BLOCK KRONECKER LINEARIZATIONS OF MATRIX POLYNOMIALS AND THEIR BACKWARD ERRORS *

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Abstract. We introduce a new family of strong linearizations of matrix polynomials—which we call "block Kronecker pencils"—and perform a backward stability analysis of complete polynomial eigenproblems. These problems are solved by applying any backward stable algorithm to a block Kronecker pencil, such as the staircase algorithm for singular pencils or the QZ algorithm for regular pencils. This stability analysis allows us to identify those block Kronecker pencils that yield a computed complete eigenstructure which is exactly that of a slightly perturbed matrix polynomial. These favorable pencils include the famous Fiedler linearizations, which are just a very particular case of block Kronecker pencils. Thus, our analysis offers the first proof available in the literature of global backward stability for Fiedler pencils. In addition, the theory developed for block Kronecker pencils is much simpler than the theory available for Fiedler pencils, especially in the case of rectangular matrix polynomials. The global backward error analysis in this work presents for the first time the following key properties: it is a rigurous analysis valid for finite perturbations (i.e., it is not a first order analysis), it provides precise bounds, it is valid simultaneously for a large class of linearizations, and it establishes a framework that may be generalized to other classes of linearizations. These features are related to the fact that block Kronecker pencils are a particular case of the new family of "strong block minimal bases pencils", which include certain perturbations of block Kronecker pencils; this will allow us to extend the results in this paper to other contexts.

Key words. Backward error analysis, polynomial eigenvalue problems, complete eigenstructure, dual minimal bases, Fiedler pencils, linearization, matrix polynomials, matrix perturbation theory, minimal indices

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1. Introduction.

Matrix polynomials are ubiquitous in a wide range of applications in engineering, mechanics, control, linear systems theory, and computer-aided geometric design. They may arise directly, as for instance in the study of dynamical problems described by systems of ordinary differential equations with constant coefficients, or they may arise indirectly, from finite element discretizations of continuous models, or as approximations of highly nonlinear eigenvalue problems. The classical works [33, 42, 58] and the modern surveys [51, 62], as well as the references therein, include detailed discussions of different applications of matrix polynomials. Those readers unfamiliar with the basics on matrix polynomials can find in Section 2 relevant definitions and explanations of the concepts mentioned in this introduction.

Square regular matrix polynomials are usually related to *polynomial eigenvalue problems* (PEPs), i.e., to the computation of all of the eigenvalues of the polynomial,

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while singular matrix polynomials are related to *complete polynomial eigenproblems* (CPEs), i.e., to the computation of all of the eigenvalues and of all of the so-called minimal indices of the polynomial. The numerical solution of PEPs and CPEs is usually performed by embedding the coefficients of the associated matrix polynomial into a larger linear matrix polynomial, or matrix pencil, called a *linearization*, and then applying well-established algorithms for matrix pencils, like the QZ [34] or the staircase algorithm [63], to the linearization. Linearizations of matrix polynomials have the same eigenvalues as the polynomial and the reliable ones should have minimal indices easily related to those of the matrix polynomial in the singular case. This approach of solving PEPs and CPEs numerically via linearizations was proposed for the first time for general matrix polynomials (i.e., regular or singular, square or rectangular) in [64]. The concept of linearization was formally introduced in [33] only for regular matrix polynomials. A formal definition of linearization of singular polynomials is given in [9] and a thorough treatment of this concept as a particular case of the most general concept of ℓ-ification can be found in [14].

The linearizations used most often to solve PEPs and CPEs are the well known Frobenius companion forms (see (4.1) and (4.2)). These linearizations are used in [64] and are also used in the command `polyeig` of MATLAB. They have many favorable properties; in particular, it was proven in [64] that they yield computed solutions of PEPs and CPEs which are exactly those of slightly perturbed matrix polynomials (i.e., from the polynomial point of view they have perfect structured backward stability).

However, it is well known that the Frobenius companion forms do not preserve the algebraic structures that are often present in the matrix polynomials arising in applications. Therefore, the rounding errors inherent to numerical computations may destroy qualitative properties of the eigenstructures of such polynomials when they are computed via the Frobenius forms. In addition, it is also known that because of their ill-conditioning, Frobenius forms do not deliver accurate solutions of PEPs when the matrix coefficients of the polynomial have very different norms; this problem has to date only been addressed in the quadratic case [36, 67]. These drawbacks have motivated an intense activity in the last few years towards the development and analysis of new classes of linearizations of matrix polynomials, with special emphasis on linearizations that preserve certain structures important in applications (see, for example: [1, 2, 4, 5, 7, 8, 11, 12, 13, 28, 38, 48, 49, 54, 66]).

There are two main families of linearizations of matrix polynomials available in the recent literature, which generalize Frobenius companion forms in different directions. A first class of linearizations was presented in [49] and further analyzed in [37, 38, 39, 48, 53]. The linearizations in this family are often referred to as linearizations in vector spaces, and are valid only for square polynomials. A second class was introduced in [28] for monic scalar polynomials, and then generalized to regular matrix polynomials in [2], to square singular matrix polynomials in [11], and to rectangular matrix polynomials in [13]. The linearizations in this second class were baptized as Fiedler companion pencils in [11], and have been generalized in several ways with the aim of constructing structure-preserving linearizations [2, 5, 6, 7, 8, 12, 54, 66].

Fiedler companion linearizations and their generalizations offer several important advantages over other families of linearizations. For instance, they are easily constructible from the coefficients of the matrix polynomial without performing any operations, and they are always strong linearizations of the matrix polynomial regardless whether the polynomial is regular or singular, square or rectangular. Moreover, the eigenvectors, minimal indices, and minimal bases of any Fiedler pencil and those of the matrix polynomial are related in very simple ways. However, the proofs of these results are quite involved, as they require keeping track of a large number of unimodular transformations, and for rectangular polynomials the very same definition of Fiedler pencils is complicated. All of these difficulties are intrinsically connected to the implicit way Fiedler pencils are defined, either in terms of products of matrices for square polynomials or as the output of a symbolic algorithm for rectangular ones.

A key open problem in this area is that global backward error analyses of PEPs and CPEs solved by linearizations in vector spaces or by Fiedler linearizations are not yet available in the literature. Only "local" residual backward error analyses valid for each particular computed eigenpair in the case of the linearizations in vector spaces have been developed to date [37, 61]. Again, the implicit definition of Fiedler pencils constitutes a major obstacle to extending the global backward error analysis for Frobenius companion forms presented originally in [64] or the ones developed very recently in [46, 47, 55] to Fiedler pencils.

In this paper, we introduce two new families of strong linearizations of general matrix polynomials—square or rectangular, regular or singular—whose minimal indices are related to those of the matrix polynomial via constant uniform shifts. We call these families the *strong block minimal bases pencils* (see Section 3), and a subfamily of it the *block Kronecker pencils* (see Section 5). Strong block minimal bases pencils are defined in an abstract way in terms of the classical concept of *dual minimal bases* [29]. This allows us to prove that they are always strong linearizations of matrix polynomials in a clean, simple, and general way and that simple relationships exist between their minimal indices and those of the matrix polynomial. These properties are inherited by the subfamily of block Kronecker pencils, which include all of the Fiedler pencils—modulo permutations—as a very particular case, and which have the key advantage of being easily described explicitly in terms of their entries. Therefore, we obtain as a corollary of our general results a simplified theory of Fiedler pencils.

The simple theory established for strong block minimal bases pencils and block Kronecker pencils enables us to perform a global backward error analysis of PEPs and CPEs solved via block Kronecker pencils. The distinctive point of this analysis is that perturbations of block Kronecker pencils lead, after some manipulations, to other strong block minimal bases pencils with similar properties. As a consequence, this error analysis has the following novel properties with respect to previous global analyses: (1) it is valid for perturbations with finite norms, in contrast to previous analyses which are valid only to first order; (2) it delivers precise bounds, in contrast to other analyses which only provide vague big-O bounds; (3) it is valid simultaneously for a very large class of linearizations, in contrast to other analyses that are specific for particular linearizations; and (4) it may be generalized to other subfamilies of strong block minimal bases pencils. We emphasize that the analysis we present solves, as a corollary, the open problem of proving that all Fiedler pencils yield computed complete eigenstructures of matrix polynomials that enjoy perfect structured backward stability from the polynomial point of view.

The paper is organized as follows. Section 2 presents a summary of basic concepts and results that are used throughout the paper. In Section 3, the strong block minimal bases pencils are introduced and their properties are established. Section 4 proves that certain permutations transform any Fiedler pencil into a strong block minimal bases pencil of a very particular type, which motivates the definition and study of block Kronecker pencils in Section 5. A fully rigorous and detailed global backward error analysis of complete polynomial eigenproblems solved by means of block Kronecker pencils is the subject of Section 6. Some conclusions and lines of future research are discussed in Section 7. Finally, Appendices A and B present long technical proofs of two results needed in the paper. For brevity, we do not present in this paper recovery procedures of eigenvectors and minimal bases of a matrix polynomial from those of its strong block minimal bases pencils and from those of its block Kronecker pencils. These results are included in Section 7 of the extended version of this manuscript available in [22].

2. Basic concepts, auxiliary results, and notation.

Throughout the paper we use the following notation. Given an arbitrary field $\mathbb{F}$, we denote by $\mathbb{F}[\lambda]$ the ring of polynomials in the variable λ with coefficients in $\mathbb{F}$ and by $\mathbb{F}(\lambda)$ the field of rational functions with coefficients in $\mathbb{F}$. The set of $m \times n$ matrices with entries in $\mathbb{F}[\lambda]$ is denoted by $\mathbb{F}[\lambda]^{m\times n}$ and is also called the set of $m \times n$ *matrix polynomials*. In this context, row or column *vector polynomials* are just matrix polynomials with $m = 1$ or $n = 1$. $\mathbb{F}(\lambda)^{m\times n}$ denotes the set of $m \times n$ rational matrices. Given two matrices A and B, $A \oplus B$ denotes their direct sum, i.e., $A \oplus B = \mathrm{diag}(A, B)$, and $A \otimes B$ denotes their Kronecker product [40]. The algebraic closure of $\mathbb{F}$ is denoted by $\overline{\mathbb{F}}$. The results in Section 6 and Subsection 2.1 assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, while the rest of results remain valid in any field.

A matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is said to have *grade* d if it is written as

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0, \quad \text{with } P_0, \ldots, P_d \in \mathbb{F}^{m\times n}, \tag{2.1}$$

where any of the coefficient matrices P_k, including P_d, may be the zero matrix. As usual, the *degree* of $P(\lambda)$, denoted by $\deg(P)$, is the maximum integer k such that P_k is a nonzero matrix. Thus, the degree of $P(\lambda)$ is fixed while its grade d is a choice that must satisfy $d \geq \deg(P)$. The concept of grade has been used previously in [14, 50] and is convenient in situations where the degree of a polynomial is not known in advance. Throughout this paper when the grade of $P(\lambda)$ is not explicitly stated, we consider its grade equal to its degree. A matrix polynomial of grade 1 is called a *matrix pencil.*

For any $d \geq \deg(P)$ the *d-reversal matrix polynomial* of $P(\lambda)$ is defined as

$$\mathrm{rev}_d P(\lambda) := \lambda^d P(\lambda^{-1}).$$

Observe that if $P(\lambda)$ is assumed to have grade d, then it is assumed that $\mathrm{rev}_d P(\lambda)$ has also grade d, but that the degree of $\mathrm{rev}_d P(\lambda)$ may be different that the degree of $P(\lambda)$, even in the case $d = \deg(P)$.

In this paper, we define the *rank* of a matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ as its rank over the field $\mathbb{F}(\lambda)$, i.e., as the size of the largest non-identically zero minor of $P(\lambda)$ [30] and is denoted by $\mathrm{rank}(P)$. This is also called the “normal rank” of $P(\lambda)$, but we avoid to use this name for brevity. Note that expressions such as $\mathrm{rank}(P(\lambda_0))$ denote the rank of the constant matrix $P(\lambda_0) \in \overline{\mathbb{F}}^{m\times n}$, i.e., of the polynomial evaluated at $\lambda_0 \in \overline{\mathbb{F}}$. We will say that $P(\lambda_0)$ has full row (resp. column) rank if $\mathrm{rank}\, P(\lambda_0) = m$ (resp. $\mathrm{rank}\, P(\lambda_0) = n$). Observe that if the constant matrix $P(\lambda_0)$ has full row (resp. column) rank, then also the matrix polynomial $P(\lambda)$ has full row (resp. column) rank.

A key distinction for matrix polynomials is between regular and singular matrix polynomials. A matrix polynomial $P(\lambda)$ is said to be *regular* if $P(\lambda)$ is square (that is, $m = n$) and $\det P(\lambda)$ is not the identically zero polynomial. Otherwise, $P(\lambda)$ is said to be *singular* (note that this includes all rectangular matrix polynomials $m \neq n$).

We refer the reader to [14, Section 2] for the precise definitions of the spectral and the singular structures of a matrix polynomial, as well as for other related concepts that are used in this paper. In addition, as in [18], the term *complete eigenstructure* of $P(\lambda)$ stands for the collection of all of the elementary divisors of $P(\lambda)$, both finite and infinite, and for the collection of all of its minimal indices, both left and right, i.e., for the union of the spectral and singular structures of $P(\lambda)$. In the next paragraph, we explain in detail the concepts of minimal bases and minimal indices, as they play an essential role in this paper.

If a matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is singular, then it has non-trivial left and/or right *rational* null spaces:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \{y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \quad \text{such that} \quad y(\lambda)^T P(\lambda) = 0\}, \\ \mathcal{N}_r(P) &:= \{x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \quad \text{such that} \quad P(\lambda)x(\lambda) = 0\}. \end{aligned} \tag{2.2}$$

These null spaces are particular examples of *rational* subspaces, i.e., subspaces over the field $\mathbb{F}(\lambda)$ formed by p-tuplas whose entries are rational functions [29]. It is not difficult to show that any rational subspace $\mathcal{V}$ has bases consisting entirely of vector polynomials. The *order* of a vector polynomial basis of $\mathcal{V}$ is defined as the sum of the degrees of its vectors [29, Definition 2]. Amongst all of the possible polynomial bases of $\mathcal{V}$, those with least order are called *minimal bases* of $\mathcal{V}$ [29, Definition 3]. There are infinitely many minimal bases of $\mathcal{V}$, but the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same [29, Remark 4, p. 497]. This list of degrees is called the list of *minimal indices* of $\mathcal{V}$. With these definitions at hand, the left (resp. right) minimal indices and bases of a matrix polynomial $P(\lambda)$ are defined as those of the rational subspace $\mathcal{N}_\ell(P)$ (resp. $\mathcal{N}_r(P)$).

The following definitions are useful when working with minimal bases in practice. The *ith row degree* of a matrix polynomial $Q(\lambda)$ is the degree of the ith row of $Q(\lambda)$.

DEFINITION 2.1. *Let $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ be a matrix polynomial with row degrees $d_1, d_2, \ldots, d_m$. The* highest row degree coefficient matrix *of $Q(\lambda)$, denoted by Q_h, is the $m \times n$ constant matrix whose jth row is the coefficient of λ^{d_j} in the jth row of $Q(\lambda)$, for $j = 1, 2, \ldots, m$. The matrix polynomial $Q(\lambda)$ is called* row reduced *if Q_h has full row rank.*

Observe that Q_h is equal to the leading coefficient $Q_d \neq 0$ in the expansion $Q(\lambda) = \sum_{i=0}^{d} Q_i \lambda^i$ if and only if all the row degrees of $Q(\lambda)$ are equal to d.

Theorem 2.2 is probably the most useful characterization of minimal bases in practice. It is a classical result that was proved in [29, Main Theorem-Part 2, p. 495], where it was stated in more abstract terms. The statement we present can be found in [18, Theorem 2.14].

THEOREM 2.2. *The rows of a matrix polynomial $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ are a minimal basis of the rational subspace they span if and only if $Q(\lambda_0) \in \overline{\mathbb{F}}^{m\times n}$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and $Q(\lambda)$ is row reduced.*

REMARK 2.3. Most of the minimal bases appearing in this work are arranged as the rows of a matrix. Therefore, throughout the paper—and with a slight abuse of notation—we say that an $m \times n$ matrix polynomial (with $m < n$) is a minimal basis if its rows form a minimal basis of the rational subspace they span.

Definition 2.1 and Theorem 2.2 admit obvious extensions "for columns", which are used occasionally in this paper.

Corollary 2.4 is a consequence of Theorem 2.2 which is relevant in this work.

COROLLARY 2.4. *If a matrix polynomial $Q(\lambda)$ is a minimal basis and I_p is the $p \times p$ identity matrix, then $Q(\lambda) \otimes I_p$ is also a minimal basis.*

Proof. It follows from Theorem 2.2 just by taking into account that $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A)\,\mathrm{rank}(B)$ [40, Theorem 4.2.15] and that $Q_h \otimes I_p$ is the highest row degree coefficient matrix of $Q(\lambda) \otimes I_p$. ☐

The concept of *dual minimal bases* is fundamental in this paper and is introduced in Definition 2.5.

DEFINITION 2.5. *Two matrix polynomials* $L(\lambda) \in \mathbb{F}[\lambda]^{m_1 \times n}$ *and* $N(\lambda) \in \mathbb{F}[\lambda]^{m_2 \times n}$ *are called* dual minimal bases *if* $L(\lambda)$ *and* $N(\lambda)$ *are both minimal bases and they satisfy* $m_1 + m_2 = n$ *and* $L(\lambda)N(\lambda)^T = 0$.

The name "dual minimal bases" and its definition were introduced in [15, Definition 2.10], but their origins can be traced back to [29]. We also use the expression "$N(\lambda)$ is a minimal basis dual to $L(\lambda)$", or vice versa, for referring to matrix polynomials $L(\lambda)$ and $N(\lambda)$ as those in Definition 2.5.

EXAMPLE 2.6. We illustrate the concept of dual minimal bases with a simple example that is important in this paper. Consider the following matrix polynomials:

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{k \times (k+1)}, \tag{2.3}$$

and

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1 \times (k+1)}, \tag{2.4}$$

where here and throughout the paper we occasionally omit some, or all, of the zero entries of a matrix. Theorem 2.2 guarantees that $L_k(\lambda)$ and $\Lambda_k(\lambda)^T$ are minimal bases. In addition, $L_k(\lambda)\Lambda_k(\lambda) = 0$ holds. Therefore, $L_k(\lambda)$ and $\Lambda_k(\lambda)^T$ are dual minimal bases. From Corollary 2.4 and the properties of the Kronecker product we get that $L_k(\lambda) \otimes I_p$ and $\Lambda_k(\lambda)^T \otimes I_p$ are also dual minimal bases.

The matrix $L_k(\lambda)$ is very well known since is a right singular block of the Kronecker Canonical Form of pencils [30, Chapter XII]. Also the *column* vector polynomial $\Lambda_k(\lambda)$ is very well known and plays an essential role, for instance, in the famous vector spaces of linearizations studied in [38, 49].

Theorem 2.7 establishes properties of minimal bases whose row degrees are all equal. These are the minimal bases of interest in this work. Part (a) of Theorem 2.7 can be obtained from more general results on row-wise reversals of minimal bases [10, 50], but a proof based on Theorem 2.2 is simpler and, therefore, is included. Theorem 2.7(b) has been used in [19] without being explicitly stated.

THEOREM 2.7.

(a) *Let* $K(\lambda)$ *be a minimal basis whose row degrees are all equal to* j. *Then* $\mathrm{rev}_j K(\lambda)$ *is also a minimal basis whose row degrees are all equal to* j.

(b) *Let* $K(\lambda)$ *and* $N(\lambda)$ *be dual minimal bases. If the row degrees of* $K(\lambda)$ *are all equal to* j *and the row degrees of* $N(\lambda)$ *are all equal to* ℓ, *then* $\mathrm{rev}_j K(\lambda)$ *and* $\mathrm{rev}_\ell N(\lambda)$ *are also dual minimal bases.*

Proof. (a) Consider the expansion $K(\lambda) = \sum_{i=0}^{j} K_i \lambda^i \in \mathbb{F}[\lambda]^{m_1 \times n}$ and note that Theorem 2.2 implies that K_0 and K_j have both full row rank, since in this case K_j is the highest row degree coefficient matrix of $K(\lambda)$. Observe that the row degrees of $Q(\lambda) := \mathrm{rev}_j K(\lambda)$ are also all equal to j because the leading degree coefficient of $Q(\lambda)$ is K_0, which is also its highest row degree coefficient. In addition: (1) $Q(0) = K_j$

has full row rank, and (2) for all nonzero $\lambda_0 \in \overline{\mathbb{F}}$, $\operatorname{rank} Q(\lambda_0) = \operatorname{rank} \lambda_0^j K(1/\lambda_0) = \operatorname{rank} K(1/\lambda_0) = m_1$, again by Theorem 2.2. Therefore, Theorem 2.2 applied to $Q(\lambda)$ proves part (a).

(b) From (a) we get that $\operatorname{rev}_j K(\lambda)$ and $\operatorname{rev}_\ell N(\lambda)$ are both minimal bases. Moreover, $K(\lambda)N(\lambda)^T = 0$ implies $K(1/\lambda)N(1/\lambda)^T = 0$ and $(\lambda^j K(1/\lambda))\,(\lambda^\ell N(1/\lambda))^T = 0$. Therefore, $\operatorname{rev}_j K(\lambda)\,(\operatorname{rev}_\ell N(\lambda))^T = 0$ and part (b) is proved. □

EXAMPLE 2.8. Theorem 2.7(b) can be applied to the dual minimal bases $L_k(\lambda)$ and $\Lambda_k(\lambda)^T$ in Example 2.6 to prove that

$$\operatorname{rev}_1 L_k(\lambda) = \begin{bmatrix} -\lambda & 1 & & & \\ & -\lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & -\lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{k\times(k+1)}$$

and

$$\operatorname{rev}_k \Lambda_k(\lambda)^T = \begin{bmatrix} 1 & \lambda & \cdots & \lambda^k \end{bmatrix} \in \mathbb{F}[\lambda]^{1\times(k+1)}$$

are also dual minimal bases. This fact follows also directly from Theorem 2.2 and matrix multiplication.

Lemma 2.9 states that any matrix polynomial $Q(\lambda)$ such that $Q(\lambda_0)$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$ can be completed into a *unimodular matrix polynomial*, i.e., a matrix polynomial with nonzero constant determinant. This is an old result that can be traced back at least to Kailath [42] (a very simple proof appears in [18, Lemma 2.16(b)]). Efficient algorithms for computing such completions can be found in Beelen and Van Dooren [3].

LEMMA 2.9. *Let $Q(\lambda)$ be a matrix polynomial over a field $\mathbb{F}$. If $Q(\lambda_0)$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$, then there exists a matrix polynomial $\widetilde{Q}(\lambda)$ such that*

$$\widehat{Q}(\lambda) = \begin{bmatrix} Q(\lambda) \\ \widetilde{Q}(\lambda) \end{bmatrix}$$

is unimodular.

Lemma 2.9 can be applied, in particular, when $Q(\lambda)$ is a minimal basis, as a consequence of Theorem 2.2. Moreover, Lemma 2.9 can be extended to Theorem 2.10, which is one of the main tools employed in Section 3. Observe that Theorem 2.10 can be applied, in particular, when $L(\lambda)$ and $N(\lambda)$ are dual minimal bases.

THEOREM 2.10. *Let $L(\lambda) \in \mathbb{F}[\lambda]^{m_1\times n}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{m_2\times n}$ be matrix polynomials such that $m_1 + m_2 = n$, $L(\lambda_0)$ and $N(\lambda_0)$ have both full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$, and $L(\lambda)N(\lambda)^T = 0$. Then, there exists a unimodular matrix polynomial $U(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ such that*

$$U(\lambda) = \begin{bmatrix} L(\lambda) \\ \widehat{L}(\lambda) \end{bmatrix} \quad \textit{and} \quad U(\lambda)^{-1} = \begin{bmatrix} \widehat{N}(\lambda)^T & N(\lambda)^T \end{bmatrix}.$$

Proof. By Lemma 2.9, there exist unimodular embeddings

$$\begin{bmatrix} L(\lambda) \\ Z_1(\lambda) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Z_2(\lambda)^T & N(\lambda)^T \end{bmatrix}.$$

Since the product of two unimodular matrix polynomials is also unimodular, from

$$\begin{bmatrix} L(\lambda) \\ Z_1(\lambda) \end{bmatrix} \begin{bmatrix} Z_2(\lambda)^T & N(\lambda)^T \end{bmatrix} = \begin{bmatrix} L(\lambda)Z_2(\lambda)^T & 0 \\ Z_1(\lambda)Z_2(\lambda)^T & Z_1(\lambda)N(\lambda)^T \end{bmatrix},$$

it follows that $L(\lambda)Z_2(\lambda)^T \in \mathbb{F}[\lambda]^{m_1\times m_1}$ and $Z_1(\lambda)N(\lambda)^T \in \mathbb{F}[\lambda]^{m_2\times m_2}$ must also be unimodular matrix polynomials, as well as their inverses. Let us now consider the following unimodular matrix polynomials

$$U(\lambda) = \begin{bmatrix} I_{m_1} & 0 \\ 0 & (Z_1(\lambda)N(\lambda)^T)^{-1} \end{bmatrix} \begin{bmatrix} L(\lambda) \\ Z_1(\lambda) \end{bmatrix}$$

and

$$V(\lambda) = \begin{bmatrix} Z_2(\lambda)^T & N(\lambda)^T \end{bmatrix} \begin{bmatrix} (L(\lambda)Z_2(\lambda)^T)^{-1} & 0 \\ 0 & I_{m_2} \end{bmatrix} \begin{bmatrix} I_{m_1} & 0 \\ -X(\lambda) & I_{m_2} \end{bmatrix},$$

where $X(\lambda) = (Z_1(\lambda)N(\lambda)^T)^{-1}Z_1(\lambda)Z_2(\lambda)^T(L(\lambda)Z_2(\lambda)^T)^{-1}$. The statement of the theorem then follows by verifying that $U(\lambda)V(\lambda) = I_n$. □

Remark 2.11. We emphasize that fixed any $L(\lambda)$ and $N(\lambda)$ that satisfy the assumptions of Theorem 2.10, there exist infinitely many unimodular matrix polynomials $U(\lambda)$ as in Theorem 2.10. For instance, if one of such $U(\lambda)$ is found, then

$$\begin{bmatrix} I_{m_1} & 0 \\ Y(\lambda) & I_{m_2} \end{bmatrix} U(\lambda)$$

has the properties established in Theorem 2.10 for any matrix polynomial $Y(\lambda)$.

Example 2.12. We illustrate Theorem 2.10 with a particular embedding of the dual minimal bases $L_k(\lambda)$ and $\Lambda_k(\lambda)^T$ introduced in Example 2.6. If e_{k+1} is the last column of I_{k+1}, then it is easily verified that

$$V_k(\lambda) = \left[\begin{array}{c} L_k(\lambda) \\ \hline e_{k+1}^T \end{array}\right] = \left[\begin{array}{ccccc} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \\ \hline 0 & \cdots & \cdots & 0 & 1 \end{array}\right] \in \mathbb{F}[\lambda]^{(k+1)\times(k+1)}$$

is unimodular and that its inverse is

$$V_k(\lambda)^{-1} = \left[\begin{array}{ccccc|c} -1 & -\lambda & -\lambda^2 & \cdots & -\lambda^{k-1} & \lambda^k \\ & -1 & -\lambda & \ddots & \vdots & \lambda^{k-1} \\ & & -1 & \ddots & -\lambda^2 & \vdots \\ & & & \ddots & -\lambda & \lambda^2 \\ & & & & -1 & \lambda \\ & & & & & 1 \end{array}\right] \in \mathbb{F}[\lambda]^{(k+1)\times(k+1)}. \qquad (2.5)$$

Note that the last column of $V_k(\lambda)^{-1}$ is $\Lambda_k(\lambda)$. Therefore, $V_k(\lambda)$ is a particular instance of a matrix $U(\lambda)$ in Theorem 2.10 for $L_k(\lambda)$ and $\Lambda_k(\lambda)^T$. Moreover, $V_k(\lambda)\otimes I_p$ is a particular instance of $U(\lambda)$ for the dual minimal bases $L_k(\lambda)\otimes I_p$ and $\Lambda_k(\lambda)^T \otimes I_p$ discussed also in Example 2.6.

We now recall the definitions of linearization and strong linearization of a matrix polynomial, which are central in this paper. These definitions were introduced in [32, 33] for regular matrix polynomials, and then extended to the singular case in [9]. We refer the reader to [14] for a thorough treatment of these concepts and their properties.

Definition 2.13. *A matrix pencil $\mathcal{L}(\lambda)$ is a* linearization *of a matrix polynomial $P(\lambda)$ of grade d if for some $s \geq 0$ there exist two unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that*

$$U(\lambda)\mathcal{L}(\lambda)V(\lambda) = \begin{bmatrix} I_s & \\ & P(\lambda) \end{bmatrix}. \tag{2.6}$$

Furthermore, a linearization $\mathcal{L}(\lambda)$ is called a strong linearization *of $P(\lambda)$ if* $\mathrm{rev}_1\mathcal{L}(\lambda)$ *is a linearization of* $\mathrm{rev}_d P(\lambda)$.

The key property of any strong linearization $\mathcal{L}(\lambda)$ of a matrix polynomial $P(\lambda)$ is that $\mathcal{L}(\lambda)$ and $P(\lambda)$ share the same finite and infinite elementary divisors [14, Theorem 4.1]. However, Definition 2.13 only guarantees that the number of left (resp. right) minimal indices of $\mathcal{L}(\lambda)$ is equal to the number of left (resp. right) minimal indices of $P(\lambda)$. In fact, except by these constraints on the numbers, $\mathcal{L}(\lambda)$ may have any set of right and left minimal indices [14, Theorem 4.11]. Therefore, in the case of singular matrix polynomials, one needs to consider *strong linearizations with the additional property* that their minimal indices allow us to recover the minimal indices of the polynomial via some simple rule. In addition, *such rule should be robust under perturbations*, in order to be reliable in numerical computations affected by rounding errors, since minimal indices of matrix polynomials may vary wildly under perturbations [25, 26, 41]. These questions about the minimal indices are carefully studied throughout this paper.

Lemma 2.14 is a very simple result that allows us to easily recognize linearizations in certain situations which are of interest in this work.

Lemma 2.14. *Let $P(\lambda)$ be an $m \times n$ matrix polynomial and $\mathcal{L}(\lambda)$ be a matrix pencil. If there exist two unimodular matrix polynomials $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ such that*

$$\widetilde{U}(\lambda)\mathcal{L}(\lambda)\widetilde{V}(\lambda) = \begin{bmatrix} Z(\lambda) & X(\lambda) & I_t \\ Y(\lambda) & P(\lambda) & 0 \\ I_s & 0 & 0 \end{bmatrix}, \tag{2.7}$$

for some $s \geq 0$ and $t \geq 0$ and for some matrix polynomials $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$, then $\mathcal{L}(\lambda)$ is a linearization of $P(\lambda)$.

Proof. Define the unimodular matrix polynomials

$$R(\lambda) = \begin{bmatrix} I_t & 0 & -Z(\lambda) \\ 0 & 0 & I_s \\ 0 & I_m & -Y(\lambda) \end{bmatrix}, \quad S(\lambda) = \begin{bmatrix} 0 & I_s & 0 \\ 0 & 0 & I_n \\ I_t & 0 & -X(\lambda) \end{bmatrix}.$$

Then equation (2.7) implies that $R(\lambda)\widetilde{U}(\lambda)\mathcal{L}(\lambda)\widetilde{V}(\lambda)S(\lambda) = \mathrm{diag}(I_t, I_s, P(\lambda))$. This proves that $\mathcal{L}(\lambda)$ is a linearization of $P(\lambda)$. ☐

2.1. Norms of matrix polynomials and their submultiplicative properties.

The study of perturbations and backward errors in Section 6 requires the use of norms of matrix polynomials. We have chosen the simple norm in Definition 2.15. In this section the polynomials are assumed to have *real or complex coefficients*, i.e.,

$\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We refer the reader to [60] for the definitions and properties of the Frobenius norm, $\| \cdot \|_F$, and the spectral norm, $\| \cdot \|_2$, of constant matrices.

DEFINITION 2.15. *Let* $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{F}[\lambda]^{m \times n}$. *Then the Frobenius norm of* $P(\lambda)$ *is*

$$\|P(\lambda)\|_F := \sqrt{\sum_{i=0}^{d} \|P_i\|_F^2} \, .$$

Obviously $\|P(\lambda)\|_F$ defines a norm on the vector space of matrix polynomials with arbitrary grade and fixed size $m \times n$. In fact, Definition 2.15 defines a *family of norms*, because we have a different vector space, and, so, a different norm for each particular selection of size $m \times n$. This is important when considering the norm of the product $P(\lambda)Q(\lambda)$ of two polynomials $P(\lambda)$ and $Q(\lambda)$, since the sizes of the two factors and the product are, in general, different. In this context, it is also important to realize that the value of $\|P(\lambda)\|_F$ is independent of the grade chosen for $P(\lambda)$. This property allows us to work with $\|P(\lambda)\|_F$ without specifying the grade of $P(\lambda)$.

The family of norms $\|P(\lambda)\|_F$ is not submultiplicative, i.e., $\|P(\lambda)\, Q(\lambda)\|_F \not\leq \|P(\lambda)\|_F \, \|Q(\lambda)\|_F$ in general. For instance, if E_2 is the 2×2 matrix with all the entries equal to 1, and $P(\lambda) = Q(\lambda) = \lambda E_2 + E_2$, then $\|P(\lambda)\|_F \, \|Q(\lambda)\|_F = 8$ and $\|P(\lambda)\, Q(\lambda)\|_F = 4\sqrt{6}$. Therefore, since in Section 6 we need to bound the norms of certain products of matrix polynomials, we prove Lemma 2.16.

LEMMA 2.16. *Let* $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i$, *let* $Q(\lambda) = \sum_{i=0}^{t} Q_i \lambda^i$, *and let* $\Lambda_k(\lambda)^T$ *be the vector polynomial defined in* (2.4). *Then the following inequalities hold:*

(a) $\displaystyle \|P(\lambda)\, Q(\lambda)\|_F \leq \sqrt{d+1} \cdot \sqrt{\sum_{i=0}^{d} \|P_i\|_2^2} \cdot \|Q(\lambda)\|_F \ ,$

(b) $\displaystyle \|P(\lambda)\, Q(\lambda)\|_F \leq \sqrt{t+1} \cdot \|P(\lambda)\|_F \cdot \sqrt{\sum_{i=0}^{t} \|Q_i\|_2^2} \ ,$

(c) $\|P(\lambda)\, Q(\lambda)\|_F \leq \min\{\sqrt{d+1}, \sqrt{t+1}\} \, \|P(\lambda)\|_F \, \|Q(\lambda)\|_F \ ,$

(d) $\|P(\lambda)\, (\Lambda_k(\lambda) \otimes I_p)\|_F \leq \min\{\sqrt{d+1}, \sqrt{k+1}\} \, \|P(\lambda)\|_F,$

(e) $\|(\Lambda_k(\lambda)^T \otimes I_p)\, Q(\lambda)\|_F \leq \min\{\sqrt{t+1}, \sqrt{k+1}\} \, \|Q(\lambda)\|_F,$

where we assume that all the products are defined.

Proof. We only prove parts (a) and (d), because (b) (resp. (e)) follows from (a) (resp. (d)) applied to the transposed product. Observe also that (c) follows from (a), (b), and the fact that $\|A\|_2 \leq \|A\|_F$ for any constant matrix A. We will also use the property $\|AB\|_F \leq \|A\|_2 \|B\|_F$ for any constant matrices A and B.

Proof of (a). Note that $P(\lambda)\, Q(\lambda) = \sum_{i=0}^{d} P_i Q(\lambda)\, \lambda^i$. Therefore:

$$\begin{aligned} \|P(\lambda)\, Q(\lambda)\|_F &\leq \sum_{i=0}^{d} \|P_i Q(\lambda)\, \lambda^i\|_F = \sum_{i=0}^{d} \|P_i Q(\lambda)\|_F = \sum_{i=0}^{d} \sqrt{\sum_{j=0}^{t} \|P_i Q_j\|_F^2} \\ &\leq \sum_{i=0}^{d} \|P_i\|_2 \sqrt{\sum_{j=0}^{t} \|Q_j\|_F^2} \leq \sqrt{d+1} \cdot \sqrt{\sum_{i=0}^{d} \|P_i\|_2^2} \cdot \|Q(\lambda)\|_F . \end{aligned}$$

Proof of (d). Note again that $P(\lambda)\, (\Lambda_k(\lambda) \otimes I_p) = \sum_{i=0}^{d} P_i (\Lambda_k(\lambda) \otimes I_p)\, \lambda^i$. Let us partition $P_i = [P_{ik}, \ldots, P_{i1}, P_{i0}]$, where each of the blocks P_{ij} has p columns. Then

$P_i(\Lambda_k(\lambda) \otimes I_p) = \sum_{j=0}^{k} P_{ij}\lambda^j$ and $P(\lambda)\,(\Lambda_k(\lambda) \otimes I_p) = \sum_{i=0}^{d}\sum_{j=0}^{k} P_{ij}\lambda^{i+j}$, which can be expressed as $P(\lambda)\,(\Lambda_k(\lambda) \otimes I_p) = \sum_{\ell=0}^{d+k} \lambda^\ell \left(\sum_{i+j=\ell} P_{ij}\right)$. Therefore,

$$\|P(\lambda)\,(\Lambda_k(\lambda) \otimes I_p)\|_F^2 = \sum_{\ell=0}^{d+k} \Big\| \sum_{i+j=\ell} P_{ij} \Big\|_F^2. \tag{2.8}$$

Since $\sum_{i+j=\ell} P_{ij}$ has at most $\min\{d+1, k+1\}$ summands, the Cauchy-Schwarz inequality implies that

$$\Big\| \sum_{i+j=\ell} P_{ij} \Big\|_F \leq \sum_{i+j=\ell} \|P_{ij}\|_F \leq \min\{\sqrt{d+1}, \sqrt{k+1}\} \sqrt{\sum_{i+j=\ell} \|P_{ij}\|_F^2}\,,$$

which, when combined with (2.8) yields the final result. □

Finally, in Section 6 we need to consider pairs of matrices (C, D) where C and D may have different sizes. Therefore, (C, D) cannot be considered as a matrix pencil. For these pairs, we introduce the corresponding Frobenius norm as:

$$\|(C, D)\|_F := \sqrt{\|C\|_F^2 + \|D\|_F^2}. \tag{2.9}$$

3. Block minimal bases linearizations.

The most important linearizations considered in this work, including those in Sections 4, 5, and 6, are particular cases of the pencils introduced in Definition 3.1.

Definition 3.1. *A matrix pencil*

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \tag{3.1}$$

is called a block minimal bases pencil *if $K_1(\lambda)$ and $K_2(\lambda)$ are both minimal bases. If, in addition, the row degrees of $K_1(\lambda)$ are all equal to* 1*, the row degrees of $K_2(\lambda)$ are all equal to* 1*, the row degrees of a minimal basis dual to $K_1(\lambda)$ are all equal, and the row degrees of a minimal basis dual to $K_2(\lambda)$ are all equal, then $\mathcal{L}(\lambda)$ is called a* strong block minimal bases pencil.

Remark 3.2. Observe in Definition 3.1 that the row degrees of any minimal basis dual to $K_1(\lambda)$ are always the same, up to permutations, since they are the right minimal indices of $K_1(\lambda)$. The same holds for $K_2(\lambda)$. Therefore, there are no ambiguities in the definition of strong block minimal bases pencils with respect to the selection of the minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$.

Next theorem reveals that (strong) block minimal bases pencils are (strong) linearizations of certain matrix polynomials.

Theorem 3.3. *Let $K_1(\lambda)$ and $N_1(\lambda)$ be a pair of dual minimal bases, and let $K_2(\lambda)$ and $N_2(\lambda)$ be another pair of dual minimal bases. Consider the matrix polynomial*

$$Q(\lambda) := N_2(\lambda)M(\lambda)N_1(\lambda)^T, \tag{3.2}$$

and the block minimal bases pencil $\mathcal{L}(\lambda)$ in (3.1)*. Then:*

(a) *$\mathcal{L}(\lambda)$ is a linearization of $Q(\lambda)$.*

(b) *If $\mathcal{L}(\lambda)$ is a strong block minimal bases pencil, then $\mathcal{L}(\lambda)$ is a strong linearization of $Q(\lambda)$, considered as a polynomial with grade $1 + \deg(N_1(\lambda)) + \deg(N_2(\lambda))$.*

Proof. (a) According to Theorem 2.10, for $i = 1, 2$, there exist unimodular matrix polynomials such that

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i(\lambda) \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix}. \tag{3.3}$$

Note that if m_i is the number of rows of $K_i(\lambda)$, for $i = 1, 2$, then (3.3) implies $K_i(\lambda)\widehat{N}_i(\lambda)^T = I_{m_i}$ and $K_i(\lambda)N_i(\lambda)^T = 0$. Keep in mind that these equalities are used in subsequent matrix products. Next, consider the unimodular matrices $U_2(\lambda)^{-T} \oplus I_{m_1}$ and $U_1(\lambda)^{-1} \oplus I_{m_2}$, and form the following matrix product:

$$\begin{aligned} &(U_2(\lambda)^{-T} \oplus I_{m_1})\, \mathcal{L}(\lambda)\, (U_1(\lambda)^{-1} \oplus I_{m_2}) \\ &\qquad = \begin{bmatrix} \widehat{N}_2(\lambda) & 0 \\ N_2(\lambda) & 0 \\ 0 & I_{m_1} \end{bmatrix} \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \begin{bmatrix} \widehat{N}_1(\lambda)^T & N_1(\lambda)^T & 0 \\ 0 & 0 & I_{m_2} \end{bmatrix} \\ &\qquad = \begin{bmatrix} Z(\lambda) & X(\lambda) & I_{m_2} \\ Y(\lambda) & Q(\lambda) & 0 \\ I_{m_1} & 0 & 0 \end{bmatrix}, \end{aligned} \tag{3.4}$$

where the expressions of the matrix polynomials $X(\lambda), Y(\lambda)$, and $Z(\lambda)$ are not of specific interest in this proof. Equation (3.4) and Lemma 2.14 prove that $\mathcal{L}(\lambda)$ is a linearization of $Q(\lambda)$.

(b) Let us denote for brevity $\ell_1 = \deg(N_1(\lambda))$ and $\ell_2 = \deg(N_2(\lambda))$. Since $\mathcal{L}(\lambda)$ is a strong block minimal bases pencil, Theorem 2.7(b) guarantees that $\mathrm{rev}_1 K_1(\lambda)$ and $\mathrm{rev}_{\ell_1} N_1(\lambda)$ are dual minimal bases, as well as $\mathrm{rev}_1 K_2(\lambda)$ and $\mathrm{rev}_{\ell_2} N_2(\lambda)$. Therefore,

$$\mathrm{rev}_1 \mathcal{L}(\lambda) = \begin{bmatrix} \mathrm{rev}_1 M(\lambda) & \mathrm{rev}_1 K_2(\lambda)^T \\ \mathrm{rev}_1 K_1(\lambda) & 0 \end{bmatrix}$$

is also a block minimal bases pencil and Theorem 3.3(a) (just proved) implies that $\mathrm{rev}_1 \mathcal{L}(\lambda)$ is a linearization of

$$\begin{aligned} &(\mathrm{rev}_{\ell_2} N_2(\lambda))\, (\mathrm{rev}_1 M(\lambda))\, (\mathrm{rev}_{\ell_1} N_1(\lambda))^T = \lambda^{\ell_2} N_2\left(\lambda^{-1}\right) \lambda\, M\left(\lambda^{-1}\right) \lambda^{\ell_1} N_1\left(\lambda^{-1}\right)^T \\ &\qquad \lambda^{1+\ell_1+\ell_2} Q(\lambda^{-1}) = \mathrm{rev}_{1+\ell_1+\ell_2} Q(\lambda), \end{aligned}$$

proving part (b). ☐

REMARK 3.4. Given a *strong* block minimal bases pencil $\mathcal{L}(\lambda)$, there are infinitely many minimal bases $N_1(\lambda)$ and $N_2(\lambda)$ dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively. Therefore, the matrix polynomial $Q(\lambda)$ is not defined uniquely by $\mathcal{L}(\lambda)$. This is connected to the following remark: the standard scenario when using linearizations is that the matrix polynomial $Q(\lambda)$ is given and one wants to construct a linearization of $Q(\lambda)$ as easily as possible, but Theorem 3.3 seems to operate in the opposite way. However, if $Q(\lambda)$ is given and $N_1(\lambda)$ and $N_2(\lambda)$ are fixed, then (3.2) can be viewed as a linear equation for the unknown pencil $M(\lambda)$. It is possible to prove that this equation is always consistent, as a consequence of the properties of the minimal bases $N_1(\lambda)$ and $N_2(\lambda)$. Despite its consistency, the equation (3.2) may be very difficult to solve for arbitrary minimal bases $N_1(\lambda)$ and $N_2(\lambda)$. We will see in Section 5 that for certain particular choices of $N_1(\lambda)$ and $N_2(\lambda)$ it is very easy to characterize all possible solutions $M(\lambda)$ and to define, in this way, a new wide class of linearizations easily constructible from

$Q(\lambda)$. This new class includes, among many others, all Fiedler linearizations, up to permutations, of square or rectangular polynomials [2, 11, 13, 28].

Remark 3.5. We include in Definition 3.1 the cases in which either $K_1(\lambda)$ or $K_2(\lambda)$ is an empty matrix. This means that $\mathcal{L}(\lambda)$ is either a 1×2 or a 2×1 block matrix, and, so, the zero block is not present. All of the proofs in this paper remain valid in these border cases with the following convention: if $K_1(\lambda)$ (resp. $K_2(\lambda)$) is an empty matrix, then $N_1(\lambda) = I_s$ (resp. $N_2(\lambda) = I_s$), where s is the number of colums (resp. rows) of $M(\lambda)$.

In the rest of this section we investigate, in the case of strong block minimal bases pencils, the relationship of the minimal indices of $Q(\lambda)$ in (3.2) with those of its strong linearization $\mathcal{L}(\lambda)$ in (3.1). For this purpose, we prove the next technical lemma that is also useful in [22, Section 7].

Lemma 3.6. *Let $\mathcal{L}(\lambda)$ be a strong block minimal bases pencil as in* (3.1)*, let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$, let $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$, let $Q(\lambda)$ be the matrix polynomial defined in* (3.2)*, and let $\widehat{N}_2(\lambda)$ be the matrix appearing in* (3.3)*. Then the following hold:*

(a) *If $h(\lambda) \in \mathcal{N}_r(Q)$, then*

$$z(\lambda) := \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h(\lambda) \in \mathcal{N}_r(\mathcal{L})\,. \tag{3.5}$$

Moreover, if $0 \neq h(\lambda) \in \mathcal{N}_r(Q)$ is a vector polynomial, then $z(\lambda)$ is also a vector polynomial and

$$\deg(z(\lambda)) = \deg(N_1(\lambda)^T\, h(\lambda)) = \deg(N_1(\lambda)) + \deg(h(\lambda)). \tag{3.6}$$

(b) *If $\{h_1(\lambda), \ldots, h_p(\lambda)\}$ is a right minimal basis of $Q(\lambda)$, then*

$$\left\{ \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_1(\lambda), \ldots, \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_p(\lambda) \right\}$$

is a right minimal basis of $\mathcal{L}(\lambda)$.

Proof. (a) It can be checked, via a direct multiplication, that the matrix $X(\lambda)$ in (3.4) satisfies $X(\lambda) = \widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T$. Then, from (3.4), we get that

$$(U_2(\lambda)^{-T} \oplus I_{m_1})\, \mathcal{L}(\lambda)\, (U_1(\lambda)^{-1} \oplus I_{m_2}) \begin{bmatrix} 0 \\ I \\ -X(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ Q(\lambda) \\ 0 \end{bmatrix},$$

where the sizes of the identity and zero blocks are conformable with the partition of the last matrix in (3.4). By using the structure of $U_1(\lambda)^{-1} \oplus I_{m_2}$ (recall (3.3)), the multiplication of the last two factors in the left-hand side of the previous equation leads to

$$(U_2(\lambda)^{-T} \oplus I_{m_1})\, \mathcal{L}(\lambda)\, \begin{bmatrix} N_1(\lambda)^T \\ -X(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ Q(\lambda) \\ 0 \end{bmatrix}. \tag{3.7}$$

This equation implies that $z(\lambda) \in \mathcal{N}_r(\mathcal{L})$ if $h(\lambda) \in \mathcal{N}_r(Q)$, and also that $z(\lambda)$ is a vector polynomial if $h(\lambda)$ is, because $N_1(\lambda)$ and $X(\lambda)$ are matrix polynomials.

It only remains to prove the degree shift property (3.6) to conclude the proof of part (a). First, take into account that all the row degrees of the minimal basis $N_1(\lambda)$ are equal and that its highest degree coefficient has full row rank. Therefore,

$$\deg(N_1(\lambda)^T g(\lambda)) = \deg(N_1(\lambda)) + \deg(g(\lambda)), \tag{3.8}$$

for any vector polynomial $g(\lambda) \neq 0$. The same argument applied to the minimal basis $K_2(\lambda)$ proves that

$$\deg(K_2(\lambda)^T y(\lambda)) = \deg(K_2(\lambda)) + \deg(y(\lambda)) = 1 + \deg(y(\lambda)), \tag{3.9}$$

for any vector polynomial $y(\lambda) \neq 0$. Next, observe that

$$\deg(z(\lambda)) = \max\{\deg(N_1(\lambda)^T h(\lambda)), \deg(X(\lambda)h(\lambda))\}. \tag{3.10}$$

Therefore (3.6) follows trivially if $X(\lambda)h(\lambda) = 0$. Finally, assume that $X(\lambda)h(\lambda) \neq 0$ and $h(\lambda) \in \mathcal{N}_r(Q)$. Then use $\mathcal{L}(\lambda)z(\lambda) = 0$, and perform the multiplication corresponding to the first block of $\mathcal{L}(\lambda)z(\lambda)$, using the expressions of $z(\lambda)$ in (3.5) and $\mathcal{L}(\lambda)$ in (3.1), to get

$$M(\lambda)N_1(\lambda)^T h(\lambda) = K_2(\lambda)^T X(\lambda)h(\lambda).$$

This equality implies, together with (3.9), that

$$\begin{aligned} 1 + \deg(X(\lambda)h(\lambda)) = \deg(K_2(\lambda)^T X(\lambda)h(\lambda)) &\leq \deg(M(\lambda)) + \deg(N_1(\lambda)^T h(\lambda)) \\ &\leq 1 + \deg(N_1(\lambda)^T h(\lambda)), \end{aligned}$$

and, so, $\deg(X(\lambda)h(\lambda)) \leq \deg(N_1(\lambda)^T h(\lambda))$. This inequality, together with (3.8) and (3.10) thus prove (3.6).

(b) Let us consider the matrix product

$$B(\lambda) := \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} [h_1(\lambda) \cdots h_p(\lambda)],$$

and let us prove that their columns are a minimal basis of the rational subspace they span by applying a version of Theorem 2.2 for columns. Note that for all $\lambda_0 \in \overline{\mathbb{F}}$, $B(\lambda_0)$ has full column rank since $N_1(\lambda_0)^T$ and $[h_1(\lambda_0) \cdots h_p(\lambda_0)]$ have both full column rank, since the columns of $N_1(\lambda)^T$ and $[h_1(\lambda) \cdots h_p(\lambda)]$ are minimal bases. Next, observe that (3.6) implies that the highest column degree coefficient matrix B_{hc} of $B(\lambda)$ has as a submatrix the highest column degree coefficient matrix C_{hc} of $C(\lambda) := N_1(\lambda)^T[h_1(\lambda) \cdots h_p(\lambda)]$. Since the column degrees of $N_1(\lambda)^T$ are all equal, we have that C_{hc} is the product of the highest column degree coefficient matrices of $N_1(\lambda)^T$ and $[h_1(\lambda) \cdots h_p(\lambda)]$, which have both full column rank because the columns of both matrices are minimal bases. So C_{hc} has full column rank, as well as B_{hc}. This implies that the columns of $B(\lambda)$ are a minimal basis of a rational subspace $\mathcal{S}$. In addition, $\mathcal{S} \subseteq \mathcal{N}_r(\mathcal{L}(\lambda))$ by part (a). Finally, note that $\mathcal{S} = \mathcal{N}_r(\mathcal{L})$ because $\dim(\mathcal{N}_r(Q)) = \dim(\mathcal{N}_r(\mathcal{L}))$, since $\mathcal{L}(\lambda)$ is a strong linearization of $Q(\lambda)$ by Theorem 3.3(b) and, then, Theorem 4.1 in [14] holds. □

As a corollary of the technical Lemma 3.6, we get the desired theorem on minimal indices.

THEOREM 3.7. *Let $\mathcal{L}(\lambda)$ be a strong block minimal bases pencil as in* (3.1)*, let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$, let $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$, and let $Q(\lambda)$ be the matrix polynomial defined in* (3.2)*. Then the following hold:*

(a) *If* $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $Q(\lambda)$*, then*

$$\varepsilon_1 + \deg(N_1(\lambda)) \leq \varepsilon_2 + \deg(N_1(\lambda)) \leq \cdots \leq \varepsilon_p + \deg(N_1(\lambda))$$

are the right minimal indices of $\mathcal{L}(\lambda)$*.*

(b) *If* $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_q$ *are the left minimal indices of* $Q(\lambda)$*, then*

$$\eta_1 + \deg(N_2(\lambda)) \leq \eta_2 + \deg(N_2(\lambda)) \leq \cdots \leq \eta_q + \deg(N_2(\lambda))$$

are the left minimal indices of $\mathcal{L}(\lambda)$*.*

Proof. Part (a) follows immediately from Lemma 3.6(b) and equation (3.6). Part (b) follows simply from applying part (a) to $\mathcal{L}(\lambda)^T$ and $Q(\lambda)^T$ after taking into account that: (i) $\mathcal{L}(\lambda)^T$ is also a strong block minimal bases pencil with the roles of $(K_1(\lambda), N_1(\lambda))$ and $(K_2(\lambda), N_2(\lambda))$ interchanged, (ii) so $\mathcal{L}(\lambda)^T$ is a strong linearization of $Q(\lambda)^T$, and (iii) for any matrix polynomial its left minimal indices are the right minimal indices of its transpose. ☐

In order to concisely refer to results like those in Theorem 3.7 we use in this paper expressions as "the right minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ shifted by $\deg(N_1(\lambda))$", whose rigorous meaning is precisely the statement of Theorem 3.7(a).

4. Fiedler pencils are strong block minimal bases pencils modulo permutations.

Loosely speaking most of the new linearizations of matrix polynomials introduced in the last decade fall into two families: vector spaces of linearizations [38, 48, 49], and Fiedler linearizations [2, 11, 13, 28], together with different extensions of Fiedler's original idea [5, 7, 8, 12, 54, 66]. Fiedler linearizations have a number of desirable properties that make them particularly interesting both in theory and in applications. For instance, they can be extended to rectangular matrix polynomials, they are strong linearizations for all matrix polynomials, their extensions share relevant structures with the original matrix polynomial, they are very easily constructible from the coefficients of the matrix polynomial without performing any arithmetic operation, etc. We emphasize that several of these properties are not satisfied by vector spaces linearizations. However, the mathematical proofs of the nice properties enjoyed by Fiedler linearizations are often not easy [11], despite of some interesting simplifying efforts [56]. The theory of Fiedler linearizations becomes especially messy in the case of rectangular matrix polynomials [13].

In this section we prove that after performing some row and column permutations all Fiedler pencils, both for square and rectangular matrix polynomials, become particular cases of strong block minimal bases pencils, with the pencils $M(\lambda)$, $K_1(\lambda)$, and $K_2(\lambda)$ in (3.1) having very simple structures that can be explicitly described in terms of their entries. Therefore, the abstract, but simple, theory developed in Section 3 can be applied to prove many of the properties of Fiedler pencils in a unified and simplified way which is simultaneously valid for square and rectangular matrix polynomials. We remark that a key point in this simplification is the explicit description of the entries of the corresponding pencil $\mathcal{L}(\lambda)$ in (3.1), which is in contrast with the original factorized description in [28] or the algorithmic description in [13].

We continue this section by revisiting in Section 4.1 the Fiedler companion pencils.

4.1. Fiedler companion pencils.

The most well known and most commonly used strong linearizations of an $m \times n$ matrix polynomial $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i$ are the *first and second Frobenius companion pencils* defined as

$$C_1(\lambda) := \lambda \begin{bmatrix} P_d & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} P_{d-1} & P_{d-2} & \cdots & P_0 \\ -I_n & & & \\ & \ddots & & \\ & & -I_n & \end{bmatrix} \tag{4.1}$$

and

$$C_2(\lambda) := \lambda \begin{bmatrix} P_d & & & \\ & I_m & & \\ & & \ddots & \\ & & & I_m \end{bmatrix} + \begin{bmatrix} P_{d-1} & -I_m & & \\ P_{d-2} & & \ddots & \\ \vdots & & & -I_m \\ P_0 & & & \end{bmatrix}, \tag{4.2}$$

respectively. The Frobenius pencils are defined for square regular complex matrix polynomials in [33]. A thorough analysis of the properties of $C_1(\lambda)$ and $C_2(\lambda)$ for general matrix polynomials, i.e., square or rectangular, regular or singular, over any field, is presented in [14, Section 5.1]. It is well known that the Frobenius companion pencils are particular instances of Fiedler pencils [2, 11, 13, 28], which, in fact, were conceived as generalizations of the Frobenius companion pencils.

A key point to be emphasized in the context of this work is that the *Frobenius companion pencils are strong block minimal bases pencils* (recall Remark 3.5). To see this note that $C_1(\lambda)$ has the structure of $\mathcal{L}(\lambda)$ in (3.1) with $M(\lambda) = [\lambda P_d + P_{d-1}, P_{d-2}, \ldots, P_0]$, $K_2(\lambda)$ empty, and $K_1(\lambda) = L_{d-1}(\lambda) \otimes I_n$, where $L_{d-1}(\lambda)$ is defined in (2.3). Example 2.6 shows that $K_1(\lambda) = L_{d-1}(\lambda) \otimes I_n$ and $\Lambda_{d-1}(\lambda)^T \otimes I_n$ are dual minimal bases, the former with all the row degrees equal to 1 and the latter with all the row degrees equal to $d-1$. As illustration, observe that the application of Theorem 3.3 proves again the very well known fact that $C_1(\lambda)$ is a strong linearization of $M(\lambda)(\Lambda_{d-1}(\lambda) \otimes I_n) = \sum_{i=0}^{d} P_i \lambda^i = P(\lambda)$. A similar discussion holds for $C_2(\lambda)$ by taking $K_1(\lambda)$ as the empty matrix. We advance that, after proper permutations, the rest of Fiedler pencils "interpolate" the extreme cases corresponding to $C_1(\lambda)$ and $C_2(\lambda)$, having the structure of $\mathcal{L}(\lambda)$ in (3.1) with nonempty $K_1(\lambda)$ and $K_2(\lambda)$ blocks given by $K_1(\lambda) = L_\varepsilon(\lambda) \otimes I_n$ and $K_2(\lambda) = L_\eta(\lambda) \otimes I_m$, where $\varepsilon + \eta = d - 1$.

To define the Fiedler companion pencils different from $C_1(\lambda)$ and $C_2(\lambda)$ associated with a *square* matrix polynomial $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{F}[\lambda]^{n\times n}$, we first need to define the following block matrices

$$A_d := \begin{bmatrix} P_d & \\ & I_{(d-1)n} \end{bmatrix}, \quad A_0 := \begin{bmatrix} I_{(d-1)n} & \\ & -P_0 \end{bmatrix},$$

and

$$A_i := \begin{bmatrix} I_{(d-i-1)n} & & & \\ & -P_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \quad \text{for } i = 1, 2, \ldots, d-1.$$

Then, given a bijection $\sigma : \{0, 1, \ldots, d-1\} \to \{1, 2, \ldots, d\}$, the Fiedler companion pencil of $P(\lambda)$ associated with the bijection σ, denoted by $F_\sigma(\lambda)$, is the pencil

$$F_\sigma(\lambda) := \lambda A_d - A_\sigma := \lambda A_d - A_{\sigma^{-1}(1)} A_{\sigma^{-1}(2)} \cdots A_{\sigma^{-1}(d)}.$$

More details can be found in [11, Section 3]. In particular, pay attention to the fact that different bijections σ may yield the same Fiedler pencil as a consequence of obvious commutativity properties of the matrices A_i.

When the matrix polynomial $P(\lambda)$ is *rectangular* the definition of Fiedler pencils in terms of a sequence of products of the matrices A_i becomes problematic, since the sizes of the very same factors A_i are not well defined [13, Section 3.2]. Therefore, the authors of [13] found more convenient to define Fiedler pencils via [13, `Algorithm 2`] that we recall below, since it is needed to prove that Fiedler pencils are particular cases of strong block minimal bases pencils. In order to describe `Algorithm 2` in [13], we need to refresh the concepts of consecutions and inversions of a bijection [11, 13].

Definition 4.1. *Let* $\sigma : \{0, 1, \ldots, d-1\} \to \{1, 2, \ldots, d\}$ *be a bijection. For* $i = 0, \ldots, d-2$, *we say that* σ *has a consecution at* i *if* $\sigma(i) < \sigma(i+1)$, *and that* σ *has an inversion at* i *if* $\sigma(i) > \sigma(i+1)$. *The total numbers of consecutions and inversions of* σ *are denoted by* $\mathfrak{c}(\sigma)$ *and* $\mathfrak{i}(\sigma)$, *respectively.*

Observe that $\mathfrak{c}(\sigma) + \mathfrak{i}(\sigma) = d - 1$. With these definitions, the Fiedler companion pencil of $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{F}[\lambda]^{m\times n}$ associated with σ is defined as the pencil

$$F_\sigma(\lambda) := \lambda \begin{bmatrix} P_d & \\ & I_{m\mathfrak{c}(\sigma)+n\mathfrak{i}(\sigma)} \end{bmatrix} - A_\sigma,$$

where the matrix A_σ is constructed with the following algorithm, which uses MATLAB notation for submatrices on block indices (see [13] for more details):

`Algorithm 2 in [13].` Given $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{F}[\lambda]^{m\times n}$ and a bijection σ, the following algorithm constructs A_σ.

```
if $\sigma$ has a consecution at 0 then
    $W_0 = \begin{bmatrix} -P_1 & I_m \\ -P_0 & 0 \end{bmatrix}$
else
    $W_0 = \begin{bmatrix} -P_1 & -P_0 \\ I_n & 0 \end{bmatrix}$
endif
for $i = 1 : d-2$
    if $\sigma$ has a consecution at $i$ then
        $W_i = \begin{bmatrix} -P_{i+1} & I_m & 0 \\ W_{i-1}(:,1) & 0 & W_{i-1}(:,2:i+1) \end{bmatrix}$
    else
        $W_i = \begin{bmatrix} -P_{i+1} & W_{i-1}(1,:) \\ I_n & 0 \\ 0 & W_{i-1}(2:i+1,:) \end{bmatrix}$
    endif
endfor
$A_\sigma = W_{d-2}$
```

4.2. Fiedler pencils and permutations revealing block minimal bases pencils. Before presenting the general results, we illustrate with Example 4.2 how adequate permutations transform Fiedler pencils into particularly simple strong block minimal bases pencils, and how it can be used to prove easily several properties of Fiedler pencils.

Example 4.2. Let us consider a Fiedler companion pencil of an $m \times n$ matrix polynomial $P(\lambda) = \sum_{i=0}^{6} P_i \lambda^i$ of grade 6 associated with any bijection σ with consecutions at 0, 2, and 4, and inversions at 1 and 3. The reader may check easily using

`Algorithm` 2 above that this Fiedler pencil is equal to

$$F_\sigma(\lambda) = \begin{bmatrix} \lambda P_6 + P_5 & -I_m & 0 & 0 & 0 & 0 \\ P_4 & \lambda I_m & P_3 & -I_m & 0 & 0 \\ -I_n & 0 & \lambda I_n & 0 & 0 & 0 \\ 0 & 0 & P_2 & \lambda I_m & P_1 & -I_m \\ 0 & 0 & -I_n & 0 & \lambda I_n & 0 \\ 0 & 0 & 0 & 0 & P_0 & \lambda I_m \end{bmatrix}. \tag{4.3}$$

We can put this Fiedler pencil into block anti-triangular form via row and column permutations, denoted by Π_r and Π_c, to obtain

$$\Pi_r F_\sigma(\lambda)\, \Pi_c = \left[\begin{array}{ccc|ccc} \lambda P_6 + P_5 & 0 & 0 & -I_m & 0 & 0 \\ P_4 & P_3 & 0 & \lambda I_m & -I_m & 0 \\ 0 & P_2 & P_1 & 0 & \lambda I_m & -I_m \\ 0 & 0 & P_0 & 0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 & 0 \end{array}\right]. \tag{4.4}$$

Note that $\Pi_r F_\sigma(\lambda)\, \Pi_c$ and $F_\sigma(\lambda)$ are strictly equivalent [14, Definition 3.1] and, therefore, they have the same complete eigenstructures. From Example 2.6 and equation (3.1), we get that $\Pi_r F_\sigma(\lambda)\, \Pi_c$ is a strong block minimal bases pencil with $K_1(\lambda) = L_2(\lambda) \otimes I_n$, $K_2(\lambda) = L_3(\lambda) \otimes I_m$, and two minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$ given, respectively, by $N_1(\lambda) = \Lambda_2(\lambda)^T \otimes I_n$ and $N_2(\lambda) = \Lambda_3(\lambda)^T \otimes I_m$. As a consequence, Theorem 3.3 implies that $F_\sigma(\lambda)$ in (4.3) is a strong linearization of

$$(\Lambda_3(\lambda)^T \otimes I_m) \begin{bmatrix} \lambda P_6 + P_5 & 0 & 0 \\ P_4 & P_3 & 0 \\ 0 & P_2 & P_1 \\ 0 & 0 & P_0 \end{bmatrix} (\Lambda_2(\lambda) \otimes I_n) = P(\lambda).$$

Moreover, Theorem 3.7 implies that the right minimal indices of $F_\sigma(\lambda)$ are those of $P(\lambda)$ shifted by 2 and that the left minimal indices of $F_\sigma(\lambda)$ are those of $P(\lambda)$ shifted by 3. We emphasize the sheer simplicity of these arguments when they are compared with the ones in [11, 13].

Finally, observe that in this example the unimodular matrices transforming the strong block minimal bases pencil $\Pi_r F_\sigma(\lambda)\, \Pi_c$ into the block anti-triangular form (3.4) can be explicitly described in terms of the matrices $V_k(\lambda)^{-1}$ in Example 2.12. More precisely

$$((V_3(\lambda)^{-T} \otimes I_m) \oplus I_{2n})\, (\Pi_r F_\sigma(\lambda)\, \Pi_c)\, ((V_2(\lambda)^{-1} \otimes I_n) \oplus I_{3m})$$

has the block anti-triangular structure in (3.4) with $Q(\lambda)$ replaced by $P(\lambda)$. This can be checked without performing any operation from the discussion in Example 2.12 and the proof of Theorem 3.3, but also via direct multiplications. This last option proves in a very simple way that Fiedler pencils are linearizations of $P(\lambda)$, without the need of the general theory developed in Section 3. However, this simple approach is not enough to deal with the perturbed linearizations considered in Section 6.

Our next goal is to prove that any Fiedler companion pencil can be transformed via permutations into a strong block minimal bases pencil with a structure similar to (4.4). The first step of this task is to provide a rigorous meaning for the vague sentence "a structure similar to (4.4)". That is, if (4.4) is partitioned into 2×2 blocks, we need to describe the structure of the $(1,1)$-block. This is the purpose of Definition 4.3, which adapts for this work concepts introduced in [16, Section 5] and [24].

DEFINITION 4.3. *Let* $Q_{d-1}(\lambda), Q_{d-2}(\lambda), \ldots, Q_0(\lambda)$ *be* $m \times n$ *matrix pencils, let* p *and* q *be positive integers such that* $p+q=d+1$*, and let* $B(\lambda)$ *be a* $(pm)\times(qn)$ *pencil which is partitioned into* $p \times q$ *blocks each of size* $m\times n$*. The pencil* $B(\lambda)$ follows a staircase pattern for $Q_{d-1}(\lambda), Q_{d-2}(\lambda), \ldots, Q_0(\lambda)$ *if it satisfies the following properties:*

(a) *the* $(1,1)$*-block-entry of* $B(\lambda)$ *is* $Q_{d-1}(\lambda)$*,*
(b) *the* (p,q)*-block-entry of* $B(\lambda)$ *is* $Q_0(\lambda)$*,*
(c) *if* $Q_j(\lambda)$*, for* $j=d-1,\ldots,1$*, is the* (s,t)*-block-entry of* $B(\lambda)$*, then* $Q_{j-1}(\lambda)$ *is either the* $(s+1,t)$*-block-entry or the* $(s,t+1)$*-block-entry of* $B(\lambda)$*,*
(d) *and the remaining entries of* $B(\lambda)$ *are zero.*

EXAMPLE 4.4. If the pencil $\Pi_r F_\sigma(\lambda)\,\Pi_c$ in (4.4) is partitioned into 2×2 blocks according to the lines displayed in (4.4), then its $(1,1)$-block follows a staircase pattern for $\lambda P_6 + P_5, P_4, P_3, P_2, P_1, P_0$ with parameters $p=4, q=3, d=6$.

With this definition at hand the most important result in this section is stated in Theorem 4.5, whose proof is postponed to Appendix A since is long.

THEOREM 4.5. *Let* $P(\lambda)=\sum_{i=0}^{d} P_i\lambda^i \in \mathbb{F}[\lambda]^{m\times n}$*, let* $F_\sigma(\lambda)$ *be a Fiedler pencil of* $P(\lambda)$ *associated with a bijection* σ *having* $\mathfrak{c}(\sigma)$ *consecutions and* $\mathfrak{i}(\sigma)$ *inversions, and let* $L_k(\lambda)$ *be the pencil defined in* (2.3)*. Then there exist two permutation matrices* Π_r *and* Π_c *such that*

$$\Pi_r F_\sigma(\lambda)\,\Pi_c = \left[\begin{array}{c|c} \lambda M_1 + M_0 & L_{\mathfrak{c}(\sigma)}(\lambda)^T \otimes I_m \\ \hline L_{\mathfrak{i}(\sigma)}(\lambda)\otimes I_n & 0 \end{array}\right], \tag{4.5}$$

where the $(\mathfrak{c}(\sigma)+1)m \times (\mathfrak{i}(\sigma)+1)n$ *pencil* $\lambda M_1 + M_0$ *follows a staircase pattern for* $\lambda P_d + P_{d-1}, P_{d-2}, \ldots, P_1, P_0$*.*

Conversely, any pencil with the structure of the right-hand side of (4.5) *and such that* $\lambda M_1 + M_0$ *follows a staircase pattern for* $\lambda P_d + P_{d-1}, P_{d-2}, \ldots, P_1, P_0$ *can be transformed via row and column permutations into a Fiedler pencil of* $P(\lambda)$ *associated with a bijection* σ *having* $\mathfrak{c}(\sigma)$ *consecutions and* $\mathfrak{i}(\sigma)$ *inversions.*

Theorem 4.5 provides a very simple explicit description of all Fiedler pencils modulo permutations, which is valid for both rectangular and square matrix polynomials. We emphasize again the simplicity of this description compared to the original description of Fiedler [28] and, much more striking, compared to that in [13]. Related results about permutations of Fiedler matrices (not pencils) corresponding to monic scalar polynomials have been recently proved in [23, 24].

In addition, Theorem 4.5 allows us to prove very easily the most interesting properties of Fiedler pencils $F_\sigma(\lambda)$, which are the same as those of the strictly equivalent pencils $\Pi_r F_\sigma(\lambda)\,\Pi_c$ [14, Section 3.1], as corollaries of the results in Section 3. Indeed, Example 2.6 and equation (3.1) imply that the pencil $\Pi_r F_\sigma(\lambda)\,\Pi_c$ in (4.5) is a strong block minimal bases pencil with $K_1(\lambda)=L_{\mathfrak{i}(\sigma)}(\lambda)\otimes I_n$, $K_2(\lambda)=L_{\mathfrak{c}(\sigma)}(\lambda)\otimes I_m$, and two minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$ given, respectively, by $N_1(\lambda)=\Lambda_{\mathfrak{i}(\sigma)}(\lambda)^T\otimes I_n$ and $N_2(\lambda)=\Lambda_{\mathfrak{c}(\sigma)}(\lambda)^T\otimes I_m$. Therefore, according to Theorem 3.3, $\Pi_r F_\sigma(\lambda)\,\Pi_c$ (and also $F_\sigma(\lambda)$) is a strong linearization of the matrix polynomial

$$Q(\lambda) = (\Lambda_{\mathfrak{c}(\sigma)}(\lambda)^T \otimes I_m)(\lambda M_1 + M_0)(\Lambda_{\mathfrak{i}(\sigma)}(\lambda)\otimes I_n)\,, \tag{4.6}$$

which is precisely $P(\lambda)$. To check this, denote by $[M_0]_{ij}$ and $[M_1]_{ij}$ the (i,j)-block-entries of M_0 and M_1 when they are partitioned into $(\mathfrak{c}(\sigma)+1)\times(\mathfrak{i}(\sigma)+1)$ blocks each of size $m \times n$, recall that $\mathfrak{c}(\sigma) + \mathfrak{i}(\sigma) = d - 1$, and multiply the right-hand side of (4.6) to get

$$\begin{aligned} Q(\lambda) &= \sum_{k=0}^{d} \lambda^k \left(\sum_{i+j=d+2-k} [M_1]_{ij} + \sum_{i+j=d+1-k} [M_0]_{ij} \right) \\ &= \sum_{k=0}^{d} \lambda^k P_k, \end{aligned} \tag{4.7}$$

where in the last equality we have used that $\lambda M_1 + M_0$ follows a staircase pattern for $\lambda P_d + P_{d-1}, P_{d-2}, \ldots, P_1, P_0$, which implies that $\lambda P_d + P_{d-1}$ is the only possibly nonzero block in the first block-antidiagonal of $\lambda M_1 + M_0$, P_{d-2} is the only possibly nonzero block in the second block-antidiagonal of $\lambda M_1 + M_0$, and so on.

From the discussion in the previous paragraph and Theorem 3.7, we obtain the following corollary of Theorem 4.5, which states well known properties of Fiedler pencils [11, 13].

Corollary 4.6. *Let $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{F}[\lambda]^{m\times n}$ and let $F_\sigma(\lambda)$ be a Fiedler pencil of $P(\lambda)$ associated with a bijection σ having $\mathfrak{c}(\sigma)$ consecutions and $\mathfrak{i}(\sigma)$ inversions. Then there exist row and column permutations that transform $F_\sigma(\lambda)$ into a strong block minimal bases pencil. Therefore, $F_\sigma(\lambda)$ is a strong linearization of $P(\lambda)$, the right minimal indices of $F_\sigma(\lambda)$ are those of $P(\lambda)$ shifted by $\mathfrak{i}(\sigma)$, and the left minimal indices of $F_\sigma(\lambda)$ are those of $P(\lambda)$ shifted by $\mathfrak{c}(\sigma)$.*

The recovery of the minimal bases and the eigenvectors of $P(\lambda)$ from those of $F_\sigma(\lambda)$ can be obtained also as a corollary of general results for strong block minimal bases pencils which are presented in [22, Section 7].

5. Block Kronecker linearizations.

In this section we introduce a subfamily of strong block minimal bases pencils $\mathcal{L}(\lambda)$ that extend the permuted Fiedler pencils in (4.5). The pencils $\mathcal{L}(\lambda)$ in this family have anti-diagonal blocks equal to those in (4.5), but the pencils $\lambda M_1 + M_0$ in the $(1,1)$-block are arbitrary, instead of following a staircase pattern. Many properties of these pencils $\mathcal{L}(\lambda)$ follow immediately as corollaries of the general theory developed in Section 3, but the pencils in the new family have an essential advantage over general strong block minimal bases pencils that is key in applications: given a matrix polynomial $P(\lambda)$ it is very easy to characterize an infinite set of $(1,1)$-blocks $\lambda M_1 + M_0$ that make $\mathcal{L}(\lambda)$ a strong linearization of precisely this $P(\lambda)$ having shifting properties for the minimal indices. Such an advantage is a consequence of the equality of (4.6) and (4.7) which holds for arbitrary pencils $\lambda M_1 + M_0$. The new family of matrix pencils is formally introduced in Definition 5.1.

Definition 5.1. *Let $L_k(\lambda)$ be the matrix pencil defined in* (2.3) *and let $\lambda M_1 + M_0$ be an arbitrary pencil. Then any matrix pencil of the form*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right] \begin{array}{l} \}\, {\scriptstyle (\eta+1)m} \\ \}\, {\scriptstyle \varepsilon n} \end{array}, \qquad \begin{array}{cc} \underbrace{\hphantom{\lambda M_1 + M_0}}_{(\varepsilon+1)n} & \underbrace{\hphantom{L_\eta(\lambda)^T \otimes I_m}}_{\eta m} \end{array} \tag{5.1}$$

is called an $(\varepsilon, n, \eta, m)$-block Kronecker pencil or, simply, a block Kronecker pencil. *The partition of $\mathcal{L}(\lambda)$ into 2×2 blocks in* (5.1) *is called the natural partition of a block Kronecker pencil.*

The name "block Kronecker pencil" is motivated by the fact that the anti-diagonal blocks of $\mathcal{L}(\lambda)$ in (5.1) are Kronecker products of singular blocks of the Kronecker canonical form of pencils [30, Chapter XII] with identity matrices.

Notice that the permuted Fiedler pencil in (4.5) is a particular case of a block Kronecker pencil with $\varepsilon = \mathfrak{i}(\sigma)$ and $\eta = \mathfrak{c}(\sigma)$. Moreover, *block Kronecker pencils are particular cases of strong block minimal bases pencils* as in (3.1), with $K_1(\lambda) = L_\varepsilon(\lambda) \otimes I_n$, $K_2(\lambda) = L_\eta(\lambda) \otimes I_m$, and, according to Example 2.6, two minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$ given, respectively, by $N_1(\lambda) = \Lambda_\varepsilon(\lambda)^T \otimes I_n$ and $N_2(\lambda) = \Lambda_\eta(\lambda)^T \otimes I_m$. Therefore, we obtain the following result for block Kronecker pencils as an immediate corollary of Theorems 3.3 and 3.7.

Theorem 5.2. *Let $\mathcal{L}(\lambda)$ be an $(\varepsilon, n, \eta, m)$-block Kronecker pencil as in* (5.1)*. Then $\mathcal{L}(\lambda)$ is a strong linearization of the matrix polynomial*

$$Q(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_m)(\lambda M_1 + M_0)(\Lambda_\varepsilon(\lambda) \otimes I_n) \in \mathbb{F}[\lambda]^{m\times n} \tag{5.2}$$

of grade $\varepsilon + \eta + 1$, the right minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ shifted by ε, and the left minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ shifted by η.

Remark 5.3. Explicit unimodular matrices that transform a block Kronecker pencil as in (5.1) into a block anti-triangular form (3.4) can be described via the matrices $V_k(\lambda)^{-1}$ in Example 2.12. In fact, an immediate corollary of the discussion in Example 2.12 and the block matrix multiplications yielding (3.4) in the proof of Theorem 3.3 is that

$$((V_\eta(\lambda)^{-T} \otimes I_m) \oplus I_{\varepsilon n})\, \mathcal{L}(\lambda)\, ((V_\varepsilon(\lambda)^{-1} \otimes I_n) \oplus I_{\eta m}) \tag{5.3}$$

has the block anti-triangular structure in (3.4). This can also be checked via a direct multiplication, which proves in a simple way that block Kronecker pencils are linearizations of $Q(\lambda)$ as a consequence of Lemma 2.14. A similar approach via explicit unimodular transformations can be used to prove that $\mathcal{L}(\lambda)$ is a strong linearization of $Q(\lambda)$. All these simple explicit unimodular transformations, which are valid simultaneously for all $(\varepsilon, n, \eta, m)$-block Kronecker pencils including the permuted Fiedler pencils, were the seed of this paper. However, they are not enough to prove the results on perturbed block Kronecker pencils that are needed in the backward error analysis presented in Section 6 and this led us to develop the general theory of Section 3.

Finally, we show what conditions on $\lambda M_1 + M_0$ are needed for a block Kronecker pencil (5.1) to be a strong linearization of a *prescribed* matrix polynomial $P(\lambda)$.

Theorem 5.4. *Let $P(\lambda) = \sum_{k=0}^{d} P_k \lambda^k \in \mathbb{F}[\lambda]^{m\times n}$, let $\mathcal{L}(\lambda)$ be an $(\varepsilon, n, \eta, m)$-block Kronecker pencil as in* (5.1) *with $\varepsilon + \eta + 1 = d$, let us consider M_0 and M_1 partitioned into $(\eta + 1) \times (\varepsilon + 1)$ blocks each of size $m \times n$, and let us denote these blocks by $[M_0]_{ij}, [M_1]_{ij} \in \mathbb{F}^{m\times n}$ for $i = 1, \ldots, \eta + 1$ and $j = 1, \ldots, \varepsilon + 1$. If*

$$\sum_{i+j=d+2-k} [M_1]_{ij} + \sum_{i+j=d+1-k} [M_0]_{ij} = P_k, \quad \text{for } k = 0, 1, \ldots, d, \tag{5.4}$$

then $\mathcal{L}(\lambda)$ is a strong linearization of $P(\lambda)$, the right minimal indices of $\mathcal{L}(\lambda)$ are those of $P(\lambda)$ shifted by ε, and the left minimal indices of $\mathcal{L}(\lambda)$ are those of $P(\lambda)$ shifted by η.

Proof. A direct multiplication, the condition $\varepsilon + \eta + 1 = d$, and some elementary manipulations of summations allow us to express $Q(\lambda)$ in (5.2) as

$$Q(\lambda) = \sum_{k=0}^{d} \lambda^k \left(\sum_{i+j=d+2-k} [M_1]_{ij} + \sum_{i+j=d+1-k} [M_0]_{ij} \right).$$

Then (5.4) implies that $Q(\lambda) = P(\lambda)$ and the result follows from Theorem 5.2. $\square$

Theorem 5.4 admits a revealing interpretation in terms of block antidiagonals of M_0 and M_1. To see this, note that equation (5.4) tells us that the sum of the blocks on the $(d-k)$th block antidiagonal of M_0 plus the sum of the blocks on the $(d-k+1)$th block antidiagonal of M_1 must be equal to the coefficient P_k of $P(\lambda)$. This implies that the upper-left block of M_1 must be equal to P_d, and that the lower-right block of M_0 must be equal to P_0, that is, the pencil $\lambda M_1 + M_0$ has the form

$$\lambda M_1 + M_0 = \begin{bmatrix} \lambda P_d + [M_0]_{11} & \iddots & \iddots \\ \iddots & \iddots & \iddots \\ \iddots & \iddots & \lambda [M_1]_{\eta+1,\varepsilon+1} + P_0 \end{bmatrix}. \tag{5.5}$$

There are infinitely many ways to select the remaining block entries of M_1 and M_0 to *synthesize* $P(\lambda)$ in the pencil $\lambda M_1 + M_0$. Notice that the staircase pattern of the permuted Fiedler pencils in Theorem 4.5 is just one possible way of doing this.

In Example 5.5 we show three different block Kronecker pencils that are all strong linearizations of a grade 5 matrix polynomial $P(\lambda)$. These three pencils have parameters $\varepsilon = \eta = 2$. Moreover, the corresponding pencils $\lambda M_1 + M_0$ in these block Kronecker pencils do not follow a staircase pattern for $\lambda P_5 + P_4, P_3, \ldots, P_0$, that is, they are not permuted Fiedler pencils.

EXAMPLE 5.5. Let $P(\lambda) = \sum_{k=0}^{5} P_k \lambda^k \in \mathbb{F}[\lambda]^{m\times n}$ and let $A, B \in \mathbb{F}^{m\times n}$ be arbitrary constant matrices. The following block Kronecker pencils

$$\left[\begin{array}{ccc|cc} \lambda P_5 + P_4 & 0 & 0 & -I_m & 0 \\ 0 & \lambda P_3 + P_2 & 0 & \lambda I_m & -I_m \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 & \lambda P_4 & \lambda P_3 & -I_m & 0 \\ 0 & 0 & \lambda P_2 & \lambda I_m & -I_m \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right], \quad \text{and}$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 & A & P_2 & -I_m & 0 \\ \lambda P_4 & -\lambda A & \lambda B + P_1 & \lambda I_m & -I_m \\ \lambda P_3 & -\lambda B & P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right]$$

are all strong linearizations of $P(\lambda)$.

6. Backward error analysis of complete polynomial eigenproblems solved via block Kronecker pencils. The problem of computing in floating point arithmetic the complete eigenstructure of a matrix polynomial $P(\lambda)$ is called in this paper the *complete polynomial eigenproblem.* The complete eigenstructure consists of all of the eigenvalues, finite and infinite, and all of the minimal indices, left and right, of $P(\lambda)$. This eigenstructure can be efficiently computed via the *staircase algorithm for matrix pencils* applied to any strong linearization $\mathcal{L}(\lambda)$ of the polynomial that allows us to recover the minimal indices of the polynomial from those of the linearizations via constant shifts (like those of Theorem 5.4 for block Kronecker pencils). The staircase algorithm for pencils was introduced for the first time in [63] and was further developed in [20, 21], where reliable software for computing such a staircase form was presented. Though problems involving singular polynomials arise very often in control theory, the matrix polynomials arising in many other applications are normally square and regular. In this case the complete eigenstructure does not include minimal indices and the algorithm of choice is the simpler *QZ algorithm* [34].

Standard backward error results guarantee that if the staircase algorithm or the QZ algorithm are applied to $\mathcal{L}(\lambda)$ in a computer with unit roundoff $\mathbf{u}$, then *the computed complete eigenstructure of* $\mathcal{L}(\lambda)$ *is the exact complete eigenstructure of a matrix pencil* $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ *such that*

$$\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F} = O(\mathbf{u}), \tag{6.1}$$

where $\|\cdot\|_F$ denotes the Frobenius norm introduced in Definition 2.15. However, (6.1) is not the desired ideal result for the original problem of computing the complete eigenstructure of the matrix polynomial $P(\lambda)$ of given grade d. The desired backward error result would be that *the computed complete eigenstructure of* $P(\lambda)$ *is the exact complete eigenstructure of a matrix polynomial* $P(\lambda)+\Delta P(\lambda)$ *also of grade* d *and such that*

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} = O(\mathbf{u}). \tag{6.2}$$

In order to establish (6.2), if possible, starting from (6.1), two results must be proved: *(i)* that the perturbed pencil $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ is a strong linearization for some matrix polynomial $P(\lambda)+\Delta P(\lambda)$ of grade d with the shifting relations between the minimal indices of $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ and $P(\lambda)+\Delta P(\lambda)$ equal to the shifting relations between the minimal indices of $\mathcal{L}(\lambda)$ and $P(\lambda)$; and *(ii)* to prove a *perturbation bound* of the type

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq C_{P,\mathcal{L}} \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}, \tag{6.3}$$

with $C_{P,\mathcal{L}}$ a moderate number depending, in principle, on $P(\lambda)$ and $\mathcal{L}(\lambda)$. We emphasize that to prove *(i)* is much easier for regular than for singular polynomials, because in the former case there are no minimal indices involved in the computations. Observe also that the minimal indices of $P(\lambda)$ are computed via the recovery rules valid for the unperturbed linearization $\mathcal{L}(\lambda)$ applied to the computed minimal indices of $\mathcal{L}(\lambda)$, that is, to the exact minimal indices of $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$. Therefore, if the recovery rules for the minimal indices of $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ were different that those of $\mathcal{L}(\lambda)$, such a method for computing the minimal indices of $P(\lambda)$ would not make any sense because the minimal indices are integer numbers.

The goal of this section is to study these questions for any block Kronecker pencil $\mathcal{L}(\lambda)$ as in (5.1) of a given polynomial $P(\lambda)$ of grade d and size $m\times n$. In plain words, we will prove that *if the block Kronecker pencil satisfies* $\|\lambda M_1 + M_0\|_F \approx \|P(\lambda)\|_F$ *and* $P(\lambda)$ *is scaled to satisfy* $\|P(\lambda)\|_F = 1$, then (6.3) holds with $C_{P,\mathcal{L}} \approx d^3\sqrt{m+n}$. Therefore, *under these two conditions, we get perfect structured backward stability from the polynomial point of view* when the block Kronecker pencils are combined with the staircase or QZ algorithms for computing the complete eigenstructure of $P(\lambda)$.

We emphasize that this is no longer true if $\|\lambda M_1 + M_0\|_F \gg \|P(\lambda)\|_F$, because in this case we will prove that $C_{P,\mathcal{L}}$ in (6.3) is huge. Note that $\|\lambda M_1 + M_0\|_F \gg \|P(\lambda)\|_F$ may happen, for instance, if in the last block Kronecker pencil of Example 5.5 the arbitrary matrices A or B have very large norms. Observe that the permuted Fiedler pencils in (4.5) satisfy $\|\lambda M_1 + M_0\|_F = \|P(\lambda)\|_F$ and, so, our analysis guarantees perfect structured polynomial backward stability for all Fiedler pencils.

Backward error analyses valid simultaneously for the complete eigenstructure, i.e., global analyses, of complete polynomial eigenproblems (and complete scalar rootfinding problems) solved by linearizations are not new in the literature. They appeared for the first time in the seminal paper [64], were studied in the influential work [27], and have received considerable attention in recent years [17, 45, 46, 47, 52, 55]. However, we stress that the analysis developed in this paper has a number of key features which are not present in any of the other analyses published so far: first, it is not a first order analysis since it holds for perturbations $\Delta\mathcal{L}(\lambda)$ of finite norm; second, it provides very detailed bounds, and not just vague big-O bounds as other analyses do; third, it is valid simultaneously for a very large class of linearizations for which backward error analyses are not yet known; and, fourth, it establishes a framework that may be generalized to other classes of linearizations.

Before proceeding, we remark that our analysis is of a completely different nature than the "local" residual backward error analyses presented in [37, 61], which are only valid for regular matrix polynomials, are based on the residual of a particular computed eigenvalue-vector pair, and find a nearby polynomial to the original one that has as exact eigenpair the particular computed one. A key difference with our analysis is that in these local analyses the nearby polynomial is different for each computed eigenpair, while in our case it is the same for the complete eigenstructure.

The main result in this section is Theorem 6.22, whose proof requires considerable efforts. The proof is split into three main steps that are briefly described in the next paragraphs in such a way that the reader may follow easily the main flow of the argument. In this section we assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Initial data. A matrix polynomial $P(\lambda) = \sum_{k=0}^{d} P_k \lambda^k \in \mathbb{F}[\lambda]^{m\times n}$ and a block Kronecker pencil $\mathcal{L}(\lambda)$ as in (5.1) such that

$$P(\lambda) = (\Lambda_\eta(\lambda)^T \otimes I_m)(\lambda M_1 + M_0)(\Lambda_\varepsilon(\lambda) \otimes I_n), \qquad \text{with } \varepsilon + \eta + 1 = d, \tag{6.4}$$

are given. A perturbation pencil $\Delta\mathcal{L}(\lambda)$ of $\mathcal{L}(\lambda)$ is also given and is partitioned conformably to the natural partition of $\mathcal{L}(\lambda)$, that is,

$$\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda) = \left[\begin{array}{c|c} \lambda M_1 + M_0 + \Delta\mathcal{L}_{11}(\lambda) & L_\eta(\lambda)^T \otimes I_m + \Delta\mathcal{L}_{12}(\lambda) \\ \hline L_\varepsilon(\lambda) \otimes I_n + \Delta\mathcal{L}_{21}(\lambda) & \Delta\mathcal{L}_{22}(\lambda) \end{array}\right]. \tag{6.5}$$

First step. We establish a bound on $\|\Delta\mathcal{L}(\lambda)\|_F$ that allows us to construct a strict equivalence transformation that returns the $(2,2)$-block of the perturbed pencil (6.5) back to zero as in $\mathcal{L}(\lambda)$:

$$\begin{aligned} &\begin{bmatrix} I_{(\eta+1)m} & 0 \\ C & I_{\varepsilon n} \end{bmatrix} (\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)) \begin{bmatrix} I_{(\varepsilon+1)n} & D \\ 0 & I_{\eta m} \end{bmatrix} \qquad (6.6)\\ &\quad = \begin{bmatrix} \lambda M_1 + M_0 + \Delta\mathcal{L}_{11}(\lambda) & L_\eta(\lambda)^T \otimes I_m + \Delta\widetilde{\mathcal{L}}_{12}(\lambda) \\ L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda) & 0 \end{bmatrix} =: \mathcal{L}(\lambda) + \Delta\widetilde{\mathcal{L}}(\lambda). \end{aligned}$$

This construction is equivalent to solving a nonlinear system of matrix equations whose unknowns are the constant matrices C and D. Moreover, we prove detailed bounds on $\|(C,D)\|_F$, $\|\Delta\widetilde{\mathcal{L}}_{12}(\lambda)\|_F$, and $\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F$ in terms of $\|\Delta\mathcal{L}(\lambda)\|_F$. It is important to remark that $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ and the pencil $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ in (6.6) have the same complete eigenstructures (including minimal indices), since they are strictly equivalent [14, Definition 3.1].

Remark 6.1. This first step is not needed if either $\varepsilon = 0$ or $\eta = 0$, which means that one of the anti-diagonal blocks and the zero block in (5.1) are not present. These border cases are important since include the first and second Frobenius companion pencils in (4.1) and (4.2).

Second step. The second step consists of establishing bounds on $\|\Delta\widetilde{\mathcal{L}}_{12}(\lambda)\|_F$ and $\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F$ that guarantee that $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ *in* (6.6) *is a strong block minimal bases pencil.* This requires two substeps: (a) to prove that $K_1(\lambda) := L_\varepsilon(\lambda)\otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda)$ and $K_2(\lambda) := L_\eta(\lambda)\otimes I_m + \Delta\widetilde{\mathcal{L}}_{12}(\lambda)^T$ are both minimal bases with their row degrees all equal to 1, and (b) to prove that there exist minimal bases

$$\Lambda_\varepsilon(\lambda)^T\otimes I_n + \Delta R_\varepsilon(\lambda)^T \quad \text{and} \quad \Lambda_\eta(\lambda)^T\otimes I_m + \Delta R_\eta(\lambda)^T$$

dual, respectively, to $K_1(\lambda)$ and $K_2(\lambda)$ with their row degrees all equal, respectively, to ε and η. In addition, we prove detailed bounds on $\|\Delta R_\varepsilon(\lambda)\|_F$ in terms of $\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F$, and on $\|\Delta R_\eta(\lambda)\|_F$ in terms of $\|\Delta\widetilde{\mathcal{L}}_{12}(\lambda)\|_F$.

Remark 6.2. Obviously, it is only needed to prove the results in the substeps (a) and (b) for $K_1(\lambda) = L_\varepsilon(\lambda)\otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda)$ and $\Lambda_\varepsilon(\lambda)^T\otimes I_n + \Delta R_\varepsilon(\lambda)^T$, since, then, the ones for $K_2(\lambda) = L_\eta(\lambda)\otimes I_m + \Delta\widetilde{\mathcal{L}}_{12}(\lambda)^T$ and $\Lambda_\eta(\lambda)^T\otimes I_m + \Delta R_\eta(\lambda)^T$ are obtained as corollaries.

Third step. Combining the first and second steps and Theorems 3.3 and 3.7, we get that $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ is a strong linearization of the matrix polynomial

$$\begin{aligned} &P(\lambda)+\Delta P(\lambda) && (6.7)\\ &\quad := \left(\Lambda_\eta(\lambda)^T\otimes I_m + \Delta R_\eta(\lambda)^T\right)(\lambda M_1 + M_0 + \Delta\mathcal{L}_{11}(\lambda))\left(\Lambda_\varepsilon(\lambda)\otimes I_n + \Delta R_\varepsilon(\lambda)\right), \end{aligned}$$

that the right minimal indices of $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ are those of $P(\lambda)+\Delta P(\lambda)$ shifted by ε, and that the left minimal indices of $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ are those of $P(\lambda)+\Delta P(\lambda)$ shifted by η, i.e., the shifting relations between the minimal indices are the same as those between the minimal indices of $\mathcal{L}(\lambda)$ and $P(\lambda)$. The rest of the proof consists of bounding $\|\Delta P(\lambda)\|_F/\|P(\lambda)\|_F$ in terms of $\|\Delta\mathcal{L}(\lambda)\|_F/\|\mathcal{L}(\lambda)\|_F$ using the bounds obtained in the first and second steps. In a last part, the consequences of this final bound are discussed.

In the rest of this section, the three steps described above are developed in detail. We use very often, without explicitly referring to, the properties of the Frobenius norm of matrix polynomials proved in Lemma 2.16 and, also, that for any matrix polynomial $P(\lambda)$ and any submatrix $B(\lambda)$ of $P(\lambda)$, the inequality $\|B(\lambda)\|_F \leq \|P(\lambda)\|_F$ holds.

6.1. First step: solving a system of quadratic Sylvester-like matrix equations for constructing the strict equivalence (6.6).

For brevity, hereafter we use the following notation for the anti-diagonal blocks of block Kronecker pencils, which are constructed from the pencil (2.3): $L_k(\lambda)\otimes I_\ell =: (\lambda F_k - E_k)\otimes I_\ell =: \lambda F_{k\ell} - E_{k\ell}$, where

$$E_{k\ell} = \begin{bmatrix} I_k & 0_{k\times 1}\end{bmatrix}\otimes I_\ell\,, \quad \text{and} \quad F_{k\ell} = \begin{bmatrix} 0_{k\times 1} & I_k\end{bmatrix}\otimes I_\ell\,. \tag{6.8}$$

In addition, the natural blocks of the perturbation $\Delta\mathcal{L}(\lambda)$ in (6.5) are denoted by

$$\Delta\mathcal{L}(\lambda) = \left[\begin{array}{c|c} \Delta\mathcal{L}_{11}(\lambda) & \Delta\mathcal{L}_{12}(\lambda) \\ \hline \Delta\mathcal{L}_{21}(\lambda) & \Delta\mathcal{L}_{22}(\lambda) \end{array}\right] =: \left[\begin{array}{c|c} \lambda\Delta B_{11} + \Delta A_{11} & \lambda\Delta B_{12} + \Delta A_{12} \\ \hline \lambda\Delta B_{21} + \Delta A_{21} & \lambda\Delta B_{22} + \Delta A_{22} \end{array}\right]. \tag{6.9}$$

According to Remark 6.1, we assume that $\varepsilon \neq 0$ and $\eta \neq 0$ throughout this subsection.

The main result of this subsection is Theorem 6.9 and the starting point is the trivial Lemma 6.3, which follows from elementary matrix operations applied to the lower-right block in (6.6).

LEMMA 6.3. *There exist constant matrices* $C \in \mathbb{F}^{\varepsilon n \times (\eta+1)m}$ *and* $D \in \mathbb{F}^{(\varepsilon+1)n \times \eta m}$ *satisfying* (6.6) *if and only if*

$$\left[\begin{array}{cc} C & I_{\varepsilon n} \end{array}\right] (\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)) \left[\begin{array}{c} D \\ I_{\eta m} \end{array}\right] = 0\,. \tag{6.10}$$

Moreover, with the notation introduced in (6.8) *and* (6.9)*, the equation* (6.10) *is equivalent to the following system of quadratic Sylvester-like matrix equations*

$$\left\{ \begin{array}{l} C(E_{\eta m}^T - \Delta A_{12}) + (E_{\varepsilon n} - \Delta A_{21})D = \Delta A_{22} + C(M_0 + \Delta A_{11})D \\ C(F_{\eta m}^T + \Delta B_{12}) + (F_{\varepsilon n} + \Delta B_{21})D = -\Delta B_{22} - C(M_1 + \Delta B_{11})D \end{array} \right. , \tag{6.11}$$

for the unknown matrices C *and* D.

The system of matrix equations (6.11) is equivalent to a system of $2\varepsilon\eta nm$ quadratic scalar equations in the $2\varepsilon\eta nm + (\varepsilon+\eta)mn$ unknown entries of C and D. Therefore, (6.11) is an underdetermined system of equations that may have infinitely many solutions. Our aim is to establish conditions on $\|\Delta\mathcal{L}(\lambda)\|_F$ that guarantee the existence of a solution (C, D) to (6.11) with $\|(C, D)\|_F \lesssim \|\Delta\mathcal{L}(\lambda)\|_F$, where the norm $\|(C, D)\|_F$ was defined in (2.9). This is done in Theorem 6.8, whose proof follows that of Stewart [59, Theorem 5.1] (see also [60, Theorem 2.11, p. 242] for a more general and more accesible result) and is based on a fixed point iteration argument. However, we emphasize that the result by Stewart is valid only for certain nonlinear matrix equations having a unique solution, while in our case there may be infinitely many solutions.

The solution of (6.11) relies upon first solving the system of linear Sylvester equations obtained by removing the quadratic terms in C and D of (6.11). Such a system is:

$$\left\{ \begin{array}{l} C(E_{\eta m}^T - \Delta A_{12}) + (E_{\varepsilon n} - \Delta A_{21})D = \Delta A_{22} \\ C(F_{\eta m}^T + \Delta B_{12}) + (F_{\varepsilon n} + \Delta B_{21})D = -\Delta B_{22} \end{array} \right. , \tag{6.12}$$

which is equivalent to the underdetermined standard linear system $(T + \Delta T)x = b$ given by

$$\left(\underbrace{\left[\begin{array}{c|c} E_{\eta m} \otimes I_{\varepsilon n} & I_{\eta m} \otimes E_{\varepsilon n} \\ \hline F_{\eta m} \otimes I_{\varepsilon n} & I_{\eta m} \otimes F_{\varepsilon n} \end{array}\right]}_{=:T} + \underbrace{\left[\begin{array}{c|c} -\Delta A_{12}^T \otimes I_{\varepsilon n} & -I_{\eta m} \otimes \Delta A_{21} \\ \hline \Delta B_{12}^T \otimes I_{\varepsilon n} & I_{\eta m} \otimes \Delta B_{21} \end{array}\right]}_{=:\Delta T} \right) \underbrace{\left[\begin{array}{c} \operatorname{vec}(C) \\ \hline \operatorname{vec}(D) \end{array}\right]}_{=:x}$$

$$= \underbrace{\left[\begin{array}{c} \operatorname{vec}(\Delta A_{22}) \\ \hline -\operatorname{vec}(\Delta B_{22}) \end{array}\right]}_{=:b}, \tag{6.13}$$

where, for any $m \times n$ matrix $M = [m_{ij}]$, the column vector $\mathrm{vec}(M)$ is the *vectorization* of M, namely, $\mathrm{vec}(M) := [m_{11} \ldots m_{m1}\, m_{12} \ldots m_{m2} \ldots m_{1n} \ldots m_{mn}]^T$ (see Horn and Johnson [40, Def. 4.2.9], for instance). For brevity, and with an abuse of notation we use expressions such as "(C, D) is a solution of (6.13)".

Lemma 6.4 proves that the matrix T in (6.13) has full row rank and provides an expression for its minimum singular value. This implies that if $\|\Delta T\|_2$ is small enough, then $T + \Delta T$ has also full row rank and the linear system (6.13) is consistent, as well as the equivalent system of matrix equations (6.12). The proof of Lemma 6.4 is long and can be found in Appendix B. Here and in the rest of the paper the *minimum singular value* of any matrix M is denoted by $\sigma_{\min}(M)$.

Lemma 6.4. *The matrix T in* (6.13) *has full row rank and its minimum singular value is given by*

$$\sigma_{\min}(T) = \begin{cases} 2\sin\frac{\pi}{4\min(\eta,\varepsilon)+2}, & \varepsilon \neq \eta \\ 2\sin\frac{\pi}{4\eta}, & \varepsilon = \eta \end{cases}. \tag{6.14}$$

The following simple lower bound on $\sigma_{\min}(T)$ is useful to get bounds that can be easily handled and are related to the grade of the original matrix polynomial.

Corollary 6.5. *Let T be the matrix in* (6.13) *and $d = \varepsilon + \eta + 1$. Then*

$$\sigma_{\min}(T) \geq \frac{\pi}{4d}.$$

Proof. It follows from (6.14) and the inequality $\sin(x) \geq x/2$ for $0 \leq x \leq 1$. ☐

Lemma 6.6 bounds the norm of the minimum 2-norm solution of (6.13) or, equivalently, of the minimum Frobenius norm solution of the matrix equation (6.12), since $\|[\mathrm{vec}(C)^T, \mathrm{vec}(D)^T]^T\|_2 = \|(C, D)\|_F$.

Lemma 6.6. *Let $(T + \Delta T)x = b$ be the underdetermined linear system* (6.13), *and let us assume that $\sigma_{\min}(T) > \|\Delta T\|_2$. Then $(T + \Delta T)x = b$ is consistent and its minimum norm solution (C_0, D_0) satisfies*

$$\|(C_0, D_0)\|_F \leq \frac{1}{\delta}\|(\Delta A_{22}, \Delta B_{22})\|_F, \tag{6.15}$$

where $\delta := \sigma_{\min}(T) - \|\Delta T\|_2$.

Proof. From Weyl's perturbation theorem for singular values [40, Theorem 3.3.16], we get $\sigma_{\min}(T + \Delta T) \geq \sigma_{\min}(T) - \|\Delta T\|_2 > 0$. Therefore, $T + \Delta T$ has full row rank and the linear system (6.13) is consistent. Its minimum norm solution, (C_0, D_0), is given by $(T + \Delta T)^\dagger b$, where $(T + \Delta T)^\dagger$ denotes the Moore-Penrose pseudoinverse of $T + \Delta T$. Then,

$$\begin{aligned}\|(C_0, D_0)\|_F \leq & \|(T + \Delta T)^\dagger\|_2 \|(\Delta A_{22}, \Delta B_{22})\|_F = \frac{1}{\sigma_{\min}(T + \Delta T)}\|(\Delta A_{22}, \Delta B_{22})\|_F \\ \leq & \frac{1}{\sigma_{\min}(T) - \|\Delta T\|_2}\|(\Delta A_{22}, \Delta B_{22})\|_F.\end{aligned}$$

☐

From Lemma 6.6, it is clear that the quantity $\delta = \sigma_{\min}(T) - \|\Delta T\|_2$ will play a relevant role in our analysis. Therefore, we establish a tractable lower bound on δ.

Lemma 6.7. *Let T and ΔT be the matrices in* (6.13), *let $\Delta\mathcal{L}(\lambda)$ be the pencil in* (6.9), *and $d = \varepsilon + \eta + 1$. If $\|\Delta\mathcal{L}(\lambda)\|_F < 1/(3d)$, then*

$$\sigma_{\min}(T) - \|\Delta T\|_2 \geq \frac{\pi}{4d}\left(1 - 3d\|\Delta\mathcal{L}(\lambda)\|_F\right) > 0.$$

Proof. Using standard properties of norms and Kronecker products [40, Chapter 4] (pay particular attention to [40, p. 247, paragraph 1]), we get

$$\begin{aligned}\|\Delta T\|_2 &\leq \left\| \left[\frac{-\Delta A_{12}^T \otimes I_{\varepsilon n}}{\Delta B_{12}^T \otimes I_{\varepsilon n}} \right] \right\|_2 + \left\| \left[\frac{-I_{\eta m} \otimes \Delta A_{21}}{I_{\eta m} \otimes \Delta B_{21}} \right] \right\|_2 \\ &= \left\| \left[\frac{-\Delta A_{12}^T}{\Delta B_{12}^T} \right] \right\|_2 + \left\| \left[\frac{-\Delta A_{21}}{\Delta B_{21}} \right] \right\|_2 \leq \left\| \left[\frac{-\Delta A_{12}^T}{\Delta B_{12}^T} \right] \right\|_F + \left\| \left[\frac{-\Delta A_{21}}{\Delta B_{21}} \right] \right\|_F \\ &\leq 2\|\Delta \mathcal{L}(\lambda)\|_F \,.\end{aligned}$$

From this inequality and Corollary 6.5, the result is obtained as follows:

$$\sigma_{\min}(T) - \|\Delta T\|_2 \geq \frac{\pi}{4d} - 2\|\Delta \mathcal{L}(\lambda)\|_F = \frac{\pi}{4d}(1 - \frac{8d}{\pi}\|\Delta \mathcal{L}(\lambda)\|_F) \geq \frac{\pi}{4d}(1 - 3d\|\Delta \mathcal{L}(\lambda)\|_F)\,.$$

□

Theorem 6.8 is the key technical result of this section. It proves that the system of quadratic Sylvester-like matrix equations (6.11) has a solution (C, D) such that $\|(C, D)\|_F \lesssim \|\Delta \mathcal{L}(\lambda)\|_F$, whenever $\|\Delta \mathcal{L}(\lambda)\|_F$ is properly upper bounded. As mentioned before, this theorem extends to underdetermined quadratic matrix equations results proved by Stewart for equations with a unique solution [59, Theorem 5.1], [60, Theorem 2.11, p. 242]. The proof of Theorem 6.8 follows those by Stewart.

THEOREM 6.8. *There exists a solution* (C, D) *of the quadratic system of Sylvester-like matrix equations* (6.11) *satisfying*

$$\|(C, D)\|_F \leq 2\frac{\theta}{\delta}\,, \tag{6.16}$$

whenever

$$\delta > 0 \quad \text{and} \quad \frac{\theta\omega}{\delta^2} < \frac{1}{4}\,, \tag{6.17}$$

where $\delta = \sigma_{\min}(T) - \|\Delta T\|_2$, $\theta := \|(\Delta A_{22}, \Delta B_{22})\|_F$, *and* $\omega := \|(M_0 + \Delta A_{11}, M_1 + \Delta B_{11})\|_F$.

Proof. Lemma 6.6 and the hypothesis $\delta > 0$ guarantee that the linear system of matrix equations (6.12) is consistent, and, even more, that is consistent for any right-hand side. Let the minimum norm solution of (6.12) be denoted by (C_0, D_0). It satisfies

$$\|(C_0, D_0)\|_F \leq \frac{1}{\delta}\|(\Delta A_{22}, \Delta B_{22})\|_F = \frac{\theta}{\delta} =: \rho_0,$$

according to Lemma 6.6. Then, let us define a sequence $\{(C_i, D_i)\}_{i=0}^{\infty}$ of pairs of matrices as follows: for $i > 0$ the pair (C_i, D_i) is the minimum norm solution of

$$\left\{ \begin{array}{l} C_i(E_{\eta m}^T - \Delta A_{12}) + (E_{\varepsilon n} - \Delta A_{21})D_i = \Delta A_{22} + C_{i-1}(M_0 + \Delta A_{11})D_{i-1} \\ C_i(F_{\eta m}^T + \Delta B_{12}) + (F_{\varepsilon n} + \Delta B_{21})D_i = -\Delta B_{22} - C_{i-1}(M_1 + \Delta B_{11})D_{i-1} \end{array} \right. . \tag{6.18}$$

Therefore, vectorizing (6.18) and using the matrix $T + \Delta T$ defined in (6.13), we get

$$\left[\begin{array}{c} \mathrm{vec}(C_i) \\ \mathrm{vec}(D_i) \end{array} \right] = \left[\begin{array}{c} \mathrm{vec}(C_0) \\ \mathrm{vec}(D_0) \end{array} \right] + (T + \Delta T)^{\dagger} \left(\left[\begin{array}{c} \mathrm{vec}(C_{i-1}(M_0 + \Delta A_{11})D_{i-1}) \\ -\mathrm{vec}(C_{i-1}(M_1 + \Delta B_{11})D_{i-1}) \end{array} \right] \right). \tag{6.19}$$

We claim that the sequence $\{(C_i, D_i)\}_{i=0}^{\infty}$ converges to a solution (C, D) of (6.11) satisfying (6.16). To prove this, we first show that the sequence $\{\|(C_i, D_i)\|_F\}_{i=0}^{\infty}$ is a bounded sequence. If $\|(C_{i-1}, D_{i-1})\|_F \leq \rho_{i-1}$, then we have from (6.19) that

$$\begin{aligned}\|(C_i, D_i)\|_F \leq & \|(C_0, D_0)\|_F \\ & + \|(T + \Delta T)^{\dagger}\|_2 \|\|(C_{i-1}, D_{i-1})\|_F^2 \|(M_0 + \Delta A_{11}, M_1 + \Delta B_{11})\|_F \\ \leq & \rho_0 + \rho_{i-1}^2 \omega \delta^{-1} =: \rho_i \, .\end{aligned}$$

We may write the quantity ρ_i in the equation above as $\rho_i = \rho_0(1 + \kappa_i)$, where κ_i satisfies the recursion

$$\begin{cases} \kappa_1 = \rho_0 \omega \delta^{-1} = \theta \omega \delta^{-2}, \\ \kappa_{i+1} = \kappa_1 (1 + \kappa_i)^2 \, . \end{cases} \tag{6.20}$$

An induction argument proves that $0 < \kappa_1 < \kappa_2 < \cdots$, i.e., that the sequence is strictly increasing. In addition, if $\kappa_1 < 1/4$, then the function $g(x) := \kappa_1(1 + x)^2$, which defines the fixed point iteration in (6.20), has two positive fixed points, one smaller than one and another larger, and satisfies $0 < g(x) < 1$ and $0 < g'(x) < 1$ for $0 < x < 1$. Therefore, standard results on fixed point iterations imply that $\lim_{i\to\infty} \kappa_i = \kappa$, where κ is the smallest fixed point of $g(x)$, i.e.,

$$\kappa = \lim_{i\to\infty} \kappa_i = \frac{2\kappa_1}{1 - 2\kappa_1 + \sqrt{1 - 4\kappa_1}} < 1,$$

and $\kappa_i < \kappa$ for all $i \geq 1$. Thus, the norms of the elements of the sequence $\{(C_i, D_i)\}_{i=0}^{\infty}$ are bounded as

$$\|(C_i, D_i)\|_F \leq \rho := \lim_{i\to\infty} \rho_i = \rho_0(1 + \kappa) \, . \tag{6.21}$$

We now show that the sequence $\{(C_i, D_i)\}_{i=0}^{\infty}$ converges provided that $2\delta^{-1}\omega\rho < 1$, which is ensured by (6.17). For this purpose, let $S_i = (S_i^{(C)}, S_i^{(D)}) := (C_{i+1} - C_i, D_{i+1} - D_i)$. Then (6.19) implies

$$\begin{aligned}\|S_i\|_F &\leq \|(T + \Delta T)^{\dagger}\|_2 \left\| \begin{bmatrix} \operatorname{vec}\,(C_i(M_0 + \Delta A_{11})D_i - C_{i-1}(M_0 + \Delta A_{11})D_{i-1}) \\ \operatorname{vec}\,(C_i(M_1 + \Delta B_{11})D_i - C_{i-1}(M_1 + \Delta B_{11})D_{i-1}) \end{bmatrix} \right\|_2 \\ &\leq \delta^{-1} \left\| \begin{bmatrix} \operatorname{vec}\left(S_{i-1}^{(C)}(M_0 + \Delta A_{11})D_i + C_{i-1}(M_0 + \Delta A_{11})S_{i-1}^{(D)}\right) \\ \operatorname{vec}\left(S_{i-1}^{(C)}(M_1 + \Delta B_{11})D_i + C_{i-1}(M_1 + \Delta B_{11})S_{i-1}^{(D)}\right) \end{bmatrix} \right\|_2 \\ &\leq 2\delta^{-1}\omega\rho\|S_{i-1}\|_F.\end{aligned}$$

Therefore, the sequence $\{(C_i, D_i)\}_{i=0}^{\infty}$ is a Cauchy sequence, since $2\delta^{-1}\omega\rho < 1$, and must have a limit $(C, D) := \lim_{i\to\infty}(C_i, D_i)$. Taking limits of both sides of (6.18), we get that (C, D) is a solution of (6.11). Finally, from (6.21), $\|(C, D)\|_F \leq \rho_0(1 + \kappa) < 2\rho_0 = 2\delta^{-1}\theta$, which concludes the proof. □

Theorem 6.9 completes the first step of the backward error analysis. Its proof follows from Theorem 6.8 and norm inequalites. The numerical constants appearing in Theorem 6.9 are not optimal and have been chosen to keep the analysis and the bounds simple.

THEOREM 6.9. *Let $\mathcal{L}(\lambda)$ be an $(\varepsilon, n, \eta, m)$-block Kronecker pencil as in (5.1), let $\varepsilon + \eta + 1 = d$, and let $\Delta\mathcal{L}(\lambda)$ be any pencil with the same size as $\mathcal{L}(\lambda)$ and such that*

$$\|\Delta\mathcal{L}(\lambda)\|_F < \left(\frac{\pi}{16}\right)^2 \frac{1}{d^2} \frac{1}{1 + \|\lambda M_1 + M_0\|_F}. \tag{6.22}$$

Then, there exist matrices $C \in \mathbb{F}^{\varepsilon n\times(\eta+1)m}$ *and* $D \in \mathbb{F}^{(\varepsilon+1)n\times\eta m}$ *that satisfy*

$$\|(C,D)\|_F \leq \frac{3d}{1-3d\|\Delta\mathcal{L}(\lambda)\|_F}\,\|\Delta\mathcal{L}(\lambda)\|_F, \tag{6.23}$$

and the equality (6.6) *with*

$$\begin{aligned}&\max\{\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F, \|\Delta\widetilde{\mathcal{L}}_{12}(\lambda)\|_F\}\\ &\qquad\leq \|\Delta\mathcal{L}(\lambda)\|_F\left(1+\frac{3d}{1-3d\|\Delta\mathcal{L}(\lambda)\|_F}\left(\|\lambda M_1+M_0\|_F+\|\Delta\mathcal{L}(\lambda)\|_F\right)\right).\end{aligned} \tag{6.24}$$

Proof. The notation in (6.9) for the blocks of $\Delta\mathcal{L}(\lambda)$ is used throughout the proof. We first prove that (6.22) implies (6.17) and, so, the existence of C and D satisfying (6.6). For this purpose, note that (6.22) implies $\|\Delta\mathcal{L}(\lambda)\|_F < 1/(6d) < 1/(3d)$ and, therefore, that Lemma 6.7 holds and that $\delta > 0$. With this, and the notation in Theorem 6.8, we get

$$\begin{aligned}\frac{\theta\omega}{\delta^2} &\leq \frac{\|\Delta\mathcal{L}(\lambda)\|_F\left(\|\lambda M_1+M_0\|_F+\|\Delta\mathcal{L}(\lambda)\|_F\right)}{\left(\frac{\pi}{4d}\right)^2(1-3d\|\Delta\mathcal{L}(\lambda)\|_F)^2} < \frac{\|\Delta\mathcal{L}(\lambda)\|_F\left(\|\lambda M_1+M_0\|_F+1\right)}{\left(\frac{\pi}{4d}\right)^2(\frac{1}{2})^2}\\ &< \frac{\left(\frac{\pi}{16}\right)^2\frac{1}{d^2}}{\left(\frac{\pi}{4d}\right)^2(\frac{1}{2})^2} = \frac{1}{4},\end{aligned}$$

and (6.17) indeed holds. Then, Theorem 6.8 implies that there exist matrices C and D satisfying (6.6) and

$$\|(C,D)\|_F \leq 2\frac{\theta}{\delta} \leq \frac{2\|\Delta\mathcal{L}(\lambda)\|_F}{\frac{\pi}{4d}(1-3d\|\Delta\mathcal{L}(\lambda)\|_F)} < \frac{3d\|\Delta\mathcal{L}(\lambda)\|_F}{1-3d\|\Delta\mathcal{L}(\lambda)\|_F},$$

which proves (6.23).

Finally, from (6.5) and (6.6), we obtain that

$$\begin{aligned}\Delta\widetilde{\mathcal{L}}_{12}(\lambda) &= (\lambda M_1+M_0+\Delta\mathcal{L}_{11}(\lambda))D+\Delta\mathcal{L}_{12}(\lambda),\\ \Delta\widetilde{\mathcal{L}}_{21}(\lambda) &= C(\lambda M_1+M_0+\Delta\mathcal{L}_{11}(\lambda))+\Delta\mathcal{L}_{21}(\lambda),\end{aligned}$$

which combined with (6.23) leads to (6.24).
□

6.2. Second step: proving that $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ in (6.6) is a strong block minimal bases pencil. The main result of this section is Theorem 6.18. From the definition of strong block minimal bases pencils, it is not surprising that part of the proof of Theorem 6.18 relies on algebraic results that characterize when a matrix polynomial is a minimal basis with all its row degrees equal and such that any minimal basis dual to it has also all its row degrees equal. In the first part of this section, we establish such characterizations. In this process, we use the complete eigenstructure of a matrix polynomial. Since it may include infinite eigenvalues, whose definition depends on which grade is chosen for the polynomial [14, Section 2], we adopt the convention in this section that anytime a complete eigenstructure is mentioned, the grade of the corresponding polynomial is equal to its degree.

A simple result that is used in this section is the next lemma.

LEMMA 6.10. *Let* $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *with* $m < n$*. Then,* $Q(\lambda)$ *is a minimal basis with all its row degrees equal if and only if the complete eigenstructure of* $Q(\lambda)$ *consists of only* $n-m$ *right minimal indices.*

Proof. It is a simple consequence of Theorem 2.2 and the fact that if all the row degrees of $Q(\lambda) = \sum_{i=0}^{q} Q_i \lambda^i$ (where $Q_q \neq 0$) are equal, then its highest row degree coefficient matrix is equal to its leading coefficient Q_q. So, if $Q(\lambda)$ is a minimal basis with all its row degrees equal, then Theorem 2.2 guarantees that $Q(\lambda)$ has no finite eigenvalues, since $Q(\lambda_0)$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$, and that $Q(\lambda)$ has no infinite eigenvalues, since it is row reduced. In addition, $Q(\lambda)$ has no left minimal indices, since it has full row rank. Therefore, the complete eigenstructure of $Q(\lambda)$ consists of only $n - m$ right minimal indices.

Conversely, if the complete eigenstructure of $Q(\lambda)$ consists of only $n - m$ right minimal indices, then $\operatorname{rank} Q_q = \operatorname{rank} Q(\lambda) = m$, because $Q(\lambda)$ has neither infinite eigenvalues nor left minimal indices. This implies, in particular, that all the row degrees of $Q(\lambda)$ are equal, since otherwise $\operatorname{rank} Q_q < m$, and that $Q(\lambda)$ is row reduced. Moreover, $\operatorname{rank} Q(\lambda_0) = m$ for all $\lambda_0 \in \overline{\mathbb{F}}$ because $Q(\lambda)$ has no finite eigenvalues, and we get from Theorem 2.2 that $Q(\lambda)$ is a minimal basis with all its row degrees equal. ☐

Convolution matrices will be useful in our characterizations of minimal bases and in a number of perturbation results. For any matrix polynomial $Q(\lambda) = \sum_{i=0}^{q} Q_i \lambda^i$ of grade q and arbitrary size, we define in the spirit of Gantmacher [30, Chapter XII] the sequence of its *convolution matrices* as follows

$$C_j(Q(\lambda)) = \underbrace{\begin{bmatrix} Q_q & & & \\ Q_{q-1} & Q_q & & \\ \vdots & Q_{q-1} & \ddots & \\ Q_0 & \vdots & \ddots & Q_q \\ & Q_0 & & Q_{q-1} \\ & & \ddots & \vdots \\ & & & Q_0 \end{bmatrix}}_{j+1 \text{ block columns}}, \quad \text{for } j = 0, 1, 2, \ldots. \tag{6.25}$$

Observe that for every j the matrix $C_j(Q(\lambda))$ is a constant matrix. In particular for $j = 0$, the matrix $C_0(Q(\lambda))$ is a block column matrix whose block entries are the matrix coefficients of the polynomial. The fundamental property of these convolution matrices is that if $Z(\lambda)$ is any matrix polynomial of grade j for which the product $Q(\lambda)Z(\lambda)$ is defined and is considered to have grade $q + j$, then

$$C_0(Q(\lambda)Z(\lambda)) = C_j(Q(\lambda))\, C_0(Z(\lambda)). \tag{6.26}$$

Another easy property of convolution matrices that we often use in this subsection is that $\|C_j(Q(\lambda))\|_F = \sqrt{j+1}\, \|Q(\lambda)\|_F$. Note also that if $S(\lambda)$ is another matrix polynomial with the same grade as $Q(\lambda)$, then $C_j(Q(\lambda) + S(\lambda)) = C_j(Q(\lambda)) + C_j(S(\lambda))$, for all j. The convolution matrices for pencils are particularly simple. For instance, for the pencil $L_\varepsilon(\lambda) \otimes I_n$ in the $(2,1)$-block of (5.1), we have with the notation in (6.8) that

$$C_{\varepsilon-1}(L_\varepsilon(\lambda) \otimes I_n) = \underbrace{\begin{bmatrix} F_{\varepsilon n} & & \\ -E_{\varepsilon n} & \ddots & \\ & \ddots & F_{\varepsilon n} \\ & & -E_{\varepsilon n} \end{bmatrix}}_{\varepsilon \text{ block columns}} \left.\vphantom{\begin{bmatrix} F \\ F \\ F \\ F \end{bmatrix}}\right\} \varepsilon + 1 \text{ block rows}. \tag{6.27}$$

Lemma 6.10 motivates us to look deeper into the right minimal indices of a matrix polynomial $Q(\lambda)$ and into the rational right null subspace $\mathcal{N}_r(Q)$ defined in (2.2). This is the purpose of Lemma 6.11, which states and proves with precision for general matrix polynomials ideas that can be found in [30, Chapter XII] only for matrix pencils.

LEMMA 6.11. *Let* $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *and let* $C_s(Q(\lambda))$, $s = 0, 1, 2, ...,$ *be the sequence of convolution matrices of* $Q(\lambda)$. *Then, the following statements hold.*

(a) *Let* $v(\lambda) \in \mathbb{F}[\lambda]^{n\times 1}$ *be a polynomial vector of grade* j. *Then,* $v(\lambda) \in \mathcal{N}_r(Q)$ *if and only if* $C_0(v(\lambda)) \in \mathcal{N}_r(C_j(Q(\lambda)))$.

(b) *The smallest right minimal index of* $Q(\lambda)$ *is* j *if and only if* $C_{j-1}(Q(\lambda))$ *has full column rank and* $C_j(Q(\lambda))$ *does not have full column rank.*

(c) *If* j *is the smallest right minimal index of* $Q(\lambda)$ *and* $\dim \mathcal{N}_r(C_j(Q(\lambda))) = p$, *then* $Q(\lambda)$ *has at least* p *minimal indices equal to* j.

Proof. Part (a) follows immediately from (6.26). Before proceeding, note that part (a) establishes the natural bijection[1] $v(\lambda) \longmapsto C_0(v(\lambda))$ between the set of polynomial vectors of grade j in $\mathcal{N}_r(Q) \subseteq \mathbb{F}(\lambda)^n$ and $\mathcal{N}_r(C_j(Q(\lambda))) \subseteq \mathbb{F}^{(j+1)n\times 1}$. Indeed $v(\lambda) \mapsto C_0(v(\lambda))$ is a bijection, since its inverse can be trivially constructed as follows: partition any $x \in \mathcal{N}_r(C_j(Q(\lambda))) \subseteq \mathbb{F}^{(j+1)n\times 1}$ as $x = [x_j^T, \dots, x_1^T, x_0^T]^T$, where $x_i \in \mathbb{F}^{n\times 1}$, and note that

$$x \longmapsto \sum_{i=0}^{j} x_i \lambda^i =: \mathcal{P}(x;\lambda) \in \mathcal{N}_r(Q) \tag{6.28}$$

is the inverse of $v(\lambda) \mapsto C_0(v(\lambda))$.

Part (b). From part (a), it is obvious that if the smallest right minimal index of $Q(\lambda)$ is j, then $C_{j-1}(Q(\lambda))$ has full column rank but $C_j(Q(\lambda))$ does not. The converse also follows from part (a) by taking into account that if $C_{j-1}(Q(\lambda))$ has full column rank, then $C_{j-2}(Q(\lambda)), \dots, C_0(Q(\lambda))$ have also full column ranks. Therefore, $\mathcal{N}_r(C_{j-1}(Q(\lambda))) = \{0\}, \dots, \mathcal{N}_r(C_1(Q(\lambda))) = \{0\}, \mathcal{N}_r(C_0(Q(\lambda))) = \{0\}$ and part (a) implies that $\mathcal{N}_r(Q)$ does not include vectors different from 0 of degree less than j, but does include vectors of degree j because $C_j(Q(\lambda))$ does not have full column rank and so $\mathcal{N}_r(C_j(Q(\lambda))) \neq \{0\}$.

The proof of part (c) requires more work. Let $\{v^{(1)}, \dots, v^{(p)}\}$ be a basis of $\mathcal{N}_r(C_j(Q(\lambda)))$ and consider, according to (6.28), the vector polynomials $\mathcal{P}(v^{(k)};\lambda) = \sum_{i=0}^{j} v_i^{(k)} \lambda^i \in \mathcal{N}_r(Q)$ for $k = 1, \dots, p$. Note that $\mathcal{P}(v^{(k)};\lambda) \neq 0$, because $v^{(k)} \neq 0$, and that $\deg(\mathcal{P}(v^{(k)};\lambda)) = j$, because otherwise $Q(\lambda)$ would have right minimal indices smaller than j. The result follows from proving that $\mathcal{P}(v^{(1)};\lambda), \dots, \mathcal{P}(v^{(p)};\lambda)$ are linearly independent. We prove this by contradiction. Assume that there exists a linear combination

$$a_1(\lambda)\mathcal{P}(v^{(1)};\lambda) + a_2(\lambda)\mathcal{P}(v^{(2)};\lambda) + \cdots + a_p(\lambda)\mathcal{P}(v^{(p)};\lambda) = 0,$$

where, without loss of generality, we assume that $a_1(\lambda), \dots, a_p(\lambda)$ are scalar polynomials not all equal to zero (if they were rational functions we may multiply the equation above by their least common denominator). The coefficient of the highest power in the equation above satisfies

$$c_1 v_j^{(1)} + c_2 v_j^{(2)} + \cdots + c_p v_j^{(p)} = 0,$$

---

[1] We emphasize that this bijection is not a linear map since the fields of the linear spaces corresponding to the domain and the codomain are different. Nevertheless, it has some obvious linear properties that can be used.

for some constants $c_1, c_2, \ldots, c_p$, where at least one of them is nonzero. Then, let us define the polynomial vector $q(\lambda) := \sum_{k=1}^{p} c_k \mathcal{P}(v^{(k)};\lambda)$. Notice that $q(\lambda) \in \mathcal{N}_r(Q)$ and that $\deg(q(\lambda)) < j$. Then $q(\lambda) = 0$, because otherwise $Q(\lambda)$ would have right minimal indices smaller than j, which implies $\sum_{k=1}^{p} c_k v^{(k)} = 0$. This is a contradiction since $\{v^{(1)}, v^{(2)}, \ldots, v^{(p)}\}$ is a linearly independent set of vectors. ☐

Next, we study when arbitrary pencils with the same size as the $(2,1)$-block of $\mathcal{L}(\lambda) + \Delta\widetilde{\mathcal{L}}(\lambda)$ in (6.6) are the corresponding block of a strong block minimal bases pencil.

THEOREM 6.12. *Let* $A + \lambda B \in \mathbb{F}[\lambda]^{\varepsilon n \times (\varepsilon+1)n}$ *and let* $C_s(A+\lambda B)$, $s = 0, 1, 2, ...,$ *be the sequence of convolution matrices of* $A + \lambda B$. *Then,* $A + \lambda B$ *is a minimal basis with all its row degrees equal to* 1 *and with all the row degrees of any minimal basis dual to it equal to* ε *if and only if* $C_{\varepsilon-1}(A + \lambda B) \in \mathbb{F}^{\varepsilon(\varepsilon+1)n \times \varepsilon(\varepsilon+1)n}$ *is nonsingular and* $C_\varepsilon(A+\lambda B) \in \mathbb{F}^{\varepsilon(\varepsilon+2)n\times(\varepsilon+1)^2 n}$ *has full row rank.*

Proof. Bear in mind that the right minimal indices of a minimal basis are the row degrees of any minimal basis dual to it. First, assume that $A+\lambda B$ is a minimal basis with all its row degrees equal to 1 and with all the row degrees of any minimal basis dual to it equal to ε. Then, the complete eigenstructure of $A + \lambda B$ consists of only n right minimal indices equal to ε, by Lemma 6.10. From Lemma 6.11(b), we get that $C_{\varepsilon-1}(A + \lambda B)$ has full column rank and, since it is square, it must be nonsingular. From Lemma 6.11(c), we get that $n \geq \dim \mathcal{N}_r(C_\varepsilon(A+\lambda B)) = (\varepsilon+1)^2 n - \operatorname{rank}(C_\varepsilon(A+\lambda B))$, which implies that $\operatorname{rank}(C_\varepsilon(A + \lambda B)) \geq (\varepsilon + 1)^2 n - n = \varepsilon(\varepsilon + 2)n$ and, finally, that $\operatorname{rank}(C_\varepsilon(A + \lambda B)) = \varepsilon(\varepsilon + 2)n$, because $C_\varepsilon(A + \lambda B)$ has $\varepsilon(\varepsilon + 2)n$ rows.

Next, assume that $C_{\varepsilon-1}(A+\lambda B)$ is nonsingular and $C_\varepsilon(A+\lambda B)$ has full row rank. Therefore, $\dim \mathcal{N}_r(C_\varepsilon(A+\lambda B)) = (\varepsilon+1)^2 n - \operatorname{rank}(C_\varepsilon(A+\lambda B)) = (\varepsilon+1)^2 n - \varepsilon(\varepsilon+2)n = n$. From Lemma 6.11(b), we get that the smallest right minimal index of $A + \lambda B$ is ε, and from Lemma 6.11(c), we get that $A+\lambda B$ has at least n right minimal indices equal to ε. Also note that the degree of $A+\lambda B$ must be 1, since otherwise its minimal indices would be all equal to zero. Combining this information with the index sum theorem [14, Theorem 6.5] applied to $A+\lambda B$ and with the obvious bound $\varepsilon n \geq \operatorname{rank}(A+\lambda B)$, we get

$$n\varepsilon \geq \operatorname{rank}(A + \lambda B) \geq n\varepsilon + \delta(A + \lambda B) + \mu_{left}(A + \lambda B), \tag{6.29}$$

where $\delta(A + \lambda B)$ is the sum of the degrees of all the elementary divisors (finite and infinite) of $A + \lambda B$ and $\mu_{left}(A + \lambda B)$ is the sum of the left minimal indices of $A + \lambda B$. The inequalities (6.29) imply that $\operatorname{rank}(A + \lambda B) = n\varepsilon$ and that $A + \lambda B$ has no elementary divisors at all. Moreover, $\operatorname{rank}(A + \lambda B) = n\varepsilon$ implies that $A + \lambda B$ has no left minimal indices and that it has exactly n right minimal indices. Therefore, the complete eigenstructure of $A + \lambda B$ consists of only n right minimal indices equal to ε, which implies, by Lemma 6.10, that $A + \lambda B$ is a minimal basis with all its row degrees equal to 1 and with all the row degrees of any minimal basis dual to it equal to ε. ☐

Theorem 6.13 is a counterpart of the previous result which is valid for matrix polynomials that may be minimal bases dual to the pencils considered in Theorem 6.12. The proof of Theorem 6.13 is omitted, since it is very similar to that of Theorem 6.12 and is based again on Lemmas 6.10 and 6.11.

THEOREM 6.13. *Let* $Q(\lambda) = \sum_{i=0}^{\varepsilon} Q_i \lambda^i \in \mathbb{F}[\lambda]^{n\times(\varepsilon+1)n}$ *and let* $C_s(Q(\lambda))$, $s = 0, 1, 2, ...,$ *be the sequence of convolution matrices of* $Q(\lambda)$. *Then,* $Q(\lambda)$ *is a minimal basis with all its row degrees equal to* ε *and with all the row degrees of any minimal basis dual to it equal to* 1 *if and only if* $C_0(Q(\lambda)) \in \mathbb{F}^{(\varepsilon+1)n\times(\varepsilon+1)n}$ *is nonsingular and*

$C_1(Q(\lambda)) \in \mathbb{F}^{(\varepsilon+2)n\times 2(\varepsilon+1)n}$ *has full row rank.*

Theorems 6.12 and 6.13 have established the characterizations of a minimal basis with all its row degrees equal and with all the row degrees of any minimal basis dual to it also equal that are needed in this paper. We now return to our perturbation problem for $\mathcal{L}(\lambda) + \Delta\widetilde{\mathcal{L}}(\lambda)$ in (6.6). In Theorem 6.14, we prove some properties of the unperturbed $(2,1)$-block of $\mathcal{L}(\lambda)$, that is, $L_\varepsilon(\lambda) \otimes I_n$, and its dual minimal basis $\Lambda_\varepsilon(\lambda)^T \otimes I_n$.

THEOREM 6.14. *Let* $L_\varepsilon(\lambda)$ *and* $\Lambda_\varepsilon(\lambda)^T$ *be the pencil and the row vector polynomial defined in* (2.3) *and* (2.4), *respectively. Then the following statements hold.*

(a) $C_{\varepsilon-1}(L_\varepsilon(\lambda) \otimes I_n) \in \mathbb{F}^{\varepsilon(\varepsilon+1)n\times\varepsilon(\varepsilon+1)n}$ *is nonsingular and* $C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n) \in \mathbb{F}^{\varepsilon(\varepsilon+2)n\times(\varepsilon+1)^2 n}$ *has full row rank.*

(b) $C_0(\Lambda_\varepsilon(\lambda)^T \otimes I_n) = I_{(\varepsilon+1)n}$ *and, therefore, is nonsingular, and* $C_1(\Lambda_\varepsilon(\lambda)^T \otimes I_n) \in \mathbb{F}^{(\varepsilon+2)n\times 2(\varepsilon+1)n}$ *has full row rank.*

(c) $\sigma_{\min}(C_{\varepsilon-1}(L_\varepsilon(\lambda) \otimes I_n)) \geq \sigma_{\min}(C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n)) \geq 2\sin\left(\dfrac{\pi}{4(\varepsilon+1)}\right)$, *and*

$\sigma_{\min}(C_{\varepsilon-1}(L_\varepsilon(\lambda) \otimes I_n)) \geq \sigma_{\min}(C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n)) \geq \dfrac{\pi}{4(\varepsilon+1)}$.

(d) $\sigma_{\min}(C_0(\Lambda_\varepsilon(\lambda)^T \otimes I_n)) = \sigma_{\min}(C_1(\Lambda_\varepsilon(\lambda)^T \otimes I_n)) = 1$.

Proof. Taking into account that $L_\varepsilon(\lambda) \otimes I_n$ and $\Lambda_\varepsilon(\lambda)^T \otimes I_n$ are dual minimal bases with all their row degrees equal, respectively, to 1 and ε, part (a) is an immediate consequence of Theorem 6.12. Part (b) can also be seen as a consequence of Theorem 6.13 (except the obvious equality $C_0(\Lambda_\varepsilon(\lambda)^T \otimes I_n) = I_{(\varepsilon+1)n}$), although it can be deduced directly because the matrices $C_0(\Lambda_\varepsilon(\lambda)^T \otimes I_n)$ and $C_1(\Lambda_\varepsilon(\lambda)^T \otimes I_n)$ are very simple.

In order to prove part (c), note that $\sigma_{\min}(C_{\varepsilon-1}(L_\varepsilon(\lambda) \otimes I_n)) \geq \sigma_{\min}(C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n))$, as a consequence of very well known properties of the singular values of submatrices of a given matrix [40, Corollary 3.1.3]. In [47, Lemma 6.1], it was proven that

$$\sigma_{\min}(C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n)) \geq \sqrt{2\left(1+\cos\left(\frac{\pi}{2}\frac{(2\varepsilon+1)}{\varepsilon+1}\right)\right)} = 2\sin\left(\frac{\pi}{4(\varepsilon+1)}\right),$$

which proves the first inequality in part (c). The second inequality follows from the inequality $\sin(x) \geq x/2$ for $0 \leq x \leq 1$.

The proof of part (d) follows from the equality $C_0(\Lambda_\varepsilon(\lambda)^T \otimes I_n) = I_{(\varepsilon+1)n}$ and the fact that an obvious column permutation Π allows us to prove that $C_1(\Lambda_\varepsilon(\lambda)^T \otimes I_n)\,\Pi = I_n \oplus (I_{\varepsilon n} \otimes [1, 1]) \oplus I_n$. Therefore, the singular values of $C_1(\Lambda_\varepsilon(\lambda)^T \otimes I_n)$ are 1 (with multiplicity $2n$) and $\sqrt{2}$ (with multiplicity εn). □

REMARK 6.15. Theorem 6.14(c) can be improved, since one can prove that

$$\sigma_{\min}(C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n)) = 2\,\sin\left(\frac{\pi}{4\varepsilon+2}\right),$$

with a considerably more complicated argument in the spirit of that in Appendix B. However, this result does not lead to significantly sharper bounds and, thus, we have not used it for the sake of brevity.

As a corollary of Theorem 6.12 and Theorem 6.14(a)-(c), we obtain the following perturbation result for the $(2,1)$-block of $\mathcal{L}(\lambda) + \Delta\widetilde{\mathcal{L}}(\lambda)$ in (6.6).

COROLLARY 6.16. *Let* $\Delta\widetilde{\mathcal{L}}_{21}(\lambda)$ *be any pencil of size* $\varepsilon n \times (\varepsilon+1)n$ *such that*

$$\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F < \frac{\pi}{4(\varepsilon+1)^{3/2}}. \tag{6.30}$$

Then, $L_\varepsilon(\lambda)\otimes I_n+\Delta\widetilde{\mathcal{L}}_{21}(\lambda)$ *is a minimal basis with all its row degrees equal to* 1 *and with all the row degrees of any minimal basis dual to it equal to* ε*.*

Proof. Observe that (6.30) implies that $\|C_{\varepsilon-1}(\Delta\widetilde{\mathcal{L}}_{21}(\lambda))\|_2\leq\|C_{\varepsilon-1}(\Delta\widetilde{\mathcal{L}}_{21}(\lambda))\|_F$ $=\sqrt{\varepsilon}\,\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F<\pi/(4(\varepsilon+1))\leq\sigma_{\min}(C_{\varepsilon-1}(L_\varepsilon(\lambda)\otimes I_n))$, where we have used Theorem 6.14(c). Therefore, $C_{\varepsilon-1}(L_\varepsilon(\lambda)\otimes I_n+\Delta\widetilde{\mathcal{L}}_{21}(\lambda))=C_{\varepsilon-1}(L_\varepsilon(\lambda)\otimes I_n)+C_{\varepsilon-1}(\Delta\widetilde{\mathcal{L}}_{21}(\lambda))$ is nonsingular, as a consequence of Theorem 6.14(a) and Weyl's perturbation theorem for singular values [40, Theorem 3.3.16]. An analogous argument proves that $C_\varepsilon(L_\varepsilon(\lambda)\otimes I_n+\Delta\widetilde{\mathcal{L}}_{21}(\lambda))$ has full row rank. The result follows from Theorem 6.12. ▯

As a corollary of Theorem 6.13 and Theorem 6.14(b)-(d), we obtain the following perturbation result for the minimal basis dual to $L_\varepsilon(\lambda)\otimes I_n$.

COROLLARY 6.17. *Let* $\Delta R_\varepsilon(\lambda)^T$ *be a matrix polynomial of size* $n\times(\varepsilon+1)n$*, grade* ε*, and such that*

$$\|\Delta R_\varepsilon(\lambda)\|_F<\frac{1}{\sqrt{2}}. \tag{6.31}$$

Then, $\Lambda_\varepsilon(\lambda)^T\otimes I_n+\Delta R_\varepsilon(\lambda)^T$ *is a minimal basis with all its row degrees equal to* ε *and with all the row degrees of any minimal basis dual to it equal to* 1*.*

Proof. Observe that (6.31) implies that $\|C_1(\Delta R_\varepsilon(\lambda)^T)\|_2\leq\|C_1(\Delta R_\varepsilon(\lambda)^T)\|_F=\sqrt{2}\,\|\Delta R_\varepsilon(\lambda)^T\|_F<1=\sigma_{\min}(C_1(\Lambda_\varepsilon(\lambda)^T\otimes I_n))$, where we have used Theorem 6.14(d). Therefore, $C_1(\Lambda_\varepsilon(\lambda)^T\otimes I_n+\Delta R_\varepsilon(\lambda)^T)=C_1(\Lambda_\varepsilon(\lambda)^T\otimes I_n)+C_1(\Delta R_\varepsilon(\lambda)^T)$ has full row rank, as a consequence of Theorem 6.14(b) and Weyl's perturbation theorem for singular values. An analogous argument proves that $C_0(\Lambda_\varepsilon(\lambda)^T\otimes I_n+\Delta R_\varepsilon(\lambda)^T)$ is nonsingular. The result follows from Theorem 6.13. ▯

Now, we are in the position of proving the main result of this section.

THEOREM 6.18. *Let* $L_\varepsilon(\lambda)$ *and* $\Lambda_\varepsilon(\lambda)^T$ *be the pencil and the row vector polynomial defined in* (2.3) *and* (2.4)*, respectively, and let* $\Delta\widetilde{\mathcal{L}}_{21}(\lambda)$ *be any pencil of size* $\varepsilon n\times(\varepsilon+1)n$ *such that*

$$\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F<\frac{\pi}{12(\varepsilon+1)^{3/2}}. \tag{6.32}$$

Then, there exists a matrix polynomial $\Delta R_\varepsilon(\lambda)^T$ *with size* $n\times(\varepsilon+1)n$ *and grade* ε *such that*

(a) $L_\varepsilon(\lambda)\otimes I_n+\Delta\widetilde{\mathcal{L}}_{21}(\lambda)$ *and* $\Lambda_\varepsilon(\lambda)^T\otimes I_n+\Delta R_\varepsilon(\lambda)^T$ *are dual minimal bases, with all the row degrees of the former equal to* 1 *and with all the row degrees of the latter equal to* ε*, and*

(b) $\|\Delta R_\varepsilon(\lambda)\|_F\leq\dfrac{6\sqrt{2}\,(\varepsilon+1)}{\pi}\,\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F<\dfrac{1}{\sqrt{2}}$.

Proof. The hypothesis (6.32) implies $\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F<\pi/(4(\varepsilon+1)^{3/2})$. Therefore, from Corollary 6.16, we get that $L_\varepsilon(\lambda)\otimes I_n+\Delta\widetilde{\mathcal{L}}_{21}(\lambda)$ is a minimal basis with all its row degrees equal to 1 and with all the row degrees of any minimal basis dual to it equal to ε, and, according to Theorem 6.12, we also have that $C_\varepsilon(L_\varepsilon(\lambda)\otimes I_n+\Delta\widetilde{\mathcal{L}}_{21}(\lambda))$ has full row rank. Using this fact, the goal of the rest of the proof is to show that there exists a matrix polynomial $\Delta R_\varepsilon(\lambda)^T$ with grade ε, that satisfies the bound in Theorem 6.18(b), and such that

$$(L_\varepsilon(\lambda)\otimes I_n+\Delta\widetilde{\mathcal{L}}_{21}(\lambda))\,(\Lambda_\varepsilon(\lambda)\otimes I_n+\Delta R_\varepsilon(\lambda))=0\,. \tag{6.33}$$

Once this is proved, the proof of Theorem 6.18 concludes by the application of Corollary 6.17.

Since $(L_\varepsilon(\lambda) \otimes I_n)\,(\Lambda_\varepsilon(\lambda) \otimes I_n) = 0$, the equation (6.33) is equivalent to the following linear equation for $\Delta R_\varepsilon(\lambda)$

$$(L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda))\,(\Delta R_\varepsilon(\lambda)) = -\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\,(\Lambda_\varepsilon(\lambda) \otimes I_n)\,. \tag{6.34}$$

Both sides of (6.34) have grade $\varepsilon+1$, therefore, by using convolution matrices, (6.34) is equivalent to $C_0((L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda))\,(\Delta R_\varepsilon(\lambda))) = -C_0(\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\,(\Lambda_\varepsilon(\lambda) \otimes I_n))$, which in turn, by using (6.26), is equivalent to

$$C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda))\; C_0(\Delta R_\varepsilon(\lambda)) = -C_0(\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\,(\Lambda_\varepsilon(\lambda) \otimes I_n))\,. \tag{6.35}$$

Observe that (6.35) is a consistent linear system for the unknown $C_0(\Delta R_\varepsilon(\lambda))$, since $C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda))$ has full row rank, which has the minimum Frobenius norm solution

$$C_0(\Delta R_\varepsilon(\lambda)) = -C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda))^\dagger\; C_0(\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\,(\Lambda_\varepsilon(\lambda) \otimes I_n))\,. \tag{6.36}$$

From (6.36), we get the bound

$$\begin{aligned}\|C_0(\Delta R_\varepsilon(\lambda))\|_F &\leq \|C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda))^\dagger\|_2\;\|C_0(\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\,(\Lambda_\varepsilon(\lambda) \otimes I_n))\|_F \\ &= \frac{1}{\sigma_{\min}(C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda)))}\;\|C_0(\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\,(\Lambda_\varepsilon(\lambda) \otimes I_n))\|_F\,.\end{aligned} \tag{6.37}$$

In the rest of the proof, the two factors in the right-hand side of (6.37) are bounded. For bounding the first factor, we use Theorem 6.14(c) and (6.32) as follows:

$$\begin{aligned}\frac{1}{\sigma_{\min}(C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda)))} &\leq \frac{1}{\sigma_{\min}(C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n)) - \|C_\varepsilon(\Delta\widetilde{\mathcal{L}}_{21}(\lambda))\|_2} \\ &\leq \frac{1}{\sigma_{\min}(C_\varepsilon(L_\varepsilon(\lambda) \otimes I_n)) - \|C_\varepsilon(\Delta\widetilde{\mathcal{L}}_{21}(\lambda))\|_F} \\ &\leq \frac{1}{\frac{\pi}{4(\varepsilon+1)} - \sqrt{\varepsilon+1}\,\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda))\|_F} \\ &\leq \frac{1}{\frac{\pi}{4(\varepsilon+1)} - \frac{\pi}{12(\varepsilon+1)}} = \frac{6(\varepsilon+1)}{\pi}\,.\end{aligned} \tag{6.38}$$

For bounding the second factor of (6.37), we use Lemma 2.16(d) with $d=1$ as follows:

$$\|C_0(\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\,(\Lambda_\varepsilon(\lambda)\otimes I_n))\|_F = \|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\,(\Lambda_\varepsilon(\lambda)\otimes I_n)\|_F \leq \sqrt{2}\,\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F\,. \tag{6.39}$$

Finally, by combining (6.37, 6.38, 6.39), the following bound is obtained

$$\|\Delta R_\varepsilon(\lambda)\|_F = \|C_0(\Delta R_\varepsilon(\lambda))\|_F \leq \frac{6\sqrt{2}\,(\varepsilon+1)}{\pi}\,\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F,$$

and the proof is finished. □

Theorem 6.18 can be applied with ε replaced by η and I_n replaced by I_m, i.e., to the transpose of the $(1,2)$-block of $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ in (6.6). This allows us to state, as a corollary of Theorem 6.18, the final conclusion of this section in Theorem 6.19.

Theorem 6.19. *Let $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ be the pencil in* (6.6) *and let $d=\varepsilon+\eta+1$. If*

$$\max\{\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F, \|\Delta\widetilde{\mathcal{L}}_{12}(\lambda)\|_F\} < \frac{\pi}{12\, d^{3/2}},$$

then $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ is a strong block minimal bases pencil. Moreover, there exist matrix polynomials $\Delta R_\varepsilon(\lambda)^T$ and $\Delta R_\eta(\lambda)^T$ of grades ε and η, respectively, such that $\Lambda_\varepsilon(\lambda)^T\otimes I_n+\Delta R_\varepsilon(\lambda)^T$ is a minimal basis dual to the $(2,1)$-block of $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ with all its row degrees equal to ε, $\Lambda_\eta(\lambda)^T\otimes I_m+\Delta R_\eta(\lambda)^T$ is a minimal basis dual to the transpose of the $(1,2)$-block of $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ with all its row degrees equal to η, and

$$\max\{\|\Delta R_\varepsilon(\lambda)\|_F, \|\Delta R_\eta(\lambda)\|_F\} \le \frac{6\sqrt{2}\, d}{\pi}\max\{\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F, \|\Delta\widetilde{\mathcal{L}}_{12}(\lambda)\|_F\} < \frac{1}{\sqrt{2}}\,.$$

The bound $\max\{\|\Delta R_\varepsilon(\lambda)\|_F, \|\Delta R_\eta(\lambda)\|_F\} < 1/\sqrt{2}$ in the equation above has the main purpose to emphasize that the hypotheses of Corollary 6.17 hold. In addition, it motivates the assumptions in Lemmas 6.20 and 6.21 that allow us to get rid of nonlinear terms in bounding $\|\Delta P(\lambda)\|_F$. Observe that in the conditions of Theorem 6.19, the sharper bound $\max\{\|\Delta R_\varepsilon(\lambda)\|_F, \|\Delta R_\eta(\lambda)\|_F\} < 1/\sqrt{2d}$ also holds. Since the use of such bound in Lemmas 6.20 and 6.21 does not lead to any significant improvement and it makes the arguments a bit more complicated, we have not used this sharper bound.

6.3. Third step: Mapping perturbations to a block Kronecker pencil onto the matrix polynomial. In this section, we combine the results in Sections 6.1 and 6.2 to obtain our main backward error (or perturbation) results, that is, Theorem 6.22 for general block Kronecker pencils as in (5.1) and Theorem 6.23 for degenerate block Kronecker pencils in which either $\varepsilon=0$ or $\eta=0$, that is, in which one of the anti-diagonal blocks and the zero block are not present. According to Remark 6.1 both cases require somewhat different treatments which makes the discussion longer.

The proofs of Theorems 6.22 and 6.23 are direct consequences of previous results, but require some delicate (although elementary) norm manipulations which are simplified if the technical Lemmas 6.20 and 6.21 are stated in advance. The relevance of these lemmas comes from the fact that the strong block minimal bases pencil $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ in Theorem 6.19 is a strong linearization of the matrix polynomial in (6.7), as a consequence of Theorem 3.3. The numerical constants appearing in Lemmas 6.20 and 6.21, and in the rest of the analysis, are not optimal but allow us to keep the analysis simple.

Lemma 6.20. *Let $P(\lambda)$ and $P(\lambda)+\Delta P(\lambda)$ be the matrix polynomials in* (6.4) *and* (6.7)*, respectively. If the matrix polynomials $\Delta R_\varepsilon(\lambda)$ and $\Delta R_\eta(\lambda)$ of grades ε and η, respectively, satisfy $\|\Delta R_\varepsilon(\lambda)\|_F < 1/\sqrt{2}$ and $\|\Delta R_\eta(\lambda)\|_F < 1/\sqrt{2}$, then*

$$\|\Delta P(\lambda)\|_F \le \sqrt{d}\,(5\|\Delta\mathcal{L}_{11}(\lambda)\|_F + 4\|\lambda M_1+M_0\|_F\max\{\|\Delta R_\varepsilon(\lambda)\|_F, \|\Delta R_\eta(\lambda)\|_F\})\ ,$$

where $d=\varepsilon+\eta+1$.

Proof. For brevity, we use in this proof the notation $\Lambda_{\varepsilon n}^T := \Lambda_\varepsilon(\lambda)^T \otimes I_n$ and omit the dependence on λ of some matrix polynomials. From (6.4) and (6.7), we get that

$$\begin{aligned}\Delta P(\lambda) =& \Delta R_\eta^T(\lambda M_1 + M_0)\Lambda_{\varepsilon n} + \Lambda_{\eta m}^T \Delta\mathcal{L}_{11}\Lambda_{\varepsilon n} + \Delta R_\eta^T \Delta\mathcal{L}_{11}\Lambda_{\varepsilon n}\\ &+ \Lambda_{\eta m}^T(\lambda M_1 + M_0)\Delta R_\varepsilon + \Delta R_\eta^T(\lambda M_1 + M_0)\Delta R_\varepsilon\\ &+ \Lambda_{\eta m}^T \Delta\mathcal{L}_{11}\Delta R_\varepsilon + \Delta R_\eta^T \Delta\mathcal{L}_{11}\Delta R_\varepsilon\,. \end{aligned} \tag{6.40}$$

The result follows from bounding the Frobenius norm of each of the terms in the right-hand side of (6.40). For this purpose, Lemma 2.16 is used and, in addition, the inequalities $\|\Delta R_\varepsilon(\lambda)\|_F < 1/\sqrt{2}$ and $\|\Delta R_\eta(\lambda)\|_F < 1/\sqrt{2}$ are used in those terms that are not linear in $\Delta\mathcal{L}_{11}(\lambda)$, $\Delta R_\varepsilon(\lambda)$, and $\Delta R_\eta(\lambda)$ for bounding them with linear terms. Let us show how to bound only one of the terms in (6.40), since the rest are bounded via similar procedures,

$$\begin{aligned}\|\Delta R_\eta^T(\lambda M_1 + M_0)\Delta R_\varepsilon\|_F &\le \sqrt{d}\,\|\Delta R_\eta\|_F\|(\lambda M_1 + M_0)\Delta R_\varepsilon\|_F\\ &\le \sqrt{2d}\,\|\Delta R_\eta\|_F\|\lambda M_1 + M_0\|_F\|\Delta R_\varepsilon\|_F\\ &\le \sqrt{d}\,\|\lambda M_1 + M_0\|_F\|\Delta R_\varepsilon\|_F\,.\end{aligned}$$

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Lemma 6.21 is the counterpart of Lemma 6.20 that is needed to deal with perturbations of degenerate block Kronecker pencils. The proof of Lemma 6.21 is omitted because it is similar to, and simpler than, the one of Lemma 6.20.

LEMMA 6.21.

(a) *Let us consider the matrix polynomials*

$$\begin{aligned}P(\lambda) &= (\lambda M_1 + M_0)(\Lambda_\varepsilon(\lambda)\otimes I_n),\\ P(\lambda) + \Delta P(\lambda) &= (\lambda M_1 + M_0 + \Delta\mathcal{L}_{11}(\lambda))\,(\Lambda_\varepsilon(\lambda)\otimes I_n + \Delta R_\varepsilon(\lambda))\,.\end{aligned}$$

If the matrix polynomial $\Delta R_\varepsilon(\lambda)$ *satisfies* $\|\Delta R_\varepsilon(\lambda)\|_F < 1/\sqrt{2}$, *then*

$$\|\Delta P(\lambda)\|_F \le 3\,\|\Delta\mathcal{L}_{11}(\lambda)\|_F + \sqrt{2}\,\|\lambda M_1 + M_0\|_F\,\|\Delta R_\varepsilon(\lambda)\|_F\,.$$

(b) *Let us consider the matrix polynomials*

$$\begin{aligned}P(\lambda) &= (\Lambda_\eta(\lambda)^T\otimes I_m)(\lambda M_1 + M_0),\\ P(\lambda) + \Delta P(\lambda) &= \left(\Lambda_\eta(\lambda)^T\otimes I_m + \Delta R_\eta(\lambda)^T\right)(\lambda M_1 + M_0 + \Delta\mathcal{L}_{11}(\lambda))\,.\end{aligned}$$

If the matrix polynomial $\Delta R_\eta(\lambda)$ *satisfies* $\|\Delta R_\eta(\lambda)\|_F < 1/\sqrt{2}$, *then*

$$\|\Delta P(\lambda)\|_F \le 3\,\|\Delta\mathcal{L}_{11}(\lambda)\|_F + \sqrt{2}\,\|\lambda M_1 + M_0\|_F\,\|\Delta R_\eta(\lambda)\|_F\,.$$

Next, we state and prove the main results of Section 6 concerning perturbations of the block Kronecker pencils defined and studied in Section 5. Recall that these pencils are strong linearizations of prescribed matrix polynomials enjoying constant shifting recovery properties for the minimal indices (see Theorems 5.2 and 5.4).

THEOREM 6.22. *Let* $P(\lambda) = \sum_{i=0}^d P_i\lambda^i \in \mathbb{F}[\lambda]^{m\times n}$ *and let* $\mathcal{L}(\lambda)$ *be an* $(\varepsilon, n, \eta, m)$-*block Kronecker pencil with* $d = \varepsilon + \eta + 1$ *such that* $P(\lambda) = (\Lambda_\eta(\lambda)^T \otimes I_m)(\lambda M_1 + M_0)(\Lambda_\varepsilon(\lambda)\otimes I_n)$, *where* $\lambda M_1 + M_0$ *is the* $(1,1)$-*block in the natural partition of* $\mathcal{L}(\lambda)$

and $\Lambda_k(\lambda)$ is the vector polynomial in (2.4). If $\Delta\mathcal{L}(\lambda)$ is any pencil with the same size as $\mathcal{L}(\lambda)$ and such that

$$\|\Delta\mathcal{L}(\lambda)\|_F < \left(\frac{\pi}{16}\right)^2 \frac{1}{d^{5/2}} \frac{1}{1+\|\lambda M_1 + M_0\|_F}, \tag{6.41}$$

then $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ is a strong linearization of a matrix polynomial $P(\lambda)+\Delta P(\lambda)$ with grade d and such that

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq 68\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1+\|\lambda M_1+M_0\|_F+\|\lambda M_1+M_0\|_F^2\right) \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}\,.$$

In addition, the right minimal indices of $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ are those of $P(\lambda)+\Delta P(\lambda)$ shifted by ε, and the left minimal indices of $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ are those of $P(\lambda)+\Delta P(\lambda)$ shifted by η.

Proof. Observe that the condition (6.41) implies that (6.22) holds. Therefore, we can apply Theorem 6.9 to $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ for proving that it is strictly equivalent to the pencil $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ in (6.6) and thus both pencils have the same complete eigenstructures. By combining (6.41), which implies $3d\|\Delta\mathcal{L}(\lambda)\|_F < 1/2$, with (6.24), we get the following bound

$$\begin{aligned}\max\{\|\Delta\widetilde{\mathcal{L}}_{21}(\lambda)\|_F, \|\Delta\widetilde{\mathcal{L}}_{12}(\lambda)\|_F\} &\leq \|\Delta\mathcal{L}(\lambda)\|_F\,(2+6\,d\,\|\lambda M_1+M_0\|_F) \\ &\leq 6\left(\frac{\pi}{16}\right)^2 \frac{1}{d^{3/2}} < \frac{\pi}{12}\,\frac{1}{d^{3/2}}\,,\end{aligned} \tag{6.42}$$

which allows us to apply Theorem 6.19 to $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$. Then, $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ is a strong block minimal bases pencil which, according to Theorem 3.3, is a strong linearization of the matrix polynomial $P(\lambda)+\Delta P(\lambda)$ in (6.7). Moreover, Theorem 3.7 guarantees that the right minimal indices of $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ are those of $P(\lambda)+\Delta P(\lambda)$ shifted by ε, and that the left minimal indices of $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$ are those of $P(\lambda)+\Delta P(\lambda)$ shifted by η. The same holds for $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$, since it is strictly equivalent to $\mathcal{L}(\lambda)+\Delta\widetilde{\mathcal{L}}(\lambda)$. It only remains to bound $\|\Delta P(\lambda)\|_F$. For this purpose, we combine Lemma 6.20 and the bound on $\max\{\|\Delta R_\varepsilon(\lambda)\|_F, \|\Delta R_\eta(\lambda)\|_F\}$ in Theorem 6.19. By using Theorem 6.19 and (6.42), the inequality

$$\max\{\|\Delta R_\varepsilon(\lambda)\|_F, \|\Delta R_\eta(\lambda)\|_F\} \leq 17\,d^2\,\|\Delta\mathcal{L}(\lambda)\|_F\,(1+\|\lambda M_1+M_0\|_F)\,,$$

is proved. If this inequality is introduced in the bound of Lemma 6.20, then we obtain

$$\|\Delta P(\lambda)\|_F \leq 68\,d^{5/2}\,\|\Delta\mathcal{L}(\lambda)\|_F\,(1+\|\lambda M_1+M_0\|_F+\|\lambda M_1+M_0\|_F^2)\,,$$

and the proof concludes. □

Next, we state and prove Theorem 6.23, which is the counterpart of Theorem 6.22 for degenerate block Kronecker pencils. For brevity, degenerate block Kronecker pencils are called either $(0,n,\eta,m)$-block Kronecker pencils when the second block row in (5.1) is missing or $(\varepsilon,n,0,m)$-block Kronecker pencils when the second block column in (5.1) is missing, i.e., they correspond to taking either $\varepsilon=0$ or $\eta=0$. We emphasize that the perturbation bound in Theorem 6.23 is smaller than the one in Theorem 6.22 because performing the strict equivalence (6.6) is not needed in the degenerate case. The most relevant difference in Theorem 6.23 with respect to the bound in Theorem 6.22 is that the term $\|\lambda M_1+M_0\|_F^2$ is not present, which is in agreement with the first order results obtained in [17] for Fiedler matrices (not pencils) of scalar monic polynomials.

THEOREM 6.23. *Let* $P(\lambda) = \sum_{i=0}^{d} P_i\lambda^i \in \mathbb{F}[\lambda]^{m\times n}$ *and let* $\mathcal{L}(\lambda)$ *be either a* $(0, n, \eta, m)$*-block Kronecker pencil with* $d = \eta + 1$ *such that* $P(\lambda) = (\Lambda_\eta(\lambda)^T \otimes I_m)(\lambda M_1 + M_0)$ *or an* $(\varepsilon, n, 0, m)$*-block Kronecker pencil with* $d = \varepsilon + 1$ *such that* $P(\lambda) = (\lambda M_1 + M_0)(\Lambda_\varepsilon(\lambda) \otimes I_n)$*, where* $\lambda M_1 + M_0$ *is the* $(1,1)$*-block in the natural partition of* $\mathcal{L}(\lambda)$ *and* $\Lambda_k(\lambda)$ *is the vector polynomial in* (2.4)*. If* $\Delta\mathcal{L}(\lambda)$ *is any pencil with the same size as* $\mathcal{L}(\lambda)$ *and such that*

$$\|\Delta\mathcal{L}(\lambda)\|_F < \frac{\pi}{12\, d^{3/2}} \ , \tag{6.43}$$

then $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ *is a strong linearization of a matrix polynomial* $P(\lambda) + \Delta P(\lambda)$ *with grade* d *and such that*

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq 4\, d\, \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F}\, (1 + \|\lambda M_1 + M_0\|_F)\, \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F} \,.$$

In addition, the right minimal indices of $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ *are those of* $P(\lambda) + \Delta P(\lambda)$ *shifted by* ε*, and the left minimal indices of* $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ *are those of* $P(\lambda) + \Delta P(\lambda)$ *shifted by* η*, where either* $\varepsilon = 0$ *or* $\eta = 0$*.*

Proof. We simply sketch the proof, since it follows the same ideas as the proof of Theorem 6.22. The reader should bear in mind Remark 3.5. In the degenerate case, we can apply Theorem 6.19 directly to $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda) = \mathcal{L}(\lambda) + \Delta\widetilde{\mathcal{L}}(\lambda)$. After that, it only remains to prove the bound on $\|\Delta P(\lambda)\|_F$. For this purpose, we combine Lemma 6.21 and the bound on $\max\{\|\Delta R_\varepsilon(\lambda)\|_F, \|\Delta R_\eta(\lambda)\|_F\}$ in Theorem 6.19 for obtaining

$$\|\Delta P(\lambda)\|_F \leq 4\, d\, \|\Delta\mathcal{L}(\lambda)\|_F\, (1 + \|\lambda M_1 + M_0\|_F) \,.$$

This ends the proof. □

Finally, we discuss when Theorems 6.22 and 6.23 guarantee backward stability of complete polynomial eigenproblems solved via the staircase or the QZ algorithms applied to a block Kronecker pencil. We restrict the discussion to nondegenerate block Kronecker pencils, since the obtained conclusions are also valid for the degenerate case. According to our discussion at the beginning of Section 6, to equation (6.3), and to Theorem 6.22, if

$$C_{P,\mathcal{L}} := 68\, d^{5/2}\, \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F}\, (1 + \|\lambda M_1 + M_0\|_F + \|\lambda M_1 + M_0\|_F^2) \tag{6.44}$$

is a moderate number, then the backward stability is guaranteed. From (6.44), it is clear that the following elementary lemma is useful for our discussion.

LEMMA 6.24. *Let* $P(\lambda) = \sum_{k=0}^{d} P_k\lambda^k \in \mathbb{F}[\lambda]^{m\times n}$ *and let* $\mathcal{L}(\lambda)$ *be an* $(\varepsilon, n, \eta, m)$*-block Kronecker pencil with* $d = \varepsilon + \eta + 1$ *such that* $P(\lambda) = (\Lambda_\eta(\lambda)^T \otimes I_m)(\lambda M_1 + M_0)(\Lambda_\varepsilon(\lambda) \otimes I_n)$*. Then:*

(a) $\dfrac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} = \sqrt{\left(\dfrac{\|\lambda M_1 + M_0\|_F}{\|P(\lambda)\|_F}\right)^2 + \dfrac{2(n\varepsilon + m\eta)}{\|P(\lambda)\|_F^2}} \geq \dfrac{1}{\sqrt{2\, d}}$.

(b) $\|\lambda M_1 + M_0\|_F \geq \|P(\lambda)\|_F/\sqrt{2\, d}$.

Proof. The equality in part (a) follows from (5.1) and Definition 2.15. The inequality follows from (5.4), which implies, for $k = 0, 1, \ldots, d$,

$$\begin{aligned}\|P_k\|_F &\leq \sum_{i+j=d+2-k} \|[M_1]_{ij}\|_F + \sum_{i+j=d+1-k} \|[M_0]_{ij}\|_F \\ &\leq \sqrt{2d}\sqrt{\sum_{i+j=d+2-k} \|[M_1]_{ij}\|_F^2 + \sum_{i+j=d+1-k} \|[M_0]_{ij}\|_F^2}\,.\end{aligned}$$

This in turn implies $\|P(\lambda)\|_F \leq \sqrt{2d}\,\|\lambda M_1 + M_0\|_F$, which is the result in part (b), and gives the inequality in part (a). ◻

From (6.44) and Lemma 6.24(a), we see that if $\|P(\lambda)\|_F \ll 1$, then $C_{P,\mathcal{L}}$ is huge, since $2(n\varepsilon + m\eta)/\|P(\lambda)\|_F^2$ is huge. Moreover, from (6.44) and Lemma 6.24(b), we see that if $\|P(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}}$ is also huge, since $\|\lambda M_1 + M_0\|_F$ is huge and $\|\mathcal{L}(\lambda)\|_F/\|P(\lambda)\|_F \geq 1/\sqrt{2\,d}$. Therefore, one should scale $P(\lambda)$ in advance in such a way that $\|P(\lambda)\|_F = 1$ to have a chance of $C_{P,\mathcal{L}}$ is moderate. But even in this case, $C_{P,\mathcal{L}}$ is large if $\|\lambda M_1 + M_0\|_F$ is large. This happens, for instance, in the last pencil in Example 5.5 if the arbitrary matrices A and/or B have huge norms.

As a consequence of the discussion above and Theorems 6.22 and 6.23, we can state the informal Corollary 6.25, which establishes sufficient conditions for the backward stability of the solution of complete polynomial eigenproblems via block Kronecker pencils (degenerate or not). For the sake of clarity and simplicity any nonessential numerical constant is omitted in Corollary 6.25.

Corollary 6.25. *Let* $P(\lambda) = \sum_{i=0}^{d} P_i\lambda^i \in \mathbb{F}[\lambda]^{m\times n}$ *with* $\|P(\lambda)\|_F = 1$. *Let* $\mathcal{L}(\lambda)$ *be an* $(\varepsilon, n, \eta, m)$*-block Kronecker pencil as in* (5.1) *with* $d = \varepsilon + \eta + 1$ *and such that* $P(\lambda) = (\Lambda_\eta(\lambda)^T \otimes I_m)(\lambda M_1 + M_0)(\Lambda_\varepsilon(\lambda) \otimes I_n)$. *Let* $\Delta\mathcal{L}(\lambda)$ *be any pencil with the same size as* $\mathcal{L}(\lambda)$ *and with* $\|\Delta\mathcal{L}(\lambda)\|_F$ *sufficiently small. If* $\|\lambda M_1 + M_0\|_F \approx \|P(\lambda)\|_F$, *then* $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ *is a strong linearization of a matrix polynomial* $P(\lambda) + \Delta P(\lambda)$ *with grade* d *and such that*

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3\,\sqrt{m+n}\,\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}\,. \tag{6.45}$$

In addition, the right minimal indices of $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ *are those of* $P(\lambda) + \Delta P(\lambda)$ *shifted by* ε, *and the left minimal indices of* $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ *are those of* $P(\lambda) + \Delta P(\lambda)$ *shifted by* η. *In particular, this corollary holds for all permuted Fiedler pencils, since for them* $\|\lambda M_1 + M_0\|_F = \|P(\lambda)\|_F$.

For degenerate block Kronecker pencils, the bound (6.45) can be improved as follows: the factor d^3 can be replaced by $d^{3/2}$, as a consequence of Theorem 6.23, and $\sqrt{m+n}$ by $\sqrt{m}$ if $\varepsilon = 0$ or by $\sqrt{n}$ if $\eta = 0$, as a consequence of Lemma 6.24(a).

Remark 6.26. We emphasize that Corollary 6.25 can be applied also to non-permuted Fiedler pencils, since the Frobenius norm is invariant under permutations and permutations preserve strong linearizations and minimal indices. Therefore, given a Fiedler pencil and a perturbation of it, we can permute both according to Theorem 4.5 and transform the corresponding perturbation problem into the problem we have solved in this section.

Remark 6.27. Consider that each block-entry of the $(1,1)$-block $\lambda M_1 + M_0$ of the block Kronecker pencil $\mathcal{L}(\lambda)$ in Theorems 6.22 and 6.23, and in Corollary 6.25, is a linear combination of the coefficients $P_d, \ldots, P_0$ of $P(\lambda)$ and some arbitrary matrices. Then, the pencil $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ in Theorems 6.22 and 6.23, and in Corollary 6.25, is strictly equivalent to a block Kronecker pencil $\widehat{\mathcal{L}}(\lambda)$ *with exactly the same structure* *as* $\mathcal{L}(\lambda)$ but for the polynomial $P(\lambda) + \Delta P(\lambda)$ instead of $P(\lambda)$. This means that each block-entry of the $(1,1)$-block of the block Kronecker pencil $\widehat{\mathcal{L}}(\lambda)$ is the same linear combination of the coefficients $P_d + \Delta P_d, \ldots, P_0 + \Delta P_0$ of $P(\lambda) + \Delta P(\lambda)$ and of the same arbitrary matrices as the corresponding block entry of $\mathcal{L}(\lambda)$ is for the coefficients $P_d, \ldots, P_0$ and the same arbitrary matrices. In particular, if $\mathcal{L}(\lambda)$ is a given permuted Fiedler pencil of $P(\lambda)$, then $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is strictly equivalent to the same permuted Fiedler pencil of $P(\lambda) + \Delta P(\lambda)$. This result follows from the fact that Theorem 5.4 guarantees that $\widehat{\mathcal{L}}(\lambda)$ has the same complete eigenstructure as $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$, and so both pencils must be strictly equivalent [30, Chapter XII]. This remark by itself does not prove that the strict equivalence transformations connecting $\widehat{\mathcal{L}}(\lambda)$ and $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ are small perturbations of identity matrices, despite the fact that $\widehat{\mathcal{L}}(\lambda)$ and $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ are indeed very close each other. However, it is clear that this remark opens the possibility of proving directly that $\widehat{\mathcal{L}}(\lambda)$ and $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ are strictly equivalent via nonsingular matrices that are very close to the identity, as it was done in [64] for the Frobenius companion linearizations. In fact, we have already successfully followed this approach. Unfortunately, at present, such alternative analysis—as the one in [64]—is only valid up to first order in the perturbation, yields vague big-O perturbation bounds, and the inclusion of finite perturbations seems messy.

7. Conclusions and future work.

The new family of strong block minimal bases pencils has been introduced and analyzed. We have proven in a simple and general way that these pencils are always strong linearizations of matrix polynomials and that their minimal indices and those of the polynomials satisfy constant uniform shifting relationships. These proofs are based on the properties of dual minimal bases—classical tools in multivariable linear system theory that have been used recently in different matrix polynomial eigenproblems. As an immediate corollary of this simple and general theory, we obtain automatically that the same results hold for the subfamily of block Kronecker pencils, which form a wide subclass of block minimal bases pencils easily constructible from the coefficients of a given but general matrix polynomial (general in the sense that it may be square or rectangular, regular or singular). It is also proven that block Kronecker pencils contain—modulo permutations—all Fiedler pencils of a matrix polynomial. Therefore, a simplified description and theory for this relevant class of strong linearizations is obtained as a by-product. A rigorous global backward error analysis of complete polynomial eigenproblems solved via block Kronecker pencils is carefully developed. The backward error bounds delivered by this analysis enjoy a number of novel features not present so far in the literature and solve, in particular, the open problem of proving that the use of any Fiedler pencil yields perfect structured polynomial backward stability in the solution of complete polynomial eigenproblems. This error analysis is based on the key idea that perturbations of block Kronecker pencils lead after some manipulations to other strong block minimal bases pencils with similar properties.

We are convinced that the results in this work will motivate further research in the area, will help to organize and clarify many of the results that have been published in the last few years on linearizations of matrix polynomials, and will allow the development of rigorous global backward error analyses for other classes of linearizations. For instance, motivated by the results in this paper some other particular subfamilies of strong block minimal bases pencils that may have important applications have been already considered very recently [44, 57]. In addition, we believe that generalized Fiedler pencils [2, 6, 12], Fiedler pencils with repetition [5, 8, 66], and generalized Fiedler pencils with repetition [7] may be transformed through proper permutations into either block Kronecker pencils, strong block minimal bases pencils, or other closely related types of pencils. This would probably enable a considerable simplification of the complicated description and messy theory of these classes of linearizations and the development of global backward error analyses for the solution of complete polynomial eigenproblems via these linearizations. We also believe that the new error analysis can be extended to the strong block minimal bases linearizations introduced recently in [44, 57]. Finally, it also seems possible to generalize the ideas presented in this work to the context of ℓ-ifications of matrix polynomials [14, 19].

Appendix A. Proof of Theorem 4.5: permuted Fiedler pencils are strong block minimal bases pencils. We use the same notation as in the statement of Theorem 4.5. Moreover, the bijection $\sigma : \{0, 1, \ldots, d-1\} \to \{1, 2, \ldots, d\}$ is also described as $\sigma = (\sigma(0), \sigma(1), \ldots, \sigma(d-1))$. This allows us to define the bijections $\sigma_k := (\sigma(0), \sigma(1), \ldots, \sigma(k+1))$, for $k = 0, 1, \ldots, d-2$. Notice that the bijection σ_k has a consecution (resp. inversion) at i if and only if σ has a consecution (resp. inversion) at i. In addition, let us define $Z_k(\lambda) := \sum_{j=0}^{k+2} P_j \lambda^j$, for $k = 0, 1, \ldots, d-2$, and let $F_{\sigma_k}(\lambda)$ be the Fiedler companion pencil of $Z_k(\lambda)$ associated with the bijection σ_k. Observe that $\sigma_{d-2} = \sigma$, $Z_{d-2}(\lambda) = P(\lambda)$, and $F_{\sigma_{d-2}}(\lambda) = F_\sigma(\lambda)$. Then, we claim that, for $k = 0, 1, \ldots, d-2$, there exist permutation matrices S_k and R_k such that

$$\begin{bmatrix} I_m & 0 \\ 0 & S_k \end{bmatrix} F_{\sigma_k}(\lambda) \begin{bmatrix} I_n & 0 \\ 0 & R_k \end{bmatrix} = \left[\begin{array}{c|c} \lambda M_1^{(k)} + M_0^{(k)} & L_{\mathfrak{c}(\sigma_k)}(\lambda)^T \otimes I_m \\ \hline L_{\mathfrak{i}(\sigma_k)}(\lambda) \otimes I_n & 0 \end{array}\right], \tag{A.1}$$

where $L_k(\lambda)$ is defined in (2.3), and where $\lambda M_1^{(k)} + M_0^{(k)}$ follows a staircase pattern for $\lambda P_{k+2} + P_{k+1}, P_k, P_{k-1}, \ldots, P_0$. Notice that this result proves the first part of Theorem 4.5 just by taking $k = d-2$.

The proof of (A.1) proceeds by induction on k. For $k = 0$ the result is obviously true since the only possible Fiedler companion pencils are either

$$F_{\sigma_0}(\lambda) = \begin{bmatrix} \lambda P_2 + P_1 & -I_m \\ P_0 & \lambda I_m \end{bmatrix} \quad \text{or} \quad F_{\sigma_0}(\lambda) = \begin{bmatrix} \lambda P_2 + P_1 & P_0 \\ -I_n & \lambda I_n \end{bmatrix},$$

if σ has either a consecution or an inversion at $k = 0$.

Let us now assume that (A.1) holds for k, with $k < d-2$, and let us prove it for $k+1$. To this end, let W_k be the matrix generated by `Algorithm 2` in Section 4.1 at step k. This algorithm iteratively induces in W_k a partition into $(k+2) \times (k+2)$ blocks, which we denote as $W_k(i, j)$ for $i, j = 1, 2, \ldots, k+2$. In terms of this partition the Fiedler companion pencil $F_{\sigma_k}(\lambda)$ is equal to

$$F_{\sigma_k}(\lambda) = \begin{bmatrix} \lambda P_{k+2} - W_k(1,1) & -W_k(1, 2:k+2) \\ -W_k(2:k+2, 1) & \lambda I_{\mathfrak{c}(\sigma_k)m + \mathfrak{i}(\sigma_k)n} - W_k(2:k+2, 2:k+2) \end{bmatrix},$$

with $W_k(1,1) = -P_{k+1}$. This expression for $F_{\sigma_k}(\lambda)$ and (A.1) allows us to get the following equalities

$$\begin{aligned} &\begin{bmatrix} I_m & 0 \\ 0 & S_k \end{bmatrix} F_{\sigma_k}(\lambda) \begin{bmatrix} I_n & 0 \\ 0 & R_k \end{bmatrix} \\ &= \begin{bmatrix} \lambda P_{k+2} - W_k(1,1) & -W_k(1, 2:k+2) R_k \\ -S_k W_k(2:k+2, 1) & S_k(\lambda I_{\mathfrak{c}(\sigma_k)m + \mathfrak{i}(\sigma_k)n} - W_k(2:k+2, 2:k+2)) R_k \end{bmatrix} \quad \text{(A.2)} \\ &= \left[\begin{array}{c|c} \lambda M_1^{(k)} + M_0^{(k)} & L_{\mathfrak{c}(\sigma_k)}(\lambda)^T \otimes I_m \\ \hline L_{\mathfrak{i}(\sigma_k)}(\lambda) \otimes I_n & 0 \end{array}\right], \quad \text{(A.3)} \end{aligned}$$

where recall that $\lambda M_1^{(k)} + M_0^{(k)}$ follows a staircase pattern for $\lambda P_{k+2} + P_{k+1}, P_k, P_{k-1}, \ldots, P_0$. Note that this implies that $M_0^{(k)}$ follows a staircase pattern for $P_{k+1}, P_k, P_{k-1}, \ldots, P_0$, and that $M_1^{(k)}$ has P_{k+2} as $(1,1)$-block while the rest of its entries are zero.

We next prove that (A.1) holds with k replaced by $k+1$, i.e., that our claim is true for $k+1$. We will assume that the bijection σ has a consecution at $k+1$. The proof when σ has an inversion at $k+1$ is similar and, so, it is omitted. In this situation, `Algorithm 2` in Section 4.1 implies that the Fiedler pencil $F_{\sigma_{k+1}}(\lambda)$ is equal to

$$F_{\sigma_{k+1}}(\lambda) = \begin{bmatrix} \lambda P_{k+3} + P_{k+2} & -I_m & 0 \\ -W_k(1,1) & \lambda I_m & -W_k(1, 2:k+2) \\ -W_k(2:k+2, 1) & 0 & \lambda I_{\mathfrak{c}(\sigma_k)m+\mathfrak{i}(\sigma_k)n} - W_k(2:k+2, 2:k+2) \end{bmatrix}.$$

Then, pre and post multiplying $F_{\sigma_{k+1}}(\lambda)$ by $\operatorname{diag}(I_m, I_m, S_k)$ and $\operatorname{diag}(I_n, I_m, R_k)$, respectively, we get

$$\begin{bmatrix} \lambda P_{k+3} + P_{k+2} & -I_m & 0 \\ -W_k(1,1) & \lambda I_m & -W_k(1, 2:k+2)R_k \\ -S_k W_k(2:k+2, 1) & 0 & S_k(\lambda I_{\mathfrak{c}(\sigma_k)m+\mathfrak{i}(\sigma_k)n} - W_k(2:k+2, 2:k+2))R_k \end{bmatrix},$$

which is strictly equivalent via column permutations, that do not affect to the first block column, to

$$\begin{bmatrix} \lambda P_{k+3} + P_{k+2} & 0 & -I_m \\ -W_k(1,1) & -W_k(1, 2:k+2)R_k & \lambda I_m \\ -S_k W_k(2:k+2, 1) & S_k(\lambda I_{\mathfrak{c}(\sigma_k)m+\mathfrak{i}(\sigma_k)n} - W_k(2:k+2, 2:k+2))R_k & 0 \end{bmatrix}. \tag{A.4}$$

The equality of (A.2) and (A.3) implies that the pencil in (A.4) is equal to

$$\left[\begin{array}{c|c|c} [\lambda P_{k+3} + P_{k+2} \ \ 0] & 0 & -I_m \\ \hline M_0^{(k)} & L_{\mathfrak{c}(\sigma_k)}(\lambda)^T \otimes I_m & \begin{bmatrix} \lambda I_m \\ 0 \end{bmatrix} \\ \hline L_{\mathfrak{i}(\sigma_k)}(\lambda) \otimes I_n & 0 & 0 \end{array}\right]. \tag{A.5}$$

Since $\mathfrak{c}(\sigma_{k+1}) = \mathfrak{c}(\sigma_k) + 1$, there exists a column permutation $\widetilde{\Pi}_k$ of adequate size such that

$$L_{\mathfrak{c}(\sigma_{k+1})}(\lambda)^T \otimes I_m = \left[\begin{array}{c|c} 0 & -I_m \\ \hline L_{\mathfrak{c}(\sigma_k)}(\lambda)^T \otimes I_m & \begin{bmatrix} \lambda I_m \\ 0 \end{bmatrix} \end{array}\right] \widetilde{\Pi}_k. \tag{A.6}$$

Moreover, observe that the pencil

$$\lambda M_1^{(k+1)} + M_0^{(k+1)} := \left[\begin{array}{c} [\lambda P_{k+3} + P_{k+2} \ \ 0] \\ \hline M_0^{(k)} \end{array}\right] \tag{A.7}$$

follows a staircase pattern for $\lambda P_{k+3} + P_{k+2}, P_{k+1}, P_k, \ldots, P_0$. Taking into account $\mathfrak{i}(\sigma_{k+1}) = \mathfrak{i}(\sigma_k)$, (A.6), (A.7), we get from (A.5) and the permutations leading to (A.4) that there exist permutation matrices S_{k+1} and R_{k+1} such that

$$\begin{bmatrix} I_m & 0 \\ 0 & S_{k+1} \end{bmatrix} F_{\sigma_{k+1}}(\lambda) \begin{bmatrix} I_n & 0 \\ 0 & R_{k+1} \end{bmatrix} = \left[\begin{array}{c|c} \lambda M_1^{(k+1)} + M_0^{(k+1)} & L_{\mathfrak{c}(\sigma_{k+1})}(\lambda)^T \otimes I_m \\ \hline L_{\mathfrak{i}(\sigma_{k+1})}(\lambda) \otimes I_n & 0 \end{array}\right].$$

This concludes the proof of the first part of Theorem 4.5.

The proof of the "converse part" is just sketched very briefly. It is based on the following remark on the structure of $\lambda M_1 + M_0$ in (4.5), which follows from (A.7): if σ has a consecution at $k+1$ and P_{k+2} is in the (s,t)-block-entry of $\lambda M_1 + M_0$, then P_{k+1} is in the $(s+1,t)$-block-entry of $\lambda M_1 + M_0$. Analogously, if σ has an inversion at $k+1$ and P_{k+2} is in the (s,t)-block-entry of $\lambda M_1 + M_0$, then P_{k+1} is in the $(s,t+1)$-block-entry of $\lambda M_1 + M_0$. Therefore, given any pencil $\lambda \widetilde{M}_1 + \widetilde{M}_0$ of size $(\mathfrak{c}(\sigma)+1)m \times (\mathfrak{i}(\sigma)+1)n$ which follows a staircase pattern for $\lambda P_d + P_{d-1}, P_{d-2}, \ldots, P_1, P_0$, it is possible to construct a bijection σ with $\mathfrak{c}(\sigma)$ consecutions and $\mathfrak{i}(\sigma)$ inversions, and such that $F_\sigma(\lambda)$ satisfies (4.5) with $\lambda \widetilde{M}_1 + \widetilde{M}_0$ in the $(1,1)$-block, by selecting a proper sequence of consecutions and inversions of σ.

Appendix B. Proof of Lemma 6.4. In this appendix, we assume that $\varepsilon \neq 0$ and $\eta \neq 0$ according to Remark 6.1. We first reduce in Lemma B.1 the problem of computing $\sigma_{\min}(T)$ to the problem of computing the minimum singular value of a matrix of size $2\varepsilon\eta \times (2\varepsilon\eta + \varepsilon + \eta)$, which is much smaller than the size of T.

LEMMA B.1. *Let T be the matrix defined in* (6.13) *and*

$$\widehat{T} := \left[\begin{array}{c|c} I_\varepsilon \otimes E_\eta & E_\varepsilon \otimes I_\eta \\ \hline I_\varepsilon \otimes F_\eta & F_\varepsilon \otimes I_\eta \end{array}\right], \tag{B.1}$$

where $\lambda F_k - E_k := L_k(\lambda)$ is the pencil in (2.3)*. Then $\sigma_{\min}(T) = \sigma_{\min}(\widehat{T})$.*

Proof. Since the Kronecker product is associative [40, Chapter 4], we may write the matrix T as

$$\begin{aligned} T &= \left[\begin{array}{c|c} E_\eta \otimes I_m \otimes I_\varepsilon \otimes I_n & I_\eta \otimes I_m \otimes E_\varepsilon \otimes I_n \\ \hline F_\eta \otimes I_m \otimes I_\varepsilon \otimes I_n & I_\eta \otimes I_m \otimes F_\varepsilon \otimes I_n \end{array}\right] \\ &= \left[\begin{array}{c|c} (E_\eta \otimes I_m) \otimes I_\varepsilon & I_{\eta m} \otimes E_\varepsilon \\ \hline (F_\eta \otimes I_m) \otimes I_\varepsilon & I_{\eta m} \otimes F_\varepsilon \end{array}\right] \otimes I_n =: \widetilde{T} \otimes I_n. \end{aligned} \tag{B.2}$$

Thus, $\sigma_{\min}(T) = \sigma_{\min}(\widetilde{T})$ by [40, Theorem 4.2.15]. Let us perform a perfect shuffle on the matrix $\widetilde{T}$ on the right of (B.2) to swap the order of the Kronecker products of its blocks. Following Van Loan [65], there exist permutation matrices S, R_1^T and R_2^T of sizes $\varepsilon\eta m \times \varepsilon\eta m$, $\varepsilon(\eta+1)m \times \varepsilon(\eta+1)m$ and $(\varepsilon+1)\eta m \times (\varepsilon+1)\eta m$, respectively, such that

$$\begin{aligned} &\left[\begin{array}{c|c} S & \\ \hline & S \end{array}\right] \left[\begin{array}{c|c} (E_\eta \otimes I_m) \otimes I_\varepsilon & I_{\eta m} \otimes E_\varepsilon \\ \hline (F_\eta \otimes I_m) \otimes I_\varepsilon & I_{\eta m} \otimes F_\varepsilon \end{array}\right] \left[\begin{array}{c|c} R_1^T & \\ \hline & R_2^T \end{array}\right] \\ &= \left[\begin{array}{c|c} I_\varepsilon \otimes (E_\eta \otimes I_m) & E_\varepsilon \otimes I_{\eta m} \\ \hline I_\varepsilon \otimes (F_\eta \otimes I_m) & F_\varepsilon \otimes I_{\eta m} \end{array}\right] = \left[\begin{array}{c|c} I_\varepsilon \otimes E_\eta & E_\varepsilon \otimes I_\eta \\ \hline I_\varepsilon \otimes F_\eta & F_\varepsilon \otimes I_\eta \end{array}\right] \otimes I_m = \widehat{T} \otimes I_m. \end{aligned}$$

Using again [40, Theorem 4.2.15], we get $\sigma_{\min}(T) = \sigma_{\min}(\widetilde{T}) = \sigma_{\min}(\widehat{T})$. ☐

Lemma B.2 reduces the problem of computing the minimum singular value of $\widehat{T}$ in (B.1) to compute the largest singular value of a matrix smaller than $\widehat{T}$, essentially with half its size, and with a simpler structure.

LEMMA B.2. *Let $\widehat{T}$ be the matrix in* (B.1)*. Then*

$$\sigma_{\min}(\widehat{T}) = \sqrt{2 - \sigma_{\max}(W_{\varepsilon,\eta})}\,, \tag{B.3}$$

where $W_{\varepsilon,\eta} = I_\varepsilon \otimes E_\eta F_\eta^T + E_\varepsilon F_\varepsilon^T \otimes I_\eta \in \mathbb{R}^{\varepsilon\eta \times \varepsilon\eta}$ and $\sigma_{\max}(W_{\varepsilon,\eta})$ denotes its maximum singular value.

Proof. The singular values of $\widehat{T}$ are the square roots of the eigenvalues of

$$\widehat{T}\widehat{T}^T = \begin{bmatrix} 2I_{\varepsilon\eta} & W_{\varepsilon,\eta} \\ W_{\varepsilon,\eta}^T & 2I_{\varepsilon\eta} \end{bmatrix} = 2\, I_{2\varepsilon\eta} + \begin{bmatrix} 0 & W_{\varepsilon,\eta} \\ W_{\varepsilon,\eta}^T & 0 \end{bmatrix},$$

where $W_{\varepsilon,\eta} = I_\varepsilon \otimes E_\eta F_\eta^T + E_\varepsilon F_\varepsilon^T \otimes I_\eta$. It is well known (see, for instance, [60, Theorem I.4.2]) that the eigenvalues of $[0\,,\, W_{\varepsilon,\eta}\,;\, W_{\varepsilon,\eta}^T\,,\, 0]$ are $\pm\sigma_1(W_{\varepsilon,\eta}), \dots, \pm\sigma_{\varepsilon\eta}(W_{\varepsilon,\eta})$, where $\sigma_1(W_{\varepsilon,\eta}) \geq \cdots \geq \sigma_{\varepsilon\eta}(W_{\varepsilon,\eta})$ are the singular values of $W_{\varepsilon,\eta}$. Therefore, the eigenvalues of $\widehat{T}\widehat{T}^T$ are $2 \pm \sigma_1(W_{\varepsilon,\eta}), \dots, 2 \pm \sigma_{\varepsilon\eta}(W_{\varepsilon,\eta})$, which implies the result. Observe that $\widehat{T}\widehat{T}^T$ is positive semidefinite and, thus, its eigenvalues are nonnegative. ☐

The advantage of the matrix $W_{\varepsilon,\eta}$ is that has a bidiagonal block Toeplitz structure with very simple blocks. This comes from the fact that

$$E_k F_k^T = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} =: J_k \in \mathbb{R}^{k\times k} \quad (\text{with } J_1 := 0_{1\times 1}),$$

which implies

$$W_{\varepsilon,\eta} = I_\varepsilon \otimes E_\eta F_\eta^T + E_\varepsilon F_\varepsilon^T \otimes I_\eta = \underbrace{\begin{bmatrix} J_\eta & & & & \\ I_\eta & J_\eta & & & \\ & I_\eta & J_\eta & & \\ & & \ddots & \ddots & \\ & & & I_\eta & J_\eta \end{bmatrix}}_{\varepsilon \text{ block columns}} \left.\vphantom{\begin{bmatrix} J \\ I \\ I \\ \ddots \\ I \end{bmatrix}}\right\} \varepsilon \text{ block rows}\,. \tag{B.4}$$

This structure will allow us to compute explicitly the largest singular value of $W_{\varepsilon,\eta}$. Without loss of generality, we assume that $\varepsilon \geq \eta$, since, otherwise, $W_{\varepsilon,\eta}$ is transformed into $W_{\eta,\varepsilon}$ with a perfect shuffle permutation , i.e., by interchanging the order of the Kronecker products in the summands of $W_{\varepsilon,\eta}$. In this situation, note that if $\eta = 1$, then $W_{1,1} = 0_{1\times 1}$ and $W_{\varepsilon,1} = J_\varepsilon$ for $\varepsilon > \eta = 1$. Therefore,

$$\sigma_{\max}(W_{1,1}) = 0 \quad \text{and} \quad \sigma_{\max}(W_{\varepsilon,1}) = 1, \quad \text{if } \varepsilon > \eta = 1. \tag{B.5}$$

If $\eta > 1$, then $\sigma_{\max}(W_{\varepsilon,\eta})$ can be computed with the help of Lemma B.3, where we show that $W_{\varepsilon,\eta}$ is permutationally equivalent to a direct sum involving the following two types of matrices

$$M_k := \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \in \mathbb{R}^{k\times k} \quad \text{and} \quad G_k := \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{(k+1)\times k}. \tag{B.6}$$

LEMMA B.3. *Let $W_{\varepsilon,\eta}$ be the matrix in* (B.4)*, let M_k and G_k be the matrices in* (B.6)*, and assume that $\varepsilon \geq \eta$. Then, there exist two permutation matrices P_1 and P_2*

such that

$$P_1 W_{\varepsilon,\eta} P_2 = \underbrace{(M_\eta \oplus M_\eta \oplus \cdots \oplus M_\eta)}_{\varepsilon-\eta \textit{ times}} \oplus (G_{\eta-1} \oplus G_{\eta-1}^T) \oplus \cdots \oplus (G_1 \oplus G_1^T) \oplus 0_{1\times 1}. \quad \text{(B.7)}$$

Proof. If $\eta = 1$, then the result follows trivially from the discussion in the two lines above (B.5) with the convention $G_0 \oplus G_0^T := 0_{1\times 1}$. Therefore, we assume in the rest of the proof that $\eta > 1$. Observe that the $0_{1\times 1}$ block is a consequence of the fact that the first row and the last column of $W_{\varepsilon,\eta}$ are both zero. Thus, permuting the first row to the last row position produces the $0_{1\times 1}$ block. A complete formal proof is rather technical, but the key ideas are easy to follow. Therefore, we restrict ourselves to describe such ideas. In order to do this in a concise way we use in this proof the following notation: the column $2^{(3)}$ of $W_{\varepsilon,\eta}$ stands for the 2nd column in the 3rd block column of $W_{\varepsilon,\eta}$. An analogous notation is used for rows and both notations are combined with the standard MATLAB's notation for submatrices.

Observe that G_1 is the submatrix of nonzero rows of $W_{\varepsilon,\eta}(:, 1^{(1)})$, which correspond to rows of $W_{\varepsilon,\eta}$ with the remaining entries equal to zero and, thus, this submatrix can be transformed via permutations into an explicit direct summand. G_2 is the submatrix of nonzero rows of $W_{\varepsilon,\eta}(:, [2^{(1)}, 1^{(2)}])$, which correspond to rows of $W_{\varepsilon,\eta}$ with the remaining entries equal to zero. G_3 is the submatrix of nonzero rows of $W_{\varepsilon,\eta}(:, [3^{(1)}, 2^{(2)}, 1^{(3)}])$, which correspond to rows of $W_{\varepsilon,\eta}$ with the remaining entries equal to zero. This process continues until we find that $G_{\eta-1}$ is the submatrix of nonzero rows of $W_{\varepsilon,\eta}(:, [(\eta-1)^{(1)}, (\eta-2)^{(2)}, \ldots, 1^{(\eta-1)}])$, which correspond to rows of $W_{\varepsilon,\eta}$ with the remaining entries equal to zero. Note that we have started each of the previous submatrices with the 1st, 2nd, ..., $(\eta-1)$th columns of the first block column of $W_{\varepsilon,\eta}$. Since $W_{\varepsilon,\eta}$ is symmetric with respect to the main antidiagonal, $G_1^T, G_2^T, \ldots, G_{\eta-1}^T$ are obtained starting from the bottom with the ηth, $(\eta-1)$th , ..., 2nd rows of the last block row of $W_{\varepsilon,\eta}$. More precisely, G_1^T comes from $W_{\varepsilon,\eta}(\eta^{(\varepsilon)}, :)$, G_2^T comes from $W_{\varepsilon,\eta}([(\eta^{(\varepsilon-1)}, (\eta-1)^{(\varepsilon)}], :)$, and so on until one gets $G_{\eta-1}^T$, which comes from $W_{\varepsilon,\eta}([\eta^{(\varepsilon-\eta+2)}, \ldots, 3^{(\varepsilon-1)}, 2^{(\varepsilon)}], :)$. In this way, the direct summands $0_{1\times 1}, G_1, G_1^T, G_2, G_2^T, \ldots, G_{\eta-1}, G_{\eta-1}^T$ have been identified for any $\varepsilon \geq \eta$. This leads directly to the proof in the case $\varepsilon = \eta$, because in this case the size and the number of entries equal to 1 of $(G_{\eta-1} \oplus G_{\eta-1}^T) \oplus \cdots \oplus (G_1 \oplus G_1^T) \oplus 0_{1\times 1}$ are equal to those of $W_{\eta,\eta}$.

Next, we prove the case $\varepsilon = \eta + 1$. The proof we propose is based on the fact that $W_{\eta,\eta}$ is the submatrix of $W_{\eta+1,\eta}$ lying in its $2, \ldots, \eta+1$ block rows and columns. This implies that the submatrices of $W_{\eta,\eta}$ are submatrices of $W_{\eta+1,\eta}$. Observe that the top rows of the submatrices of $W_{\eta,\eta}$ corresponding to $G_1, G_2, \ldots, G_{\eta-1}$, that is, the submatrices formed by the nonzero rows of $W_{\eta,\eta}(:, 1^{(1)}), W_{\eta,\eta}(:, [2^{(1)}, 1^{(2)}]), \ldots, W_{\eta,\eta}(:, [(\eta-1)^{(1)}, (\eta-2)^{(2)}, \ldots, 1^{(\eta-1)}])$, when viewed as submatrices of $W_{\eta+1,\eta}$ correspond to rows of $W_{\eta+1,\eta}$ that have nonzero entries to the left of such submatrices and, thus, these submatrices cannot be transformed via permutations into block summands. In fact, $W_{\eta+1,\eta}(:, 2^{(1)})$ combined with the G_1 of $W_{\eta,\eta}$ (not permuted) leads to the G_2 in $W_{\eta+1,\eta}$, $W_{\eta+1,\eta}(:, 3^{(1)})$ combined with the G_2 of $W_{\eta,\eta}$ leads to the G_3 in $W_{\eta+1,\eta}$, and so on until we get that $W_{\eta+1,\eta}(:, (\eta-1)^{(1)})$ combined with the $G_{\eta-2}$ of $W_{\eta,\eta}$ leads to the $G_{\eta-1}$ in $W_{\eta+1,\eta}$. The column $W_{\eta+1,\eta}(:, \eta^{(1)})$ is different that the previous ones of $W_{\eta+1,\eta}$, since it has exactly one entry equal to 1, while the previous ones have two entries equal to 1. Therefore, $W_{\eta+1,\eta}(:, \eta^{(1)})$ combined with the $G_{\eta-1}$ of $W_{\eta,\eta}$ leads to an M_η block in $W_{\eta+1,\eta}$. The missing G_1 direct summand of $W_{\eta+1,\eta}$ comes from the nonzero rows of $W_{\eta+1,\eta}(:,1^{(1)})$. To summarize, we have identified the direct sum $M_\eta \oplus (G_{\eta-1} \oplus G_{\eta-1}^T) \oplus \cdots \oplus (G_1 \oplus G_1^T) \oplus 0_{1\times 1}$ inside $W_{\eta+1,\eta}$. The proof is completed by noting that the sizes and the numbers of 1s of these two matrices are equal.

The last part of the proof is an induction argument that follows exactly the steps explained in the previous paragraph. Let us assume that the result is true for any $W_{\varepsilon,\eta}$ with $\varepsilon > \eta$ and let us prove it for $W_{\varepsilon+1,\eta}$. As in the previous paragraph $W_{\varepsilon,\eta}$ is viewed as the submatrix of $W_{\varepsilon+1,\eta}$ lying in its $2,\ldots,\varepsilon+1$ block rows and columns. Also as in the previous paragraph, $W_{\varepsilon+1,\eta}(:,2^{(1)})$ combined with the G_1 of $W_{\varepsilon,\eta}$ leads to the G_2 in $W_{\varepsilon+1,\eta}$, $W_{\varepsilon+1,\eta}(:,3^{(1)})$ combined with the G_2 of $W_{\varepsilon,\eta}$ leads to the G_3 in $W_{\varepsilon+1,\eta}$, ... ,$W_{\varepsilon+1,\eta}(:,(\eta-1)^{(1)})$ combined with the $G_{\eta-2}$ of $W_{\varepsilon,\eta}$ leads to the $G_{\eta-1}$ in $W_{\varepsilon+1,\eta}$, and $W_{\varepsilon+1,\eta}(:,\eta^{(1)})$ combined with the $G_{\eta-1}$ of $W_{\varepsilon,\eta}$ leads to an M_η block in $W_{\varepsilon+1,\eta}$. The G_1 direct summand of $W_{\varepsilon+1,\eta}$ comes from the nonzero rows of $W_{\varepsilon+1,\eta}(:,1^{(1)})$. The rest of submatrices of $W_{\varepsilon,\eta}$ producing direct summands lie in its $2,...,\varepsilon$ block rows, i.e., in the $3,\ldots,\varepsilon+1$ block rows and $2,\ldots,\varepsilon+1$ block columns of $W_{\varepsilon+1,\eta}$, and, thus, do not interact with other nonzero entries of $W_{\varepsilon+1,\eta}$, which implies that they remain as direct summands of $W_{\varepsilon+1,\eta}$, To summarize, we have identified the direct sum

$$\underbrace{(M_\eta \oplus M_\eta \oplus \cdots \oplus M_\eta)}_{\varepsilon+1-\eta \text{ times}} \oplus (G_{\eta-1} \oplus G_{\eta-1}^T) \oplus \cdots \oplus (G_1 \oplus G_1^T) \oplus 0_{1\times 1}$$

inside $W_{\varepsilon+1,\eta}$. The proof concludes by noting that the sizes and the numbers of 1s of this direct sum and $W_{\varepsilon+1,\eta}$ are equal. □

Now, we are in the position of computing $\sigma_{\max}(W_{\varepsilon,\eta})$.

PROPOSITION B.4. *Let* $W_{\varepsilon,\eta}$ *be the matrix in* (B.4). *Then*

$$\sigma_{\max}(W_{\varepsilon,\eta}) = \begin{cases} 2\cos\frac{\pi}{2\min\{\varepsilon,\eta\}+1}, & \textit{if } \varepsilon \neq \eta, \\ 2\cos\frac{\pi}{2\eta}, & \textit{if } \varepsilon = \eta. \end{cases} \tag{B.8}$$

Proof. As explained after the equation (B.4), we may assume without loss of generality that $\varepsilon \geq \eta$. In addition, if $\eta = 1$, then the result follows immediately from (B.5). Thus, the rest of the proof assumes $\varepsilon \geq \eta > 1$.

Let us consider first the case $\varepsilon = \eta > 1$. Lemma B.3 implies that $\sigma_{\max}(W_{\eta,\eta}) = \max\{\sigma_{\max}(G_{\eta-1}),...,\sigma_{\max}(G_2),\sigma_{\max}(G_1)\}$. In addition, since G_k is a submatrix of G_{k+1}, we have that $\sigma_{\max}(G_{\eta-1}) \geq \cdots \geq \sigma_{\max}(G_2) \geq \sigma_{\max}(G_1)$ [40, Corollary 3.1.3]. Therefore, $\sigma_{\max}(W_{\eta,\eta}) = \sigma_{\max}(G_{\eta-1})$. The singular values of $G_{\eta-1}$ are the square roots of the eigenvalues of

$$G_{\eta-1}^T G_{\eta-1} = \begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ & & & 1 & 2 \end{bmatrix} \in \mathbb{R}^{(\eta-1)\times(\eta-1)},$$

which are known at least from the 1940s [31, p. 111]. They are

$$\lambda_j = 2\left(1 - \cos\frac{\pi j}{\eta}\right), \quad \text{for } j = 1,2,\ldots,\eta-1.$$

Therefore the maximum of these eigenvalues is

$$\lambda_{\eta-1} = 2\left(1 - \cos\frac{\pi(\eta-1)}{\eta}\right) = 2\left(1 + \cos\frac{\pi}{\eta}\right) = 4\cos^2\frac{\pi}{2\eta}\,.$$

The result follows from $\sigma_{\max}(W_{\eta,\eta}) = \sigma_{\max}(G_{\eta-1}) = \sqrt{\lambda_{\eta-1}}$.

Next, we consider the case $\varepsilon > \eta > 1$. In this situation, Lemma B.3 implies that $\sigma_{\max}(W_{\varepsilon,\eta}) = \max\{\sigma_{\max}(M_\eta), \sigma_{\max}(G_{\eta-1}), ..., \sigma_{\max}(G_1)\} = \sigma_{\max}(M_\eta)$, where we have used again that G_k is a submatrix of G_{k+1} and that $G_{\eta-1}$ is a submatrix of M_η. The singular values of M_η are the square roots of the eigenvalues of $M_\eta M_\eta^T$, i.e., the square roots of the roots of the characteristic equation

$$\det(\lambda I - M_\eta M_\eta^T) = \det\begin{bmatrix} (\lambda-2) & -1 & & & \\ -1 & (\lambda-2) & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & (\lambda-2) & -1 \\ & & & -1 & (\lambda-1) \end{bmatrix} = 0.$$

With the change of variable $\lambda = 2\mu + 2$, the equation above becomes

$$\det\begin{bmatrix} 2\mu & -1 & & & \\ -1 & 2\mu & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2\mu & -1 \\ & & & -1 & 2\mu+1 \end{bmatrix} = U_\eta(\mu) + U_{\eta-1}(\mu) = 0, \tag{B.9}$$

where $U_\ell(\mu)$ is the degree-ℓ Chebyshev polynomial of the second kind. The first equality in (B.9) can be obtained directly from [43, eq. (11)] by applying the recurrence relation of the Chebyshev polynomials of the second kind[2]. It can also be easily established from results in [35]. Observe that Gershgorin Circle Theorem [34, Theorem 7.2.1] implies that the eigenvalues of $M_\eta M_\eta^T$ satisfy $0 \le \lambda \le 4$. Therefore, the roots of (B.9) satisfy $-1 \le \mu \le 1$. Moreover, we also have that 1 and -1 are not roots of (B.9) since $U_\eta(1) + U_{\eta-1}(1) = 2\eta + 1 \neq 0$ and $U_\eta(-1) + U_{\eta-1}(-1) = (-1)^\eta \neq 0$. Thus, the roots of (B.9) satisfy $-1 < \mu < 1$. With the change of variable $\mu = \cos\theta$, we get the equation

$$U_\eta(\mu) + U_{\eta-1}(\mu) = \frac{1}{\sin\theta}(\sin(\eta+1)\theta + \sin\eta\theta) = 2\,\frac{\cos\frac{\theta}{2}}{\sin\theta}\,\sin\frac{(2\eta+1)\theta}{2} = 0,$$

whose roots are $\theta_j = 2\pi j/(2\eta+1)$, $j = 1, \dots, \eta$ in the interval $0 < \theta < \pi$. We finally obtain that the eigenvalues of $M_\eta M_\eta^T$ are

$$\lambda_j = 2 + 2\cos\frac{2j\pi}{2\eta+1}, \quad \text{for } j = 1, 2, \dots, \eta.$$

The largest one is λ_1, which implies

$$\sigma_{\max}(W_{\varepsilon,\eta}) = \sigma_{\max}(M_\eta) = \sqrt{2 + 2\cos\frac{2\pi}{2\eta+1}} = 2\cos\frac{\pi}{2\eta+1}.$$

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[2] The reader should take into account that in [43] the characteristic polynomial is defined as $\det(M_\eta M_\eta^T - \lambda I)$ and the change of variable is slightly different.

□

Finally, Lemma 6.4 follows from combining Lemmas B.1 and B.2, Proposition B.4 and a elementary trigonometric identity. Observe that $\sigma_{\min}(T) \neq 0$, which implies that T has full row rank.

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