Generic (low rank) spectral perturbation results for matrix polynomials

Froilán M. Dopico Fernando De Terán

Departamento de Matemáticas, Universidad Carlos III de Madrid, Spain

16th Conference of the International Linear Algebra Society,
Pisa, Italy, June 21-25, 2010

Abstract (I)

Goal

- Given a **fixed** $n \times n$ **complex matrix polynomial** $P(\lambda)$,
- we want to determine which are "generically" the elementary divisors of $P(\lambda)$ that are preserved in the perturbed polynomial

$$P(\lambda) + E(\lambda),$$

- where $E(\lambda)$ belongs to a set

$$\mathcal{C} = \{E(\lambda) \,:\, E(\lambda) \text{ is a polynomial of } \textbf{low rank} \text{ with } deg(E) \leq deg(P)\}.$$

- The precise meaning of low rank will be defined later.

Abstract (I)

Goal

- Given a **fixed** $n \times n$ **complex matrix polynomial** $P(\lambda)$,
- we want to determine which are "generically" the elementary divisors of $P(\lambda)$ that are preserved in the perturbed polynomial

$$P(\lambda) + E(\lambda),$$

- where $E(\lambda)$ belongs to a set

$$\mathcal{C} = \{E(\lambda) \; : \; E(\lambda) \text{ is a polynomial of } \textbf{low rank} \text{ with } deg(E) \leq deg(P)\}.$$

- The precise meaning of low rank will be defined later.

Abstract (I)

Goal

- Given a **fixed $n \times n$ complex matrix polynomial $P(\lambda)$**,
- we want to determine which are "generically" the elementary divisors of $P(\lambda)$ that are preserved in the perturbed polynomial

$$P(\lambda) + E(\lambda),$$

- where $E(\lambda)$ belongs to a set

$$\mathcal{C} = \{E(\lambda) \ : \ E(\lambda) \text{ is a polynomial of } \textbf{low rank} \text{ with } deg(E) \leq deg(P)\}.$$

- The precise meaning of low rank will be defined later.

Abstract (I)

Goal

- Given a **fixed** $n \times n$ **complex matrix polynomial** $P(\lambda)$,
- we want to determine which are "generically" the elementary divisors of $P(\lambda)$ that are preserved in the perturbed polynomial

$$P(\lambda) + E(\lambda),$$

- where $E(\lambda)$ belongs to a set

$$\mathcal{C} = \{E(\lambda) \, : \, E(\lambda) \text{ is a polynomial of } \textbf{low rank} \text{ with } deg(E) \leq deg(P)\}.$$

- The precise meaning of low rank will be defined later.

Abstract (II)

We deal with

$$P(\lambda) + E(\lambda), \qquad \text{with } E(\lambda) \in \mathcal{C}$$

$$\mathcal{C} = \{E(\lambda) \,:\, E(\lambda) \text{ is a polynomial of } \textbf{low rank} \text{ with } deg(E) \le deg(P)\}.$$

The rigorous meaning of "generic"

- If $deg(P) = \ell$, then the set $\{E(\lambda) \,:\, deg(E) \le deg(P)\}$ can be identified with $\mathbb{C}^{n^2(\ell+1)}$ through the coefficients of $E(\lambda)$.
- A nonempty subset $\mathcal{G} \subset \mathcal{C}$ is generic if its complement in $\mathcal{C}$ is contained in a proper algebraic submanifold of $\mathcal{C}$, i.e., in the set of common zeros of some multivariate polynomials in the entries of the coefficients of $E(\lambda)$ that are nonzero for some elements of $\mathcal{C}$.
- The elementary divisors of $\{P(\lambda) + E(\lambda) \,:\, E(\lambda) \in \mathcal{G}\}$ are "generically" the elementary divisors of $P(\lambda) + E(\lambda)$ when $E(\lambda) \in \mathcal{C}$.

Abstract (II)

We deal with

$$P(\lambda) + E(\lambda), \qquad \text{with } E(\lambda) \in \mathcal{C}$$

$$\mathcal{C} = \{E(\lambda) \,:\, E(\lambda) \text{ is a polynomial of } \textbf{low rank} \text{ with } deg(E) \leq deg(P)\}.$$

The rigorous meaning of "generic"

- If $deg(P) = \ell$, then the set $\{E(\lambda) \,:\, deg(E) \leq deg(P)\}$ can be identified with $\mathbb{C}^{n^2(\ell+1)}$ through the coefficients of $E(\lambda)$.
- A nonempty subset $\mathcal{G} \subset \mathcal{C}$ is generic if its complement in $\mathcal{C}$ is contained in a proper algebraic submanifold of $\mathcal{C}$, i.e., in the set of common zeros of some multivariate polynomials in the entries of the coefficients of $E(\lambda)$ that are nonzero for some elements of $\mathcal{C}$.
- The elementary divisors of $\{P(\lambda) + E(\lambda) \,:\, E(\lambda) \in \mathcal{G}\}$ are "generically" the elementary divisors of $P(\lambda) + E(\lambda)$ when $E(\lambda) \in \mathcal{C}$.

Abstract (III)

Main reference

- F. De Terán and D., *Low rank perturbation of regular matrix polynomials*, LAA 430 (2009) 579-586.

Generic low rank perturbations are in fashion...

- **Unstructured for matrices and matrix pencils**: Hörmander-Mellin (1994), Moro-D. (2003), Savchenko (2003-04), De Terán-D. (2007), De Terán-D-Moro (2008).
- **Structured for structured matrices**: Mehl-Mehrmann-Ran-Rodman, very recent, to appear. See C. Mehl's web page.
- **Related with classical nongeneric (classification) results for matrix polynomials**: Marques de Sà (1979), Thompson (1979-80).

Where do low rank perturbation arise?

Perturbations of any magnitude that affect a few degrees of freedom in problems modeled by matrix polynomials.

Abstract (III)

Main reference

- F. De Terán and D., *Low rank perturbation of regular matrix polynomials*, LAA 430 (2009) 579-586.

Generic low rank perturbations are in fashion...

- **Unstructured for matrices and matrix pencils**: Hörmander-Mellin (1994), Moro-D. (2003), Savchenko (2003-04), De Terán-D. (2007), De Terán-D-Moro (2008).
- **Structured for structured matrices**: Mehl-Mehrmann-Ran-Rodman, very recent, to appear. See C. Mehl's web page.
- **Related with classical nongeneric (classification) results for matrix polynomials**: Marques de Sà (1979), Thompson (1979-80).

Where do low rank perturbation arise?

Perturbations of any magnitude that affect a few degrees of freedom in problems modeled by matrix polynomials.

Abstract (III)

Main reference

- F. De Terán and D., *Low rank perturbation of regular matrix polynomials*, LAA 430 (2009) 579-586.

Generic low rank perturbations are in fashion...

- **Unstructured for matrices and matrix pencils**: Hörmander-Mellin (1994), Moro-D. (2003), Savchenko (2003-04), De Terán-D. (2007), De Terán-D-Moro (2008).
- **Structured for structured matrices**: Mehl-Mehrmann-Ran-Rodman, very recent, to appear. See C. Mehl's web page.
- **Related with classical nongeneric (classification) results for matrix polynomials**: Marques de Sà (1979), Thompson (1979-80).

Where do low rank perturbation arise?

Perturbations of any magnitude that affect a few degrees of freedom in problems modeled by matrix polynomials.

Outline

1. Antecedents: Matrices and regular matrix pencils
2. Low rank perturbation of regular matrix polynomials
3. Keys of the proof through Thompson's result
4. Low rank perturbation of singular pencils
5. Conclusions

Outline

1. **Antecedents: Matrices and regular matrix pencils**
2. Low rank perturbation of regular matrix polynomials
3. Keys of the proof through Thompson's result
4. Low rank perturbation of singular pencils
5. Conclusions

Recall the...

...following basic important fact

In matrices and matrix pencils elementary divisors are, respectively, in one to one correspondence with Jordan blocks in the Jordan Canonical Form and in the Kronecker Canonical Form:

$$(\lambda - \mu)^p \longleftrightarrow J_p(\mu) \equiv p \times p \text{ Jordan block of eigenvalue } \mu$$

Perturbation of Jordan Canonical Form (JCF) of matrices: Example

Let A be a matrix with only two different eigenvalues 9 and -3 and JCF:

$$\begin{aligned}\text{JCF of } A \quad &= \quad J_5(9) \oplus J_5(9) \oplus J_5(9) \oplus J_3(9) \oplus \\ &\quad\; J_7(-3) \oplus J_6(-3) \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)\end{aligned}$$

Then generically for perturbations E such that $\operatorname{rank}(E) = 2$:

$$\begin{aligned}\text{JCF of } A + E \quad &= \quad * \oplus \cdots \oplus * \oplus J_5(9) \oplus J_3(9) \oplus \\ &\quad\; * \oplus \cdots \oplus * \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)\end{aligned}$$

In the $* \oplus \cdots \oplus *$ of the JCF of $A + E$ there are no Jordan blocks associated to the eigenvalues 9 and -3. Besides, generically, it contains only 1×1 Jordan blocks corresponding to distinct eigenvalues.

Perturbation of Jordan Canonical Form (JCF) of matrices: Example

Let A be a matrix with only two different eigenvalues 9 and -3 and JCF:

$$\begin{aligned} \text{JCF of } A &= J_5(9) \oplus J_5(9) \oplus J_5(9) \oplus J_3(9) \oplus \\ & \quad J_7(-3) \oplus J_6(-3) \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3) \end{aligned}$$

Then generically for perturbations E such that $\operatorname{rank}(E) = 2$:

$$\begin{aligned} \text{JCF of } A + E &= * \oplus \cdots \oplus * \oplus J_5(9) \oplus J_3(9) \oplus \\ & \quad * \oplus \cdots \oplus * \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3) \end{aligned}$$

In the $* \oplus \cdots \oplus *$ of the JCF of $A + E$ there are no Jordan blocks associated to the eigenvalues 9 and -3. Besides, generically, it contains only 1×1 Jordan blocks corresponding to distinct eigenvalues.

Perturbation of Jordan Canonical Form (JCF) of matrices: Example

Let A be a matrix with only two different eigenvalues 9 and -3 and JCF:

$$\begin{aligned}\text{JCF of } A &= J_5(9) \oplus J_5(9) \oplus J_5(9) \oplus J_3(9) \oplus \\ &\quad J_7(-3) \oplus J_6(-3) \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)\end{aligned}$$

Then generically for perturbations E such that $\text{rank}(E) = 2$:

$$\begin{aligned}\text{JCF of } A+E &= * \oplus \cdots \oplus * \oplus J_5(9) \oplus J_3(9) \oplus \\ &\quad * \oplus \cdots \oplus * \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)\end{aligned}$$

In the $* \oplus \cdots \oplus *$ of the JCF of $A+E$ there are no Jordan blocks associated to the eigenvalues 9 and -3. Besides, generically, it contains only 1×1 Jordan blocks corresponding to distinct eigenvalues.

Let A be a matrix with only two different eigenvalues 9 and -3 and JCF:

$$\begin{aligned}\text{JCF of } A \quad &= \quad J_5(9) \oplus J_5(9) \oplus J_5(9) \oplus J_3(9) \oplus \\ &\quad\ J_7(-3) \oplus J_6(-3) \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)\end{aligned}$$

Then generically for perturbations E such that $\mathrm{rank}(E) = 2$:

$$\begin{aligned}\text{JCF of } A + E \quad &= \quad * \oplus \cdots \oplus * \oplus J_5(9) \oplus J_3(9) \oplus \\ &\quad\ * \oplus \cdots \oplus * \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)\end{aligned}$$

In the $* \oplus \cdots \oplus *$ of the JCF of $A + E$ there are no Jordan blocks associated to the eigenvalues 9 and -3. Besides, generically, it contains only 1×1 Jordan blocks corresponding to distinct eigenvalues.

Perturbation of Jordan Canonical Form (JCF) of matrices: Example

Let A be a matrix with only two different eigenvalues 9 and -3 and JCF:

$$\begin{aligned}\text{JCF of } A \quad = \quad & J_5(9) \oplus J_5(9) \oplus J_5(9) \oplus J_3(9) \oplus \\ & J_7(-3) \oplus J_6(-3) \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)\end{aligned}$$

Then generically for perturbations E such that $\operatorname{rank}(E) = 2$:

$$\begin{aligned}\text{JCF of } A + E \quad = \quad & * \oplus \cdots \oplus * \oplus J_5(9) \oplus J_3(9) \oplus \\ & * \oplus \cdots \oplus * \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)\end{aligned}$$

In the $* \oplus \cdots \oplus *$ of the JCF of $A + E$ there are no Jordan blocks associated to the eigenvalues 9 and -3. Besides, generically, it contains only 1×1 Jordan blocks corresponding to distinct eigenvalues.

Low rank perturbation of JCF of matrices: Theorem

Theorem

Let $A \in \mathbb{C}^{n\times n}$ and λ_0 be an eigenvalue of A with g_0 Jordan blocks in the JCF of A. Let r be a fixed integer number such that $r < g_0$.

Then *generically with respect perturbations $E \in \mathbb{C}^{n\times n}$ such that*

$$\operatorname{rank}(E) \leq r \quad \textit{(low rank condition for } \lambda_0\textit{).}$$

- *The Jordan blocks of $A+E$ with eigenvalue λ_0 are precisely the $g_0 - r$ smallest Jordan blocks of A with eigenvalue λ_0.*

  (Hörmander-Mellin (1994), Moro-D. (2003), Savchenko (2003-04))

- *The eigenvalues of $A+E$ that are different from the eigenvalues of A are all simple.*

  (Melh-Mehrmann-Ran-Rodman (2010))

Theorem

Let $A \in \mathbb{C}^{n\times n}$ and λ_0 be an eigenvalue of A with g_0 Jordan blocks in the JCF of A. Let r be a fixed integer number such that $r < g_0$.
Then *generically with respect perturbations $E \in \mathbb{C}^{n\times n}$ such that*

$$\operatorname{rank}(E) \leq r \quad \textit{(low rank condition for } \lambda_0\textit{).}$$

- *The Jordan blocks of $A + E$ with eigenvalue λ_0 are precisely the $g_0 - r$ smallest Jordan blocks of A with eigenvalue λ_0.*

  (Hörmander-Mellin (1994), Moro-D. (2003), Savchenko (2003-04))

- *The eigenvalues of $A + E$ that are different from the eigenvalues of A are all simple.*

  (Melh-Mehrmann-Ran-Rodman (2010))

Theorem

Let $A \in \mathbb{C}^{n\times n}$ and λ_0 be an eigenvalue of A with g_0 Jordan blocks in the JCF of A. Let r be a fixed integer number such that $r < g_0$.
Then *generically with respect perturbations $E \in \mathbb{C}^{n\times n}$ such that*

$$\operatorname{rank}(E) \leq r \quad \textit{(low rank condition for } \lambda_0\textit{).}$$

- *The Jordan blocks of $A + E$ with eigenvalue λ_0 are precisely the $g_0 - r$ smallest Jordan blocks of A with eigenvalue λ_0.*

  (Hörmander-Mellin (1994), Moro-D. (2003), Savchenko (2003-04))

- *The eigenvalues of $A + E$ that are different from the eigenvalues of A are all simple.*

  (Melh-Mehrmann-Ran-Rodman (2010))

Low rank perturbation of JCF of matrices: Theorem

Theorem

Let $A \in \mathbb{C}^{n\times n}$ and λ_0 be an eigenvalue of A with g_0 Jordan blocks in the JCF of A. Let r be a fixed integer number such that $r < g_0$.
Then *generically with respect perturbations $E \in \mathbb{C}^{n\times n}$ such that*

$$\operatorname{rank}(E) \leq r \quad \textit{(low rank condition for } \lambda_0\textit{).}$$

- *The Jordan blocks of $A+E$ with eigenvalue λ_0 are precisely the $g_0 - r$ smallest Jordan blocks of A with eigenvalue λ_0.*

  (Hörmander-Mellin (1994), Moro-D. (2003), Savchenko (2003-04))
- *The eigenvalues of $A+E$ that are different from the eigenvalues of A are all simple.*

  (Melh-Mehrmann-Ran-Rodman (2010))

Low rank perturbation of JCF of matrices: Theorem

Theorem

Let $A \in \mathbb{C}^{n\times n}$ and λ_0 be an eigenvalue of A with g_0 Jordan blocks in the JCF of A. Let r be a fixed integer number such that $r < g_0$.
Then *generically with respect perturbations $E \in \mathbb{C}^{n\times n}$ such that*

$$\operatorname{rank}(E) \leq r \quad \textit{(low rank condition for } \lambda_0\textit{).}$$

- *The Jordan blocks of $A+E$ with eigenvalue λ_0 are precisely the $g_0 - r$ smallest Jordan blocks of A with eigenvalue λ_0.*

  (Hörmander-Mellin (1994), Moro-D. (2003), Savchenko (2003-04))

- *The eigenvalues of $A+E$ that are different from the eigenvalues of A are all simple.*

  (Melh-Mehrmann-Ran-Rodman (2010))

Low rank perturbation of JCF of matrices: Theorem

Theorem

Let $A \in \mathbb{C}^{n\times n}$ and λ_0 be an eigenvalue of A with g_0 Jordan blocks in the JCF of A. Let r be a fixed integer number such that $r < g_0$.
Then *generically with respect perturbations $E \in \mathbb{C}^{n\times n}$ such that*

$$\operatorname{rank}(E) \leq r \quad \textit{(low rank condition for } \lambda_0\textit{).}$$

- *The Jordan blocks of $A + E$ with eigenvalue λ_0 are precisely the $g_0 - r$ smallest Jordan blocks of A with eigenvalue λ_0.*

  (Hörmander-Mellin (1994), Moro-D. (2003), Savchenko (2003-04))

- *The eigenvalues of $A + E$ that are different from the eigenvalues of A are all simple.*

  (Melh-Mehrmann-Ran-Rodman (2010))

Perturbation of Kronecker (Weierstrass) Canonical form (KCF) of REGULAR matrix pencils: Example

Let $(A_0 + \lambda A_1)$ with two distinct eigenvalues $-5, -1$ and KCF

$$(J_5(5) \oplus J_4(5) \oplus J_3(5) \oplus J_2(5)) + (J_7(1) \oplus J_6(1) \oplus J_3(1)) + \lambda I$$

Then generically with respect perturbations $E(\lambda) = E_0 + \lambda E_1$ such that

$$\operatorname{rank}(E(-5)) = 2, \quad \operatorname{rank}(E(-1)) = 1, \quad \text{and} \quad \operatorname{rank}(E_1) = 1$$

the KCF of $(A_0 + E_0) + \lambda(A_1 + E_1)$ is

$$(* \oplus * \oplus J_1(5) \oplus J_2(5)) + (* \oplus J_1(1) \oplus J_3(1)) + \lambda I$$

Remarks

- It is possible to have different generic behaviors for different eigenvalues as in the example, but this is not usual because "generically"

$$\operatorname{nrank}(E(\lambda)) = \operatorname{rank}(E(-5)) = \operatorname{rank}(E(-1)).$$

- Note the separate role of $\operatorname{rank}(E_1)$ which produces a very different behavior than the one for matrices.

Perturbation of Kronecker (Weierstrass) Canonical form (KCF) of REGULAR matrix pencils: Example

Let $(A_0 + \lambda A_1)$ with two distinct eigenvalues $-5, -1$ and KCF

$$(J_5(5) \oplus J_4(5) \oplus J_3(5) \oplus J_2(5)) + (J_7(1) \oplus J_6(1) \oplus J_3(1)) + \lambda I$$

Then generically with respect perturbations $E(\lambda) = E_0 + \lambda E_1$ such that

$$\operatorname{rank}(E(-5)) = 2, \quad \operatorname{rank}(E(-1)) = 1, \quad \text{and} \quad \operatorname{rank}(E_1) = 1$$

the KCF of $(A_0 + E_0) + \lambda(A_1 + E_1)$ is

$$(* \oplus * \oplus J_1(5) \oplus J_2(5)) + (* \oplus J_1(1) \oplus J_3(1)) + \lambda I$$

Remarks

- It is possible to have different generic behaviors for different eigenvalues as in the example, but this is not usual because "generically"

$$\operatorname{nrank}(E(\lambda)) = \operatorname{rank}(E(-5)) = \operatorname{rank}(E(-1)).$$

- Note the separate role of $\operatorname{rank}(E_1)$ which produces a very different behavior than the one for matrices.

Perturbation of Kronecker (Weierstrass) Canonical form (KCF) of REGULAR matrix pencils: Example

Let $(A_0 + \lambda A_1)$ with two distinct eigenvalues $-5, -1$ and KCF

$$(J_5(5) \oplus J_4(5) \oplus J_3(5) \oplus J_2(5)) + (J_7(1) \oplus J_6(1) \oplus J_3(1)) + \lambda I$$

Then generically with respect perturbations $E(\lambda) = E_0 + \lambda E_1$ such that

$$\text{rank}(E(-5)) = 2, \quad \text{rank}(E(-1)) = 1, \quad \text{and} \quad \text{rank}(E_1) = 1$$

the KCF of $(A_0 + E_0) + \lambda(A_1 + E_1)$ is

$$(* \oplus * \oplus J_1(5) \oplus J_2(5)) + (* \oplus J_1(1) \oplus J_3(1)) + \lambda I$$

Remarks

- It is possible to have different generic behaviors for different eigenvalues as in the example, but this is not usual because "generically"

$$\text{nrank}(E(\lambda)) = \text{rank}(E(-5)) = \text{rank}(E(-1)).$$

- Note the separate role of $\text{rank}(E_1)$ which produces a very different behavior than the one for matrices.

Perturbation of Kronecker (Weierstrass) Canonical form (KCF) of REGULAR matrix pencils: Example

Let $(A_0 + \lambda A_1)$ with two distinct eigenvalues $-5, -1$ and KCF

$$(J_5(5) \oplus J_4(5) \oplus J_3(5) \oplus J_2(5)) + (J_7(1) \oplus J_6(1) \oplus J_3(1)) + \lambda I$$

Then generically with respect perturbations $E(\lambda) = E_0 + \lambda E_1$ such that

$$\text{rank}(E(-5)) = 2, \quad \text{rank}(E(-1)) = 1, \quad \text{and} \quad \text{rank}(E_1) = 1$$

the KCF of $(A_0 + E_0) + \lambda(A_1 + E_1)$ is

$$(* \oplus * \oplus J_1(5) \oplus J_2(5)) + (* \oplus J_1(1) \oplus J_3(1)) + \lambda I$$

Remarks

- It is possible to have different generic behaviors for different eigenvalues as in the example, but this is not usual because “generically”

$$\text{nrank}(E(\lambda)) = \text{rank}(E(-5)) = \text{rank}(E(-1)).$$

- Note the separate role of $\text{rank}(E_1)$ which produces a very different behavior than the one for matrices.

Perturbation of KCF of REGULAR matrix pencils: Theorem

Theorem (De Terán-D-Moro, SIMAX, 2008)

Let λ_0 *be an eigenvalue of the regular* $n \times n$ *complex pencil* $A_0 + \lambda A_1$ *with* g_0 *Jordan blocks in the KCF. Let* ρ_0, ρ_1 *be fixed integer numbers such that* $\rho_0 < g_0$ *and* $\rho_1 \leq n$.

Then *generically with respect perturbation pencils* $E_0 + \lambda E_1$ *such that*

$$\operatorname{rank}(E_0 + \lambda_0 E_1) \leq \rho_0 < g_0 \qquad \textit{and} \qquad \operatorname{rank}(E_1) \leq \rho_1 ,$$

- *There are* $g_0 - \rho_0$ *Jordan blocks associated with* λ_0 *in the KCF of* $A_0 + E_0 + \lambda(A_1 + E_1)$*, and*
- *If* $g_0 - \rho_0 - \rho_1 > 0$*, then they are*
  1. *the* $g_0 - \rho_0 - \rho_1$ *smallest Jordan Blocks for* λ_0 *in the KCF of* $A_0 + \lambda A_1$,
  2. *together with* ρ_1 *Jordan* 1×1 *blocks for* λ_0.
- *Otherwise, all of them are* 1×1 *Jordan blocks for* λ_0.

Perturbation of KCF of REGULAR matrix pencils: Theorem

Theorem (De Terán-D-Moro, SIMAX, 2008)

Let λ_0 be an eigenvalue of the regular $n \times n$ complex pencil $A_0 + \lambda A_1$ with g_0 Jordan blocks in the KCF. Let ρ_0, ρ_1 be fixed integer numbers such that $\rho_0 < g_0$ and $\rho_1 \leq n$.

Then *generically with respect perturbation pencils $E_0 + \lambda E_1$ such that*

$$\operatorname{rank}(E_0 + \lambda_0 E_1) \leq \rho_0 < g_0 \quad \textit{and} \quad \operatorname{rank}(E_1) \leq \rho_1 \,,$$

- *There are $g_0 - \rho_0$ Jordan blocks associated with λ_0 in the KCF of $A_0 + E_0 + \lambda(A_1 + E_1)$, and*
- *If $g_0 - \rho_0 - \rho_1 > 0$, then they are*
  1. *the $g_0 - \rho_0 - \rho_1$ smallest Jordan Blocks for λ_0 in the KCF of $A_0 + \lambda A_1$,*
  2. *together with ρ_1 Jordan 1×1 blocks for λ_0.*
- *Otherwise, all of them are 1×1 Jordan blocks for λ_0.*

Perturbation of KCF of REGULAR matrix pencils: Theorem

Theorem (De Terán-D-Moro, SIMAX, 2008)

Let λ_0 be an eigenvalue of the regular $n \times n$ complex pencil $A_0 + \lambda A_1$ with g_0 Jordan blocks in the KCF. Let ρ_0, ρ_1 be fixed integer numbers such that $\rho_0 < g_0$ and $\rho_1 \leq n$.

Then *generically with respect perturbation pencils $E_0 + \lambda E_1$ such that*

$$\mathrm{rank}(E_0 + \lambda_0 E_1) \leq \rho_0 < g_0 \qquad \textit{and} \qquad \mathrm{rank}(E_1) \leq \rho_1 \,,$$

- *There are $g_0 - \rho_0$ Jordan blocks associated with λ_0 in the KCF of $A_0 + E_0 + \lambda(A_1 + E_1)$, and*
- *If $g_0 - \rho_0 - \rho_1 > 0$, then they are*
  1. *the $g_0 - \rho_0 - \rho_1$ smallest Jordan Blocks for λ_0 in the KCF of $A_0 + \lambda A_1$,*
  2. *together with ρ_1 Jordan 1×1 blocks for λ_0.*
- *Otherwise, all of them are 1×1 Jordan blocks for λ_0.*

Theorem (De Terán-D-Moro, SIMAX, 2008)

Let λ_0 *be an eigenvalue of the regular* $n \times n$ *complex pencil* $A_0 + \lambda A_1$ *with* g_0 *Jordan blocks in the KCF. Let* ρ_0, ρ_1 *be fixed integer numbers such that* $\rho_0 < g_0$ *and* $\rho_1 \leq n$.

Then *generically with respect perturbation pencils* $E_0 + \lambda E_1$ *such that*

$$\operatorname{rank}(E_0 + \lambda_0 E_1) \leq \rho_0 < g_0 \qquad \textit{and} \qquad \operatorname{rank}(E_1) \leq \rho_1 \,,$$

- *There are* $g_0 - \rho_0$ *Jordan blocks associated with* λ_0 *in the KCF of* $A_0 + E_0 + \lambda(A_1 + E_1)$, *and*
- *If* $g_0 - \rho_0 - \rho_1 > 0$, *then they are*
  1. *the* $g_0 - \rho_0 - \rho_1$ *smallest Jordan Blocks for* λ_0 *in the KCF of* $A_0 + \lambda A_1$,
  2. *together with* ρ_1 *Jordan* 1×1 *blocks for* λ_0.
- *Otherwise, all of them are* 1×1 *Jordan blocks for* λ_0.

Perturbation of KCF of REGULAR matrix pencils: Theorem

Theorem (De Terán-D-Moro, SIMAX, 2008)

Let λ_0 be an eigenvalue of the regular $n \times n$ complex pencil $A_0 + \lambda A_1$ with g_0 Jordan blocks in the KCF. Let ρ_0, ρ_1 be fixed integer numbers such that $\rho_0 < g_0$ and $\rho_1 \leq n$.

Then *generically with respect perturbation pencils $E_0 + \lambda E_1$ such that*

$$\mathrm{rank}(E_0 + \lambda_0 E_1) \leq \rho_0 < g_0 \qquad \text{and} \qquad \mathrm{rank}(E_1) \leq \rho_1 \,,$$

- *There are $g_0 - \rho_0$ Jordan blocks associated with λ_0 in the KCF of $A_0 + E_0 + \lambda(A_1 + E_1)$, and*
- *If $g_0 - \rho_0 - \rho_1 > 0$, then they are*
  1. *the $g_0 - \rho_0 - \rho_1$ smallest Jordan Blocks for λ_0 in the KCF of $A_0 + \lambda A_1$,*
  2. *together with ρ_1 Jordan 1×1 blocks for λ_0.*
- *Otherwise, all of them are 1×1 Jordan blocks for λ_0.*

Outline

Perturbation of elementary divisors (Smith normal form) of REGULAR matrix polynomials: Example (1)

Let $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ be a regular poly with two distinct eigenvalues $-5, -1$ and elementary divisors

$$\begin{array}{lllll} (\lambda+5)^5, & (\lambda+5)^4, & (\lambda+5)^3, & (\lambda+5)^2, & \\ (\lambda+1)^7, & (\lambda+1)^6, & (\lambda+1)^3, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Then generically with respect perturbations $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ such that

$$\begin{array}{ll} \mathrm{rank}(E(-5)) = 2, & \mathrm{nrank}(E(\lambda) - E(-5)) = 1 \\ \mathrm{rank}(E(-1)) = 1, & \mathrm{nrank}(E(\lambda) - E(-1)) = 2 \end{array}$$

the elementary divisors of $P(\lambda) + E(\lambda)$ associated with eigenvalues -5 and -1 are

$$\begin{array}{lllll} *, & *, & (\lambda+5)^1, & (\lambda+5)^2, & \\ *, & (\lambda+1)^1, & (\lambda+1)^1, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Perturbation of elementary divisors (Smith normal form) of REGULAR matrix polynomials: Example (1)

Let $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ be a regular poly with two distinct eigenvalues $-5, -1$ and elementary divisors

$$\begin{array}{lllll} (\lambda+5)^5, & (\lambda+5)^4, & (\lambda+5)^3, & (\lambda+5)^2, & \\ (\lambda+1)^7, & (\lambda+1)^6, & (\lambda+1)^3, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Then generically with respect perturbations $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ such that

$$\begin{array}{ll} \operatorname{rank}(E(-5)) = 2, & \operatorname{nrank}(E(\lambda) - E(-5)) = 1 \\ \operatorname{rank}(E(-1)) = 1, & \operatorname{nrank}(E(\lambda) - E(-1)) = 2 \end{array}$$

the elementary divisors of $P(\lambda) + E(\lambda)$ associated with eigenvalues -5 and -1 are

$$\begin{array}{lllll} *, & *, & (\lambda+5)^1, & (\lambda+5)^2, & \\ *, & (\lambda+1)^1, & (\lambda+1)^1, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Perturbation of elementary divisors (Smith normal form) of REGULAR matrix polynomials: Example (1)

Let $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ be a regular poly with two distinct eigenvalues $-5, -1$ and elementary divisors

$$\begin{array}{lllll} (\lambda+5)^5, & (\lambda+5)^4, & (\lambda+5)^3, & (\lambda+5)^2, & \\ (\lambda+1)^7, & (\lambda+1)^6, & (\lambda+1)^3, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Then generically with respect perturbations $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ such that

$$\begin{array}{ll} \mathrm{rank}(E(-5)) = 2, & \mathrm{nrank}(E(\lambda) - E(-5)) = 1 \\ \mathrm{rank}(E(-1)) = 1, & \mathrm{nrank}(E(\lambda) - E(-1)) = 2 \end{array}$$

the elementary divisors of $P(\lambda) + E(\lambda)$ associated with eigenvalues -5 and -1 are

$$\begin{array}{lllll} *, & *, & (\lambda+5)^1, & (\lambda+5)^2, & \\ *, & (\lambda+1)^1, & (\lambda+1)^1, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Perturbation of elementary divisors (Smith normal form) of REGULAR matrix polynomials: Example (1)

Let $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ be a regular poly with two distinct eigenvalues $-5, -1$ and elementary divisors

$$\begin{array}{lllll} (\lambda+5)^5, & (\lambda+5)^4, & (\lambda+5)^3, & (\lambda+5)^2, & \\ (\lambda+1)^7, & (\lambda+1)^6, & (\lambda+1)^3, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Then generically with respect perturbations $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ such that

$$\begin{array}{ll} \operatorname{rank}(E(-5)) = 2, & \operatorname{nrank}(E(\lambda) - E(-5)) = 1 \\ \operatorname{rank}(E(-1)) = 1, & \operatorname{nrank}(E(\lambda) - E(-1)) = 2 \end{array}$$

the elementary divisors of $P(\lambda) + E(\lambda)$ associated with eigenvalues -5 and -1 are

$$\begin{array}{lllll} *, & *, & (\lambda+5)^1, & (\lambda+5)^2, & \\ *, & (\lambda+1)^1, & (\lambda+1)^1, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Perturbation of elementary divisors (Smith normal form) of REGULAR matrix polynomials: Example (1)

Let $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ be a regular poly with two distinct eigenvalues $-5, -1$ and elementary divisors

$$\begin{array}{lllll} (\lambda+5)^5, & (\lambda+5)^4, & (\lambda+5)^3, & (\lambda+5)^2, & \\ (\lambda+1)^7, & (\lambda+1)^6, & (\lambda+1)^3, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Then generically with respect perturbations $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ such that

$$\begin{array}{ll} \mathrm{rank}(E(-5)) = 2, & \mathrm{nrank}(E(\lambda) - E(-5)) = 1 \\ \mathrm{rank}(E(-1)) = 1, & \mathrm{nrank}(E(\lambda) - E(-1)) = 2 \end{array}$$

the elementary divisors of $P(\lambda) + E(\lambda)$ associated with eigenvalues -5 and -1 are

$$\begin{array}{lllll} *, & *, & (\lambda+5)^1, & (\lambda+5)^2, & \\ *, & (\lambda+1)^1, & (\lambda+1)^1, & (\lambda+1)^2, & (\lambda+1) \end{array}$$

Remarks on the example

- It is possible to have different generic behaviors for different eigenvalues as in the example, but this is not usual because "generically"

$$\operatorname{nrank}(E(\lambda)) = \operatorname{rank}(E(-5)) = \operatorname{rank}(E(-1))$$
$$\operatorname{nrank}(E(\lambda) - E(-5)) = \operatorname{nrank}(E(\lambda) - E(-1)).$$

- Note that for each eigenvalue λ_0 the number of elementary divisors that are transformed into elementary divisors of degree one is

$$\rho_1 = \operatorname{nrank}(E(\lambda) - E(\lambda_0)),$$

not the rank of the coefficient of the highest degree term as in pencils.

- $E(\lambda) = E_0 + \lambda E_1 \Longrightarrow \rho_1 = \operatorname{nrank}((\lambda - \lambda_0)E_1) = \operatorname{rank}(E_1)$.
- In perturbation of matrices we deal with the matrix polys $A - \lambda I$ and $(A + E_0) - \lambda I$, so $E(\lambda) = E_0$ and $\rho_1 = \operatorname{nrank}(E(\lambda) - E(\lambda_0)) = 0$.

Remarks on the example

- It is possible to have different generic behaviors for different eigenvalues as in the example, but this is not usual because "generically"

$$\begin{aligned} \operatorname{nrank}(E(\lambda)) &= \operatorname{rank}(E(-5)) = \operatorname{rank}(E(-1)) \\ \operatorname{nrank}(E(\lambda) - E(-5)) &= \operatorname{nrank}(E(\lambda) - E(-1)). \end{aligned}$$

- Note that for each eigenvalue λ_0 the number of elementary divisors that are transformed into elementary divisors of degree one is

$$\rho_1 = \operatorname{nrank}(E(\lambda) - E(\lambda_0)),$$

  not the rank of the coefficient of the highest degree term as in pencils.

- $E(\lambda) = E_0 + \lambda E_1 \Longrightarrow \rho_1 = \operatorname{nrank}((\lambda - \lambda_0)E_1) = \operatorname{rank}(E_1)$.
- In perturbation of matrices we deal with the matrix polys $A - \lambda I$ and $(A + E_0) - \lambda I$, so $E(\lambda) = E_0$ and $\rho_1 = \operatorname{nrank}(E(\lambda) - E(\lambda_0)) = 0$.

Remarks on the example

- It is possible to have different generic behaviors for different eigenvalues as in the example, but this is not usual because "generically"

$$\mathrm{nrank}(E(\lambda)) = \mathrm{rank}(E(-5)) = \mathrm{rank}(E(-1))$$
$$\mathrm{nrank}(E(\lambda) - E(-5)) = \mathrm{nrank}(E(\lambda) - E(-1)).$$

- Note that for each eigenvalue λ_0 the number of elementary divisors that are transformed into elementary divisors of degree one is

$$\rho_1 = \mathrm{nrank}(E(\lambda) - E(\lambda_0)),$$

not the rank of the coefficient of the highest degree term as in pencils.

- $E(\lambda) = E_0 + \lambda E_1 \Longrightarrow \rho_1 = \mathrm{nrank}((\lambda - \lambda_0)E_1) = \mathrm{rank}(E_1).$

- In perturbation of matrices we deal with the matrix polys $A - \lambda I$ and $(A + E_0) - \lambda I$, so $E(\lambda) = E_0$ and $\rho_1 = \mathrm{nrank}(E(\lambda) - E(\lambda_0)) = 0$.

Remarks on the example

- It is possible to have different generic behaviors for different eigenvalues as in the example, but this is not usual because "generically"

$$\operatorname{nrank}(E(\lambda)) = \operatorname{rank}(E(-5)) = \operatorname{rank}(E(-1))$$
$$\operatorname{nrank}(E(\lambda) - E(-5)) = \operatorname{nrank}(E(\lambda) - E(-1)).$$

- Note that for each eigenvalue λ_0 the number of elementary divisors that are transformed into elementary divisors of degree one is

$$\rho_1 = \operatorname{nrank}(E(\lambda) - E(\lambda_0)),$$

  not the rank of the coefficient of the highest degree term as in pencils.

- $E(\lambda) = E_0 + \lambda E_1 \Longrightarrow \rho_1 = \operatorname{nrank}((\lambda - \lambda_0)E_1) = \operatorname{rank}(E_1).$
- In perturbation of matrices we deal with the matrix polys $A - \lambda I$ and $(A + E_0) - \lambda I$, so $E(\lambda) = E_0$ and $\rho_1 = \operatorname{nrank}(E(\lambda) - E(\lambda_0)) = 0$.

Low rank perturbation of elementary divisors of REGULAR complex matrix polynomials: Theorem

Theorem (De Terán-D., LAA, 2009)

Let λ_0 be eigenvalue of the regular $n \times n$ poly $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with g_0 elementary divisors. Let ρ_0, ρ_1 be fixed integers s.t. $\rho_0 < g_0$ and $\rho_1 \leq n$.

Then *generically with respect perturbation polys $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ such that*

$$\operatorname{rank}(E(\lambda_0)) \leq \rho_0 < g_0 \qquad \textit{and} \qquad \operatorname{nrank}(E(\lambda) - E(\lambda_0)) \leq \rho_1 .$$

- *There are $g_0 - \rho_0$ elementary divisors associated with λ_0 in $P(\lambda) + E(\lambda)$, and*
- *If $g_0 - \rho_0 - \rho_1 > 0$, then they are*
  1. *the $g_0 - \rho_0 - \rho_1$ elementary divisors with smallest degree associated with λ_0 in $P(\lambda)$,*
  2. *together with ρ_1 elementary divisors $(\lambda - \lambda_0)$ of degree 1.*
- *Otherwise, all of them are equal to $(\lambda - \lambda_0)$.*

Low rank perturbation of elementary divisors of REGULAR complex matrix polynomials: Theorem

Theorem (De Terán-D., LAA, 2009)

Let λ_0 be eigenvalue of the regular $n \times n$ poly $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with g_0 elementary divisors. Let ρ_0, ρ_1 be fixed integers s.t. $\rho_0 < g_0$ and $\rho_1 \leq n$.

Then *generically with respect perturbation polys $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ such that*

$$\operatorname{rank}(E(\lambda_0)) \leq \rho_0 < g_0 \qquad \textit{and} \qquad \operatorname{nrank}(E(\lambda) - E(\lambda_0)) \leq \rho_1 \,.$$

- *There are $g_0 - \rho_0$ elementary divisors associated with λ_0 in $P(\lambda) + E(\lambda)$, and*
- *If $g_0 - \rho_0 - \rho_1 > 0$, then they are*
  1. *the $g_0 - \rho_0 - \rho_1$ elementary divisors with smallest degree associated with λ_0 in $P(\lambda)$,*
  2. *together with ρ_1 elementary divisors $(\lambda - \lambda_0)$ of degree 1.*
- *Otherwise, all of them are equal to $(\lambda - \lambda_0)$.*

Low rank perturbation of elementary divisors of REGULAR complex matrix polynomials: Theorem

Theorem (De Terán-D., LAA, 2009)

Let λ_0 *be eigenvalue of the regular* $n \times n$ *poly* $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ *with* g_0 *elementary divisors. Let* ρ_0, ρ_1 *be fixed integers s.t.* $\rho_0 < g_0$ *and* $\rho_1 \leq n$.

Then *generically with respect perturbation polys* $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ *such that*

$$\operatorname{rank}(E(\lambda_0)) \leq \rho_0 < g_0 \qquad \textit{and} \qquad \operatorname{nrank}(E(\lambda) - E(\lambda_0)) \leq \rho_1 .$$

- *There are* $g_0 - \rho_0$ *elementary divisors associated with* λ_0 *in* $P(\lambda) + E(\lambda)$*, and*
- *If* $g_0 - \rho_0 - \rho_1 > 0$*, then they are*
  1. *the* $g_0 - \rho_0 - \rho_1$ *elementary divisors with smallest degree associated with* λ_0 *in* $P(\lambda)$*,*
  2. *together with* ρ_1 *elementary divisors* $(\lambda - \lambda_0)$ *of degree* 1.
- *Otherwise, all of them are equal to* $(\lambda - \lambda_0)$.

Low rank perturbation of elementary divisors of REGULAR complex matrix polynomials: Theorem

Theorem (De Terán-D., LAA, 2009)

Let λ_0 *be eigenvalue of the regular* $n \times n$ *poly* $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ *with* g_0 *elementary divisors. Let* ρ_0, ρ_1 *be fixed integers s.t.* $\rho_0 < g_0$ *and* $\rho_1 \leq n$.

Then *generically with respect perturbation polys* $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ *such that*

$$\operatorname{rank}(E(\lambda_0)) \leq \rho_0 < g_0 \qquad \text{and} \qquad \operatorname{nrank}(E(\lambda) - E(\lambda_0)) \leq \rho_1 \,.$$

- *There are* $g_0 - \rho_0$ *elementary divisors associated with* λ_0 *in* $P(\lambda) + E(\lambda)$, *and*
- *If* $g_0 - \rho_0 - \rho_1 > 0$, *then they are*
  1. *the* $g_0 - \rho_0 - \rho_1$ *elementary divisors with smallest degree associated with* λ_0 *in* $P(\lambda)$,
  2. *together with* ρ_1 *elementary divisors* $(\lambda - \lambda_0)$ *of degree* 1.
- *Otherwise, all of them are equal to* $(\lambda - \lambda_0)$.

Low rank perturbation of elementary divisors of REGULAR complex matrix polynomials: Theorem

Theorem (De Terán-D., LAA, 2009)

Let λ_0 be eigenvalue of the regular $n \times n$ poly $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with g_0 elementary divisors. Let ρ_0, ρ_1 be fixed integers s.t. $\rho_0 < g_0$ and $\rho_1 \leq n$.

Then *generically with respect perturbation polys $E(\lambda) = \sum_{j=0}^{\ell} \lambda^j E_j$ such that*

$$\operatorname{rank}(E(\lambda_0)) \leq \rho_0 < g_0 \qquad \text{and} \qquad \operatorname{nrank}(E(\lambda) - E(\lambda_0)) \leq \rho_1 \,.$$

- *There are $g_0 - \rho_0$ elementary divisors associated with λ_0 in $P(\lambda) + E(\lambda)$, and*
- *If $g_0 - \rho_0 - \rho_1 > 0$, then they are*
  1. *the $g_0 - \rho_0 - \rho_1$ elementary divisors with smallest degree associated with λ_0 in $P(\lambda)$,*
  2. *together with ρ_1 elementary divisors $(\lambda - \lambda_0)$ of degree 1.*
- *Otherwise, all of them are equal to $(\lambda - \lambda_0)$.*

Outline

1. Antecedents: Matrices and regular matrix pencils
2. Low rank perturbation of regular matrix polynomials
3. **Keys of the proof through Thompson's result**
4. Low rank perturbation of singular pencils
5. Conclusions

Thompson's result on rank one perturbations

Theorem (Thompson, Can. J. Math, 1980)

Let $P(\lambda)$ *be an* $n \times n$ *complex matrix polynomial with invariant factors*

$$h_n(P)|h_{n-1}(P)|\ldots|h_1(P),$$

(*Smith form* $U(\lambda)P(\lambda)V(\lambda) = \mathrm{diag}(h_n(P), h_{n-1}(P), \ldots, h_1(P))$)

let $Z(\lambda)$ *be a matrix polynomial with* $\mathrm{nrank}\, Z(\lambda) \leq 1$*, and*

$$M(\lambda) = P(\lambda) + Z(\lambda).$$

Then the achievable invariant factors $h_n(M)|h_{n-1}(M)|\ldots|h_1(M)$ *of* $M(\lambda)$ *as* $Z(\lambda)$ *ranges over all matrix polynomials with* $\mathrm{nrank}\, Z(\lambda) \leq 1$ *are precisely those polynomials that satisfy*

$$h_n(P)|h_{n-1}(M)|h_{n-2}(P)|h_{n-3}(M)|\ldots$$
$$h_n(M)|h_{n-1}(P)|h_{n-2}(M)|h_{n-3}(P)|\ldots$$

Thompson's result on rank one perturbations

Theorem (Thompson, Can. J. Math, 1980)

Let $P(\lambda)$ *be an* $n \times n$ *complex matrix polynomial with invariant factors*

$$h_n(P)|h_{n-1}(P)|\ldots|h_1(P),$$

(*Smith form* $U(\lambda)P(\lambda)V(\lambda) = \mathrm{diag}(h_n(P), h_{n-1}(P), \ldots, h_1(P))$)

let $Z(\lambda)$ *be a matrix polynomial with* $\mathrm{nrank}\, Z(\lambda) \leq 1$*, and*

$$M(\lambda) = P(\lambda) + Z(\lambda).$$

Then the achievable invariant factors $h_n(M)|h_{n-1}(M)|\ldots|h_1(M)$ *of* $M(\lambda)$ *as* $Z(\lambda)$ *ranges over all matrix polynomials with* $\mathrm{nrank}\, Z(\lambda) \leq 1$ *are precisely those polynomials that satisfy*

$$h_n(P)|h_{n-1}(M)|h_{n-2}(P)|h_{n-3}(M)|\ldots$$
$$h_n(M)|h_{n-1}(P)|h_{n-2}(M)|h_{n-3}(P)|\ldots$$

Thompson's result on rank one perturbations

Theorem (Thompson, Can. J. Math, 1980)

Let $P(\lambda)$ be an $n \times n$ complex matrix polynomial with invariant factors

$$h_n(P)|h_{n-1}(P)|\ldots|h_1(P),$$

(*Smith form* $U(\lambda)P(\lambda)V(\lambda) = \mathrm{diag}(h_n(P), h_{n-1}(P), \ldots, h_1(P))$)

let $Z(\lambda)$ be a matrix polynomial with $\mathrm{nrank}\, Z(\lambda) \leq 1$, and

$$M(\lambda) = P(\lambda) + Z(\lambda).$$

Then the achievable invariant factors $h_n(M)|h_{n-1}(M)|\ldots|h_1(M)$ of $M(\lambda)$ as $Z(\lambda)$ ranges over all matrix polynomials with $\mathrm{nrank}\, Z(\lambda) \leq 1$ are precisely those polynomials that satisfy

$$h_n(P)|h_{n-1}(M)|h_{n-2}(P)|h_{n-3}(M)|\ldots$$
$$h_n(M)|h_{n-1}(P)|h_{n-2}(M)|h_{n-3}(P)|\ldots$$

Thompson's result on rank one perturbations

Theorem (Thompson, Can. J. Math, 1980)

Let $P(\lambda)$ be an $n \times n$ complex matrix polynomial with invariant factors

$$h_n(P)|h_{n-1}(P)|\ldots|h_1(P),$$

(*Smith form* $U(\lambda)P(\lambda)V(\lambda) = \mathrm{diag}(h_n(P), h_{n-1}(P), \ldots, h_1(P))$)

let $Z(\lambda)$ be a matrix polynomial with $\mathrm{nrank}\, Z(\lambda) \leq 1$, and

$$M(\lambda) = P(\lambda) + Z(\lambda).$$

Then the achievable invariant factors $h_n(M)|h_{n-1}(M)|\ldots|h_1(M)$ of $M(\lambda)$ as $Z(\lambda)$ ranges over all matrix polynomials with $\mathrm{nrank}\, Z(\lambda) \leq 1$ are precisely those polynomials that satisfy

$$h_n(P)|h_{n-1}(M)|h_{n-2}(P)|h_{n-3}(M)|\ldots$$
$$h_n(M)|h_{n-1}(P)|h_{n-2}(M)|h_{n-3}(P)|\ldots$$

Remarks on Thompson's result for rank one perturbations

- It describes all possible elementary divisors under rank-one polynomial perturbations, but not the generic behavior.
- It's valid for singular polys $(h_n(P)|h_{n-1}(P)|\ldots|h_k(P)|0|0|\ldots|0)$
- We will use it only for **regular polynomials**. If there are g_0 elementary divisors of $P(\lambda)$ associated with λ_0, we denote their degrees, as

$$0 = m_n = \cdots = m_{g_0+1} < m_{g_0} \leq \cdots \leq m_2 \leq m_1 .$$

Analogously, $\widetilde{m}_n \leq \cdots \leq \widetilde{m}_1$ denote the degrees of the λ_0-elementary divisors of $M(\lambda) = P(\lambda) + Z(\lambda)$.

- $$\left. \begin{array}{l} h_n(P)|h_{n-1}(M)|h_{n-2}(P)|h_{n-3}(M)|\ldots \\ h_n(M)|h_{n-1}(P)|h_{n-2}(M)|h_{n-3}(P)|\ldots \end{array} \right\} \Longrightarrow m_{i+1} \leq \widetilde{m}_i$$

Remarks on Thompson's result for rank one perturbations

- It describes all possible elementary divisors under rank-one polynomial perturbations, but not the generic behavior.
- It's valid for singular polys $(h_n(P)|h_{n-1}(P)|\dots|h_k(P)|0|0|\dots|0)$
- We will use it only for **regular polynomials**. If there are g_0 elementary divisors of $P(\lambda)$ associated with λ_0, we denote their degrees, as

$$0 = m_n = \cdots = m_{g_0+1} < m_{g_0} \leq \cdots \leq m_2 \leq m_1 .$$

  Analogously, $\widetilde{m}_n \leq \cdots \leq \widetilde{m}_1$ denote the degrees of the λ_0-elementary divisors of $M(\lambda) = P(\lambda) + Z(\lambda)$.

- $$\left. \begin{array}{l} h_n(P)|h_{n-1}(M)|h_{n-2}(P)|h_{n-3}(M)|\dots \\ h_n(M)|h_{n-1}(P)|h_{n-2}(M)|h_{n-3}(P)|\dots \end{array} \right\} \Longrightarrow m_{i+1} \leq \widetilde{m}_i$$

Remarks on Thompson's result for rank one perturbations

- It describes all possible elementary divisors under rank-one polynomial perturbations, but not the generic behavior.
- It's valid for singular polys $(h_n(P)|h_{n-1}(P)|\dots|h_k(P)|0|0|\dots|0)$
- We will use it only for **regular polynomials**. If there are g_0 elementary divisors of $P(\lambda)$ associated with λ_0, we denote their degrees, as

$$0 = m_n = \cdots = m_{g_0+1} < m_{g_0} \leq \cdots \leq m_2 \leq m_1 .$$

  Analogously, $\widetilde{m}_n \leq \cdots \leq \widetilde{m}_1$ denote the degrees of the λ_0-elementary divisors of $M(\lambda) = P(\lambda) + Z(\lambda)$.

-

$$\left. \begin{array}{l} h_n(P)|h_{n-1}(M)|h_{n-2}(P)|h_{n-3}(M)|\dots \\ h_n(M)|h_{n-1}(P)|h_{n-2}(M)|h_{n-3}(P)|\dots \end{array} \right\} \Longrightarrow m_{i+1} \leq \widetilde{m}_i$$

Remarks on Thompson's result for rank one perturbations

- It describes all possible elementary divisors under rank-one polynomial perturbations, but not the generic behavior.
- It's valid for singular polys $(h_n(P)|h_{n-1}(P)|\dots|h_k(P)|0|0|\dots|0)$
- We will use it only for **regular polynomials**. If there are g_0 elementary divisors of $P(\lambda)$ associated with λ_0, we denote their degrees, as

  $$0 = m_n = \cdots = m_{g_0+1} < m_{g_0} \le \cdots \le m_2 \le m_1.$$

  Analogously, $\widetilde{m}_n \le \cdots \le \widetilde{m}_1$ denote the degrees of the λ_0-elementary divisors of $M(\lambda) = P(\lambda) + Z(\lambda)$.
- $$\left.\begin{array}{l} h_n(P)|h_{n-1}(M)|h_{n-2}(P)|h_{n-3}(M)|\dots \\ h_n(M)|h_{n-1}(P)|h_{n-2}(M)|h_{n-3}(P)|\dots \end{array}\right\} \Longrightarrow \boxed{m_{i+1} \le \widetilde{m}_i}$$

Proof "generic" behavior for $\lambda_0 = 0$ and rank 1-1 perturbations (I)

Let $\lambda_0 = 0$ be eigenvalue of $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with g_0 elementary divisors.

Consider perturbations $E(\lambda) = E_0 + \lambda E_1 + \cdots + \lambda^\ell E_\ell$ s.t.

$$\mathrm{rank}(E(0)) = \mathrm{rank}(E_0) = 1 \quad \text{and}$$
$$\mathrm{nrank}(E(\lambda) - E(0)) = \mathrm{nrank}(\lambda\, E_1 + \cdots + \lambda^\ell\, E_\ell) = 1.$$

Write

$$P(\lambda) + E(\lambda) = (P(\lambda) + E_0) + \lambda\, E_1 + \cdots + \lambda^\ell\, E_\ell,$$

as a sequence of TWO rank one perturbations and apply Thompson's twice, together with

(# elem. divisors for $\lambda_0 = 0$ for any regular $Q(\lambda)) = \dim \mathrm{Ker}\, Q(0)$,

then, as $P(0) + E(0) = P(0) + E_0$, **for $\lambda_0 = 0$**

(# elem. divisors of $P(\lambda) + E(\lambda)) =$ (# elem. divisors of $P(\lambda) + E_0$),

and generically

(# elem. divisors of $P(\lambda) + E_0$) = (# elem. divisors of $P(\lambda)$) - 1 ,

Proof "generic" behavior for $\lambda_0 = 0$ and rank 1-1 perturbations (I)

Let $\lambda_0 = 0$ be eigenvalue of $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with g_0 elementary divisors. Consider perturbations $E(\lambda) = E_0 + \lambda E_1 + \cdots + \lambda^\ell E_\ell$ s.t.

$$\mathrm{rank}(E(0)) = \mathrm{rank}(E_0) = 1 \quad \text{and}$$
$$\mathrm{nrank}(E(\lambda) - E(0)) = \mathrm{nrank}(\lambda E_1 + \cdots + \lambda^\ell E_\ell) = 1.$$

Write

$$P(\lambda) + E(\lambda) = (P(\lambda) + E_0) + \lambda E_1 + \cdots + \lambda^\ell E_\ell,$$

as a sequence of TWO rank one perturbations and apply Thompson's twice, together with

(# elem. divisors for $\lambda_0 = 0$ for any regular $Q(\lambda)) = \dim \mathrm{Ker}\, Q(0)$,

then, as $P(0) + E(0) = P(0) + E_0$, **for $\lambda_0 = 0$**

(# elem. divisors of $P(\lambda) + E(\lambda)$) = (# elem. divisors of $P(\lambda) + E_0$),

and generically

(# elem. divisors of $P(\lambda) + E_0$) = (# elem. divisors of $P(\lambda)$) - 1 ,

Proof "generic" behavior for $\lambda_0 = 0$ and rank 1-1 perturbations (I)

Let $\lambda_0 = 0$ be eigenvalue of $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with g_0 elementary divisors. Consider perturbations $E(\lambda) = E_0 + \lambda E_1 + \cdots + \lambda^\ell E_\ell$ s.t.

$\mathrm{rank}(E(0)) = \mathrm{rank}(E_0) = 1$ and
$\mathrm{nrank}(E(\lambda) - E(0)) = \mathrm{nrank}(\lambda\, E_1 + \cdots + \lambda^\ell\, E_\ell) = 1.$

Write

$$P(\lambda) + E(\lambda) = (P(\lambda) + E_0) + \lambda\, E_1 + \cdots + \lambda^\ell\, E_\ell,$$

as a sequence of TWO rank one perturbations and apply Thompson's twice, together with

(# elem. divisors for $\lambda_0 = 0$ for any regular $Q(\lambda)$) $= \dim \mathrm{Ker}\, Q(0)$,

then, as $P(0) + E(0) = P(0) + E_0$, **for** $\lambda_0 = 0$

(# elem. divisors of $P(\lambda) + E(\lambda)$) = (# elem. divisors of $P(\lambda) + E_0$),

and generically

(# elem. divisors of $P(\lambda) + E_0$) = (# elem. divisors of $P(\lambda)$) - 1 ,

Proof "generic" behavior for $\lambda_0 = 0$ and rank 1-1 perturbations (I)

Let $\lambda_0 = 0$ be eigenvalue of $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with g_0 elementary divisors. Consider perturbations $E(\lambda) = E_0 + \lambda E_1 + \cdots + \lambda^\ell E_\ell$ s.t.

$\operatorname{rank}(E(0)) = \operatorname{rank}(E_0) = 1$ and
$\operatorname{nrank}(E(\lambda) - E(0)) = \operatorname{nrank}(\lambda E_1 + \cdots + \lambda^\ell E_\ell) = 1.$

Write

$$P(\lambda) + E(\lambda) = (P(\lambda) + E_0) + \lambda E_1 + \cdots + \lambda^\ell E_\ell,$$

as a sequence of TWO rank one perturbations and apply Thompson's twice, together with

(# elem. divisors for $\lambda_0 = 0$ for any regular $Q(\lambda)$) $= \dim \operatorname{Ker} Q(0)$,

then, as $P(0) + E(0) = P(0) + E_0$, **for $\lambda_0 = 0$**

(# elem. divisors of $P(\lambda) + E(\lambda)$) = (# elem. divisors of $P(\lambda) + E_0$),

and generically

(# elem. divisors of $P(\lambda) + E_0$) = (# elem. divisors of $P(\lambda)$) - 1 ,

Proof "generic" behavior for $\lambda_0 = 0$ and rank 1-1 perturbations (I)

Let $\lambda_0 = 0$ be eigenvalue of $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with g_0 elementary divisors. Consider perturbations $E(\lambda) = E_0 + \lambda E_1 + \cdots + \lambda^\ell E_\ell$ s.t.

$$\operatorname{rank}(E(0)) = \operatorname{rank}(E_0) = 1 \quad \text{and}$$
$$\operatorname{nrank}(E(\lambda) - E(0)) = \operatorname{nrank}(\lambda E_1 + \cdots + \lambda^\ell E_\ell) = 1.$$

Write

$$P(\lambda) + E(\lambda) = (P(\lambda) + E_0) + \lambda E_1 + \cdots + \lambda^\ell E_\ell,$$

as a sequence of TWO rank one perturbations and apply Thompson's twice, together with

(# elem. divisors for $\lambda_0 = 0$ for any regular $Q(\lambda)$) $= \dim \operatorname{Ker} Q(0)$,

then, as $P(0) + E(0) = P(0) + E_0$, **for $\lambda_0 = 0$**

(# elem. divisors of $P(\lambda) + E(\lambda)$) = (# elem. divisors of $P(\lambda) + E_0$),

and generically

(# elem. divisors of $P(\lambda) + E_0$) = (# elem. divisors of $P(\lambda)$) - 1 ,

Proof "generic" behavior for $\lambda_0 = 0$ and rank 1-1 perturbations (I)

Let $\lambda_0 = 0$ be eigenvalue of $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with g_0 elementary divisors. Consider perturbations $E(\lambda) = E_0 + \lambda E_1 + \cdots + \lambda^\ell E_\ell$ s.t.

$$\operatorname{rank}(E(0)) = \operatorname{rank}(E_0) = 1 \quad \text{and}$$
$$\operatorname{nrank}(E(\lambda) - E(0)) = \operatorname{nrank}(\lambda\, E_1 + \cdots + \lambda^\ell\, E_\ell) = 1.$$

Write

$$P(\lambda) + E(\lambda) = (P(\lambda) + E_0) + \lambda\, E_1 + \cdots + \lambda^\ell\, E_\ell,$$

as a sequence of TWO rank one perturbations and apply Thompson's twice, together with

(# elem. divisors for $\lambda_0 = 0$ for any regular $Q(\lambda)$) $= \dim \operatorname{Ker} Q(0)$,

then, as $P(0) + E(0) = P(0) + E_0$, **for $\lambda_0 = 0$**

(# elem. divisors of $P(\lambda) + E(\lambda)$) = (# elem. divisors of $P(\lambda) + E_0$),

and generically

(# elem. divisors of $P(\lambda) + E_0$) = (# elem. divisors of $P(\lambda)$) - 1 ,

- **First perturbation**: $P(\lambda) \longrightarrow P(\lambda) + E_0 (\equiv \widetilde{P}(\lambda)\,)$.

	$P(\lambda)$	$\widetilde{P}(\lambda)$
deg	$m_{g_0} \leq \cdots \leq m_1$	$\widetilde{m}_{g_0-1} \leq \cdots \leq \widetilde{m}_1$

$$\Longrightarrow \quad \begin{array}{c} m_2 \leq \widetilde{m}_1 \\ \vdots \\ m_{g_0-1} \leq \widetilde{m}_{g_0-2} \\ m_{g_0} \leq \widetilde{m}_{g_0-1} \end{array}$$

- **Second perturbation**: $\widetilde{P}(\lambda) \longrightarrow \widetilde{P}(\lambda) + \lambda\, E_1 + \cdots + \lambda^\ell\, E_\ell (\equiv \widehat{P}(\lambda)\,)$.

	$\widetilde{P}(\lambda)$	$\widehat{P}(\lambda)$
deg	$\widetilde{m}_{g_0-1} \leq \cdots \leq \widetilde{m}_1$	$\widehat{m}_{g_0-1} \leq \cdots \leq \widehat{m}_1$

$$\Longrightarrow \quad \begin{array}{c} \widetilde{m}_2 \leq \widehat{m}_1 \\ \vdots \\ \widetilde{m}_{g_0-1} \leq \widehat{m}_{g_0-2} \\ 0 = \widetilde{m}_{g_0} \leq \widehat{m}_{g_0-1} \end{array}$$

- **Generically**, we get the lowest possible values in both steps

$$\mathbf{1} = \widehat{m}_{g_0-1}, \quad \mathbf{m_{g_0}} = \widetilde{m}_{g_0-1} = \widehat{m}_{g_0-2}, \quad \ldots, \quad \mathbf{m_3} = \widetilde{m}_2 = \widehat{m}_1$$

Proof "generic" behavior for $\lambda_0 = 0$ and rank 1-1 perturbations (II)

- **First perturbation**: $P(\lambda) \longrightarrow P(\lambda) + E_0 (\equiv \widetilde{P}(\lambda))$.

	$P(\lambda)$	$\widetilde{P}(\lambda)$
deg	$m_{g_0} \le \cdots \le m_1$	$\widetilde{m}_{g_0-1} \le \cdots \le \widetilde{m}_1$

$$\Longrightarrow \quad \begin{matrix} m_2 \le \widetilde{m}_1 \\ \vdots \\ m_{g_0-1} \le \widetilde{m}_{g_0-2} \\ m_{g_0} \le \widetilde{m}_{g_0-1} \end{matrix}$$

- **Second perturbation**: $\widetilde{P}(\lambda) \longrightarrow \widetilde{P}(\lambda) + \lambda E_1 + \cdots + \lambda^\ell E_\ell (\equiv \widehat{P}(\lambda))$.

	$\widetilde{P}(\lambda)$	$\widehat{P}(\lambda)$
deg	$\widetilde{m}_{g_0-1} \le \cdots \le \widetilde{m}_1$	$\widehat{m}_{g_0-1} \le \cdots \le \widehat{m}_1$

$$\Longrightarrow \quad \begin{matrix} \widetilde{m}_2 \le \widehat{m}_1 \\ \vdots \\ \widetilde{m}_{g_0-1} \le \widehat{m}_{g_0-2} \\ 0 = \widetilde{m}_{g_0} \le \widehat{m}_{g_0-1} \end{matrix}$$

- **Generically**, we get the lowest possible values in both steps

$$1 = \widehat{m}_{g_0-1}, \quad \mathbf{m_{g_0}} = \widetilde{m}_{g_0-1} = \widehat{m}_{g_0-2}, \quad \ldots, \quad \mathbf{m_3} = \widetilde{m}_2 = \widehat{m}_1$$

Proof "generic" behavior for $\lambda_0 = 0$ and rank 1-1 perturbations (II)

- **First perturbation**: $P(\lambda) \longrightarrow P(\lambda) + E_0 (\equiv \widetilde{P}(\lambda))$.

	$P(\lambda)$	$\widetilde{P}(\lambda)$
deg	$m_{g_0} \leq \cdots \leq m_1$	$\widetilde{m}_{g_0-1} \leq \cdots \leq \widetilde{m}_1$

$$\Longrightarrow \begin{array}{c} m_2 \leq \widetilde{m}_1 \\ \vdots \\ m_{g_0-1} \leq \widetilde{m}_{g_0-2} \\ m_{g_0} \leq \widetilde{m}_{g_0-1} \end{array}$$

- **Second perturbation**: $\widetilde{P}(\lambda) \longrightarrow \widetilde{P}(\lambda) + \lambda E_1 + \cdots + \lambda^\ell E_\ell (\equiv \widehat{P}(\lambda))$.

	$\widetilde{P}(\lambda)$	$\widehat{P}(\lambda)$
deg	$\widetilde{m}_{g_0-1} \leq \cdots \leq \widetilde{m}_1$	$\widehat{m}_{g_0-1} \leq \cdots \leq \widehat{m}_1$

$$\Longrightarrow \begin{array}{c} \widetilde{m}_2 \leq \widehat{m}_1 \\ \vdots \\ \widetilde{m}_{g_0-1} \leq \widehat{m}_{g_0-2} \\ 0 = \widetilde{m}_{g_0} \leq \widehat{m}_{g_0-1} \end{array}$$

- **Generically**, we get the lowest possible values in both steps

$$\mathbf{1} = \widehat{m}_{g_0-1}, \quad \mathbf{m_{g_0}} = \widetilde{m}_{g_0-1} = \widehat{m}_{g_0-2}, \quad \ldots, \quad \mathbf{m_3} = \widetilde{m}_2 = \widehat{m}_1$$

Outline

No elementary divisors are destroyed...

Theorem (De Terán and D., SIMAX, 2007)

Let $P(\lambda)$ be an $m \times n$ complex matrix pencil s.t. $\operatorname{nrank}(P(\lambda)) < \min\{m,n\}$, *and λ_0 be an eigenvalue of $P(\lambda)$. Let ρ be a positive integer number such that*

$$\operatorname{nrank}(P(\lambda)) + \rho \leq \min\{m,n\}.$$

Then generically with respect $m \times n$ perturbation pencils $E(\lambda)$ such that

$$\operatorname{nrank}(E(\lambda)) \leq \rho,$$

the elementary divisors associated with λ_0 of $P(\lambda)$ and $P(\lambda)+E(\lambda)$ **are exactly the same.**

Remark

It seems possible to extend the proof from pencils to higher degree singular matrix polynomials.

No elementary divisors are destroyed...

Theorem (De Terán and D., SIMAX, 2007)

Let $P(\lambda)$ be an $m \times n$ complex matrix pencil s.t. $\operatorname{nrank}(P(\lambda)) < \min\{m, n\}$, *and λ_0 be an eigenvalue of $P(\lambda)$. Let ρ be a positive integer number such that*

$$\operatorname{nrank}(P(\lambda)) + \rho \leq \min\{m, n\}.$$

Then generically with respect $m \times n$ perturbation pencils $E(\lambda)$ such that

$$\operatorname{nrank}(E(\lambda)) \leq \rho,$$

the elementary divisors associated with λ_0 of $P(\lambda)$ and $P(\lambda) + E(\lambda)$ **are exactly the same**.

Remark

It seems possible to extend the proof from pencils to higher degree singular matrix polynomials.

No elementary divisors are destroyed...

Theorem (De Terán and D., SIMAX, 2007)

Let $P(\lambda)$ *be an* $m \times n$ *complex matrix pencil s.t.* $\mathrm{nrank}(P(\lambda)) < \min\{m, n\}$*, and* λ_0 *be an eigenvalue of* $P(\lambda)$*. Let* ρ *be a positive integer number such that*

$$\mathrm{nrank}(P(\lambda)) + \rho \leq \min\{m, n\}.$$

Then generically with respect $m \times n$ *perturbation pencils* $E(\lambda)$ *such that*

$$\mathrm{nrank}(E(\lambda)) \leq \rho,$$

the elementary divisors associated with λ_0 *of* $P(\lambda)$ *and* $P(\lambda) + E(\lambda)$ ***are exactly the same****.*

Remark

It seems possible to extend the proof from pencils to higher degree singular matrix polynomials.

No elementary divisors are destroyed...

Theorem (De Terán and D., SIMAX, 2007)

Let $P(\lambda)$ be an $m \times n$ complex matrix pencil s.t. $\mathrm{nrank}(P(\lambda)) < \min\{m, n\}$, and λ_0 be an eigenvalue of $P(\lambda)$. Let ρ be a positive integer number such that

$$\mathrm{nrank}(P(\lambda)) + \rho \leq \min\{m, n\}.$$

Then generically with respect $m \times n$ perturbation pencils $E(\lambda)$ such that

$$\mathrm{nrank}(E(\lambda)) \leq \rho,$$

the elementary divisors associated with λ_0 of $P(\lambda)$ and $P(\lambda) + E(\lambda)$ **are exactly the same**.

Remark

It seems possible to extend the proof from pencils to higher degree singular matrix polynomials.

No elementary divisors are destroyed...

Theorem (De Terán and D., SIMAX, 2007)

Let $P(\lambda)$ be an $m \times n$ complex matrix pencil s.t. $\mathrm{nrank}(P(\lambda)) < \min\{m, n\}$, *and λ_0 be an eigenvalue of $P(\lambda)$. Let ρ be a positive integer number such that*

$$\mathrm{nrank}(P(\lambda)) + \rho \leq \min\{m, n\}.$$

Then generically with respect $m \times n$ perturbation pencils $E(\lambda)$ such that

$$\mathrm{nrank}(E(\lambda)) \leq \rho,$$

the elementary divisors associated with λ_0 of $P(\lambda)$ and $P(\lambda) + E(\lambda)$ **are exactly the same**.

Remark

It seems possible to extend the proof from pencils to higher degree singular matrix polynomials.

Outline

Conclusions

- The generic (and nongeneric) low rank perturbation behavior of **elementary divisors** of regular matrix polynomials can be easily understood with the help of Thompson's result.
- The problem is still open for **singular matrix polynomials**, although we have the strong hope that the techniques used by De Terán-D. for singular pencils can be extended.
- The generic (and nongeneric) low rank perturbation behavior of elementary divisors of **structured matrix polynomials** (alternating, palindromic,...) under structured low rank perturbations is an open problem.
- It may require to develop structured versions of Thompson's result...