LOW RANK PERTURBATIONS OF SPECTRAL CANONICAL FORMS

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Joint work with Fernando de Terán and Julio Moro (Universidad Carlos III)
• Low rank perturbations of matrices arise frequently in applications and in theory.

• They appear when a system with many degrees of freedom is controlled with actions on a small subset of the degrees of freedom.

• Well-known example: Sherman-Morrison-Woodbury formula.

\[(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^TA^{-1}U)^{-1}V^TA^{-1}\]

\[A \ n \times n, \ U, V \ n \times k, \ \text{rank}(U) = \text{rank}(V) = k.\]
Our Goal: How are typically modified spectral canonical forms by low rank perturbations?

- **Jordan** canonical form (JCF) of $A \in \mathbb{C}^{n \times n}$.

- **Weierstrass** canonical form (WCF) of regular matrix pencils $A + \lambda B$, $A, B \in \mathbb{C}^{n \times n}$ and $\det(A + \lambda B)$ does not vanish identically. Generalized eigenvalue problem

\[(A + \lambda B)v = 0\]

- **Kronecker** canonical form (KCF) of singular matrix pencils $A + \lambda B$, $A, B \in \mathbb{C}^{m \times n}$ or $A, B \in \mathbb{C}^{n \times n}$ and $\det(A + \lambda B) = 0$ for all $\lambda$.
Many different things may happen

\[
A + E_1 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A + E_2 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Our goal is to describe the GENERIC or most frequent behavior. This will be a behavior that holds for all perturbations \( E \) except those in a set of zero Lebesgue measure. We are able to describe explicitly this set (HARD AND NOT EASY).
Notation

\[ J_k(\lambda) = \begin{bmatrix} \lambda & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \lambda \end{bmatrix} \in \mathbb{C}^{k \times k} \]

\[ J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_p}(\lambda_p) = \begin{bmatrix} J_{k_1}(\lambda_1) & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & J_{k_p}(\lambda_p) \end{bmatrix} \]

Direct sum or block diagonal matrix of Jordan blocks.
Perturbation of Jordan canonical form: an example (I)

\[
\text{JCF of } A = J_5(9) \oplus J_5(9) \oplus J_5(9) \oplus J_3(9) \oplus \\
J_7(-3) \oplus J_6(-3) \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)
\]

Notice that \( A \) has only two different eigenvalues 9 and \(-3\).

Let \( E \) be such that \( \text{rank}(E) = 2 \). Then generically

\[
\text{JCF of } A + E = * \oplus \ldots \oplus * \oplus J_5(9) \oplus J_3(9) \oplus \\
* \oplus \ldots \oplus * \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)
\]

In the \(* \oplus \ldots \oplus *\) of the JCF of \( A + E \) there are no Jordan blocks associated to the eigenvalues 9 and \(-3\). Besides, in general, it contains only \( 1 \times 1 \) Jordan blocks.
Perturbation of Jordan canonical form: an example (II)

\[
\text{JCF of } A = J_5(9) \oplus J_5(9) \oplus J_5(9) \oplus J_3(9) \oplus J_7(-3) \oplus J_6(-3) \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)
\]

\[\text{rank}(E) = 2.\]

\[
\text{JCF of } A + E = * \oplus \ldots \oplus * \oplus J_5(9) \oplus J_3(9) \oplus *
\]
\[\oplus \ldots \oplus * \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)\]

For every eigenvalue of \( A \) the perturbation \( E \) destroys the \( 2 = \text{rank}(E) \) largest Jordan blocks. The other Jordan blocks of \( A \) remain as Jordan blocks of \( A + E \).
Theorem: Let \( A \in \mathbb{C}^{n \times n} \) and \( \lambda_0 \) be an eigenvalue of \( A \) with \( g_0 \) Jordan blocks in the JCF of \( A \). Let \( E \in \mathbb{C}^{n \times n} \) with \( \text{rank}(E) \leq g_0 \).

Then the Jordan blocks in the JCF of \( A + E \) with eigenvalue \( \lambda_0 \) are just the \( g_0 - \text{rank}(E) \) smallest Jordan blocks of \( A \) with eigenvalue \( \lambda_0 \) if and only if \( E \) does not belong to a certain algebraic manifold of codimension one in the matrix space \( \mathbb{C}^{n \times n} \).
1. The perturbations $E$ are not small.

2. A perturbation matrix $E$ can satisfy the assumptions of the Theorem for one eigenvalue but not for others.

3. Condition $\text{rank}(E) \leq g_0$ defines what we understand by “low rank” in this context. It depends on the eigenvalue we consider.

4. Let $A = PJP^{-1}$ be a Jordan Canonical factorization. If $J$ and $P$ are given then we are able to give an explicit equation for the algebraic manifold mentioned in the theorem in terms of some minors of $P^{-1}EP$. We have a different equation for each eigenvalue $\lambda_0$ of $A$.

5. We will explain this manifold at the end of the talk if we have time. Some additional notation is needed.
Some intuitions

- $\#\lambda_0$-Jordan blocks of $A = \dim \text{Nul}(A - \lambda_0 I) \equiv g_0$.

- $\text{rank}(A + E - \lambda_0 I) \leq \text{rank}(A - \lambda_0 I) + \text{rank}(E) \leq n$

- $\dim \text{Nul}(C) = n - \text{rank}(C)$

Then $g_0 - \text{rank}(E) \leq \dim \text{Nul}(A + E - \lambda_0 I)$

Generically

$\text{rank}(A + E - \lambda_0 I) = \text{rank}(A - \lambda_0 I) + \text{rank}(E)$

and

$g_0 - \text{rank}(E) = \dim \text{Nul}(A + E - \lambda_0 I)$

Why and when the smallest Jordan Blocks?
Theorem (Weierstrass) Let $A, B \in \mathbb{C}^{n \times n}$ such that the polynomial $p(\lambda) = \det(A + \lambda B)$ does not vanish identically. Then there exist two nonsingular matrices $R$ and $S$ such that

$$R(A + \lambda B)S = \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix} + \lambda \begin{pmatrix} I_p & 0 \\ 0 & N \end{pmatrix},$$

$J$ is in Jordan canonical form, and $N$ is in Jordan canonical form with all its eigenvalues equal to zero. $J$ and $N$ are unique up to permutations of the diagonal Jordan blocks. This is called the Weierstrass canonical form of the pencil $A + \lambda B$. 

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Weierstrass canonical form: summary (II)

\[ R(A + \lambda B)S = \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix} + \lambda \begin{pmatrix} I_p & 0 \\ 0 & N \end{pmatrix}, \]

1. The WCF contains all the spectral information of the generalized eigenvalue problem \((A + \lambda B)v = 0\)

2. \(J\) shows the Jordan structure of the finite eigenvalues of \(A + \lambda B\).

3. \(N\) shows the Jordan structure of the infinite eigenvalue of \(A + \lambda B\).

4. The Jordan structure of the infinite eigenvalue of \(A + \lambda B\) is the Jordan structure of the zero eigenvalue of \(B + \lambda A\).

5. Related to systems of algebraic-differential equations

\[ B \frac{dx(t)}{dt} = Ax(t) \]
Perturbation of Weierstrass canonical form: example (I)

Part WCF of \((A + \lambda B)\) for \((\lambda = -5)\) is

\[ J_5(5) \oplus J_4(5) \oplus J_3(5) \oplus J_2(5) \]

Let us consider a perturbation \(E_A + \lambda E_B\) such that

\[
\text{rank}(E_A - 5E_B) = 2 \quad \text{and} \quad \text{rank}(E_B) = 1
\]

Then generically

Part WCF of \((A + E_A + \lambda(B + E_B))\) for \((\lambda = -5)\) is

\[ J_1(5) \oplus J_2(5) \]
Perturbation Weierstrass canonical form: example (II)

WCF of \((A + \lambda B)\) is \(J_5(5) \oplus J_4(5) \oplus J_3(5) \oplus J_2(5)\)

\[
\text{rank}(E_A - 5E_B) = 2 \quad \text{and} \quad \text{rank}(E_B) = 1
\]

WCF of \((A + E_A + \lambda(B + E_B))\) is \(J_1(5) \oplus J_2(5)\)

The \(2 = \text{rank}(E_A - 5E_B)\) largest Jordan blocks are destroyed.

The \(1 = \text{rank}(E_B)\) following largest Jordan blocks turn into \(1 \times 1\) blocks.

Only the \(4 - \text{rank}(E_A - 5E_B) - \text{rank}(E_B)\) smallest Jordan blocks remain unchanged.
**Theorem:** Let $\lambda_0$ be an eigenvalue of the regular pencil $A + \lambda B$ with $g_0$ Jordan blocks in the WCF. Let $E_A + \lambda E_B$ be another pencil such that $\text{rank}(E_A + \lambda_0 E_B) < g_0$. Let us define

$$\rho_0 = \text{rank}(E_A + \lambda_0 E_B), \quad \rho_1 = \text{rank}(E_B).$$

Then for all pencils $E_A + \lambda E_B$ except those in an algebraic manifold of codimension one:

1. There are $g_0 - \rho_0$ Jordan blocks for $\lambda_0$ in the WCF of $A + E_A + \lambda(B + E_B)$, and

2. they are the $g_0 - \rho_0 - \rho_1$ smallest Jordan Blocks for $\lambda_0$ in the WCF of $A + \lambda B$,

3. together with $\rho_1$ $1 \times 1$ Jordan blocks for $\lambda_0$. 
WCF: Comments on the Theorem

1. Similar remarks to those of JCF.

2. In the case $\text{rank}(E_A) + \text{rank}(E_B) \leq n$ and $\lambda_0 \neq 0$ generically

\[
\text{rank}(E_A + \lambda_0 E_B) = \text{rank}(E_A) + \text{rank}(E_B),
\]

and the number of destroyed and preserved Jordan Blocks “does not” depend on the particular $\lambda_0$.

3. Again an elementary rank argument show that

\[
g_0 - \text{rank}(E_A + \lambda_0 E_B) \leq \dim \text{Nul}(A + E_A + \lambda_0(B + E_B)) = \text{Number of } \lambda_0\text{-Jordan blocks in the WCF}
\]
Intuition on differences WCF vs. JCF

**JCF.** Problem: \((A - \lambda_0 I)v = 0\). Jordan chain corresponding to a \(k \times k\) Jordan block

\[(A - \lambda_0 I)v_1 = 0, \quad (A - \lambda_0 I)v_j = v_{j-1} \quad j = 2 : k\]

\[E\]

**WCF.** Problem: \((A + \lambda_0 B)v = 0\). Jordan chain corresponding to a \(k \times k\) Jordan block

\[(A + \lambda_0 B)v_1 = 0, \quad (A + \lambda_0 B)v_j = Bu_{j-1} \quad j = 2 : k\]

\[E_A + \lambda_0 E_B \quad \quad \quad E_B\]
**Theorem (Kronecker)** Let $A, B \in \mathbb{C}^{m \times n}$. Then there exist two nonsingular matrices $R$ and $S$ such that

\[
R (A + \lambda B) S = L_{\epsilon_1} (\lambda) \oplus \ldots \oplus L_{\epsilon_p} (\lambda) \oplus L_{\eta_1}^T (\lambda) \oplus \ldots \oplus L_{\eta_q}^T (\lambda) \oplus (J + \lambda I) \oplus (I + \lambda N),
\]

$J$ is square and is in Jordan canonical form,

$N$ is square and is in Jordan canonical form with all its eigenvalues equal to zero,
Continuation Kronecker’s Let $A, B \in \mathbb{C}^{m \times n}$

$$R(A + \lambda B)S = L_{\epsilon_1}(\lambda) \oplus \ldots \oplus L_{\epsilon_p}(\lambda) \oplus L_{\eta_1}^T(\lambda) \oplus \ldots \oplus L_{\eta_q}^T(\lambda) \oplus (J + \lambda I) \oplus (I + \lambda N),$$

$$L_{\epsilon_i}(\lambda) = \begin{bmatrix} \lambda & 1 \\ \lambda & 1 \\ & \ddots & \ddots \\ & & \lambda & 1 \end{bmatrix} \in \mathbb{C}^{\epsilon_i \times (\epsilon_i+1)}$$

$0 \leq \epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_p$ are the column minimal indices.

$0 \leq \eta_1 \leq \eta_2 \leq \ldots \leq \eta_q$ are the row minimal indices.

$L_{\epsilon_i}(\lambda)$ ($L_{\eta_i}^T(\lambda)$) are called column (row) singular blocks.
Continuation Kronecker’s Let $A, B \in \mathbb{C}^{m \times n}$

$$R(A + \lambda B)S = L_{\epsilon_1}^{\top}(\lambda) \oplus \cdots \oplus L_{\epsilon_p}^{\top}(\lambda) \oplus L_{\eta_1}^{\top}(\lambda) \oplus \cdots \oplus L_{\eta_q}^{\top}(\lambda) \oplus (J + \lambda I) \oplus (I + \lambda N)$$

1. $0 \leq \epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_p$ and $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_q$ are unique.

2. $J$ and $N$ are unique up to permutations of the Jordan diagonal blocks.

3. $(J + \lambda I) \oplus (I + \lambda N)$ is the regular part of the pencil.

4. $\text{rank}(A + \lambda B) = n - p = m - q$

5. $R(A + \lambda B)S = (J + \lambda I) \oplus (I + \lambda N)$ if and only if $\det(A + \lambda B) \neq 0$.

6. KCF application in control theory.
1. We will consider three singular $m \times n$ pencils with rank less than $\min\{m, n\}$:

   Unperturbed: $P(\lambda) = A + \lambda B$
   Perturbation: $E(\lambda) = E_A + \lambda E_B$
   Perturbed: $(P + E)(\lambda) = (A + E_A) + \lambda(B + E_B)$

2. We will assume

   $\text{rank}(P + E)(\lambda) = \text{rank } P(\lambda) + \text{rank } E(\lambda) < \min\{m, n\}$

3. Therefore, we have the global low rank condition

   $\rho \equiv \text{rank } E(\lambda) < \min\{p, q\}$
Number of minimal indices

\[ \text{rank}(P + E)(\lambda) = \text{rank } P(\lambda) + \text{rank } E(\lambda) < \min\{m, n\} \]

and

\[ \rho \equiv \text{rank } E(\lambda) < \min\{p, q\} \]

implies

Number of column (row) singular blocks of \((P + E)(\lambda)\) is equal to \(p - \text{rank}(E)\) \((q - \text{rank}(E))\),

What are their dimensions?
How is the regular part?
Relevant data of unperturbed and perturbation pencil

\[ P(\lambda) = A + \lambda B \]

\[ 0 \leq \epsilon_1 \leq \ldots \leq \epsilon_p \text{ and } 0 \leq \eta_1 \leq \ldots \leq \eta_q \]

\[ (J + \lambda I) \oplus (I + \lambda N) \]

\[ E(\lambda) = E_A + \lambda E_B \]

\[ 0 \leq \bar{\epsilon}_1 \leq \ldots \leq \bar{\epsilon}_\bar{p} \text{ and } 0 \leq \bar{\eta}_1 \leq \ldots \leq \bar{\eta}_\bar{q} \]

\[ \bar{\epsilon} \equiv \bar{\epsilon}_1 + \ldots + \bar{\epsilon}_{\bar{p}} \text{ and } \bar{\eta} \equiv \bar{\eta}_1 + \ldots + \bar{\eta}_{\bar{q}} \]

\[ (J_E + \lambda I) \oplus (I + \lambda N_E) \]
Definitions:

\[ d_k = \left\lfloor \frac{\sum_{i=1}^{k} \epsilon_i + \tilde{\epsilon}}{k - \rho} \right\rfloor \quad k = (\rho + 1) : p \]

\[ h_k = \left\lfloor \frac{\sum_{i=1}^{k} \eta_i + \tilde{\eta}}{k - \rho} \right\rfloor \quad k = (\rho + 1) : q \]

\[ d_{\min} = \min_k d_k \quad \text{and} \quad h_{\min} = \min_k h_k \]

Lemma: There exists only one index \( s \) (or \( t \)) such that

1. \( d_s = d_{\min} \quad (h_t = h_{\min}) \)
2. \( d_s \geq \epsilon_s \geq \ldots \geq \epsilon_1 \quad (h_t \geq \eta_t \geq \ldots \geq \eta_1) \)
3. If \( k > s \) (or \( k > t \)) then \( \epsilon_k > d_k \geq d_s \quad (\eta_k > h_k \geq h_t) \)
**Theorem:** Let $\gamma_s (\mu_t)$ be the remainder of the integer division of $\sum_{i=1}^{s} \varepsilon_i + \tilde{\varepsilon}$ by $s - \rho$ (of $\sum_{i=1}^{t} \eta_i + \tilde{\eta}$ by $(t - \rho)$), where $\rho = \text{rank}(E(\lambda))$. Then under certain generic conditions the KCF of $(P + E)(\lambda)$ is determined by

1. $s - \rho - \gamma_s$ column minimal indices equal to $d_s$,
   $\gamma_s$ column minimal indices equal to $d_s + 1$,
   $p - s$ column minimal indices equal to $\varepsilon_{s+1}, \ldots, \varepsilon_p$.

2. $t - \rho - \mu_t$ row minimal indices equal to $h_t$,
   $\mu_t$ row minimal indices equal to $h_t + 1$,
   $q - t$ row minimal indices equal to $\eta_{t+1}, \ldots, \eta_q$.

3. The regular part is
   \[(J + \lambda I) \oplus (J_E + \lambda I) \oplus (\lambda I + N) \oplus (\lambda I + N_E)\]
Low rank perturbation of KCF: Three Main Ideas

- The blocks of the regular parts of $P(\lambda)$ and $E(\lambda)$ remain unchanged in the sum $(P + E)(\lambda)$ and no more regular blocks appear.

- The largest $p-s$ column and $q-t$ row singular blocks of $P(\lambda)$ remain unchanged as singular blocks of $(P + E)(\lambda)$.

- The smallest $s$ column and $t$ row singular blocks of $P(\lambda)$ are destroyed or transformed into larger blocks (but not larger than the unchanged ones).
**Theorem:** Let $A \in \mathbb{C}^{n \times n}$ and $\lambda_0$ be an eigenvalue of $A$ with $g_0$ Jordan blocks in the JCF of $A$. Let $E \in \mathbb{C}^{n \times n}$ with $\text{rank}(E) \leq g_0$.

Then the Jordan blocks in the JCF of $A + E$ with eigenvalue $\lambda_0$ are just the $g_0 - \text{rank}(E)$ smallest Jordan blocks of $A$ with eigenvalue $\lambda_0$ if and only if $E$ does not belong to a certain algebraic manifold of codimension one in the matrix space $\mathbb{C}^{n \times n}$. 

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JCF: generic conditions. Example (I)

\[ A + E = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\bullet & \cdot & \clubsuit & \cdot & \spadesuit & \cdot & \cdot \\
\spadesuit & \cdot & \heartsuit & \cdot & \spadesuit & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix} \]

\[ \text{rank}(E) = 2 \text{ and } \lambda_0 = 1 \]

\[ C_0 = \det \begin{bmatrix}
\bullet & \clubsuit \\
\spadesuit & \clubsuit
\end{bmatrix} + \det \begin{bmatrix}
\bullet & \spadesuit \\
\spadesuit & \spadesuit
\end{bmatrix} \]
JCF: generic conditions. Example (II)

\[ JCF \text{ of } A + E = \begin{bmatrix}
* & * & * \\
* & * & * \\
* & * & \\
\hline \\
* & * \\
* & * \\
\hline \\
0 & 1 \\
0 & 0 \\
\hline \\
* \\
\end{bmatrix} \]

if and only if

\[ C_0 = \det \begin{bmatrix}
\spadesuit & \clubsuit \\
\clubsuit & \clubsuit \\
\end{bmatrix} + \det \begin{bmatrix}
\spadesuit & \spadesuit \\
\spadesuit & \spadesuit \\
\end{bmatrix} \neq 0 \]
Theorem: Let \( \text{rank}(E) = \rho \) and the JCF of \( A \) be

\[
P^{-1}AP = J_{n_1}(\lambda_0) \oplus \ldots \oplus J_{n_\rho}(\lambda_0) \oplus J_{n_{\rho+1}}(\lambda_0) \oplus \ldots J_{n_{g_0}}(\lambda_0) \oplus \mathbf{\hat{J}},
\]

with \( n_1 \geq \ldots \geq n_{g_0} \) and \( \det(\mathbf{\hat{J}} - \lambda_0 I) \neq 0 \).

1. If \( n_\rho > n_{\rho+1} \) and \( \Phi_\rho \) is the minor of \( P^{-1}EP \) corresponding to the lower left positions of the \( \rho \) largest Jordan blocks of \( P^{-1}AP \) then

   Generic behavior if and only if \( \Phi_\rho \neq 0 \).

2. If \( n_\rho = n_{\rho+1} \) and \( \Phi_\rho \) is ANY minor of \( P^{-1}EP \) corresponding to the lower left positions of \( \rho \) largest Jordan blocks of \( P^{-1}AP \) then

   Generic behavior if and only if \( \sum \Phi_\rho \neq 0 \).
OUR WORK:

- JCF. Moro and FMD, SIMAX 2003.
- WCF. De Terán, FMD, Moro, submitted.
- KCF. De Terán, FMD, in preparation (one month!)
- KCF (Singular goes to Full Rank). De Terán, FMD, still in progress.

RELATED WORK: ONLY JCF, GENERIC CONDITION NOT GIVEN