Complementary Bases in Symplectic Matrices and a Proof that their Determinant is One

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Definition and Basic Properties

Let I_n be the identity matrix and J be

$$J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}.$$

Definition: A matrix $S \in \mathbb{R}^{2n\times 2n}$ is called symplectic if $S^T JS = J$.

Symplectic matrices have applications in linear control theory for discrete-time systems.

- The product of two symplectic matrices is also symplectic.
- If S is symplectic then S^{-1} and S^T are symplectic.
- If $\lambda \in \sigma(S)$ then $\lambda^{-1} \in \sigma(S)$.

From the definition, it is obvious that $\det S = \pm 1$.

However, if S is a symplectic matrix then always $\det S = 1$.
This is equivalent to the fact that the algebraic multiplicities of the eigenvalues 1 and -1 are both even.

No elementary proof available

The fact that $\det S = 1$ for symplectic matrices has long been known, but no proof seems entirely elementary.

Let us quote, for instance

"It is somewhat nontrivial to prove that the determinant itself is 1, and we will accomplish this by expressing the condition for a matrix to be symplectic in terms of a differential form....Another proof may be found in Arnold [Mathematical methods in classical mechanics (1978)]."

from D. McDuff and D. Salamon: *Introduction to Symplectic Topology*, Clarendon Press 1995.

Our purpose is to give a straightforward, matrix theoretic proof that $\det S = 1$ when S is symplectic. But in the process, we give some new information about the patterns of linearly independent rows (columns) among the blocks of S.

A summary of two classical proofs

First Proof, appearing in the books by Arnold, or by R. Abraham and J. Marsden, *Foundations of Mechanics*, 2nd. Edition, Addison-Wesley (1978).

- Use tensor algebra. More precisely antisymmetric tensor products.
- The goal is to prove that every symplectomorphism preserves the volume form.

Second Proof, appearing in E. Artin, *Geometric Algebra*, Interscience Publishers (1957).

- Define a *symplectic transvection* as a symplectic matrix S such that $Sx = x$ for all the vectors x in a given hyperplane.
- Any symplectic matrix is a product of symplectic transvections.
- The determinant of any simplectic transvection is one.

Determinants, Inverses and Schur Complements

For any square partitioned nonsingular matrix A

$$\underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & C \end{bmatrix},$$

with $C = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Using the previous factorization, it is trivial that

$$A^{-1} = \begin{bmatrix} * & * \\ * & C^{-1} \end{bmatrix}.$$

As a consequence:

Theorem: *Let A and A^{-1} be conformally partitioned*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

If A_{11} is nonsingular then

$$\det A = \frac{\det A_{11}}{\det B_{22}}.$$

Block properties of symplectic matrices

Theorem: *Let $S \in \mathbb{R}^{2n\times 2n}$ be partitioned as*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{where} \quad S_{11}, S_{12}, S_{21}, S_{22} \in \mathbb{R}^{n\times n}.$$

S is symplectic if and only if

$$S^{-1} = \begin{bmatrix} S_{22}^T & -S_{12}^T \\ -S_{21}^T & S_{11}^T \end{bmatrix}.$$

Proof: First, consider that S is symplectic if and only if $S^{-1} = J^{-1}S^TJ = -JS^TJ$, where we have used $J^{-1} = J^T = -J$. And then expand the block product of $-JS^TJ$.

The easy case: S_{11} nonsingular

$$\det S = \frac{\det S_{11}}{\det S_{11}^T} = 1.$$

This may also be easily obtained if some of the other blocks are nonsingular, by using the block structure of the inverse modulo some permutations.

The four blocks can be singular

Unfortunately the four blocks of a symplectic matrix may be singular, as the following example shows:

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right].$$

The notion of complementary bases allows us to relax the restriction of at least one block being nonsingular in the previous proof of $\det S = 1$, making it entirely general.

The Complementary Bases Theorem (I)

Let us consider again the symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$ partitioned as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{where} \quad S_{11}, S_{12}, S_{21}, S_{22} \in \mathbb{R}^{n\times n}.$$

Theorem: *Suppose that* $\mathrm{rank}(S_{11}) = k$ *and that the columns of* S_{11} *indexed by* α, $\alpha \subseteq \{1, \ldots, n\}$ *and* $|\alpha| = k$, *are linearly independent. Then the columns of* S_{12} *indexed by* α', *the complement of* α, *together with the columns* α *of* S_{11} *constitute a basis of* $\mathbb{R}^n$, *i.e., they constitute a nonsingular* $n \times n$ *matrix.*

Idea: $\mathrm{rank}(S_{11}) = 2$, $\{a_1, a_4\}$ linearly independent.

$$S = \left[\begin{array}{ccccc|ccccc} a_1 & a_2 & a_3 & a_4 & a_5 & b_1 & b_2 & b_3 & b_4 & b_5 \\ \hline & & S_{21} & & & & & S_{22} & & \end{array}\right]$$

$$[a_1\ b_2\ b_3\ a_4\ b_5] \in \mathbb{R}^{5\times 5} \quad \text{nonsingular}$$

The Complementary Bases Theorem (II)

THERE ARE EIGHT SIMILAR THEOREMS.

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{where} \quad S_{11}, S_{12}, S_{21}, S_{22} \in \mathbb{R}^{n\times n}.$$

Theorem: *Suppose that rank$(S_{pq}) = k$, $p, q \in \{1, 2\}$, and that the rows (columns) of S_{pq} indexed by α, $\alpha \subseteq \{1, \ldots, n\}$ and $|\alpha| = k$, are linearly independent. Then the rows (columns) of $S_{p'q}$ ($S_{pq'}$) indexed by α', the complement of α, together with the rows (columns) α of S_{pq} constitute a basis of $\mathbb{R}^n$, i.e., they constitute a nonsingular $n \times n$ matrix.*

Once the result is proven for (S_{11}, S_{12}), and, (S_{12}, S_{11}), the other six results are consequence of S^T, S^{-1}, and S^{-T} being symplectic matrices.

The complementary bases theorem remains valid for COMPLEX symplectic matrices: $S^*JS = J$.

Proof of the Complementary Bases Theorem (I)

Only the result for a maximal linearly independent set of columns of S_{11} is proven.

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{where} \quad S_{11}, S_{12}, S_{21}, S_{22} \in \mathbb{R}^{n \times n}.$$

Step 1: Select a permutation matrix $P \in \mathbb{R}^{n \times n}$ to move the maximal linearly independent set of columns of S_{11} we are considering to the first k positions using:

$$\underbrace{S' := S \underbrace{\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}}_{\text{symplectic}}}_{\text{symplectic}} = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}$$

Step 2: There exists a nonsingular matrix $Y \in \mathbb{R}^{n \times n}$ such that

$$Y S'_{11} = \begin{bmatrix} I_k & Z \\ 0 & 0 \end{bmatrix}$$

Proof of the Complementary Bases Theorem (II)

$$S'' := \underbrace{\underbrace{\begin{bmatrix} Y & 0 \\ 0 & Y^{-T} \end{bmatrix}}_{\text{symplectic}} \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}}_{\text{symplectic}} = \begin{bmatrix} S''_{11} & S''_{12} \\ S''_{21} & S''_{22} \end{bmatrix}$$

where

$$S''_{11} = \begin{bmatrix} I_k & Z \\ 0 & 0 \end{bmatrix} \text{ and } S''_{12} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

Step 3: We have to prove that the $(n-k) \times (n-k)$ matrix X_{22} is nonsingular. Remember that

$$S''^{-1} = \begin{bmatrix} S''^T_{22} & -S''^T_{12} \\ -S''^T_{21} & S''^T_{11} \end{bmatrix}, \quad \text{then}$$

$$0 = (S''\, S''^{-1})_{12} = -S''_{11}\, S''^T_{12} + S''_{12}\, S''^T_{11}$$

and this implies

$$X_{21} = -X_{22} Z^T.$$

Finally

$$n - k = \operatorname{rank}\,[X_{21} \;\; X_{22}] = \operatorname{rank} X_{22}[-Z^T \;\; I] = \operatorname{rank} X_{22}$$

Corollary I: zero columns (rows) in a block

Corollary: *Suppose that the rows (columns) of* S_{pq}, $p, q \in \{1, 2\}$, *indexed by* β, $\beta \subseteq \{1, \ldots, n\}$, *are zero. Then the rows (columns) of* $S_{p'q}$ ($S_{pq'}$) *indexed by* β *are linearly independent.*

Idea:

$$S = \left[\begin{array}{ccccc|ccccc} 0 & a_2 & a_3 & 0 & a_5 & b_1 & b_2 & b_3 & b_4 & b_5 \\ \hline \multicolumn{5}{c|}{S_{21}} & \multicolumn{5}{c}{S_{22}} \end{array}\right]$$

$$[b_1 \ b_4] \in \mathbb{R}^{5\times 5} \ \text{ are linearly independent}$$

Corollary II: $\det S = 1$ for $S \in \mathbb{R}^{2n\times 2n}$ symplectic (1)

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} S_{22}^T & -S_{12}^T \\ -S_{21}^T & S_{11}^T \end{bmatrix}.$$

Let us assume $\operatorname{rank}(S_{11}) = k$ and that the first k columns of S_{11} are linearly independent. (Otherwise $S(P \oplus P)$, with P permutation matrix, $P \oplus P$ symplectic, and $\det(P \oplus P) = (\det P)^2 = 1$). Then

$$[S_{11} \mid S_{12}] = [\underbrace{E_1}_{k} \ \underbrace{E_2}_{n-k} \mid \underbrace{F_1}_{k} \ \underbrace{F_2}_{n-k}]$$

with $[E_1 \ F_2]$ nonsingular.

Let $\Pi_j \in \mathbb{R}^{2n\times 2n}$ be the matrix that interchanges the column j of S_{11} with the column j of S_{12}. Then

$$S\Pi_{k+1}\cdots\Pi_n = \left[\begin{array}{c|c} E_1 \ F_2 & * \\ \hline * & * \end{array}\right]$$

$$(S\Pi_{k+1}\cdots\Pi_n)^{-1} = \Pi_n\cdots\Pi_{k+1}S^{-1} = \left[\begin{array}{c|c} * & * \\ \hline * & \begin{array}{c} E_1^T \\ -F_2^T \end{array} \end{array}\right]$$

Corollary II: $\det S = 1$ for $S \in \mathbb{R}^{2n\times 2n}$ symplectic (2)

Finally

$$(-1)^{n-k}\det S = \det S\Pi_{k+1}\cdots\Pi_n = \frac{\det[E_1\ F_2]}{\det\begin{bmatrix} E_1^T \\ -F_2^T \end{bmatrix}}$$

$$= (-1)^{n-k}\frac{\det[E_1\ F_2]}{\det[E_1\ F_2]^T} = (-1)^{n-k}$$

Then

$$\det S = 1$$

QED