrix pencils of a fixed rank in Theorem 3.1, show its irreducibility in Lemma 3.3, and count its codimension in (7). These results will be used in Section 5 for obtaining similar results about skew-symmetric matrix polynomials. In order to keep the arguments concise, we often use expressions as “the pencil  $\mathcal{P}_1$  is more generic than the pencil  $\mathcal{P}_2$ ”, whose precise meaning is that  $\overline{O^e}(\mathcal{P}_1) \supset O^e(\mathcal{P}_2)$  or  $\overline{O^c}(\mathcal{P}_1) \supset O^c(\mathcal{P}_2)$ , depending of the context. In this language, the most generic skew-symmetric pencil with rank  $2w$  is a pencil such that the closure of its congruence orbit includes the congruence orbit of any other skew-symmetric pencil with different KCF and with rank at most  $2w$ . We denote the vector space of skew-symmetric matrix pencils of size  $n \times n$  by  $\text{PENCIL}_{n \times n}^{ss}$ . A distance in  $\text{PENCIL}_{n \times n}^{ss}$  can be defined as  $d(\lambda A - B, \lambda C - D) := \sqrt{\|A - C\|_F^2 + \|B - D\|_F^2}$ , as in the case of general pencils. With this distance at hand, we can consider closures of subsets in  $\text{PENCIL}_{n \times n}^{ss}$ , as well as any other topological concept.

**Theorem 3.1.** *Let  $n$  and  $w$  be integers such that  $n \geq 2$  and  $2 \leq 2w \leq n - 1$ . The set of  $n \times n$  complex skew-symmetric matrix pencils with rank at most  $2w$  is a closed subset of  $\text{PENCIL}_{n \times n}^{ss}$  equal to  $\overline{O^e}(\mathcal{W})$ , where*

$$\mathcal{W} = \text{diag}(\underbrace{\mathcal{M}_{\alpha+1}, \dots, \mathcal{M}_{\alpha+1}}_s, \underbrace{\mathcal{M}_{\alpha}, \dots, \mathcal{M}_{\alpha}}_{n-2w-s}) \quad (3)$$

with  $\alpha = \lfloor w/(n - 2w) \rfloor$  and  $s = w \bmod (n - 2w)$ .

*Proof.* Taking into account [22, Lem. 3.8] (or the stronger result [22, Thm. 3.1]), in this proof we work with KCF rather than with the skew-symmetric KCF under congruence of Theorem 2.3, but we always apply rules from Theorem 2.2 in pairs, such that the corresponding change of the skew-symmetric KCF is obvious.

Each  $n \times n$  skew-symmetric matrix pencil of the rank  $2r_1$  less than or equal to  $2w$  has the following KCF:

$$\text{diag}(\mathcal{L}_{\gamma_1}, \dots, \mathcal{L}_{\gamma_{n-2r_1}}, \mathcal{L}_{\gamma_1}^T, \dots, \mathcal{L}_{\gamma_{n-2r_1}}^T, \mathcal{J}, \mathcal{J}), \quad (4)$$

since left and right singular blocks are paired up according to Theorem 2.3, the number of left singular blocks is equal to the dimension of the rational left null-space of the pencil, i.e., is equal to  $n - 2r_1$ , and the blocks  $\mathcal{J}$  of the regular part are also paired up.

Note that KCF of a generic skew-symmetric matrix pencil of rank less than or equal to  $2w$  can not contain any regular part (otherwise rules 3 and 4 from Theorem 2.2 can be applied to obtain a more generic skew-symmetric pencil). Moreover, the rank of any of the most generic skew-symmetric matrix pencils of rank at most  $2w$  is exactly  $2w$  (otherwise, there is an excess of singular blocks and rule 6 from Theorem 2.2 can be applied twice to obtain a more generic skew-symmetric matrix pencil with strictly larger rank). Therefore any of the most generic skew-symmetric matrix pencils of rank at most  $2w$  must have KCF consisting of  $n - 2w$  blocks  $\mathcal{L}$  and  $n - 2w$  blocks  $\mathcal{L}^T$ :

$$\text{diag}(\mathcal{L}_{\gamma_1}, \dots, \mathcal{L}_{\gamma_{n-2w}}, \mathcal{L}_{\gamma_1}^T, \dots, \mathcal{L}_{\gamma_{n-2w}}^T). \quad (5)$$

To each pencil (5) we can apply only rules 1 and 2 from Theorem 2.2 in order to get more generic skew-symmetric matrix pencils of rank at most  $2w$ , since applying rule 6 would increase the rank and rules 3–5 involve regular blocks. Thus we can only apply rule 1 to the blocks  $\mathcal{L}$  of (5) (and, simultaneously for preserving the skew-symmetric structure of the KCF, rule 2 to the blocks  $\mathcal{L}^T$  of (5), respectively), which corresponds to finding the most generic  $w \times (n-w)$  matrix pencil (and the most generic  $(n-w) \times w$  matrix pencil, respectively). For the latter we refer to, e.g., [20, Thm. 2.6], see also [18, 27, 46]<sup>1</sup>. Therefore the most generic skew-symmetric matrix pencil of rank at most  $2w$  has the KCF

$$\text{diag}(\underbrace{\mathcal{L}_{\alpha+1}, \dots, \mathcal{L}_{\alpha+1}}_s, \underbrace{\mathcal{L}_{\alpha}, \dots, \mathcal{L}_{\alpha}}_{n-2w-s}, \underbrace{\mathcal{L}_{\alpha+1}^T, \dots, \mathcal{L}_{\alpha+1}^T}_s, \underbrace{\mathcal{L}_{\alpha}^T, \dots, \mathcal{L}_{\alpha}^T}_{n-2w-s}), \quad (6)$$

or equivalently, has the skew-symmetric KCF (3).  $\square$

**Remark 3.2.** *The KCF in (6) is also one of the most generic KCFs of the  $n \times n$  general matrix pencils of rank less than or equal to  $2w$ , obtained in [9, Thm. 3.2]. Therefore proving the genericity of (3) could have been done using [9, Thm. 3.2] and [22, Thm. 3.1] but we would still have to explain why there are no more generic pencils (essentially, to repeat the proof of Theorem 3.1).*

Observe that Theorem 3.1 does not cover the case  $2w = n$ , which generically corresponds to regular skew-symmetric matrix pencils with even number of rows (and columns). In this easy case, the generic pencils clearly

---

<sup>1</sup>An equivalent, and more self-contained, argument is the following: since rule 1 (and rule 2) in Theorem 2.2 can be applied to get more generic skew-symmetric pencils as long as there are blocks  $\mathcal{L}_{\gamma_i}$  and  $\mathcal{L}_{\gamma_j}$  (and  $\mathcal{L}_{\gamma_i}^T$  and  $\mathcal{L}_{\gamma_j}^T$ ) with  $|\gamma_i - \gamma_j| \geq 2$ , the KCF of the most generic skew-symmetric matrix pencils of rank at most  $2w$  must have  $n - 2w$  blocks  $\mathcal{L}_{\gamma_i}$  and  $n - 2w$  blocks  $\mathcal{L}_{\gamma_i}^T$ ,  $i = 1, 2, \dots, n - 2w$ , with  $|\gamma_i - \gamma_j| \leq 1$ , which is only possible if the KCF is (6).

have  $w$  different eigenvalues which are paired up according to Theorem 2.3, i.e. the generic KCF is  $\text{diag}(\mathcal{H}_1(\mu_1), \dots, \mathcal{H}_1(\mu_w))$ , where  $\mu_i \neq \mu_j$  if  $i \neq j$ , and  $i, j \in \{1, 2, \dots, w\}$ .

Similarly to results included in [9, 48] for closures of orbits under strict equivalence, next we show that for any  $n \times n$  skew-symmetric pencil  $\mathcal{P}$  the set  $\overline{O^c(\mathcal{P})}$  (where the closure is taken in principle in the Euclidean metric defined before Theorem 3.1) is irreducible in the Zariski topology of  $\text{PENCIL}_{n \times n}^{ss}$ . The proof of such result also establishes that the closures of  $O^c(\mathcal{P})$  in the Euclidean and in the Zariski topologies are equal. In particular, this irreducibility implies the connectivity of  $\overline{O^c(\mathcal{P})}$  in both the Zariski and Euclidean topologies (recall that we work over  $\mathbb{C}$ ). Note also that by [6, Closed Orbit Lemma, p. 53],  $O^c(\mathcal{P})$  is open in its closure again in both the Zariski and Euclidean topologies. These properties hold, of course, for the generic pencil  $\mathcal{W}$  in (3). The proof of Lemma 3.3 essentially repeats the proof of [9, Lem. 3.4] where the analogous result for the general matrix pencils under strict equivalence is derived.

**Lemma 3.3.** *The closures in the Euclidean topology and in the Zariski topology of the congruence orbit  $O^c(\mathcal{P})$  of any  $n \times n$  complex skew-symmetric matrix pencil  $\mathcal{P}$  are equal in  $\text{PENCIL}_{n \times n}^{ss}$ . Moreover,  $\overline{O^c(\mathcal{P})}$  is an irreducible manifold in the Zariski topology of  $\text{PENCIL}_{n \times n}^{ss}$ .*

*Proof.* We identify the space of skew-symmetric matrix pencils of size  $n \times n$  with  $\mathbb{C}^{n^2-n}$  (for the pencil  $\mathcal{P} = \lambda A - B$  we have  $(n^2 - n)/2$  parameters in each of  $A$  and  $B$ ). All the nonsingular matrices  $S \in \mathbb{C}^{n \times n}$  form a dense open set  $U$  of  $\mathbb{C}^{n^2}$  both in the Zariski and in the Euclidean topology. We consider the polynomial mapping  $\phi_{\mathcal{M}}$  from  $\mathbb{C}^{n^2}$  to  $\mathbb{C}^{n^2-n}$  defined by sending  $S$  to  $\lambda S^T A S - S^T B S$ . Thus  $\phi_{\mathcal{M}}(U) = O^c(\mathcal{P})$ . Notably, in any topology, for a set  $V$  and a continuous mapping  $\phi$ , we have  $\phi(\overline{V}) \subset \overline{\phi(V)}$ , which implies  $\overline{\phi(\overline{V})} \subset \overline{\phi(V)}$ . Therefore, we have  $\overline{\phi_{\mathcal{M}}(U)} = \overline{\phi_{\mathcal{M}}(\mathbb{C}^{n \times n})}$ , where the closures can be taken either in the Zariski or the Euclidean topology, since in this particular case they are equal. The remaining part of the proof is to note that  $\overline{\phi_{\mathcal{M}}(\mathbb{C}^{n^2})}$  is an irreducible set in the Zariski topology, by [48, Sect. 1], and is equal to  $\overline{O^c(\mathcal{P})}$ , by the discussion above.  $\square$

For an  $n \times n$  skew-symmetric pencil  $\lambda A - B$ , define the *dimension* of  $O^c(\lambda A - B)$  to be the dimension of the tangent space to this orbit

$$T_{\lambda A - B}^c := \{\lambda(X^T A + A X) - (X^T B + B X) : X \in \mathbb{C}^{n \times n}\}$$

at the point  $\lambda A - B$ . The orthogonal complement (with respect to the Frobenius inner product) to  $T_{\lambda A - B}^c$ , is called the normal space to  $O^c(\lambda A - B)$  at



the point  $\lambda A - B$ . The dimension of the normal space is the *codimension* of the congruence orbit of  $\lambda A - B$  and is equal to  $n(n-1)$  minus the dimension of the congruence orbit of  $\lambda A - B$ . Explicit expressions for the codimensions of congruence orbits of skew-symmetric pencils in  $\text{PENCIL}_{n \times n}^{ss}$  are presented in [23] and implemented in the MCS (Matrix Canonical Structure) Toolbox [21, 33]. Using [23, Thm. 3], see also [21, Thm. 2.8], applied to the skew-symmetric pencil  $\mathcal{W}$  in (3), the codimension of  $O^c(\mathcal{W})$  in  $\text{PENCIL}_{n \times n}^{ss}$  is  $\text{cod } O^c(\mathcal{W}) = \sum_{i < j} (2 \max\{m_i, m_j\} + \varepsilon_{ij})$ , where  $m_i$  and  $m_j$  are the indices of the  $\mathcal{M}$  blocks (either  $\alpha$  or  $\alpha + 1$ ),  $\varepsilon_{ij} = 2$  if  $m_i = m_j$  and  $\varepsilon_{ij} = 1$  otherwise, resulting in:

$$\begin{aligned}
\text{cod } O^c(\mathcal{W}) &= (n - 2w - s)(n - 2w - s - 1)(2\alpha + 2)/2 \\
&\quad + s(s - 1)(2(\alpha + 1) + 2)/2 + s(n - 2w - s)(2(\alpha + 1) + 1) \\
&= (n - 2w - s)(n - 2w - s - 1)(\alpha + 1) + s(s - 1)(\alpha + 1) + s(s - 1) \\
&\quad + 2s(n - 2w - s)(\alpha + 1) + s(n - 2w - s) \\
&= (\alpha + 1)((n - 2w - s)(n - 2w - 1) + s(n - 2w - 1)) + s(n - 2w - 1) \\
&= (n - 2w)(n - 2w - 1)(\alpha + 1) + s(n - 2w - 1) \\
&= (n - 2w - 1)((n - 2w)\alpha + s + n - 2w) = (n - 2w - 1)(n - w).
\end{aligned} \tag{7}$$

#### 4. Preliminaries on skew-symmetric matrix polynomials

We consider skew-symmetric  $m \times m$  matrix polynomials  $P(\lambda)$  of grade  $d$ , i.e., of degree less than or equal to  $d$ , over  $\mathbb{C}$ :

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \quad A_i^T = -A_i, \quad A_i \in \mathbb{C}^{m \times m} \text{ for } i = 0, \dots, d.$$

Denote the vector space of the  $m \times m$  skew-symmetric matrix polynomials of the grade  $d$  by  $\text{POL}_{d, m \times m}^{ss}$ . Observe that  $\text{POL}_{1, n \times n}^{ss}$  is  $\text{PENCIL}_{n \times n}^{ss}$ . If there is no risk of confusion we will write  $\text{POL}$  instead of  $\text{POL}_{d, m \times m}^{ss}$ . As in the case of pencils, see Sections 2 and 3, by using the Frobenius matrix norm of complex matrices [32] a distance on  $\text{POL}_{d, m \times m}^{ss}$  is defined as  $d(P, P') = (\sum_{i=0}^d \|A_i - A'_i\|_F^2)^{\frac{1}{2}}$ , making  $\text{POL}_{d, m \times m}^{ss}$  to a metric space with the induced Euclidean topology. For convenience, define the Frobenius norm of the matrix polynomial  $P$  as  $\|P(\lambda)\|_F = (\sum_{i=0}^d \|A_i\|_F^2)^{\frac{1}{2}}$ .

Two matrix polynomials  $P(\lambda)$  and  $Q(\lambda)$  are called *unimodularly congruent* if  $F(\lambda)^T P(\lambda) F(\lambda) = Q(\lambda)$  for some unimodular matrix polynomial  $F(\lambda)$  (i.e.  $\det F(\lambda) \in \mathbb{C} \setminus \{0\}$ ), see also [40].

**Theorem 4.1.** [40] *Let  $P(\lambda)$  be a skew-symmetric  $m \times m$  matrix polynomial. Then there exists  $r \in \mathbb{N}$  with  $2r \leq m$  and a unimodular matrix polynomial*

$F(\lambda)$  such that

$$F(\lambda)^T P(\lambda) F(\lambda) = \begin{bmatrix} 0 & g_1(\lambda) \\ -g_1(\lambda) & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & g_r(\lambda) \\ -g_r(\lambda) & 0 \end{bmatrix} \oplus 0_{m-2r} =: S(\lambda),$$

where  $g_j$  is monic for  $j = 1, \dots, r$  and  $g_j(\lambda)$  divides  $g_{j+1}(\lambda)$  for  $j = 1, \dots, r-1$ . Moreover, the canonical form  $S(\lambda)$  is unique.

Similarly to the unimodular congruence, two matrix polynomials  $P(\lambda)$  and  $Q(\lambda)$  are called *unimodularly equivalent* if  $U(\lambda)P(\lambda)V(\lambda) = Q(\lambda)$  for some unimodular matrix polynomials  $U(\lambda)$  and  $V(\lambda)$  (i.e.  $\det U(\lambda), \det V(\lambda) \in \mathbb{C} \setminus \{0\}$ ). Note that the canonical form in Theorem 4.1 is the skew-symmetric version of the well-known Smith form for matrix polynomials under unimodular equivalence [29, 40].

Define the (normal) *rank* of the matrix polynomial  $P(\lambda)$  to be the integer  $2r$  from Theorem 4.1. All  $g_j(\lambda), j = 1, \dots, r$ , are called *invariant polynomials* of  $P(\lambda)$ , and each of them can be uniquely factored as

$$g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \dots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}},$$

where  $l_j \geq 0$ ,  $\delta_{j1}, \dots, \delta_{jl_j} > 0$  are integers. If  $l_j = 0$  then  $g_j(\lambda) = 1$ . The complex numbers  $\alpha_1, \dots, \alpha_{l_j}$  are finite eigenvalues of  $P(\lambda)$ . The *elementary divisors* of  $P(\lambda)$  associated with each finite eigenvalue  $\alpha_k$  is the collection of factors  $(\lambda - \alpha_k)^{\delta_{jk}}$  (possibly with repetitions).

If zero is an eigenvalue of  $\text{rev } P(\lambda) := \lambda^d P(1/\lambda)$  then we say that  $\lambda = \infty$  is an eigenvalue of the matrix polynomial  $P(\lambda)$  of grade  $d$ . The elementary divisors  $\lambda^{\gamma_k}, \gamma_k > 0$ , for the zero eigenvalue of  $\text{rev } P(\lambda)$  are the elementary divisors associated with the infinite eigenvalue of  $P(\lambda)$ .

For an  $m \times n$  matrix polynomial  $P(\lambda)$ , define the left and right null-spaces, over the field of rational functions  $\mathbb{C}(\lambda)$ , as follows:

$$\begin{aligned} \mathcal{N}_{\text{left}}(P) &:= \{y(\lambda)^T \in \mathbb{C}(\lambda)^{1 \times m} : y(\lambda)^T P(\lambda) = 0_{1 \times n}\}, \\ \mathcal{N}_{\text{right}}(P) &:= \{x(\lambda) \in \mathbb{C}(\lambda)^{n \times 1} : P(\lambda)x(\lambda) = 0_{m \times 1}\}. \end{aligned}$$

Each subspace  $\mathcal{V}$  of  $\mathbb{C}(\lambda)^n$  has bases consisting entirely of vector polynomials. A basis of  $\mathcal{V}$  consisting of vector polynomials whose sum of degrees is minimal among all bases of  $\mathcal{V}$  consisting of vector polynomials is called a *minimal basis* of  $\mathcal{V}$ . The ordered list of degrees of the vector polynomials in any minimal basis of  $\mathcal{V}$  is always the same. These degrees are called the *minimal indices* of  $\mathcal{V}$  [28, 35]. This allows us to define the minimal indices of a matrix polynomial: let the sets  $\{y_1(\lambda)^T, \dots, y_{m-r}(\lambda)^T\}$  and  $\{x_1(\lambda), \dots, x_{n-r}(\lambda)\}$  be minimal bases of  $\mathcal{N}_{\text{left}}(P)$  and  $\mathcal{N}_{\text{right}}(P)$ , respectively, ordered so that  $0 \leq$

$\deg(y_1) \leq \dots \leq \deg(y_{m-r})$  and  $0 \leq \deg(x_1) \leq \dots \leq \deg(x_{n-r})$ . Let  $\eta_k = \deg(y_k)$  for  $k = 1, \dots, m-r$  and  $\varepsilon_k = \deg(x_k)$  for  $k = 1, \dots, n-r$ . Then the scalars  $0 \leq \eta_1 \leq \eta_2 \leq \dots \leq \eta_{m-r}$  and  $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_{n-r}$  are, respectively, the *left* and *right minimal indices* of  $P(\lambda)$ . Note also that for a skew-symmetric matrix polynomial we have that  $x_i(\lambda) = y_i(\lambda)$  and thus  $\eta_i = \varepsilon_i$ , for  $i = 1, \dots, m-r$ .

Define the *complete eigenstructure* of a matrix polynomial  $P(\lambda)$  as all the finite and infinite eigenvalues, the corresponding elementary divisors, and the left and right minimal indices of  $P(\lambda)$ . The set of matrix polynomials of the same size, grade, and with the same complete eigenstructure as  $P(\lambda)$  is called an *orbit* of  $P(\lambda)$ , denoted  $O(P)$ . In this paper, if  $P(\lambda)$  is skew-symmetric, then  $O(P)$  contains all the skew-symmetric polynomials of the same size, grade, and with the same complete eigenstructure as  $P(\lambda)$ .

A matrix pencil  $\mathcal{L}_{P(\lambda)}$  is called a *linearization* of a matrix polynomial  $P(\lambda)$  if they have the same finite elementary divisors, the same number of left minimal indices, and the same number of right minimal indices [15]. If in addition,  $\text{rev } \mathcal{L}_{P(\lambda)}$  is a linearization of  $\text{rev } P(\lambda)$  then  $\mathcal{L}_{P(\lambda)}$  is called a *strong linearization* of  $P(\lambda)$  and, then,  $\mathcal{L}_{P(\lambda)}$  and  $P(\lambda)$  have also the same infinite elementary divisors. Linearizations are powerful tools for investigation of matrix polynomials [15, 31].

From now on we restrict to skew-symmetric matrix polynomials of odd grades. The reason is that there is no skew-symmetric linearization-template (i.e., a skew-symmetric companion form in the language of [15, Sects. 5 and 7]) for skew-symmetric matrix polynomials of even grades [15, 40].

The following pencil-template is known to be a skew-symmetric strong linearization of all the skew-symmetric  $m \times m$  matrix polynomials  $P(\lambda)$  of odd grade  $d$  [40], see also [1, 38]:

$$\mathcal{L}_{P(\lambda)}(i, i) = \begin{cases} \lambda A_{d-i+1} + A_{d-i} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases}$$

$$\mathcal{L}_{P(\lambda)}(i, i+1) = \begin{cases} -I_m & \text{if } i \text{ is odd,} \\ -\lambda I_m & \text{if } i \text{ is even,} \end{cases} \quad \mathcal{L}_{P(\lambda)}(i+1, i) = \begin{cases} I_m & \text{if } i \text{ is odd,} \\ \lambda I_m & \text{if } i \text{ is even,} \end{cases}$$

where  $\mathcal{L}_{P(\lambda)}(j, k)$  denotes an  $m \times m$  matrix pencil which is at the position  $(j, k)$  of the block pencil  $\mathcal{L}_{P(\lambda)}$  and  $j, k = 1, \dots, d$ . The blocks of  $\mathcal{L}_{P(\lambda)}$  in positions which are not specified above are zero. We rewrite this strong



We emphasize that all the elements of  $O(\mathcal{L}_P)$  have the block structure of the elements of  $\text{GSYL}_{d,m \times m}^{ss}$ . Thus, in particular,  $O(P) = f^{-1}(O(\mathcal{L}_P))$ , as a consequence of the properties of strong linearizations and Theorem 4.2, and  $\overline{O}(P) = f^{-1}(\overline{O}(\mathcal{L}_P))$ , as a consequence of  $f$  being a homeomorphism. Moreover, we also have that for any  $m \times m$  skew-symmetric matrix polynomials  $P, Q$  of odd grade  $d$ ,  $\overline{O}(P) \supseteq \overline{O}(Q)$  if and only if  $\overline{O}(\mathcal{L}_P) \supseteq \overline{O}(\mathcal{L}_Q)$ , where it is essential to note that the closures are taken in the metric spaces POL and GSYL, respectively, defined above. Note that similarly to the matrix pencil case,  $O(\mathcal{L}_P)$  is open in its closure in the relative Euclidean topology, and so is  $O(P)$  since  $f$  is a homeomorphism.

We will use in Section 5 the fact that for any  $m \times m$  skew-symmetric matrix polynomial  $P(\lambda)$  a sufficiently small arbitrary skew-symmetric perturbation of the pencil  $\mathcal{L}_{P(\lambda)}$  produces another pencil that although, in general, is not in  $\text{GSYL}_{d,m \times m}^{ss}$  is congruent to a pencil in  $\text{GSYL}_{d,m \times m}^{ss}$  that is very close to  $\mathcal{L}_{P(\lambda)}$ , see [19]. This type of results for the general matrix polynomials is presented in e.g., [24, 34, 47].

**Theorem 4.3** (Theorems 8 and 9 in [19]). *Let  $P(\lambda)$  be an  $m \times m$  skew-symmetric matrix polynomial with odd grade  $d$  and let  $\mathcal{L}_{P(\lambda)}$  be its skew-symmetric linearization (8). If  $\mathcal{E}$  is any arbitrarily small (entrywise)  $md \times md$  skew-symmetric pencil, then there exist a nonsingular matrix  $C$  and an arbitrarily small (entrywise)  $m \times m$  skew-symmetric polynomial  $F(\lambda)$  with grade  $d$  such that*

$$C^T(\mathcal{L}_{P(\lambda)} + \mathcal{E})C = \mathcal{L}_{P(\lambda)+F(\lambda)}.$$

It is interesting to emphasize that the extension of Theorem 4.3 for symmetric, Hermitian, skew-Hermitian, palindromic, anti-palindromic, and alternating matrix polynomials of odd grade and for the linearization (8) follows immediately from the proof of Theorem 8 in [19]. An extension of Theorem 4.3 to all the structures mentioned above and to a wider class of structure-preserving companion linearizations which include (8) can be obtained as a corollary of a much more general result recently obtained in [25, Thm. 6.12].

## 5. Generic skew-symmetric matrix polynomials with bounded rank and fixed grade

In this section we present the complete eigenstructure of the generic  $m \times m$  skew-symmetric matrix polynomial of a fixed rank  $2r$  and odd grade  $d$ . As in the case for general matrix polynomials [20, Lem. 3.1] we present the following lemma revealing a key relation between  $\overline{O}(\mathcal{L}_P)$ , where the closure is taken in  $\text{GSYL}_{d,m \times m}^{ss}$ , and  $\overline{O}^c(\mathcal{L}_P)$ , where the closure is taken in  $\text{PENCIL}_{n \times n}^{ss}$ , with  $n = md$ .

**Lemma 5.1.** *Let  $P$  be an  $m \times m$  skew-symmetric matrix polynomial with odd grade  $d$  and  $\mathcal{L}_P$  be its linearization (8) then  $\overline{\mathcal{O}(\mathcal{L}_P)} = \overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \text{GSYL}_{d,m \times m}^{ss}$ .*

*Proof.* By definition  $\mathcal{O}(\mathcal{L}_P) = \mathcal{O}^c(\mathcal{L}_P) \cap \text{GSYL}$  and thus  $\overline{\mathcal{O}(\mathcal{L}_P)} = \overline{\mathcal{O}^c(\mathcal{L}_P) \cap \text{GSYL}}$  (the closure here is taken in the space  $\text{GSYL}$ ). For any  $\mathcal{L}_Q \in \overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \text{GSYL}$  there exists an arbitrarily small (entrywise) skew-symmetric pencil  $\mathcal{E}$  such that  $(\mathcal{L}_Q + \mathcal{E}) \in \mathcal{O}^c(\mathcal{L}_P)$ . Therefore by Theorem 4.3, there exists an arbitrarily small (entrywise) skew-symmetric matrix polynomial  $F$  with grade  $d$  such that  $\mathcal{L}_{Q+F} \in \mathcal{O}^c(\mathcal{L}_P)$ . Thus  $\mathcal{L}_Q \in \overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \text{GSYL}$ , and  $\overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \text{GSYL} \subseteq \overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \text{GSYL}$ . Since  $\mathcal{O}^c(\mathcal{L}_P) \cap \text{GSYL} \subseteq \overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \text{GSYL}$ , we have that  $\overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \text{GSYL} = \overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \text{GSYL}$ , and the result is proved.  $\square$

Lemma 5.1 is the key to prove our main result about skew-symmetric matrix polynomials which uses and generalizes Theorem 3.1.

**Theorem 5.2.** *Let  $m, r$  and  $d$  be integers such that  $m \geq 2$ ,  $d \geq 1$  is odd, and  $2 \leq 2r \leq (m-1)$ . The set of  $m \times m$  complex skew-symmetric matrix polynomials of grade  $d$  with rank at most  $2r$  is a closed subset of  $\text{POL}_{d,m \times m}^{ss}$  equal to  $\overline{\mathcal{O}(W)}$ , where  $W$  is an  $m \times m$  complex skew-symmetric matrix polynomial of degree exactly  $d$  and rank exactly  $2r$  with no elementary divisors at all, with  $t$  left minimal indices equal to  $(\beta+1)$  and with  $(m-2r-t)$  left minimal indices equal to  $\beta$ , where  $\beta = \lfloor rd/(m-2r) \rfloor$  and  $t = rd \bmod (m-2r)$ , and with the right minimal indices equal to the left minimal indices.*

*Proof.* Note that Theorem 5.2 for  $d = 1$  coincides with Theorem 3.1.

Denote the complete eigenstructure from the statement of Theorem 5.2 (consisting only of the left and right minimal indices) by

$$\mathbf{W} : \left\{ \underbrace{\overbrace{\beta+1, \dots, \beta+1, \beta, \dots, \beta}^{\text{left minimal indices}}}_t, \underbrace{\overbrace{\beta+1, \dots, \beta+1, \beta, \dots, \beta}^{\text{right minimal indices}}}_{m-2r-t} \right\}. \quad (11)$$

First we show that there exists an  $m \times m$  skew-symmetric matrix polynomial  $W$  of degree exactly  $d$  and rank exactly  $2r$  that has the complete eigenstructure  $\mathbf{W}$  (11). By [19, Thm. 3.3] it is enough to show that the sum of the left (or right) minimal indices of  $\mathbf{W}$  is equal to  $rd$ :

$$\sum_1^t (\beta+1) + \sum_1^{m-2r-t} \beta = \sum_1^{m-2r} \beta + t = (m-2r) \lfloor rd/(m-2r) \rfloor + t = rd.$$

For every  $m \times m$  matrix polynomial  $P$  of grade  $d$  and rank at most  $2r$ , the linearization  $\mathcal{L}_P$  has rank at most  $2r + m(d-1)$ , because  $\mathcal{L}_P$  is unimodularly

equivalent to  $P \oplus I_{m(d-1)}$ . The linearization  $\mathcal{L}_W$  of the matrix polynomial  $W$  is an  $md \times md$  skew-symmetric matrix pencil with the rank  $m(d-1) + 2r$  and by Theorem 4.2 the KCF of  $\mathcal{L}_W$  is the direct sum of the following blocks:

$$\{\underbrace{\mathcal{L}_{\beta+\eta+1}, \dots, \mathcal{L}_{\beta+\eta+1}}_t, \underbrace{\mathcal{L}_{\beta+\eta}, \dots, \mathcal{L}_{\beta+\eta}}_{m-2r-t}, \underbrace{\mathcal{L}_{\beta+\eta+1}^T, \dots, \mathcal{L}_{\beta+\eta+1}^T}_t, \underbrace{\mathcal{L}_{\beta+\eta}^T, \dots, \mathcal{L}_{\beta+\eta}^T}_{m-2r-t}\}, \quad (12)$$

where  $\eta = \frac{1}{2}(d-1)$ . We show that the KCF of  $\mathcal{L}_W$  coincides with the KCF of the most generic skew-symmetric matrix pencil  $\mathcal{W}$  of rank  $2w = m(d-1) + 2r$  and size  $n \times n$ , where  $n = md$ , given in Theorem 3.1:

$$\{\underbrace{\mathcal{L}_{\alpha+1}, \dots, \mathcal{L}_{\alpha+1}}_s, \underbrace{\mathcal{L}_{\alpha}, \dots, \mathcal{L}_{\alpha}}_{n-2w-s}, \underbrace{\mathcal{L}_{\alpha+1}^T, \dots, \mathcal{L}_{\alpha+1}^T}_s, \underbrace{\mathcal{L}_{\alpha}^T, \dots, \mathcal{L}_{\alpha}^T}_{n-2w-s}\}. \quad (13)$$

Or equivalently, we show that the numbers and the sizes of the  $\mathcal{L}$  and  $\mathcal{L}^T$  blocks in (12) and (13) coincide, i.e.,  $\beta + (d-1)/2 = \alpha$ ,  $s = t$ , and  $m - 2r - t = n - 2w - s$ .

For the sizes of the blocks we have

$$\beta + \frac{d-1}{2} = \left\lfloor \frac{rd}{m-2r} \right\rfloor + \frac{d-1}{2} = \left\lfloor \frac{(m-2r)(d-1) + 2rd}{2(m-2r)} \right\rfloor \quad (14)$$

$$= \left\lfloor \frac{m(d-1) + 2r}{2(md - (m(d-1) + 2r))} \right\rfloor = \left\lfloor \frac{2w}{2(n-2w)} \right\rfloor = \alpha. \quad (15)$$

For the numbers of the blocks, we have

$$\begin{aligned} t &= rd \pmod{m-2r} = ((m-2r)(d-1)/2 + rd) \pmod{m-2r} \\ &= (m(d-1) + 2r)/2 \pmod{md - (m(d-1) + 2r)} \\ &= w \pmod{n-2w} = s \end{aligned}$$

and

$$m - 2r - t = md - m(d-1) - 2r - t = n - 2w - s.$$

Thus  $\mathcal{L}_W$  is congruent to the most generic skew-symmetric matrix pencil  $\mathcal{W}$  of rank  $2w$  obtained in Theorem 3.1, since they both are skew-symmetric and have the same KCF. Therefore  $\mathcal{O}^c(\mathcal{L}_W) = \mathcal{O}^c(\mathcal{W})$ .

By Theorem 3.1 for each  $m \times m$  skew-symmetric matrix polynomial  $P$  of grade  $d$  and rank at most  $2r$  we have that  $\overline{\mathcal{O}^c(\mathcal{W})} \supseteq \overline{\mathcal{O}^c(\mathcal{L}_P)}$ , thus  $\overline{\mathcal{O}^c(\mathcal{L}_W)} \supseteq \overline{\mathcal{O}^c(\mathcal{L}_P)}$ . Therefore  $\overline{\mathcal{O}^c(\mathcal{L}_W)} \cap \overline{\text{GSYL}} \supseteq \overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \overline{\text{GSYL}}$ , which is equivalent to  $\overline{\mathcal{O}^c(\mathcal{L}_W)} \cap \overline{\text{GSYL}} \supseteq \overline{\mathcal{O}^c(\mathcal{L}_P)} \cap \overline{\text{GSYL}}$  by Lemma 5.1, and, by definition, is also equivalent to  $\overline{\mathcal{O}(\mathcal{L}_W)} \supseteq \overline{\mathcal{O}(\mathcal{L}_P)}$ , which according to the discussion after (10), is equivalent to  $\overline{\mathcal{O}(W)} \supseteq \overline{\mathcal{O}(P)}$ . Therefore any  $m \times m$  skew-symmetric matrix polynomial of grade  $d$  with rank at most  $2r$  is in the closed set  $\overline{\mathcal{O}(W)}$ .  $\square$

For any  $P \in \text{POL}_{d,m \times m}^{ss}$ , we define the codimension of  $\text{O}(P) \subset \text{POL}_{d,m \times m}^{ss}$  of the set of all the skew-symmetric matrix polynomials with the same eigenstructure as  $P$  to be  $\text{cod O}(P) := \text{cod O}(\mathcal{L}_P)$ . By [19, Sect. 6],  $\text{O}(\mathcal{L}_P)$  is a manifold in the space of skew-symmetric matrix pencils  $\text{PENCIL}_{n \times n}^{ss}$ , where  $n = md$ , and  $\text{cod O}(\mathcal{L}_P) = \text{cod O}^c(\mathcal{L}_P)$ , where the codimension of  $\text{O}(\mathcal{L}_P)$  is considered in the space  $\text{GSYL}_{d,m \times m}^{ss}$  and the codimension of  $\text{O}^c(\mathcal{L}_P)$  in  $\text{PENCIL}_{n \times n}^{ss}$ . Therefore, for the generic  $m \times m$  skew-symmetric polynomial  $W$  with grade  $d$  identified in Theorem 5.2, we deduce from the proof of this theorem that  $\text{cod O}(W) = \text{cod O}^c(\mathcal{L}_W) = \text{cod O}^c(\mathcal{W})$ , where  $\mathcal{W}$  is the pencil in Theorem 3.1 with the identifications  $n = md$  and  $w = (m(d-1) + 2r)/2$ . Therefore, we get from (7)

$$\begin{aligned} \text{cod O}(W) &= (md - m(d-1) - 2r - 1) \left( md - m \frac{d-1}{2} - r \right) \\ &= \frac{1}{2} (m - 2r - 1)(m(d+1) - 2r). \end{aligned}$$

## 6. Conclusions and future work

This paper establishes that *there is only one* generic complete eigenstructure for skew-symmetric matrix polynomials of rank at most  $2r$  and odd grade  $d$ . In order to obtain this result, first, the corresponding result for skew-symmetric pencils is proved, i.e., when  $d = 1$ , and, second, this result is extended to skew-symmetric matrix polynomials of arbitrary odd grades by means of a structure preserving strong-linearization template (also known as a structure preserving companion linearization [15]) and a delicate translation of the topological properties of the space of skew-symmetric pencils into the space of skew-symmetric matrix polynomials. To the best of our knowledge, this is the first result of this kind obtained for a class of structured matrix polynomials of fixed (bounded) rank and fixed grade and it is in stark contrast to the results for general (unstructured) matrix polynomials and pencils available in the literature [9, 12, 20], which establish that there are  $rd + 1$  generic complete eigenstructures for arbitrary matrix polynomials of rank at most  $r$  and grade  $d$ , instead of only one. This striking reduction in the number of generic eigenstructures is related, among the other reasons, to the equality of the left and the right minimal indices of skew-symmetric matrix polynomials.

The results in this paper call to natural extensions to other classes of structured matrix polynomials (and pencils), but there are a number of obstacles that make such extensions challenging and nontrivial. The most obvious extension one could have in mind is to skew-symmetric matrix polynomials



of rank at most  $2r$  and *even grade*  $d$ , but in this case a structured companion linearization is not available in the literature and, in fact, it has been proved that it does not exist for  $m \times m$  skew-symmetric matrix polynomials when  $m$  is odd [15, Thm. 7.21]. Therefore, the techniques used in this paper cannot be used for skew-symmetric matrix polynomials of even grade. For several other structured classes of matrix polynomials of odd grade there are indeed structure preserving companion linearizations [16, 38, 39] (see also [15, Sect. 7] and [41], and the references therein), but there are no stratification results available in the literature for these structures. Since in this paper the stratification results for skew-symmetric pencils and polynomials previously developed in [19, 22] have played a key role, we see again that the techniques of this paper cannot be directly used to get similar results for other structured matrix polynomials.

## Acknowledgements

The work of Andrii Dmytryshyn was supported by the Swedish Research Council (VR) under grant E0485301, and by eSSENCE (essence-ofscience.se), a strategic collaborative e-Science programme funded by the Swedish Research Council.

The work of Froilán M. Dopico was supported by “Ministerio de Economía, Industria y Competitividad of Spain” and “Fondo Europeo de Desarrollo Regional (FEDER) of EU” through grants MTM-2015-68805-REDT and MTM-2015-65798-P (MINECO/FEDER, UE).

## References

- [1] E. Antoniou and S. Vologiannidis. A new family of companion forms of polynomial matrices. *Electron. J. Linear Algebra*, 11:78–87, 2004.
- [2] L. Batzke. Generic rank-one perturbations of structured regular matrix pencils. *Linear Algebra Appl.*, 458:638–670, 2014.
- [3] L. Batzke. Sign characteristics of regular Hermitian matrix pencils under generic rank-1 perturbations and a certain class of generic rank-2 perturbations. *Electron. J. Linear Algebra*, 30:760–794, 2015.
- [4] L. Batzke. Generic rank-two perturbations of structured regular matrix pencils. *Oper. Matrices*, 10(1):83–112, 2016.
- [5] D. Boley. The algebraic structure of pencils and block Toeplitz matrices. *Linear Algebra Appl.*, 279:255–279, 1998.

- [6] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [7] T. J. Bridges and S. Reich. Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity. *Phys. Lett. A*, 284(4–5):184–193, 2001.
- [8] F. De Terán and F. Dopico. Low rank perturbation of Kronecker structures without full rank. *SIAM J. Matrix Anal. Appl.*, 29(2):496–529, 2007.
- [9] F. De Terán and F. Dopico. A note on generic Kronecker orbits of matrix pencils with fixed rank. *SIAM J. Matrix Anal. Appl.*, 30(2):491–496, 2008.
- [10] F. De Terán and F. Dopico. Low rank perturbation of regular matrix polynomials. *Linear Algebra Appl.*, 430(1):579–586, 2009.
- [11] F. De Terán and F. Dopico. Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations. *SIAM J. Matrix Anal. Appl.*, 37(3):823–835, 2016.
- [12] F. De Terán, F. Dopico, and J. Landsberg. An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank. *Linear Algebra Appl.*, 520:80–103, 2017.
- [13] F. De Terán, F. Dopico, and D. S. Mackey. Fiedler companion linearizations and the recovery of minimal indices. *SIAM J. Matrix Anal. Appl.*, 31(4):2181–2204, 2010.
- [14] F. De Terán, F. Dopico, and D. S. Mackey. Fiedler companion linearizations for rectangular matrix polynomials. *Linear Algebra Appl.*, 437(3):957–991, 2012.
- [15] F. De Terán, F. Dopico, and D. S. Mackey. Spectral equivalence of matrix polynomials and the Index Sum Theorem. *Linear Algebra Appl.*, 459(Complete):264–333, 2014.
- [16] F. De Terán, F. M. Dopico, and D. S. Mackey. Palindromic companion forms for matrix polynomials of odd degree. *J. Comput. Appl. Math.*, 236(6):1464–1480, 2011.
- [17] F. De Terán, F. M. Dopico, and J. Moro. Low rank perturbation of Weierstrass structure. *SIAM J. Matrix Anal. Appl.*, 30(2):538–547, 2008.
- [18] J. Demmel and A. Edelman. The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms. *Linear Algebra Appl.*, 230:61–87, 1995.

- [19] A. Dmytryshyn. Structure preserving stratification of skew-symmetric matrix polynomials. Technical Report UMINF 15.16, Department of Computing Science, Umeå University, Sweden, 2015.
- [20] A. Dmytryshyn and F. Dopico. Generic matrix polynomials with fixed rank and fixed degree. Technical Report UMINF 16.19, Dept. of Computing Science, Umeå University, Sweden, 2016.
- [21] A. Dmytryshyn, S. Johansson, and B. Kågström. Codimension computations of congruence orbits of matrices, symmetric and skew-symmetric matrix pencils using Matlab. Technical Report UMINF 13.18, Department of Computing Science, Umeå University, Sweden, 2013.
- [22] A. Dmytryshyn and B. Kågström. Orbit closure hierarchies of skew-symmetric matrix pencils. *SIAM J. Matrix Anal. Appl.*, 35(4):1429–1443, 2014.
- [23] A. Dmytryshyn, B. Kågström, and V. Sergeichuk. Skew-symmetric matrix pencils: Codimension counts and the solution of a pair of matrix equations. *Linear Algebra Appl.*, 438(8):3375–3396, 2013.
- [24] F. Dopico, P. Lawrence, J. Pérez, and P. Van Dooren. Block Kronecker linearizations of matrix polynomials and their backward errors. MIMS EPrint 2016.34, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2016.
- [25] F. Dopico, J. Pérez, and P. Van Dooren. Structured backward error analysis of linearized structured polynomial eigenvalue problems. arXiv:1612.07011, arXiv, 2016.
- [26] A. Edelman, E. Elmroth, and B. Kågström. A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations. *SIAM J. Matrix Anal. Appl.*, 18(3):653–692, 1997.
- [27] A. Edelman, E. Elmroth, and B. Kågström. A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm. *SIAM J. Matrix Anal. Appl.*, 20(3):667–669, 1999.
- [28] G. Forney. Minimal bases of rational vector spaces with applications to multivariable linear systems. *SIAM J. Control Optim.*, 13(3):493–520, 1975.
- [29] F. Gantmacher. *The theory of matrices, Vol. I and II (transl.)*. Chelsea, New York, 1959.
- [30] H. Gernandt and C. Trunk. Eigenvalue placement for regular matrix pencils with rank one perturbations. *SIAM J. Matrix Anal. Appl.*, 38(1):134–154, 2017.

- [31] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. SIAM Publications, 2009. Originally published: New York, Academic Press, 1982.
- [32] N. J. Higham. *Accuracy and stability of numerical algorithms*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2002.
- [33] P. Johansson. Matrix Canonical Structure Toolbox. Technical Report UMINF 06.15, Department of Computing Science, Umeå University, Sweden, 2006.
- [34] S. Johansson, B. Kågström, and P. Van Dooren. Stratification of full rank polynomial matrices. *Linear Algebra Appl.*, 439:1062–1090, 2013.
- [35] T. Kailath. *Linear Systems*. Prentice Hall, New Jersey, 1980.
- [36] P. Y. Li and R. Horowitz. Passive velocity field control of mechanical manipulators. *IEEE Trans. Robotics Automation*, 15(4):751–763, 1999.
- [37] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28(4):1029–1051, 2006.
- [38] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Jordan structures of alternating matrix polynomials. *Linear Algebra Appl.*, 432(4):867–891, 2010.
- [39] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Smith forms of palindromic matrix polynomials. *Electron. J. Linear Algebra*, 22:53–91, 2011.
- [40] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Skew-symmetric matrix polynomials and their Smith forms. *Linear Algebra Appl.*, 438(12):4625–4653, 2013.
- [41] D. S. Mackey, N. Mackey, and F. Tisseur. Polynomial Eigenvalue Problems: Theory, Computation, and Structure. In *Numerical Algebra, Matrix Theory, Differential-Algebraic Equations and Control Theory*, pages 319–348. Springer, 2015.
- [42] C. Mehl, V. Mehrmann, and M. Wojtylak. On the distance to singularity via low rank perturbations. *Oper. Matrices*, 9(4):733–772, 2015.
- [43] C. Mehl, V. Mehrmann, and M. Wojtylak. Parameter-dependent rank-one perturbations of singular Hermitian or symmetric pencils. *SIAM J. Matrix Anal. Appl.*, 38(1):72–95, 2017.
- [44] P. J. Olver. Canonical Forms for Compatible BiHamiltonian Systems. In I. Antoniou and F. J. Lambert, editors, *Solitons and Chaos*, pages 171–179. Springer Berlin Heidelberg, Berlin, Heidelberg, 1991.

- [45] R. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 147:323–371, 1991.
- [46] P. Van Dooren. *The generalized eigenstructure problem: Applications in linear system theory*. PhD thesis, Kath. Univ. Leuven, Leuven, Belgium, 1979.
- [47] P. Van Dooren and P. Dewilde. The eigenstructure of a polynomial matrix: Computational aspects. *Linear Algebra Appl.*, 50:545–579, 1983.
- [48] W. Waterhouse. The codimension of singular matrix pairs. *Linear Algebra Appl.*, 57:227–245, 1984.