Fast and accurate computations with some classes of Quasiseparable Matrices

Froilán M. Dopico\(^1\)

Tom Bella\(^2\)  Vadim Olshevsky\(^3\)

\(^1\)Departamento de Matemáticas, Universidad Carlos III de Madrid
\(^2\)Department of Mathematics, University of Rhode Island
\(^3\)Department of Mathematics, University of Connecticut

SIAM Conference on Applied Linear Algebra, Monterey, October 26-29, 2009
Abstract

- There exists a long tradition connecting bidiagonal factorizations with fast algorithms for solving linear systems whose coefficient matrix has a particular structure.

- Some classic and new examples are:

<table>
<thead>
<tr>
<th>Matrices</th>
<th>Algorithms</th>
<th>Error analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quasi-separable</td>
<td>Gemignani (2008)</td>
<td>This Talk</td>
</tr>
</tbody>
</table>

- For Vandermonde and Cauchy satisfactory error bounds are only obtained in Totally Nonnegative (TN) case. Same for quasiseparable.

- We only deal with (1, 1)-quasiseparable matrices (Gemignani general case) and, in addition error analysis, we obtain simple characterizations of TN (1, 1)-quasiseparable matrices.
There exists a long tradition connecting bidiagonal factorizations with fast algorithms for solving linear systems whose coefficient matrix has a particular structure.

Some classic and new examples are:

<table>
<thead>
<tr>
<th>Matrices</th>
<th>Algorithms</th>
<th>Error analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quasi-separable</td>
<td>Gemignani (2008)</td>
<td>This Talk</td>
</tr>
</tbody>
</table>

For Vandermonde and Cauchy satisfactory error bounds are only obtained in Totally Nonnegative (TN) case. Same for quasiseparable.

We only deal with (1,1)-quasiseparable matrices (Gemignani general case) and, in addition error analysis, we obtain simple characterizations of TN (1,1)-quasiseparable matrices.
Abstract

- There exists a long tradition connecting bidiagonal factorizations with fast algorithms for solving linear systems whose coefficient matrix has a particular structure.
- Some classic and new examples are:

<table>
<thead>
<tr>
<th>Matrices</th>
<th>Algorithms</th>
<th>Error analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quasi-separable</td>
<td>Gemignani (2008)</td>
<td>This Talk</td>
</tr>
</tbody>
</table>

- For Vandermonde and Cauchy satisfactory error bounds are only obtained in Totally Nonnegative (TN) case. Same for quasiseparable.
- We only deal with \((1, 1)\)-quasiseparable matrices (Gemignani general case) and, in addition error analysis, we obtain simple characterizations of TN \((1, 1)\)-quasiseparable matrices.
Abstract

- There exists a long tradition connecting bidiagonal factorizations with fast algorithms for solving linear systems whose coefficient matrix has a particular structure.

- Some classic and new examples are:

<table>
<thead>
<tr>
<th>Matrices</th>
<th>Algorithms</th>
<th>Error analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quasi-separable</td>
<td>Gemignani (2008)</td>
<td>This Talk</td>
</tr>
</tbody>
</table>

- For Vandermonde and Cauchy satisfactory error bounds are only obtained in Totally Nonnegative (TN) case. Same for quasiseparable.

- We only deal with \((1, 1)\)-quasiseparable matrices (Gemignani general case) and, in addition error analysis, we obtain simple characterizations of TN \((1, 1)\)-quasiseparable matrices.
Abstract

- There exists a long tradition connecting bidiagonal factorizations with fast algorithms for solving linear systems whose coefficient matrix has a particular structure.
- Some classic and new examples are:

<table>
<thead>
<tr>
<th>Matrices</th>
<th>Algorithms</th>
<th>Error analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quasi-separable</td>
<td>Gemignani (2008)</td>
<td>This Talk</td>
</tr>
</tbody>
</table>

- For Vandermonde and Cauchy satisfactory error bounds are only obtained in Totally Nonnegative (TN) case. Same for quasiseparable.
- We only deal with \((1, 1)\)-quasiseparable matrices (Gemignani general case) and, in addition error analysis, we obtain simple characterizations of TN \((1, 1)\)-quasiseparable matrices.
1. Introduction and goals

2. Neville elimination, TN and quasiseparable matrices

3. Totally Nonnegative (TN) quasiseparable matrices

4. Error analysis for quasiseparable linear systems

5. Other results in accurate quasiseparable computations

6. Conclusions and future work
Outline

1. Introduction and goals
2. Neville elimination, TN and quasiseparable matrices
3. Totally Nonnegative (TN) quasiseparable matrices
4. Error analysis for quasiseparable linear systems
5. Other results in accurate quasiseparable computations
6. Conclusions and future work
Definition

A square matrix $C$ is **quasiseparable of order $(1, 1)$** if
- every submatrix of $C$ entirely located in the **strictly lower or upper triangular part** of $C$ have rank at most 1, and
- at least one of these submatrices has rank equal to 1.

Remark

In this talk, for brevity, the simple term **quasiseparable** is used instead of $(1, 1)$-quasiseparable.

It is necessary and sufficient that the following submatrices have rank at most 1:
A square matrix $C$ is **quasiseparable of order $(1, 1)$** if

- every submatrix of $C$ entirely located in the **strictly lower or upper triangular part** of $C$ have rank at most $1$, and
- at least one of these submatrices has rank equal to $1$.

**Remark**

In this talk, for brevity, the simple term **quasiseparable** is used instead of $(1, 1)$-quasiseparable.

It is necessary and sufficient that the following submatrices have rank at most $1$:
Quasiseparable matrices (I): Definition

**Definition**

A square matrix $C$ is **quasiseparable of order** $(1,1)$ if

- every submatrix of $C$ entirely located in the **strictly lower or upper triangular part** of $C$ have **rank at most** 1, and
- at least one of these submatrices has rank equal to 1.

**Remark**

In this talk, for brevity, the simple term **quasiseparable** is used instead of $(1,1)$-quasiseparable.

It is necessary and sufficient that the following submatrices have rank at most 1:
Quasiseparable matrices (II)

\[ C = \begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\end{bmatrix} \]
Quasiseparable matrices (II)

\[ C = \left[ \begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\end{array} \right] \]
Quasiseparable matrices (II)

\[ C = \begin{bmatrix} \times & \times & \times \times \times \times \times \times \\ \times & \times & \times \times \times \times \times \times \\ \times & \times & \times \times \times \times \times \times \\ \times & \times & \times \times \times \times \times \times \\ \end{bmatrix} \]
Quasiseparable matrices (II)

\[ C = \begin{bmatrix}
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
\end{bmatrix} \]
Quasiseparable matrices (II)

\[
C = \begin{bmatrix}
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\end{bmatrix}
\]
A square matrix $G$ is **Green’s quasiseparable of order $(1, 1)$** if

- every submatrix of $G$ entirely located in the **lower or upper triangular part** (including the diagonal) of $G$ have rank at most 1, and
- at least one of these submatrices has rank equal to 1.

It is necessary and sufficient that the following submatrices have rank at most 1:
Definition (Green’s quasiseparable matrices)

A square matrix $G$ is **Green’s quasiseparable of order $(1, 1)$** if

- every submatrix of $G$ entirely located in the **lower or upper triangular part (including the diagonal)** of $G$ have rank at most 1, and

- at least one of these submatrices has rank equal to 1.

It is necessary and sufficient that the following submatrices have rank at most 1:
Green’s quasiseparable matrices (II)

\[
G = \begin{bmatrix}
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\end{bmatrix}
\]
Green’s quasiseparable matrices (II)

\[
G = \begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\end{bmatrix}
\]
Green’s quasiseparable matrices (II)

\[
G = \begin{bmatrix}
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\end{bmatrix}
\]
Green’s quasiseparable matrices (II)

\[ G = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \]
Green’s quasiseparable matrices (II)

\[ G = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix} \]
Green’s quasiseparable matrices (II)

\[ G = \begin{bmatrix} \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \end{bmatrix} \]
Green’s quasiseparable matrices (II)

\[ G = \begin{bmatrix}
  \times & \times & \times & \times & \times \\
  \times & \times & \times & \times & \times \\
  \times & \times & \times & \times & \times \\
  \times & \times & \times & \times & \times \\
\end{bmatrix} \]
Green’s quasiseparable matrices (II)

\[
G = \begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\end{bmatrix}
\]
Green’s quasiseparable matrices (II)

\[ G = \begin{bmatrix}
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\end{bmatrix} \]
Green’s quasiseparable matrices (II)

\[ G = \begin{bmatrix}
** & ** & ** & ** & ** & x \\
** & ** & ** & ** & ** & x \\
** & ** & ** & ** & ** & x \\
** & ** & ** & ** & ** & x \\
** & ** & ** & ** & ** & x \\
** & ** & ** & ** & ** & x \\
\end{bmatrix} \]
Green’s quasiseparable matrices (II)

\[ G = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \]
Theorem (Eidelman and Gohberg (1999))

The set of $n \times n$ quasiseparable matrices can be parameterized in terms of $7n - 8$ independent parameters or generators.

Example (Every $5 \times 5$ quasiseparable matrix is of the form)

$$C = \begin{bmatrix}
\begin{array}{cccc}
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\end{array} &
\begin{array}{cccc}
\end{array} &
\begin{array}{cccc}
\end{array} &
\begin{array}{cccc}
\end{array} &
\begin{array}{cccc}
\end{array} \\
\end{array}
\end{bmatrix}$$
Parametrization of Green's quasiseparable theorem

The set of $n \times n$ Green's quasiseparable matrices can be parameterized in terms of $6n - 2$ parameters with the constraints $p_i q_i = g_i h_i$ for $i = 1 : n$.

Example (Every $5 \times 5$ Green's quasiseparable matrix is of the form)

$$G = \begin{bmatrix}
    p_1 q_1 & g_1 b_1 h_2 & g_1 b_1 b_2 h_3 & g_1 b_1 b_2 b_3 h_4 & g_1 b_1 b_2 b_3 b_4 h_5 \\
p_2 a_1 q_1 & p_2 q_2 & g_2 b_2 h_3 & g_2 b_2 b_3 h_4 & g_2 b_2 b_3 b_4 h_5 \\
p_3 a_2 a_1 q_1 & p_3 a_2 q_2 & p_3 q_3 & g_3 b_3 h_4 & g_3 b_3 b_4 h_5 \\
p_4 a_3 a_2 a_1 q_1 & p_4 a_3 a_2 q_2 & p_4 a_3 q_3 & p_4 q_4 & g_4 b_4 h_5 \\
p_5 a_4 a_3 a_2 a_1 q_1 & p_5 a_4 a_3 q_3 & p_5 a_4 q_4 & p_5 q_5 &
\end{bmatrix}$$
A main line of research has been the development of structured fast algorithms by using the low number of parameters defining this class. There are many algorithms and their costs are:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Cost of traditional algorithms</th>
<th>Cost of structured quasiseparable algs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>systems of equations</td>
<td>$O(n^3)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>eigenvalues</td>
<td>$O(n^3)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>singular values</td>
<td>$O(n^3)$</td>
<td>$O(n^2)$</td>
</tr>
</tbody>
</table>

The stability of these algorithms is not guaranteed and, as far as we know, error analysis have not been developed even for the most simple cases.
A main line of research has been the development of structured fast algorithms by using the low number of parameters defining this class. There are many algorithms and their costs are:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Cost of traditional algorithms</th>
<th>Cost of structured quasiseparable algs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>systems of equations</td>
<td>$O(n^3)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>eigenvalues</td>
<td>$O(n^3)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>singular values</td>
<td>$O(n^3)$</td>
<td>$O(n^2)$</td>
</tr>
</tbody>
</table>

The stability of these algorithms is not guaranteed and, as far as we know, error analysis have not been developed even for the most simple cases.
### Definition
A matrix $A$ is totally nonnegative if all its minors are nonnegative.

### TN and accurate computations
TN matrices are a classical set of matrices that appear in many applications and are amenable for guaranteed accurate computations (Boros-Kailath-Olshevsky, Demmel, D., Koev, Higham, Peña,...)
An historical TN-quasiseparable connection

In *Oscillation Matrices (1941)* by Gantmacher and Krein a particular class of symmetric Green’s quasiseparable matrices is considered. These are called single-pair matrices and are defined as

\[
S = \begin{bmatrix}
p_1q_1 & q_1p_2 & q_1p_3 & \cdots \\
p_2q_1 & p_2q_2 & q_2p_3 & \cdots \\
p_3q_1 & p_3q_2 & p_3q_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix} = \text{tril}(pq^T) + \text{strict-triu}(qp^T),
\]

where all the numbers \( p = [p_1, \ldots, p_n]^T, q = [q_1, \ldots, q_n]^T \) are nonzero.

These matrices are obtained from general Green’s quasiseparable matrices by taking \( a_i = b_i = 1, g_i = q_i, \) and \( h_i = p_i. \)

**Theorem (Gantmacher and Krein (1941))**

\( S \) is TN if and only if all the numbers \( p_1, \ldots, p_n, q_1, \ldots, q_n \) have the same sign and

\[
\frac{q_1}{p_1} \leq \frac{q_2}{p_2} \leq \cdots \leq \frac{q_n}{p_n}
\]
Goals of the talk

- We initiate the study of **stability of fast algorithms for quasiseparable matrices**, by presenting **rounding error analysis** of the solution of **quasiseparable linear systems** by using a **bidiagonal factorization** followed by a **Björck-Pereyra** type algorithm.

- We prove that this algorithm is **componentwise backward stable in a strong sense** in the **TN-quasiseparable case**.

- For **TN-Green’s quasiseparable matrices** simple **forward error bounds** for this algorithm are presented and we show that it is **frequently accurate**, independently of the traditional condition number of the matrix.

- We characterize the **set of nonsingular TN-quasiseparable matrices** through the **quasiseparable generators, the entries and the bidiagonal factorizations**. This extends Gantmacher and Krein’s result.

- We briefly mention other results on accurate computations with **quasiseparable matrices**.
Goals of the talk

- We initiate the study of **stability of fast algorithms for quasiseparable matrices**, by presenting **rounding error analysis** of the solution of **quasiseparable linear systems** by using a **bidiagonal factorization** followed by a **Björck-Pereyra** type algorithm.

- We prove that this algorithm is **componentwise backward stable in a strong sense** in the **TN-quasiseparable** case.

- For **TN-Green’s quasiseparable matrices** simple **forward error bounds** for this algorithm are presented and we show that it is **frequently accurate**, independently of the traditional condition number of the matrix.

- We characterize the **set of nonsingular TN-quasiseparable matrices** through the **quasiseparable generators**, the entries and the **bidiagonal factorizations**. This extends Gantmacher and Krein’s result.

- We briefly mention other results on accurate computations with **quasiseparable matrices**.
Goals of the talk

- We initiate the study of **stability of fast algorithms for quasiseparable matrices**, by presenting **rounding error analysis** of the solution of **quasiseparable linear systems** by using a **bidiagonal factorization** followed by a **Björck-Pereyra** type algorithm.

- We prove that this algorithm is **componentwise backward stable in a strong sense** in the **TN-quasiseparable** case.

- For **TN-Green’s quasiseparable matrices** **simple forward error bounds** for this algorithm are presented and we show that it is **frequently accurate**, independently of the traditional condition number of the matrix.

- We characterize the set of nonsingular **TN-quasiseparable matrices** through the **quasiseparable generators, the entries and the bidiagonal factorizations**. This extends Gantmacher and Krein’s result.

- We briefly mention other results on accurate computations with quasiseparable matrices.
Goals of the talk

- We initiate the study of **stability of fast algorithms for quasiseparable matrices**, by presenting **rounding error analysis** of the solution of **quasiseparable linear systems** by using a **bidiagonal factorization** followed by a **Björck-Pereyra** type algorithm.

- We prove that this algorithm is **componentwise backward stable in a strong sense** in the **TN-quasiseparable** case.

- For **TN-Green’s quasiseparable matrices**, simple **forward error bounds** for this algorithm are presented and we show that it is **frequently accurate**, independently of the traditional condition number of the matrix.

- We characterize **the set of nonsingular TN-quasiseparable matrices** through **the quasiseparable generators, the entries and the bidiagonal factorizations**. This extends Gantmacher and Krein’s result.

- We briefly mention other results on accurate computations with **quasiseparable matrices**.
Goals of the talk

- We initiate the study of stability of fast algorithms for quasiseparable matrices, by presenting rounding error analysis of the solution of quasiseparable linear systems by using a bidiagonal factorization followed by a Björck-Pereyra type algorithm.

- We prove that this algorithm is componentwise backward stable in a strong sense in the TN-quasiseparable case.

- For TN-Green’s quasiseparable matrices simple forward error bounds for this algorithm are presented and we show that it is frequently accurate, independently of the traditional condition number of the matrix.

- We characterize the set of nonsingular TN-quasiseparable matrices through the quasiseparable generators, the entries and the bidiagonal factorizations. This extends Gantmacher and Krein’s result.

- We briefly mention other results on accurate computations with quasiseparable matrices.
1. Introduction and goals

2. Neville elimination, TN and quasiseparable matrices

3. Totally Nonnegative (TN) quasiseparable matrices

4. Error analysis for quasiseparable linear systems

5. Other results in accurate quasiseparable computations

6. Conclusions and future work
It is a classic procedure to create zeros in a matrix by adding to a row (resp. column) a multiple of the previous row (resp. column).

Without interchanges, it was carefully analyzed by Gasca and Peña in a series of seminal papers in the 90s. In particular, its matricial description in terms of bidiagonal factorizations and its fundamental relationship with total nonnegativity were established.

**Theorem (Gasca and Peña (1994))**

A nonsingular matrix $A$ is TN if and only if complete Neville elimination can be performed on $A$ without row or column exchanges, with nonnegative multipliers and positive diagonal pivots.

Neville elimination with interchanges was generalized for rectangular and singular (TN) matrices by Gassó and Torregrosa (2002, 04, 08).
Brief summary on Neville elimination (I)

- It is a classic **procedure to create zeros in a matrix by adding to a row (resp. column) a multiple of the previous row (resp. column).**

- **Without interchanges**, it was carefully analyzed by **Gasca and Peña** in a series of seminal papers in the 90s. In particular, its **matricial description** in terms of **bidiagonal factorizations** and its fundamental relationship with **total nonnegativity** were established.

---

**Theorem (Gasca and Peña (1994))**

A nonsingular matrix $A$ is TN if and only if complete Neville elimination can be performed on $A$ without row or column exchanges, with nonnegative multipliers and positive diagonal pivots.

---

- Neville elimination with interchanges was generalized for rectangular and singular (TN) matrices by **Gassó and Torregrosa** (2002, 04, 08).
Brief summary on Neville elimination (I)

- It is a classic procedure to create zeros in a matrix by adding to a row (resp. column) a multiple of the previous row (resp. column).

- Without interchanges, it was carefully analyzed by Gasca and Peña in a series of seminal papers in the 90s. In particular, its matricial description in terms of bidiagonal factorizations and its fundamental relationship with total nonnegativity were established.

Theorem (Gasca and Peña (1994))

A nonsingular matrix $A$ is TN if and only if complete Neville elimination can be performed on $A$ without row or column exchanges, with nonnegative multipliers and positive diagonal pivots.

- Neville elimination with interchanges was generalized for rectangular and singular (TN) matrices by Gassó and Torregrosa (2002, 04, 08).
Brief summary on Neville elimination (I)

- It is a classic procedure to create zeros in a matrix by adding to a row (resp. column) a multiple of the previous row (resp. column).

- Without interchanges, it was carefully analyzed by Gasca and Peña in a series of seminal papers in the 90s. In particular, its matricial description in terms of bidiagonal factorizations and its fundamental relationship with total nonnegativity were established.

Theorem (Gasca and Peña (1994))

A nonsingular matrix $A$ is TN if and only if complete Neville elimination can be performed on $A$ without row or column exchanges, with nonnegative multipliers and positive diagonal pivots.

- Neville elimination with interchanges was generalized for rectangular and singular (TN) matrices by Gassó and Torregrosa (2002, 04, 08).
Neville elimination without exchanges adapts very well to the quasiseparable structure (Gemignani 2008).

In this talk, Neville elimination is never applied numerically. It is a theoretical way to get formulae, in terms of the generators, for the bidiagonal factors of the matrix. These formulae are then used

1. to develop fast algorithms,
2. to perform their error analysis,
3. and, to establish simple characterization of TN-quasiseparable matrices.
Neville elimination without exchanges adapts very well to the quasiseparable structure (Gemignani 2008).

In this talk, Neville elimination is never applied numerically. It is a theoretical way to get formulae, in terms of the generators, for the bidiagonal factors of the matrix. These formulae are then used

1. to develop fast algorithms,
2. to perform their error analysis,
3. and, to establish simple characterization of TN-quasiseparable matrices.
Bidiagonal factorizations (I)

Theorem (Gasca and Peña (1996))

If complete Neville elimination runs without exchanges on a nonsingular $n \times n$ square matrix $A$ then

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D U^{(n-1)} \cdots U^{(2)} U^{(1)},$$

where $D$ is diagonal and

$$L^{(k)} = \begin{bmatrix}
1 \\
. \\
\ell_1^{(k)} & 1 \\
\ell_2^{(k)} & . & . \\
. & . & . \\
\ell_k^{(k)} & \cdots & . & 1
\end{bmatrix}$$

This is known as a bidiagonal factorization of $A$ and all the factors amount to $n^2$ nontrivial entries.
Bidiagonal factorizations (I)

Theorem (Gasca and Peña (1996))

*If complete Neville elimination runs without exchanges on a nonsingular $n \times n$ square matrix $A$ then*

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D U^{(n-1)} \cdots U^{(2)} U^{(1)},$$

*where $D$ is diagonal and*

$$U^{(k)} = \begin{bmatrix}
1 &  &  & \\
& \ddots & u_1^{(k)} & \\
&  & 1 & u_2^{(k)} \\
&  & & 1 & \ddots \\
&  & & & \ddots & u_k^{(k)} \\
&  & & & & 1
\end{bmatrix}$$

*This is known as a bidiagonal factorization of $A$ and all the factors amount to $n^2$ nontrivial entries.*
Example (Bidiagonal factorization of a $5 \times 5$ matrix)

$$A = L^{(1)} L^{(2)} L^{(3)} L^{(4)} D U^{(4)} U^{(3)} U^{(2)} U^{(1)},$$

$$D = \begin{bmatrix}
\times \\
\times & \times \\
\times & \times & \times \\
\end{bmatrix}$$
Bidiagonal factorizations (II)

Example (Bidiagonal factorization of a $5 \times 5$ matrix)

\[ A = L^{(1)} L^{(2)} L^{(3)} L^{(4)} D U^{(4)} U^{(3)} U^{(2)} U^{(1)}, \]

\[
L^{(4)} = \begin{bmatrix} 1 \\ \times \\ 1 \\ \times \\ 1 \\ \times \end{bmatrix} \quad \quad U^{(4)} = \begin{bmatrix} 1 \\ \times \\ 1 \\ \times \\ 1 \\ \times \end{bmatrix}
\]
Example (Bidiagonal factorization of a $5 \times 5$ matrix)

\[ A = L^{(1)} L^{(2)} L^{(3)} L^{(4)} D U^{(4)} U^{(3)} U^{(2)} U^{(1)}, \]

\[
L^{(3)} = \begin{bmatrix}
1 \\
1 \\
\times 1 \\
\times 1 \\
\times 1
\end{bmatrix}
\]

\[
U^{(3)} = \begin{bmatrix}
1 \\
1 \\
\times 1 \\
\times 1 \\
\times 1
\end{bmatrix}
\]
Bidiagonal factorizations (II)

Example (Bidiagonal factorization of a $5 \times 5$ matrix)

\[ A = L^{(1)} L^{(2)} L^{(3)} L^{(4)} D U^{(4)} U^{(3)} U^{(2)} U^{(1)}, \]

\[
L^{(2)} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
\times 1 & \times 1 & 
\end{bmatrix} \quad U^{(2)} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
\times 1 & \times 1 & 1 
\end{bmatrix}
\]
Example (Bidiagonal factorization of a $5 \times 5$ matrix)

$$A = L^{(1)} L^{(2)} L^{(3)} L^{(4)} D U^{(4)} U^{(3)} U^{(2)} U^{(1)},$$

$$L^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & \times 1 \end{bmatrix} \quad U^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & \times 1 \end{bmatrix}$$
Example (Bidiagonal factorization of a $5 \times 5$ matrix)

$$A = L^{(1)} L^{(2)} L^{(3)} L^{(4)} D U^{(4)} U^{(3)} U^{(2)} U^{(1)},$$

$$L^{(1)} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \times 1 \end{bmatrix} \quad U^{(1)} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \times 1 \end{bmatrix}$$

Theorem (Gasca and Peña (1996))

A nonsingular matrix $A$ is TN if and only if all the nontrivial entries of its bidiagonal factors are nonnegative ($D > 0$).
One more piece of standard notation...

\[ E_i(\alpha) = \begin{bmatrix} 1 \\ \vdots \\ 1 \alpha 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \text{where } \alpha \text{ is in } (i, i - 1) \text{ entry} \]

Remark

In the rest of the talk, we assume that Neville elimination runs without exchanges. This is not a restriction for nonsingular TN matrices.
One more piece of standard notation...

\[ E_i(\alpha) = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \alpha \\ 1 \\ \vdots \\ 1 \end{bmatrix} , \quad \text{where } \alpha \text{ is in } (i, i - 1) \text{ entry} \]

**Remark**

In the rest of the talk, we assume that **Neville elimination runs without exchanges**. This is not a restriction for nonsingular TN matrices.
Neville on Green’s quasiseparable: $E_5(-\ell_5)G$

\[ \ell_5 := \begin{cases} \frac{p_5a_4}{p_4} \left( = \frac{g_{51}}{g_{41}} \right) & \text{if } p_4 \neq 0 \\ 0 & \text{if } p_4 = 0 \end{cases} \]

\[ \rightarrow E_5(-\ell_5) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -\ell_5 \end{bmatrix} \]

\[ G = \begin{bmatrix} p_1q_1 & g_1b_1h_2 & g_1b_1b_2h_3 & g_1b_1b_2b_3h_4 & g_1b_1b_2b_3b_4h_5 \\ p_2a_1q_1 & p_2q_2 & g_2b_2h_3 & g_2b_2b_3h_4 & g_2b_2b_3b_4h_5 \\ p_3a_2a_1q_1 & p_3a_2q_2 & p_3q_3 & g_3b_3h_4 & g_3b_3b_4h_5 \\ p_4a_3a_2a_1q_1 & p_4a_3a_2q_2 & p_4a_3q_3 & p_4q_4 & g_4b_4h_5 \\ p_5a_4a_3a_2a_1q_1 & p_5a_4a_3a_2q_2 & p_5a_4a_3q_3 & p_5a_4q_4 & p_5q_5 \end{bmatrix} \]

rank one matrix \[ \Rightarrow \frac{g_{51}}{g_{41}} = \frac{g_{52}}{g_{42}} = \frac{g_{53}}{g_{43}} = \frac{g_{54}}{g_{44}} \]
Neville on Green’s quasiseparable: $E_5(-\ell_5)G$

\[
\ell_5 := \begin{cases} 
\frac{p_5 a_4}{p_4} \left(= \frac{g_5}{g_{41}}\right) & \text{if } p_4 \neq 0 \\
0 & \text{if } p_4 = 0 
\end{cases}
\]

\[
E_5(-\ell_5) = \begin{bmatrix} 
1 & 1 & 1 \\
0 & 0 & -\ell_5 
\end{bmatrix}
\]

\[
E_5(-\ell_5)G = \begin{bmatrix} 
p_{11}q_1 & g_1 b_1 h_2 & g_1 b_2 b_2 h_3 & g_1 b_2 b_3 b_4 h_4 & g_1 b_2 b_3 b_4 h_5 \\
p_2 a_1 q_1 & p_2 q_2 & g_2 b_2 b_3 h_4 & g_2 b_2 b_3 h_4 & g_2 b_2 b_3 h_4 \\
p_3 a_2 a_1 q_1 & p_3 a_2 q_2 & p_3 q_3 & g_3 b_3 h_4 & g_3 b_3 h_4 \\
p_4 a_3 a_2 a_1 q_1 & p_4 a_3 a_2 q_2 & p_4 a_3 q_3 & p_4 q_4 & g_4 b_4 h_5 \\
0 & 0 & 0 & 0 & g_5' h_5 
\end{bmatrix}
\]

\[
p_5 q_5 = g_5 h_5 \implies g_5' = g_5 - \ell_5 g_4 b_4
\]
Bidiagonal factorization of Green’s quasiseparable

**Theorem**

Complete Neville elimination runs without exchanges on a nonsingular $n \times n$ Green’s quasiseparable matrix $G$ specified by its generators if and only if

$$G = E_n(\ell_n) \cdots E_3(\ell_3) E_2(\ell_2) D E_2(u_2)^T E_3(u_3)^T \cdots E_n(u_n)^T,$$

where $D = \text{diag}(d_1, \ldots, d_n)$, and

$$\ell_i := \begin{cases} \frac{p_i a_{i-1}}{p_{i-1}} & \text{if } p_{i-1} \neq 0 \\ 0 & \text{if } p_{i-1} = 0 \end{cases} \quad u_i := \begin{cases} \frac{h_i b_{i-1}}{h_{i-1}} & \text{if } h_{i-1} \neq 0 \\ 0 & \text{if } h_{i-1} = 0 \end{cases}$$

$$d_1 = p_1 q_1, \quad d_i = p_i q_i - \ell_i u_i p_{i-1} q_{i-1} \quad \text{for } i = 2 : n$$

**Remarks**

- The bidiagonal factorization of $G$ is sparse: $3n - 2$ nontrivial entries.
- The bidiagonal factorization of $G$ can be computed through explicit formulae from generators (also from entries) in $O(n)$ flops.
Bidiagonal factorization of Green’s quasiseparable

**Theorem**

Complete Neville elimination runs without exchanges on a nonsingular $n \times n$ Green’s quasiseparable matrix $G$ specified by its generators if and only if

$$G = E_n(\ell_n) \cdots E_3(\ell_3) E_2(\ell_2) D E_2(u_2)^T E_3(u_3)^T \cdots E_n(u_n)^T,$$

where $D = \text{diag}(d_1, \ldots, d_n)$, and

$$\ell_i := \begin{cases} \frac{p_i a_{i-1}}{p_{i-1}} & \text{if } p_{i-1} \neq 0 \\ 0 & \text{if } p_{i-1} = 0 \end{cases} \quad \text{and} \quad u_i := \begin{cases} \frac{h_i b_{i-1}}{h_{i-1}} & \text{if } h_{i-1} \neq 0 \\ 0 & \text{if } h_{i-1} = 0 \end{cases}$$

$$d_1 = p_1 q_1, \quad d_i = p_i q_i - \ell_i u_i p_{i-1} q_{i-1} \quad \text{for } i = 2 : n$$

**Remarks**

- The **bidiagonal factorization of $G$ is sparse**: $3n - 2$ nontrivial entries
- The bidiagonal factorization of $G$ **can be computed through explicit formulae from generators** (also from entries) in $O(n)$ flops.
Neville on quasiseparable: $E_5(-\ell_5)C$

$$\ell_5 := \begin{cases} \frac{p_5a_4}{p_4} \left( = \frac{c_{51}}{c_{41}} \right) & \text{if } p_4 \neq 0 \\ \frac{0}{0} & \text{if } p_4 = 0 \end{cases} \quad \rightarrow \quad E_5(-\ell_5) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -\ell_5 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix}$$

rank one matrix $\Rightarrow \frac{c_{51}}{c_{41}} = \frac{c_{52}}{c_{42}} = \frac{c_{53}}{c_{43}}$
Neville on quasiseparable: $E_5(-\ell_5)C$

$$\ell_5 := \begin{cases} \frac{p_5 a_4}{p_4} \left( = \frac{c_{51}}{c_{41}} \right) & \text{if } p_4 \neq 0 \\ 0 & \text{if } p_4 = 0 \end{cases} \rightarrow E_5(-\ell_5) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -\ell_5 \\ 1 \end{bmatrix}$$

$$E_5(-\ell_5)C = \begin{bmatrix} \times \times \times \times \times \\ \times \times \times \times \times \\ \times \times \times \times \times \\ \times \times \times \times \times \\ 0 \ 0 \ 0 \ 0 \ \times \times \end{bmatrix}$$
Bidiagonal factorization of a quasiseparable matrix (I)

Theorem

Complete Neville elimination runs without exchanges on a nonsingular \( n \times n \) quasiseparable matrix \( C \) specified by its generators if and only if

\[
C = E_n(\ell_n) \cdots E_4(\ell_4) E_3(\ell_3) T E_3(u_3)^T E_4(u_4)^T \cdots E_n(u_n)^T,
\]

where

\[
\ell_i := \begin{cases} \frac{p_i a_{i-1}}{p_i - 1} & \text{if } p_i - 1 \neq 0 \\ 0 & \text{if } p_i - 1 = 0 \end{cases}, \quad u_i := \begin{cases} \frac{h_i b_{i-1}}{h_i - 1} & \text{if } h_i - 1 \neq 0 \\ 0 & \text{if } h_i - 1 = 0 \end{cases}
\]

and

\[
T = \begin{bmatrix}
y_1 & z_2 \\
x_2 & y_2 & z_3 \\
& \ddots & \ddots & \ddots \\
x_{n-1} & y_{n-1} & z_n \\
x_n & y_n
\end{bmatrix},
\]

has \( LDU \) factorization, where
Theorem (continued)

\[ x_2 = p_2 q_1, \quad x_j = p_j q_{j-1} - \ell_j d_{j-1} \quad \text{for } j = 3 : n \]
\[ y_1 = d_1, \quad y_2 = d_2, \quad y_j = d_j - \ell_j g_{j-1} h_j - u_j p_j q_{j-1} + u_j \ell_j d_{j-1} \quad \text{for } j = 3 : n \]
\[ z_2 = g_1 h_2, \quad z_j = g_{j-1} h_j - u_j d_{j-1} \quad \text{for } j = 3 : n \]

Remarks

- We can compute through formulae from the generators (or entries)

\[ C = E_n(\ell_n) \cdots E_4(\ell_4) E_3(\ell_3) T E_3(u_3)^T E_4(u_4)^T \cdots E_n(u_n)^T, \]

- but, to get the bidiagonal factorization, it remains to compute with usual Gaussian (Neville) elimination on a tridiagonal matrix,

\[ T = L^{(n-1)} D U^{(n-1)}. \]

- Total cost is \( O(n) \) flops and the bidiagonal factorization of \( C \) is sparse: \( 5n - 6 \) nontrivial entries
Bidiagonal factorization of a quasiseparable matrix (II)

Theorem (continued)

\[ x_2 = p_2 q_1, \quad x_j = p_j q_{j-1} - \ell_j d_{j-1} \quad \text{for } j = 3 : n \]
\[ y_1 = d_1, \quad y_2 = d_2, \quad y_j = d_j - \ell_j g_{j-1} h_j - u_j p_j q_{j-1} + u_j \ell_j d_{j-1} \quad \text{for } j = 3 : n \]
\[ z_2 = g_1 h_2, \quad z_j = g_{j-1} h_j - u_j d_{j-1} \quad \text{for } j = 3 : n \]

Remarks

We can compute through formulae from the generators (or entries)

\[ C = E_n(\ell_n) \cdots E_4(\ell_4) E_3(\ell_3) T E_3(u_3)^T E_4(u_4)^T \cdots E_n(u_n)^T, \]

but, to get the bidiagonal factorization, it remains to compute with usual Gaussian (Neville) elimination on a tridiagonal matrix,

\[ T = L^{(n-1)} D U^{(n-1)}. \]

Total cost is \( O(n) \) flops and the bidiagonal factorization of \( C \) is sparse: \( 5n - 6 \) nontrivial entries
Outline

1. Introduction and goals
2. Neville elimination, TN and quasiseparable matrices
3. Totally Nonnegative (TN) quasiseparable matrices
4. Error analysis for quasiseparable linear systems
5. Other results in accurate quasiseparable computations
6. Conclusions and future work
**Theorem**

An $n \times n$ nonsingular matrix $C$ is TN and quasiseparable if and only if

$$C = E_n(\ell_n) \cdots E_3(\ell_3) L^{(n-1)} D U^{(n-1)} E_3(u_3)^T \cdots E_n(u_n)^T,$$

with all the bidiagonal factors nonnegative and the diagonal entries of $D$ positive.

**Theorem**

An $n \times n$ nonsingular matrix $G$ is TN and Green’s quasiseparable if and only if

$$G = E_n(\ell_n) \cdots E_2(\ell_2) D E_2(u_2)^T \cdots E_n(u_n)^T,$$

with all the bidiagonal factors nonnegative and the diagonal entries of $D$ positive.
Theorem

An $n \times n$ nonsingular matrix $C$ is TN and quasiseparable if and only if

$$C = E_n(\ell_n) \cdots E_3(\ell_3) L^{(n-1)} D U^{(n-1)} E_3(u_3)^T \cdots E_n(u_n)^T,$$

with all the bidiagonal factors nonnegative and the diagonal entries of $D$ positive.

Theorem

An $n \times n$ nonsingular matrix $G$ is TN and Green's quasiseparable if and only if

$$G = E_n(\ell_n) \cdots E_2(\ell_2) D E_2(u_2)^T \cdots E_n(u_n)^T,$$

with all the bidiagonal factors nonnegative and the diagonal entries of $D$ positive.
TN-Green’s quasiseparable matrices (I)

**Theorem (characterization in terms of the parameters)**

Let $G$ be an $n \times n$ Green’s quasiseparable matrix specified by its generators. Then

$$G \text{ is nonsingular and TN } \iff \begin{cases} p_1 q_1 > 0, \quad \text{and} \\ p_i q_i - \left( \frac{p_i a_{i-1}}{p_{i-1}} \right) \left( \frac{h_i b_{i-1}}{h_{i-1}} \right) p_{i-1} q_{i-1} > 0, \\ \frac{p_i a_{i-1}}{p_{i-1}} \geq 0, \quad \frac{h_i b_{i-1}}{h_{i-1}} \geq 0, \quad \text{for } 2 \leq i \leq n \end{cases}$$

These conditions can be checked in $O(n)$ flops.
Theorem (characterization in terms of the entries)

Let $G$ be an $n \times n$ Green’s quasiseparable matrix. Then

$$G \text{ is nonsingular and TN } \iff \begin{cases}
g_{ii} > 0, & 1 \leq i \leq n \\
g_{i,i-1} \geq 0, & g_{i-1,i} \geq 0, & 2 \leq i \leq n \\
\det G(i - 1 : i, i - 1 : i) > 0, & 2 \leq i \leq n
\end{cases}$$

These conditions can be checked in $O(n)$ flops.
Theorem (characterization in terms of the entries)

Let $G$ be an $n \times n$ Green’s quasiseparable matrix. Then

$G$ is nonsingular and TN $\iff$

$$
\begin{align*}
g_{ii} &> 0, & &1 \leq i \leq n \\
g_{i,i-1} &\geq 0, & &2 \leq i \leq n \\
g_{i-1,i} &\geq 0, & &2 \leq i \leq n \\
\det G(i-1 : i, i-1 : i) &> 0, & &2 \leq i \leq n
\end{align*}
$$

These conditions can be checked in $O(n)$ flops.
Theorem (characterization in terms of the entries)

Let $G$ be an $n \times n$ Green's quasiseparable matrix. Then

$G$ is nonsingular and TN $\iff$

\[
\begin{align*}
g_{ii} &> 0, & & 1 \leq i \leq n \\
g_{i,i-1} &\geq 0, & & g_{i-1,i} \geq 0, & & 2 \leq i \leq n \\
\det G(i-1 : i, i-1 : i) &> 0, & & 2 \leq i \leq n
\end{align*}
\]

These conditions can be checked in $O(n)$ flops.
Theorem (characterization in terms of the entries)

Let $G$ be an $n \times n$ Green’s quasiseparable matrix. Then

$G$ is nonsingular and TN $\iff$

\[
\begin{array}{l}
g_{ii} > 0, & 1 \leq i \leq n \\
g_{i,i-1} \geq 0, & 2 \leq i \leq n \\
g_{i-1,i} \geq 0, & 2 \leq i \leq n \\
\det G(i-1:i, i-1:i) > 0, & 2 \leq i \leq n 
\end{array}
\]

These conditions can be checked in $O(n)$ flops.
TN-Green’s quasiseparable matrices (II)

Theorem (characterization in terms of the entries)

Let $G$ be an $n \times n$ Green’s quasiseparable matrix. Then

$G$ is nonsingular and $TN$ $\iff$

\[
\begin{align*}
g_{ii} &> 0, & 1 \leq i \leq n \\
g_{i,i-1} &\geq 0, & 2 \leq i \leq n \\
g_{i-1,i} &\geq 0, & 2 \leq i \leq n \\
det G(i-1 : i, i-1 : i) &> 0, & 2 \leq i \leq n
\end{align*}
\]

These conditions can be checked in $O(n)$ flops.
Theorem (characterization in terms of the entries)

Let $G$ be an $n \times n$ Green’s quasiseparable matrix. Then

$G$ is nonsingular and $TN$ $\iff$

\[
\begin{align*}
g_{ii} & > 0, & 1 \leq i \leq n \\
g_{i,i-1} & \geq 0, & 2 \leq i \leq n \\
g_{i-1,i} & \geq 0, & 2 \leq i \leq n \\
det G(i-1:i, i-1:i) & > 0, & 2 \leq i \leq n
\end{align*}
\]

These conditions can be checked in $O(n)$ flops.

$$C = \begin{bmatrix} 
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times 
\end{bmatrix}$$
Theorem (characterization in terms of the entries)

Let $G$ be an $n \times n$ Green’s quasiseparable matrix. Then

$G$ is nonsingular and $TN \iff$

$$
\begin{align*}
g_{ii} &> 0, & 1 \leq i \leq n \\
g_{i,i-1} &\geq 0, & g_{i-1,i} \geq 0, & 2 \leq i \leq n \\
\det G(i-1:i, i-1:i) &> 0, & 2 \leq i \leq n
\end{align*}
$$

These conditions can be checked in $O(n)$ flops.
Theorem (characterization in terms of the entries)

Let $G$ be an $n \times n$ Green's quasiseparable matrix. Then $G$ is nonsingular and $TN$ if

$$
\begin{align*}
&g_{ii} > 0, &1 \leq i \leq n \\
g_{i,i-1} \geq 0, &g_{i-1,i} \geq 0, &2 \leq i \leq n \\
&\det G(i - 1 : i, i - 1 : i) > 0, &2 \leq i \leq n
\end{align*}
$$

These conditions can be checked in $O(n)$ flops.
**Theorem**

Let $C$ be an $n \times n$ quasiseparable matrix specified by its generators. Define

$$\ell_i := \begin{cases} \frac{p_{ia_i-1}}{p_{i-1}} & \text{if } p_{i-1} \neq 0 \\ 0 & \text{if } p_{i-1} = 0 \end{cases} \quad u_i := \begin{cases} \frac{h_{ib_i-1}}{h_{i-1}} & \text{if } h_{i-1} \neq 0 \\ 0 & \text{if } h_{i-1} = 0 \end{cases}$$

for $i = 3 : n$,

and

$$T = \begin{bmatrix} y_1 & z_2 & & & \\ x_2 & y_2 & z_3 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & y_{n-1} & z_n \\ & & & x_n & y_n \end{bmatrix},$$

with

$$x_2 = p_2 q_1, \quad x_j = p_j q_{j-1} - \ell_j d_{j-1} \quad \text{for } j = 3 : n$$

$$y_1 = d_1, \quad y_2 = d_2, \quad y_j = d_j - \ell_j g_{j-1} h_{j-1} - u_j p_j q_{j-1} + u_j \ell_j d_{j-1} \quad \text{for } j = 3 : n$$

$$z_2 = g_1 h_2, \quad z_j = g_{j-1} h_j - u_j d_{j-1} \quad \text{for } j = 3 : n$$
Then, $C$ is nonsingular and TN if and only if

- $\ell_i \geq 0$ and $u_i \geq 0$ for $i = 3 : n$.

- The tridiagonal matrix $T$ is nonsingular and TN.

- If $p_j q_{j-1} = 0$, for some $j$, then $C(j : n, 1 : j - 1) = 0$.

- If $g_{j-1} h_j = 0$, for some $j$, then $C(1 : j - 1, j : n) = 0$.

These conditions can be checked in $O(n)$ flops.

$$C = \begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ p_3 a_2 q_1 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ p_4 a_3 a_2 q_1 & p_4 a_3 q_2 & p_4 q_3 = 0 & d_4 & g_4 h_5 \\ p_5 a_4 a_3 a_2 q_1 & p_5 a_4 a_3 q_2 & p_5 a_4 q_3 & p_5 q_4 & d_5 \end{bmatrix}$$
Theorem (continued)

Then, $C$ is nonsingular and TN if and only if

- $\ell_i \geq 0$ and $u_i \geq 0$ for $i = 3 : n$.
- The tridiagonal matrix $T$ is nonsingular and TN.
- If $p_j q_{j-1} = 0$, for some $j$, then $C(j:n, 1:j-1) = 0$.
- If $g_{j-1} h_j = 0$, for some $j$, then $C(1:j-1, j:n) = 0$.

These conditions can be checked in $O(n)$ flops.
Theorem (continued)

Then, \( C \) is nonsingular and TN if and only if

1. \( \ell_i \geq 0 \) and \( u_i \geq 0 \) for \( i = 3 : n \).
2. The tridiagonal matrix \( T \) in nonsingular and TN.
3. If \( p_j q_{j-1} = 0 \), for some \( j \), then \( C(j : n, 1 : j - 1) = 0 \).
4. If \( g_{j-1} h_j = 0 \), for some \( j \), then \( C(1 : j - 1, j : n) = 0 \).

These conditions can be checked in \( O(n) \) flops.

\[
C = \begin{bmatrix}
  d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\
p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\
p_3 a_2 q_1 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\
p_4 a_3 a_2 q_1 & p_4 a_3 q_2 & p_4 q_3 = 0 & d_4 & g_4 h_5 \\
p_5 a_4 a_3 a_2 q_1 & p_5 a_4 a_3 q_2 & 0 & p_5 q_4 & d_5
\end{bmatrix}
\]
Theorem (continued)

Then, $C$ is nonsingular and TN if and only if

- $\ell_i \geq 0$ and $u_i \geq 0$ for $i = 3 : n$.
- The tridiagonal matrix $T$ in nonsingular and TN.
- If $p_j q_{j-1} = 0$, for some $j$, then $C(j : n, 1 : j - 1) = 0$.
- If $g_{j-1} h_j = 0$, for some $j$, then $C(1 : j - 1, j : n) = 0$.

These conditions can be checked in $O(n)$ flops.

\[
C = \begin{bmatrix}
       d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\
p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\
p_3 a_2 q_1 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\
0 & 0 & p_4 q_3 = 0 & d_4 & g_4 h_5 \\
p_5 a_4 a_3 a_2 q_1 & p_5 a_4 a_3 q_2 & p_5 a_4 q_3 & p_5 q_4 & d_5
\end{bmatrix}
\]
Then, \( C \) is nonsingular and TN if and only if

- \( \ell_i \geq 0 \) and \( u_i \geq 0 \) for \( i = 3 : n \).
- The tridiagonal matrix \( T \) in nonsingular and TN.
- If \( p_j q_{j-1} = 0 \), for some \( j \), then \( C(j : n, 1 : j - 1) = 0 \).
- If \( g_{j-1} h_j = 0 \), for some \( j \), then \( C(1 : j - 1, j : n) = 0 \).

These conditions can be checked in \( O(n) \) flops.
Outline

1. Introduction and goals
2. Neville elimination, TN and quasiseparable matrices
3. Totally Nonnegative (TN) quasiseparable matrices
4. Error analysis for quasiseparable linear systems
5. Other results in accurate quasiseparable computations
6. Conclusions and future work
Solving linear systems given a bidiagonal factorization...

Assume that for a general matrix $A$:

- We know a bidiagonal factorization

$$A = L^{(1)} L^{(2)} \ldots L^{(n-1)} D U^{(n-1)} \ldots U^{(2)} U^{(1)}.$$ 

- We want to solve $Ax = b$.

Then, we solve the sequence of systems

$$L^{(1)} x^{(1)} = b \rightarrow L^{(2)} x^{(2)} = x^{(1)} \rightarrow \ldots \rightarrow U^{(1)} x = x^{(2n-2)}$$

Observe that for quasiseparable matrices many of the bidiagonal systems are very simple and can be solved in two flops:

$$E_i(\alpha) z = y \iff z = E_i(-\alpha) y.$$ 

In fact, all are of this type for Green’s quasiseparable matrices.
Solving linear systems given a bidiagonal factorization...

Assume that for a general matrix $A$:

- We know a bidiagonal factorization
  
  $$A = L^{(1)}L^{(2)} \cdots L^{(n-1)}DU^{(n-1)} \cdots U^{(2)}U^{(1)}.$$

- We want to solve $Ax = b$.

Then, we solve the sequence of systems

$$L^{(1)}x^{(1)} = b \rightarrow L^{(2)}x^{(2)} = x^{(1)} \rightarrow \cdots \rightarrow U^{(1)}x = x^{(2n-2)}$$

Observe that for quasiseparable matrices many of the bidiagonal systems are very simple and can be solved in two flops:

$$E_i(\alpha)z = y \iff z = E_i(-\alpha)y.$$

In fact, all are of this type for Green’s quasiseparable matrices.
Solving linear systems given a bidiagonal factorization...

Assume that for a general matrix $A$:

- We know a bidiagonal factorization

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D U^{(n-1)} \cdots U^{(2)} U^{(1)}.$$ 

- We want to solve $Ax = b$.

Then, we solve the sequence of systems

$$L^{(1)} x^{(1)} = b \rightarrow L^{(2)} x^{(2)} = x^{(1)} \rightarrow \cdots \rightarrow U^{(1)} x = x^{(2n-2)}$$

Observe that for quasiseparable matrices many of the bidiagonal systems are very simple and can be solved in two flops:

$$E_i(\alpha) z = y \iff z = E_i(-\alpha)y.$$ 

In fact, all are of this type for Green’s quasiseparable matrices.
Solving linear systems given a bidiagonal factorization...

Assume that for a general matrix $A$:

- We know a bidiagonal factorization
  
  $$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D U^{(n-1)} \cdots U^{(2)} U^{(1)}.$$
  
- We want to solve $Ax = b$.

Then, we solve the sequence of systems

$$L^{(1)} x^{(1)} = b \rightarrow L^{(2)} x^{(2)} = x^{(1)} \rightarrow \cdots \rightarrow U^{(1)} x = x^{(2n-2)}$$

Observe that for quasiseparable matrices many of the bidiagonal systems are very simple and can be solved in two flops:

$$E_i(\alpha) z = y \iff z = E_i(-\alpha) y.$$ 

In fact, all are of this type for Green’s quasiseparable matrices.
Solving linear systems given a bidiagonal factorization...

Assume that for a general matrix $A$:

- We know a bidiagonal factorization

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D U^{(n-1)} \cdots U^{(2)} U^{(1)}.$$ 

- We want to solve $Ax = b$.

Then, we solve the sequence of systems

$$L^{(1)} x^{(1)} = b \rightarrow L^{(2)} x^{(2)} = x^{(1)} \rightarrow \cdots \rightarrow U^{(1)} x = x^{(2n-2)}$$

Observe that for quasiseparable matrices many of the bidiagonal systems are very simple and can be solved in two flops:

$$E_i(\alpha) z = y \iff z = E_i(-\alpha) y.$$ 

In fact, all are of this type for Green’s quasiseparable matrices.
The complete $O(n)$ quasi-separable algorithm

**ALGORITHM 1**

**INPUT:** Generators of $C$ (resp. $G$) $n \times n$ quasi-separable (resp. Green’s quasi-separable) matrix and vector $b$

**OUTPUT:** Solution of $Cx = b$ (resp. $Gx = b$)

- **Compute bidiagonal factorization with formulae** as in the first part of the talk:

  $$C = E_n(\ell_n) \cdots E_3(\ell_3) L^{(n-1)} D U^{(n-1)} E_3(u_3)^T \cdots E_n(u_n)^T,$$

  (resp. $G = E_n(\ell_n) \cdots E_2(\ell_2) D E_2(u_2)^T \cdots E_n(u_n)^T$)

- **Solve a sequence of bidiagonal systems** to get $x$ as in the previous slide.
Backward errors for general quasiseparable matrix

**Theorem**

If Algorithm 1 is applied to solve $Cx = b$, where $C$ is an $n \times n$ quasiseparable matrix, and

$$E_n(\hat{\ell}_n), \ldots, E_3(\hat{\ell}_3), \hat{L}^{(n-1)}, \hat{D}, \hat{U}^{(n-1)}, E_3(\hat{u}_3)^T, \ldots, E_n(\hat{u}_n)^T,$$

are the computed bidiagonal factors of $C$ with unit roundoff $\epsilon$, then the computed solution $\hat{x}$ satisfies

$$(C + E)\hat{x} = b,$$

where

- $(C + E)$ is quasiseparable, and

$$|E| \leq \frac{27n\epsilon}{1 - 27n\epsilon} E_n(|\hat{\ell}_n|) \cdots E_3(|\hat{\ell}_3|) |\hat{L}^{(n-1)}| |\hat{D}| |\hat{U}^{(n-1)}| E_3(|\hat{u}_3|)^T \cdots E_n(|\hat{u}_n|)^T$$
Comments on this backward error analysis

- Similar result for **Green’s quasiseparable matrices**, preserving the Green’s structure.

- It is very **tricky**.

- It requires a **delicate way to evaluate the formulae** for the bidiagonal/tridiagonal factors.

- It combines
  
  1. **mixed backward-forward errors** in terms of parameters and bidiagonal factors, with
  2. **backward errors** in terms of entries.

- The bound may be not satisfactory if

\[
E_{n}(|\widehat{l}_{n}|) \cdots E_{3}(|\widehat{l}_{3}|) |\widehat{L}^{(n-1)}| |\widehat{D}| |\widehat{U}^{(n-1)}| E_{3}(|\widehat{u}_{3}|)^T \cdots E_{n}(|\widehat{u}_{n}|)^T >> |C|
\]

- No way to incorporate pivoting to improve bounds because it destroys quasiseparable structure and it does not match well Neville elimination.
Comments on this backward error analysis

- Similar result for **Green’s quasiseparable matrices**, preserving the Green’s structure.

- It is very **tricky**.

- It requires a **delicate way to evaluate the formulae** for the bidiagonal/tridiagonal factors.

- It combines
  1. **mixed backward-forward errors** in terms of parameters and bidiagonal factors, with
  2. **backward errors** in terms of entries.

- The bound may be not satisfactory if

\[
E_n(|\hat{\ell}_n|) \cdots E_3(|\hat{\ell}_3|) |\hat{L}^{(n-1)}| |\hat{D}| |\hat{U}^{(n-1)}| E_3(|\hat{u}_3|)^T \cdots E_n(|\hat{u}_n|)^T >> |C|
\]

- No way to incorporate pivoting to improve bounds because it destroys quasiseparable structure and it does not match well Neville elimination.
Comments on this backward error analysis

- Similar result for **Green’s quasiseparable matrices**, preserving the Green’s structure.
- It is very **tricky**.
- It requires a **delicate way to evaluate the formulae** for the bidiagonal/tridiagonal factors.
- It combines
  1. **mixed backward-forward errors in terms of parameters and bidiagonal factors**, with
  2. **backward errors in terms of entries**.
- The **bound may be not satisfactory** if
  \[ E_n(|\hat{\ell}_n|) \cdots E_3(|\hat{\ell}_3|) |\hat{L}^{(n-1)}| |\hat{D}| |\hat{U}^{(n-1)}| E_3(|\hat{u}_3|)^T \cdots E_n(|\hat{u}_n|)^T >> |C| \]
- No way to incorporate pivoting to improve bounds because it destroys quasiseparable structure and it does not match well Neville elimination.
Comments on this backward error analysis

- Similar result for **Green’s quasiseparable matrices**, preserving the Green’s structure.

- It is very **tricky**.

- It requires a **delicate way to evaluate the formulae** for the bidiagonal/tridiagonal factors

- It combines
  1. **mixed backward-forward errors in terms of parameters and bidiagonal factors**, with
  2. **backward errors in terms of entries**.

- The **bound may be not satisfactory** if

\[
E_n(|\hat{\ell}_n|) \cdots E_3(|\hat{\ell}_3|) |\hat{L}^{(n-1)}| |\hat{D}| |\hat{U}^{(n-1)}| E_3(|\hat{u}_3|)^T \cdots E_n(|\hat{u}_n|)^T >> |C|
\]

- No way to incorporate pivoting to improve bounds because it destroys quasiseparable structure and it does not match well Neville elimination.
Comments on this backward error analysis

- Similar result for Green’s quasiseparable matrices, preserving the Green’s structure.
- It is very tricky.
- It requires a delicate way to evaluate the formulae for the bidiagonal/tridiagonal factors.
- It combines
  1. mixed backward-forward errors in terms of parameters and bidiagonal factors, with
  2. backward errors in terms of entries.
- The bound may be not satisfactory if
  \[ E_n(|\hat{L}_n|) \cdots E_3(|\hat{L}_3|) |\hat{L}^{(n-1)}| |\hat{D}| |\hat{U}^{(n-1)}| E_3(|\hat{u}_3|)^T \cdots E_n(|\hat{u}_n|)^T \gg |C| \]
- No way to incorporate pivoting to improve bounds because it destroys quasiseparable structure and it does not match well Neville elimination.
Comments on this backward error analysis

- Similar result for Green’s quasiseparable matrices, preserving the Green’s structure.
- It is very tricky.
- It requires a delicate way to evaluate the formulae for the bidiagonal/tridiagonal factors.
- It combines
  1. mixed backward-forward errors in terms of parameters and bidiagonal factors, with
  2. backward errors in terms of entries.
- The bound may be not satisfactory if
  \[ E_n(|\hat{\ell}_n|) \cdots E_3(|\hat{\ell}_3|) |\hat{L}^{(n-1)}| |\hat{D}| |\hat{U}^{(n-1)}| E_3(|\hat{u}_3|)^T \cdots E_n(|\hat{u}_n|)^T >> |C| \]
- No way to incorporate pivoting to improve bounds because it destroys quasiseparable structure and it does not match well Neville elimination.
Comments on this backward error analysis

• Similar result for Green’s quasiseparable matrices, preserving the Green’s structure.

• It is very tricky.

• It requires a delicate way to evaluate the formulae for the bidiagonal/tridiagonal factors

• It combines

1. mixed backward-forward errors in terms of parameters and bidiagonal factors, with
2. backward errors in terms of entries.

• The bound may be not satisfactory if

\[ E_n(|\hat{\ell}_n|) \cdots E_3(|\hat{\ell}_3|) |\hat{L}^{(n-1)}| |\hat{D}| |\hat{U}^{(n-1)}| E_3(|\hat{u}_3|)^T \cdots E_n(|\hat{u}_n|)^T \gg |C| \]

• No way to incorporate pivoting to improve bounds because it destroys quasiseparable structure and it does not match well Neville elimination.
Theorem

If Algorithm 1 is applied to solve $Cx = b$, where $C$ is an $n \times n$ quasiseparable matrix, and all the computed bidiagonal factors of $C$ are nonnegative (diag $\hat{D} > 0$), then the computed solution $\hat{x}$ satisfies

$$(C + E)\hat{x} = b,$$

where

- $(C + E)$ is TN-quasiseparable, and
- $|E| \leq \frac{27n\varepsilon}{1 - 54n\varepsilon}|C|$.

Similar for Green's quasiseparable matrices.
Satisfactory backward errors for TN-quasiseparable matrices

**Theorem**

If Algorithm 1 is applied to solve $C\mathbf{x} = \mathbf{b}$, where $C$ is $n \times n$ quasiseparable matrix, and all the computed bidiagonal factors of $C$ are nonnegative ($\text{diag} \hat{D} > 0$), then the computed solution $\hat{\mathbf{x}}$ satisfies

$$(C + E)\hat{\mathbf{x}} = \mathbf{b},$$

where

- $(C + E)$ is TN-quasiseparable, and
- $|E| \leq \frac{27n\epsilon}{1 - 54n\epsilon} |C|$

Similar for Green’s quasiseparable matrices.
Forward errors for TN-Green’s quasiseparable (I)

Theorem

Let $G$ be an $n \times n$ Green’s quasiseparable matrix and define

$$\kappa_{GQ}(G) = \max_{2 \leq i \leq n} \left| \frac{g_{i,i} g_{i-1,i-1}}{g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}} \right| + \left| \frac{g_{i,i} - g_{i-1,i} g_{i-1,i}}{g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}} \right|.$$ 

Assume that Algorithm 1 is applied to solve $Gx = b$ with unit roundoff $\epsilon$ and that the computed bidiagonal factors of $G$ are nonnegative ($\hat{D}$ nonsingular). If

$$\frac{9\epsilon}{1 - 9\epsilon} \kappa_{GQ}(G) < \frac{1}{2},$$

then

- $G$ is nonsingular and TN.
- The computed solution $\hat{x}$ satisfies

$$|x - \hat{x}| \leq 2 \left( \frac{8n\epsilon}{1 - 8n\epsilon} + \kappa_{GQ}(G) \frac{9\epsilon}{1 - 9\epsilon} \right) |G^{-1}| \|b\|.$$
Forward errors for TN-Green’s quasiseparable (II)

\[ |x - \hat{x}| \leq 2 \left( \frac{8n\epsilon}{1 - 8n\epsilon} + \kappa_{GQ}(G) \frac{9\epsilon}{1 - 9\epsilon} \right) |G^{-1}||b|

If \( \epsilon \kappa_{GQ}(G) \ll 1 \), this is a very satisfactory bound, because

\[
\frac{\|G^{-1}\|b\|_\infty}{\|x\|_\infty} = \frac{\|G^{-1}\|b\|_\infty}{\|G^{-1}b\|_\infty}
\]

is moderate except for particular \( b \)'s.

\[
\kappa_{GQ}(G) = \max_{2 \leq i \leq n} \frac{|g_{i,i} g_{i-1,i-1}| + |g_{i,i-1} g_{i-1,i}|}{|g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}|},
\]

is a condition number for this problem.

Note that

\[
\det G(i - 1 : i, i - 1 : i) = g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}.
\]
Forward errors for TN-Green’s quasiseparable (II)

\[ |x - \hat{x}| \leq 2 \left( \frac{8n\epsilon}{1 - 8n\epsilon} + \kappa_{\text{GQ}}(G) \frac{9\epsilon}{1 - 9\epsilon} \right) |G^{-1}||b|
\]

- If \( \epsilon \kappa_{\text{GQ}}(G) \ll 1 \), this is a very satisfactory bound, because

\[ \frac{\|G^{-1}\|b\|\|}{\|x\|\|} = \frac{\|G^{-1}\|b\|\|}{\|G^{-1}b\|\|} \text{ is moderate except for particular } b\text{'s.}
\]

\[ \kappa_{\text{GQ}}(G) = \max_{2 \leq i \leq n} \frac{|g_{i,i} g_{i-1,i-1}| + |g_{i,i-1} g_{i-1,i}|}{|g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}|}, \]

is a condition number for this problem.

- Note that

\[ \det G(i-1:i, i-1:i) = g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}. \]
Forward errors for TN-Green’s quasiseparable (II)

\[ |x - \hat{x}| \leq 2 \left( \frac{8n\epsilon}{1 - 8n\epsilon} + \kappa_{GQ}(G) \frac{9\epsilon}{1 - 9\epsilon} \right) |G^{-1}||b|

- If \( \epsilon \kappa_{GQ}(G) \ll 1 \), this is a very satisfactory bound, because
  \[ \frac{||G^{-1}||b||}{||x||} = \frac{||G^{-1}||b||}{||G^{-1}b||} \] is moderate except for particular \( b \)'s.

- \( \kappa_{GQ}(G) = \max_{2 \leq i \leq n} \frac{|g_{i,i} g_{i-1,i-1}| + |g_{i,i-1} g_{i-1,i}|}{|g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}|} \)
  is a condition number for this problem.

- Note that
  \[ \det G(i - 1 : i, i - 1 : i) = g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i} . \]
Forward errors for TN-Green’s quasiseparable (II)

\[ |x - \hat{x}| \leq 2 \left( \frac{8n\epsilon}{1 - 8n\epsilon} + \kappa_{GQ}(G) \frac{9\epsilon}{1 - 9\epsilon} \right) |G^{-1}||b| \]

- If \( \epsilon \kappa_{GQ}(G) \ll 1 \), this is a very satisfactory bound, because
  \[ \frac{||G^{-1}||b||}{||x||\infty} = \frac{||G^{-1}||b||}{||G^{-1}b||\infty} \]
  is moderate except for particular \( b \)'s.

\[ \kappa_{GQ}(G) = \max_{2 \leq i \leq n} \frac{|g_{i,i} g_{i-1,i-1}| + |g_{i,i-1} g_{i-1,i}|}{|g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}|} \]

is a condition number for this problem.

- Note that

\[ \det G(i - 1 : i, i - 1 : i) = g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}. \]
Forward errors for TN-Green’s quasiseparable (II)

\[ |x - \hat{x}| \leq 2 \left( \frac{8n\epsilon}{1 - 8n\epsilon} + \kappa_{GQ}(G) \frac{9\epsilon}{1 - 9\epsilon} \right) |G^{-1}||b| \]

- If \( \epsilon \kappa_{GQ}(G) \ll 1 \), this is a very satisfactory bound, because

\[ \frac{||G^{-1}||_b||}{||x||_\infty} = \frac{||G^{-1}||_b||}{||G^{-1}b||_\infty} \]

is moderate except for particular \( b \)'s.

- \( \kappa_{GQ}(G) = \max_{2 \leq i \leq n} \left| \frac{g_{i,i} g_{i-1,i-1}}{g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i}} \right| \)

is a condition number for this problem.

- Note that

\[ \det G(i - 1 : i, i - 1 : i) = g_{i,i} g_{i-1,i-1} - g_{i,i-1} g_{i-1,i} \]
Outline

1. Introduction and goals
2. Neville elimination, TN and quasiseparable matrices
3. Totally Nonnegative (TN) quasiseparable matrices
4. Error analysis for quasiseparable linear systems
5. Other results in accurate quasiseparable computations
6. Conclusions and future work
Other results for Green’s quasiseparable

- We know how to compute in $O(n^2)$ flops eigenvalues and singular values of TN-Green’s quasiseparable matrices with relative errors
  
  $$O(\epsilon \kappa_{\text{GQ}}(G))$$

- We have shown perfect componentwise backward stability in solving linear systems through bidiagonalization for diagonally dominant Green’s quasiseparable matrices.

- We know how to compute with cost $O(n^2)$ accurate eigenvalues of skew-symmetric real Green’s quasiseparable matrices and of symmetric Green’s quasiseparable matrices with zero diagonal.
1. Introduction and goals

2. Neville elimination, TN and quasiseparable matrices

3. Totally Nonnegative (TN) quasiseparable matrices

4. Error analysis for quasiseparable linear systems

5. Other results in accurate quasiseparable computations

6. Conclusions and future work
Conclusions and future work

- **Bidiagonalization + solving a sequence of bidiagonal systems is fast** on quasiseparable matrices **but not backward stable**.
- It is fast and backward stable on TN-quasiseparable.
- Simple **forward error bounds** for TN Green’s quasiseparable matrices available.
- **Next step:** error analysis of quasiseparable structured algorithms for QR factorization and its use for solving linear systems.
- **Error analysis of structured rank algorithms** for quasiseparable matrices is an open area of research full of challenging problems...
Conclusions and future work

- Bidiagonalization + solving a sequence of bidiagonal systems is **fast** on quasiseparable matrices **but not backward stable**.
- It is **fast and backward stable** on TN-quasiseparable.
- Simple **forward error bounds** for TN Green’s quasiseparable matrices available.
- Next step: error analysis of quasiseparable structured algorithms for QR factorization and its use for solving linear systems.
- Error analysis of structured rank algorithms for quasiseparable matrices is an open area of research full of challenging problems...
Conclusions and future work

- Bidiagonalization + solving a sequence of bidiagonal systems is **fast** on quasiseparable matrices but not **backward stable**.
- It is fast and backward stable on TN-quasiseparable.
- Simple **forward error bounds for TN Green’s quasiseparable matrices** available.
- Next step: error analysis of quasiseparable structured algorithms for QR factorization and its use for solving linear systems.
- Error analysis of structured rank algorithms for quasiseparable matrices is an open area of research full of challenging problems...
Bidiagonalization + solving a sequence of bidiagonal systems is **fast** on quasiseparable matrices **but not backward stable**.

It is **fast and backward stable** on TN-quasiseparable.

Simple **forward error bounds** for TN Green’s quasiseparable matrices available.

**Next step:** error analysis of quasiseparable structured algorithms for QR factorization and its use for solving linear systems.

Error analysis of structured rank algorithms for quasiseparable matrices is an open area of research full of challenging problems...
Conclusions and future work

- Bidiagonalization + solving a sequence of bidiagonal systems is **fast** on quasiseparable matrices **but not backward stable**.
- **It is fast and backward stable on TN-quasiseparable.**
- Simple **forward error bounds for TN Green’s quasiseparable matrices** available.
- **Next step:** error analysis of quasiseparable structured algorithms for QR factorization and its use for solving linear systems.
- **Error analysis of structured rank algorithms for quasiseparable matrices** is an open area of research full of challenging problems...