Recovery of eigenvectors of Matrix Polynomials from (generalized) Fiedler linearizations

Froilán M. Dopico

ICMAT and Departamento de Matemáticas, Universidad Carlos III de Madrid, Spain

joint work with

María I. Bueno (University of California at Santa Barbara),
Fernando De Terán (Universidad Carlos III de Madrid), and
D. Steven Mackey (Western Michigan University)

SC2011. Special session on Structured Linear Algebra.
S. Margherita di Pula, Italy, October 10-14, 2011

Basic definitions

We consider a matrix polynomial of degree k

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i = \lambda^k A_k + \cdots + \lambda A_1 + A_0 \,, \quad A_i \in \mathbb{F}^{n\times n} . \quad A_k \neq 0.$$

A linearization for $P(\lambda)$ is an $nk \times nk$ **linear matrix poly (pencil)** $L(\lambda)$ s. t.

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{n(k-1)} & \\ & P(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

$L(\lambda)$ is "strong linearization" if, in addition, $\operatorname{rev} L(\lambda)$ is a linearization for $\operatorname{rev} P(\lambda)$, where

$$\operatorname{rev} P(\lambda) := \lambda^k A_0 + \ldots + \lambda A_{k-1} + A_k$$

REMARK

MATLAB command `polyeig` solves polynomial eigenproblems

$$P(\lambda_0)x = 0$$

via (companion) linearizations.

We consider a matrix polynomial of degree k

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i = \lambda^k A_k + \cdots + \lambda A_1 + A_0 \,, \quad A_i \in \mathbb{F}^{n\times n} . \quad A_k \neq 0.$$

A linearization for $P(\lambda)$ is an $nk \times nk$ **linear matrix poly (pencil)** $L(\lambda)$ s. t.

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{n(k-1)} & \\ & P(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

$L(\lambda)$ is "strong linearization" if, in addition, $\operatorname{rev} L(\lambda)$ is a linearization for $\operatorname{rev} P(\lambda)$, where

$$\operatorname{rev} P(\lambda) := \lambda^k A_0 + \ldots + \lambda A_{k-1} + A_k$$

REMARK

MATLAB command `polyeig` solves polynomial eigenproblems

$$P(\lambda_0)x = 0$$

via (companion) linearizations.

Basic definitions

We consider a matrix polynomial of degree k

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i = \lambda^k A_k + \cdots + \lambda A_1 + A_0 \,, \quad A_i \in \mathbb{F}^{n\times n} . \quad A_k \neq 0.$$

A linearization for $P(\lambda)$ is an $nk \times nk$ **linear matrix poly (pencil)** $L(\lambda)$ s. t.

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{n(k-1)} & \\ & P(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

$L(\lambda)$ is "strong linearization" if, in addition, $\operatorname{rev} L(\lambda)$ is a linearization for $\operatorname{rev} P(\lambda)$, where

$$\operatorname{rev} P(\lambda) := \lambda^k A_0 + \ldots + \lambda A_{k-1} + A_k$$

REMARK

MATLAB command `polyeig` solves polynomial eigenproblems

$$P(\lambda_0)x = 0$$

via (companion) linearizations.

Advantages and disadvantages of the use of linearizations

- Strong linearizations preserve the finite and infinite elementary divisors of $P(\lambda)$, but NOT the eigenvectors and minimal indices/minimal bases.
- Good numerical methods for computing eigenvalues/vectors and minimal indices/bases of pencils are available (QZ, GUPTRI (Staircase form)).
- Standard linearizations do not preserve structures that $P(\lambda)$ may have.
- Conditioning of eigenvalues in linearizations may be much larger than in $P(\lambda)$. Backward errors?
- These difficulties have motivated an intense research on linearizations in the last years by different groups of several countries (Amiraslani, Antoniou, Bueno, Corless, De Terán, D, Grammont, Higham, Lancaster, R-C. Li, Mackey^2, Mehl, Merhmann, Tisseur, Vologiannidis, ...)
- In this talk, we review advances for one of the most relevant classes of linearizations developed in the last years.
- A "good" linearization for applications should allow to recover easily eigenvectors, minimal indices and bases of $P(\lambda)$, should preserve structures, and should be easily constructible.

Advantages and disadvantages of the use of linearizations

- Strong linearizations preserve the finite and infinite elementary divisors of $P(\lambda)$, but NOT the eigenvectors and minimal indices/minimal bases.
- Good numerical methods for computing eigenvalues/vectors and minimal indices/bases of pencils are available (QZ, GUPTRI (Staircase form)).
- Standard linearizations do not preserve structures that $P(\lambda)$ may have.
- Conditioning of eigenvalues in linearizations may be much larger than in $P(\lambda)$. Backward errors?
- These difficulties have motivated an intense research on linearizations in the last years by different groups of several countries (Amiraslani, Antoniou, Bueno, Corless, De Terán, D, Grammont, Higham, Lancaster, R-C. Li, Mackey^2, Mehl, Merhmann, Tisseur, Vologiannidis, ...)
- In this talk, we review advances for one of the most relevant classes of linearizations developed in the last years.
- A "good" linearization for applications should allow to recover easily eigenvectors, minimal indices and bases of $P(\lambda)$, should preserve structures, and should be easily constructible.

Advantages and disadvantages of the use of linearizations

- Strong linearizations preserve the finite and infinite elementary divisors of $P(\lambda)$, but NOT the eigenvectors and minimal indices/minimal bases.
- Good numerical methods for computing eigenvalues/vectors and minimal indices/bases of pencils are available (QZ, GUPTRI (Staircase form)).
- Standard linearizations do not preserve structures that $P(\lambda)$ may have.
- Conditioning of eigenvalues in linearizations may be much larger than in $P(\lambda)$. Backward errors?
- These difficulties have motivated an intense research on linearizations in the last years by different groups of several countries (Amiraslani, Antoniou, Bueno, Corless, De Terán, D, Grammont, Higham, Lancaster, R-C. Li, Mackey^2, Mehl, Merhmann, Tisseur, Vologiannidis, ...)
- In this talk, we review advances for one of the most relevant classes of linearizations developed in the last years.
- A "good" linearization for applications should allow to recover easily eigenvectors, minimal indices and bases of $P(\lambda)$, should preserve structures, and should be easily constructible.

- Strong linearizations preserve the finite and infinite elementary divisors of $P(\lambda)$, but NOT the eigenvectors and minimal indices/minimal bases.
- Good numerical methods for computing eigenvalues/vectors and minimal indices/bases of pencils are available (QZ, GUPTRI (Staircase form)).
- Standard linearizations do not preserve structures that $P(\lambda)$ may have.
- Conditioning of eigenvalues in linearizations may be much larger than in $P(\lambda)$. Backward errors?
- These difficulties have motivated an intense research on linearizations in the last years by different groups of several countries (Amiraslani, Antoniou, Bueno, Corless, De Terán, D, Grammont, Higham, Lancaster, R-C. Li, Mackey[2], Mehl, Merhmann, Tisseur, Vologiannidis, ...)
- In this talk, we review advances for one of the most relevant classes of linearizations developed in the last years.
- A "good" linearization for applications should allow to recover easily eigenvectors, minimal indices and bases of $P(\lambda)$, should preserve structures, and should be easily constructible.

Advantages and disadvantages of the use of linearizations

- Strong linearizations preserve the finite and infinite elementary divisors of $P(\lambda)$, but NOT the eigenvectors and minimal indices/minimal bases.
- Good numerical methods for computing eigenvalues/vectors and minimal indices/bases of pencils are available (QZ, GUPTRI (Staircase form)).
- Standard linearizations do not preserve structures that $P(\lambda)$ may have.
- Conditioning of eigenvalues in linearizations may be much larger than in $P(\lambda)$. Backward errors?
- These difficulties have motivated an intense research on linearizations in the last years by different groups of several countries (Amiraslani, Antoniou, Bueno, Corless, De Terán, D, Grammont, Higham, Lancaster, R-C. Li, Mackey^2, Mehl, Merhmann, Tisseur, Vologiannidis, ...)
- In this talk, we review advances for one of the most relevant classes of linearizations developed in the last years.
- A "good" linearization for applications should allow to recover easily eigenvectors, minimal indices and bases of $P(\lambda)$, should preserve structures, and should be easily constructible.

Advantages and disadvantages of the use of linearizations

- Strong linearizations preserve the finite and infinite elementary divisors of $P(\lambda)$, but NOT the eigenvectors and minimal indices/minimal bases.
- Good numerical methods for computing eigenvalues/vectors and minimal indices/bases of pencils are available (QZ, GUPTRI (Staircase form)).
- Standard linearizations do not preserve structures that $P(\lambda)$ may have.
- Conditioning of eigenvalues in linearizations may be much larger than in $P(\lambda)$. Backward errors?
- These difficulties have motivated an intense research on linearizations in the last years by different groups of several countries (Amiraslani, Antoniou, Bueno, Corless, De Terán, D, Grammont, Higham, Lancaster, R-C. Li, Mackey^2, Mehl, Merhmann, Tisseur, Vologiannidis, ...)
- In this talk, we review advances for one of the most relevant classes of linearizations developed in the last years.
- A "good" linearization for applications should allow to recover easily eigenvectors, minimal indices and bases of $P(\lambda)$, should preserve structures, and should be easily constructible.

Advantages and disadvantages of the use of linearizations

- Strong linearizations preserve the finite and infinite elementary divisors of $P(\lambda)$, but NOT the eigenvectors and minimal indices/minimal bases.
- Good numerical methods for computing eigenvalues/vectors and minimal indices/bases of pencils are available (QZ, GUPTRI (Staircase form)).
- Standard linearizations do not preserve structures that $P(\lambda)$ may have.
- Conditioning of eigenvalues in linearizations may be much larger than in $P(\lambda)$. Backward errors?
- These difficulties have motivated an intense research on linearizations in the last years by different groups of several countries (Amiraslani, Antoniou, Bueno, Corless, De Terán, D, Grammont, Higham, Lancaster, R-C. Li, Mackey^2, Mehl, Merhmann, Tisseur, Vologiannidis, ...)
- In this talk, we review advances for one of the most relevant classes of linearizations developed in the last years.
- A "good" linearization for applications should allow to recover easily eigenvectors, minimal indices and bases of $P(\lambda)$, should preserve structures, and should be easily constructible.

Fiedler pencils of $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ satisfy:

- They are **strong linearizations**, $\lambda X + Y$, **for any** $P(\lambda)$, regular or singular ($\det P(\lambda) \equiv 0$), over an arbitrary field and
- even for **rectangular matrix polynomials**.
- They allow **to recover very easily** eigenvectors, minimal indices, and minimal bases of $P(\lambda)$.
- **They are easily constructible**: If the matrices X and Y are partitioned into $k \times k$ blocks of size $n \times n$, then **each block of X and Y is either 0_n or $\pm I_n$ or $\pm A_i$ for $i = 0 : k$.**
- They are privileged among the new classes of linearizations that have been developed recently.

Fiedler pencils of $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ satisfy:

- They are **strong linearizations**, $\lambda X + Y$, **for any** $P(\lambda)$, regular or singular ($\det P(\lambda) \equiv 0$), over an arbitrary field and
- even for **rectangular matrix polynomials**.
- They allow **to recover very easily** eigenvectors, minimal indices, and minimal bases of $P(\lambda)$.
- **They are easily constructible**: If the matrices X and Y are partitioned into $k \times k$ blocks of size $n \times n$, then **each block of X and Y is either 0_n or $\pm I_n$ or $\pm A_i$ for $i = 0 : k$.**
- They are privileged among the new classes of linearizations that have been developed recently.

Fiedler pencils of $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ satisfy:

- They are **strong linearizations**, $\lambda X + Y$, **for any** $P(\lambda)$, regular or singular ($\det P(\lambda) \equiv 0$), over an arbitrary field and
- even for **rectangular matrix polynomials**.
- They allow **to recover very easily** eigenvectors, minimal indices, and minimal bases of $P(\lambda)$.
- **They are easily constructible**: If the matrices X and Y are partitioned into $k \times k$ blocks of size $n \times n$, then **each block of X and Y is either 0_n or $\pm I_n$ or $\pm A_i$ for $i = 0 : k$.**
- They are privileged among the new classes of linearizations that have been developed recently.

Fiedler pencils of $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ satisfy:

- They are **strong linearizations**, $\lambda X + Y$, **for any** $P(\lambda)$, regular or singular ($\det P(\lambda) \equiv 0$), over an arbitrary field and
- even for **rectangular matrix polynomials**.
- They allow **to recover very easily** eigenvectors, minimal indices, and minimal bases of $P(\lambda)$.
- **They are easily constructible**: If the matrices X and Y are partitioned into $k \times k$ blocks of size $n \times n$, then **each block of X and Y is either 0_n or $\pm I_n$ or $\pm A_i$ for $i = 0 : k$.**
- They are privileged among the new classes of linearizations that have been developed recently.

Fiedler pencils of $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ satisfy:

- They are **strong linearizations**, $\lambda X + Y$, **for any** $P(\lambda)$, regular or singular ($\det P(\lambda) \equiv 0$), over an arbitrary field and
- even for **rectangular matrix polynomials**.
- They allow **to recover very easily** eigenvectors, minimal indices, and minimal bases of $P(\lambda)$.
- **They are easily constructible**: If the matrices X and Y are partitioned into $k \times k$ blocks of size $n \times n$, then **each block of X and Y is either 0_n or $\pm I_n$ or $\pm A_i$ for $i = 0 : k$.**
- They are privileged among the new classes of linearizations that have been developed recently.

Fiedler pencils of $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ satisfy:

- They are **strong linearizations**, $\lambda X + Y$, **for any** $P(\lambda)$, regular or singular ($\det P(\lambda) \equiv 0$), over an arbitrary field and
- even for **rectangular matrix polynomials**.
- They allow **to recover very easily** eigenvectors, minimal indices, and minimal bases of $P(\lambda)$.
- **They are easily constructible**: If the matrices X and Y are partitioned into $k \times k$ blocks of size $n \times n$, then **each block of X and Y is either 0_n or $\pm I_n$ or $\pm A_i$ for $i = 0 : k$.**
- They are privileged among the new classes of linearizations that have been developed recently.

Fiedler pencils (II): a bit of history

- Introduced by Fiedler (LAA, 2003) for scalar polynomials.
- Extended to regular matrix polynomials (and generalized to preserve symmetry for odd degree) by Antoniou and Vologiannidis (ELA, 2004, 2006), where it is proved that they are strong linearizations.
- De Terán, D, Mackey (SIMAX, 2010) proved that they are strong linearizations for any square matrix poly in any field and found easy recovery procedures for e-vectors and minimal indices/bases of $P(\lambda)$.
- Palindromic strong linearizations for odd degree polynomials based on Fiedler pencils were introduced by De Terán, D, Mackey (JCAM 2011).
- Recovery of e-vectors and minimal indices/bases from generalized Fiedler linears. established by Bueno, De Terán, D (SIMAX, 2011)
- Further generalized with repeated factors by Vologiannidis and Antoniou (MCSS, 2011), related again to Fiedler's work (LAA, 2008).
- Extended to rectangular matrix polynomials by De Terán, D, Mackey (to be submitted soon)...

Fiedler pencils (II): a bit of history

- Introduced by Fiedler (LAA, 2003) for scalar polynomials.
- Extended to regular matrix polynomials (and generalized to preserve symmetry for odd degree) by Antoniou and Vologiannidis (ELA, 2004, 2006), where it is proved that they are strong linearizations.
- De Terán, D, Mackey (SIMAX, 2010) proved that they are strong linearizations for any square matrix poly in any field and found easy recovery procedures for e-vectors and minimal indices/bases of $P(\lambda)$.
- Palindromic strong linearizations for odd degree polynomials based on Fiedler pencils were introduced by De Terán, D, Mackey (JCAM 2011).
- Recovery of e-vectors and minimal indices/bases from generalized Fiedler linears. established by Bueno, De Terán, D (SIMAX, 2011)
- Further generalized with repeated factors by Vologiannidis and Antoniou (MCSS, 2011), related again to Fiedler's work (LAA, 2008).
- Extended to rectangular matrix polynomials by De Terán, D, Mackey (to be submitted soon)...

Fiedler pencils (II): a bit of history

- Introduced by Fiedler (LAA, 2003) for scalar polynomials.
- Extended to regular matrix polynomials (and generalized to preserve symmetry for odd degree) by Antoniou and Vologiannidis (ELA, 2004, 2006), where it is proved that they are strong linearizations.
- De Terán, D, Mackey (SIMAX, 2010) proved that they are strong linearizations for any square matrix poly in any field and found easy recovery procedures for e-vectors and minimal indices/bases of $P(\lambda)$.
- Palindromic strong linearizations for odd degree polynomials based on Fiedler pencils were introduced by De Terán, D, Mackey (JCAM 2011).
- Recovery of e-vectors and minimal indices/bases from generalized Fiedler linears. established by Bueno, De Terán, D (SIMAX, 2011)
- Further generalized with repeated factors by Vologiannidis and Antoniou (MCSS, 2011), related again to Fiedler's work (LAA, 2008).
- Extended to rectangular matrix polynomials by De Terán, D, Mackey (to be submitted soon)...

Fiedler pencils (II): a bit of history

- Introduced by Fiedler (LAA, 2003) for scalar polynomials.
- Extended to regular matrix polynomials (and generalized to preserve symmetry for odd degree) by Antoniou and Vologiannidis (ELA, 2004, 2006), where it is proved that they are strong linearizations.
- De Terán, D, Mackey (SIMAX, 2010) proved that they are strong linearizations for any square matrix poly in any field and found easy recovery procedures for e-vectors and minimal indices/bases of $P(\lambda)$.
- Palindromic strong linearizations for odd degree polynomials based on Fiedler pencils were introduced by De Terán, D, Mackey (JCAM 2011).
- Recovery of e-vectors and minimal indices/bases from generalized Fiedler linears. established by Bueno, De Terán, D (SIMAX, 2011)
- Further generalized with repeated factors by Vologiannidis and Antoniou (MCSS, 2011), related again to Fiedler's work (LAA, 2008).
- Extended to rectangular matrix polynomials by De Terán, D, Mackey (to be submitted soon)...

Fiedler pencils (II): a bit of history

- Introduced by Fiedler (LAA, 2003) for scalar polynomials.
- Extended to regular matrix polynomials (and generalized to preserve symmetry for odd degree) by Antoniou and Vologiannidis (ELA, 2004, 2006), where it is proved that they are strong linearizations.
- De Terán, D, Mackey (SIMAX, 2010) proved that they are strong linearizations for any square matrix poly in any field and found easy recovery procedures for e-vectors and minimal indices/bases of $P(\lambda)$.
- Palindromic strong linearizations for odd degree polynomials based on Fiedler pencils were introduced by De Terán, D, Mackey (JCAM 2011).
- Recovery of e-vectors and minimal indices/bases from generalized Fiedler linears. established by Bueno, De Terán, D (SIMAX, 2011)
- Further generalized with repeated factors by Vologiannidis and Antoniou (MCSS, 2011), related again to Fiedler's work (LAA, 2008).
- Extended to rectangular matrix polynomials by De Terán, D, Mackey (to be submitted soon)...

Fiedler pencils (II): a bit of history

- Introduced by Fiedler (LAA, 2003) for scalar polynomials.
- Extended to regular matrix polynomials (and generalized to preserve symmetry for odd degree) by Antoniou and Vologiannidis (ELA, 2004, 2006), where it is proved that they are strong linearizations.
- De Terán, D, Mackey (SIMAX, 2010) proved that they are strong linearizations for any square matrix poly in any field and found easy recovery procedures for e-vectors and minimal indices/bases of $P(\lambda)$.
- Palindromic strong linearizations for odd degree polynomials based on Fiedler pencils were introduced by De Terán, D, Mackey (JCAM 2011).
- Recovery of e-vectors and minimal indices/bases from generalized Fiedler linears. established by Bueno, De Terán, D (SIMAX, 2011)
- Further generalized with repeated factors by Vologiannidis and Antoniou (MCSS, 2011), related again to Fiedler's work (LAA, 2008).
- Extended to rectangular matrix polynomials by De Terán, D, Mackey (to be submitted soon)...

- Introduced by Fiedler (LAA, 2003) for scalar polynomials.
- Extended to regular matrix polynomials (and generalized to preserve symmetry for odd degree) by Antoniou and Vologiannidis (ELA, 2004, 2006), where it is proved that they are strong linearizations.
- De Terán, D, Mackey (SIMAX, 2010) proved that they are strong linearizations for any square matrix poly in any field and found easy recovery procedures for e-vectors and minimal indices/bases of $P(\lambda)$.
- Palindromic strong linearizations for odd degree polynomials based on Fiedler pencils were introduced by De Terán, D, Mackey (JCAM 2011).
- Recovery of e-vectors and minimal indices/bases from generalized Fiedler linears. established by Bueno, De Terán, D (SIMAX, 2011)
- Further generalized with repeated factors by Vologiannidis and Antoniou (MCSS, 2011), related again to Fiedler's work (LAA, 2008).
- Extended to rectangular matrix polynomials by De Terán, D, Mackey (to be submitted soon)...

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Second companion form:

$$C_2(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & I_n & & & & \\ -A_4 & & I_n & & & \\ -A_3 & & & I_n & & \\ -A_2 & & & & I_n & \\ -A_1 & & & & & I_n \\ -A_0 & & & & & \end{bmatrix}$$

Fiedler pencils (III): Examples

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Second companion form:

$$C_2(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & I_n & & & & \\ -A_4 & & I_n & & & \\ -A_3 & & & I_n & & \\ -A_2 & & & & I_n & \\ -A_1 & & & & & I_n \\ -A_0 & & & & & \end{bmatrix}$$

Fiedler pencils (III): Examples

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Second companion form:

$$C_2(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & I_n & & & & \\ -A_4 & & I_n & & & \\ -A_3 & & & I_n & & \\ -A_2 & & & & I_n & \\ -A_1 & & & & & I_n \\ -A_0 & & & & & \end{bmatrix}$$

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Another Fiedler pencil:

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

Fiedler pencils (III): Examples

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Another Fiedler pencil:

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Another Fiedler pencil:

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

Structural property 1 of Fiedler Pencils

The one-degree coefficient of every Fiedler Pencil is always the same. The zero-degree coefficient of every Fiedler Pencil has exactly the same blocks as the first companion form but they are in different positions.

Fiedler pencils (III): Examples and structural properties

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Another Fiedler pencil:

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

Structural property 1 of Fiedler Pencils

The one-degree coefficient of every Fiedler Pencil is always the same. The zero-degree coefficient of every Fiedler Pencil has exactly the same blocks as the first companion form but they are in different positions.

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & -A_0 \\ & & & I_n & 0 & 0 \end{bmatrix}$$

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & -A_0 \\ & & & I_n & 0 & 0 \end{bmatrix}$$

Fiedler pencils (III): Examples and structural properties

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & -A_0 \\ & & & I_n & 0 & 0 \end{bmatrix}$$

Structural property 2 of Fiedler pencils

Companion forms are the Fiedler pencils with largest banwidth. Pentadiagonal Fiedler pencils are the ones with smallest bandwith.

Fiedler pencils (III): Examples and structural properties

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & -A_0 \\ 0 & 0 & 0 & I_n & 0 & 0 \end{bmatrix}$$

Structural property 3 of Fiedler pencils

The zero degree coefficient of every Fiedler pencil satisfies:

- The identity blocks are never in the main block diagonal.

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & -A_0 \\ 0 & 0 & 0 & I_n & 0 & 0 \end{bmatrix}$$

Structural property 4 of Fiedler pencils

The zero degree coefficient of every Fiedler pencil satisfies:

- If an I_n block is at block-entry (i, j), then either the ith block-row or the jth block column has the remaining blocks equal to 0_n.

Fiedler pencils (III): Examples and structural properties

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & -A_0 \\ 0 & 0 & 0 & I_n & 0 & 0 \end{bmatrix}$$

Structural property 4 of Fiedler pencils

The zero degree coefficient of every Fiedler pencil satisfies:

- If an I_n block is at block-entry (i, j), then either the ith block-row or the jth block column has at least one matrix $-A_0, -A_1, \ldots, -A_{k-1}$.

Outline

Outline

Let $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{F}^{n\times n}$. We define $nk \times nk$ matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix}, \quad j = 1, \ldots, k-1\,,$$

$$M_0 := \begin{bmatrix} I_{n(k-1)} & \\ & -A_0 \end{bmatrix}, \quad M_k := \begin{bmatrix} A_k & \\ & I_{n(k-1)} \end{bmatrix}.$$

Given any **permutation** $\sigma = (j_0, j_1, \ldots, j_{k-1})$ of $(0, 1, \ldots, k-1)$, the **Fiedler pencil associated with** σ is

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

Examples: Companion forms-Pentadiagonal Fiedler pencils

$$C_1(\lambda) = \lambda M_k - M_{k-1} \cdots M_1 M_0$$
$$C_2(\lambda) = \lambda M_k - M_0 M_1 \cdots M_{k-1}$$
$$T(\lambda) = \lambda M_k - (M_1 M_3 M_5 \cdots)(M_2 M_4 M_6 \cdots) M_0$$

Definition (Fiedler, 2003–Antoniou & Vologiannidis, 2004)

Let $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0 \,, \quad A_i \in \mathbb{F}^{n\times n}$. We define $nk \times nk$ matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix}, \quad j = 1, \ldots, k-1\,,$$

$$M_0 := \begin{bmatrix} I_{n(k-1)} & \\ & -A_0 \end{bmatrix}, \quad M_k := \begin{bmatrix} A_k & \\ & I_{n(k-1)} \end{bmatrix}.$$

Given any **permutation** $\sigma = (j_0, j_1, \ldots, j_{k-1})$ of $(0, 1, \ldots, k-1)$, the **Fiedler pencil associated with** σ is

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

Examples: Companion forms-Pentadiagonal Fiedler pencils

$$C_1(\lambda) = \lambda M_k - M_{k-1} \cdots M_1 M_0$$

$$C_2(\lambda) = \lambda M_k - M_0 M_1 \cdots M_{k-1}$$

$$T(\lambda) = \lambda M_k - (M_1 M_3 M_5 \cdots)(M_2 M_4 M_6 \cdots) M_0$$

Definition (Fiedler, 2003–Antoniou & Vologiannidis, 2004)

Let $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{F}^{n\times n}$. We define $nk \times nk$ matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix}, \quad j = 1, \ldots, k-1\,,$$

$$M_0 := \begin{bmatrix} I_{n(k-1)} & \\ & -A_0 \end{bmatrix}, \quad M_k := \begin{bmatrix} A_k & \\ & I_{n(k-1)} \end{bmatrix}.$$

Given any **permutation** $\sigma = (j_0, j_1, \ldots, j_{k-1})$ of $(0, 1, \ldots, k-1)$, the **Fiedler pencil associated with** σ is

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

Examples: Companion forms-Pentadiagonal Fiedler pencils

$$C_1(\lambda) = \lambda M_k - M_{k-1} \cdots M_1 M_0$$
$$C_2(\lambda) = \lambda M_k - M_0 M_1 \cdots M_{k-1}$$
$$T(\lambda) = \lambda M_k - (M_1 M_3 M_5 \cdots)(M_2 M_4 M_6 \cdots) M_0$$

Observe that $M_i M_j = M_j M_i$ for $|i - j| \neq 1$. This implies:

Lemma

Let $P(\lambda)$ be an arbitrary matrix polynomial of degree k. Then there exist 2^{k-1} distinct Fiedler pencils associated with $P(\lambda)$.

Consequences:

- **Quadratic polys: Fiedler pencils are the two companion forms.**
- For degree $k = 3$, there are two more Fiedler pencils:
- The potential applications of Fiedler pencils are in degrees $k \geq 3$.

Observe that $M_i M_j = M_j M_i$ for $|i-j| \neq 1$. This implies:

Lemma

Let $P(\lambda)$ be an arbitrary matrix polynomial of degree k. Then there exist 2^{k-1} distinct Fiedler pencils associated with $P(\lambda)$.

Consequences:

- **Quadratic polys: Fiedler pencils are the two companion forms.**
- For degree $k=3$, there are two more Fiedler pencils:
- The potential applications of Fiedler pencils are in degrees $k \geq 3$.

Observe that $M_i M_j = M_j M_i$ for $|i - j| \neq 1$. This implies:

Lemma

Let $P(\lambda)$ be an arbitrary matrix polynomial of degree k. Then there exist 2^{k-1} distinct Fiedler pencils associated with $P(\lambda)$.

Consequences:

- **Quadratic polys: Fiedler pencils are the two companion forms.**
- For degree $k = 3$, there are two more Fiedler pencils:

$$\lambda \begin{bmatrix} A_3 & & \\ & I_n & \\ & & I_n \end{bmatrix} - \begin{bmatrix} -A_2 & -A_1 & I_n \\ I_n & & \\ & -A_0 & \end{bmatrix}$$

- The potential applications of Fiedler pencils are in degrees $k \geq 3$.

Observe that $M_i M_j = M_j M_i$ for $|i-j| \neq 1$. This implies:

Lemma

Let $P(\lambda)$ be an arbitrary matrix polynomial of degree k. Then there exist 2^{k-1} distinct Fiedler pencils associated with $P(\lambda)$.

Consequences:

- **Quadratic polys: Fiedler pencils are the two companion forms.**
- For degree $k = 3$, there are two more Fiedler pencils:

$$\lambda \begin{bmatrix} A_3 & & \\ & I_n & \\ & & I_n \end{bmatrix} - \begin{bmatrix} -A_2 & I_n & \\ -A_1 & & -A_0 \\ I_n & & \end{bmatrix}$$

- The potential applications of Fiedler pencils are in degrees $k \geq 3$.

Observe that $M_i M_j = M_j M_i$ for $|i - j| \neq 1$. This implies:

Lemma

Let $P(\lambda)$ be an arbitrary matrix polynomial of degree k. Then there exist 2^{k-1} distinct Fiedler pencils associated with $P(\lambda)$.

Consequences:

- **Quadratic polys: Fiedler pencils are the two companion forms.**
- For degree $k = 3$, there are two more Fiedler pencils:

$$\lambda \begin{bmatrix} A_3 & & \\ & I_n & \\ & & I_n \end{bmatrix} - \begin{bmatrix} -A_2 & I_n & \\ -A_1 & & -A_0 \\ I_n & & \end{bmatrix}$$

- The potential applications of Fiedler pencils are in degrees $k \geq 3$.

Consecutions and inversions

Let us consider the Fiedler pencil associated to $\sigma = (j_0, j_1, \dots, j_{k-1})$, permutation of $(0, 1, \dots, k-1)$, i.e.,

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

For $i = 0, 1, \dots k-2$, we say that $F_\sigma(\lambda)$ has a

- **consecution at** i, if the product $M_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is of the form

$$M_\sigma = \cdots M_i \cdots M_{i+1} \cdots$$

- **inversion at** i, if the product $M_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is of the form

$$M_\sigma = \cdots M_{i+1} \cdots M_i \cdots$$

We say that $F_\sigma(\lambda)$ has $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at

$$0, 1, 2, \dots, \mathfrak{c}_0 - 1,$$

but not at $\mathfrak{c}_0$. Analogous for $\mathfrak{i}_0$ **initial inversions**.

Consecutions and inversions

Let us consider the Fiedler pencil associated to $\sigma = (j_0, j_1, \ldots, j_{k-1})$, permutation of $(0, 1, \ldots, k-1)$, i.e.,

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

For $i = 0, 1, \ldots k-2$, we say that $F_\sigma(\lambda)$ has a

- **consecution at i**, if the product $M_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is of the form

$$M_\sigma = \cdots M_i \cdots M_{i+1} \cdots$$

- **inversion at i**, if the product $M_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is of the form

$$M_\sigma = \cdots M_{i+1} \cdots M_i \cdots$$

We say that $F_\sigma(\lambda)$ has $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at

$$0, 1, 2, \ldots, \mathfrak{c}_0 - 1,$$

but not at $\mathfrak{c}_0$. Analogous for $\mathfrak{i}_0$ **initial inversions**.

Consecutions and inversions

Let us consider the Fiedler pencil associated to $\sigma = (j_0, j_1, \ldots, j_{k-1})$, permutation of $(0, 1, \ldots, k-1)$, i.e.,

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

For $i = 0, 1, \ldots k-2$, we say that $F_\sigma(\lambda)$ has a

- **consecution at** i, if the product $M_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is of the form

$$M_\sigma = \cdots M_i \cdots M_{i+1} \cdots$$

- **inversion at** i, if the product $M_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is of the form

$$M_\sigma = \cdots M_{i+1} \cdots M_i \cdots$$

We say that $F_\sigma(\lambda)$ has $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at

$$0, 1, 2, \ldots, \mathfrak{c}_0 - 1,$$

but not at $\mathfrak{c}_0$. Analogous for $\mathfrak{i}_0$ **initial inversions**.

Consecutions and inversions

Let us consider the Fiedler pencil associated to $\sigma = (j_0, j_1, \ldots, j_{k-1})$, permutation of $(0, 1, \ldots, k-1)$, i.e.,

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

For $i = 0, 1, \ldots k-2$, we say that $F_\sigma(\lambda)$ has a

- **consecution at i**, if the product $M_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is of the form

$$M_\sigma = \cdots M_i \cdots M_{i+1} \cdots$$

- **inversion at i**, if the product $M_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is of the form

$$M_\sigma = \cdots M_{i+1} \cdots M_i \cdots$$

We say that $F_\sigma(\lambda)$ has $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at

$$0, 1, 2, \ldots, \mathfrak{c}_0 - 1,$$

but not at $\mathfrak{c}_0$. Analogous for $\mathfrak{i}_0$ **initial inversions**.

Example of consecutions and inversions

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$F_\sigma(\lambda) = \lambda M_6 - M_5 M_4 M_3 M_0 M_1 M_2$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

$F_\sigma(\lambda)$ has

- Consecutions at $0, 1$,
- Inversions at $2, 3, 4$,
- $\mathfrak{c}_0 = 2$, and
- $\mathfrak{i}_0 = 0$.

Example of consecutions and inversions

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$F_\sigma(\lambda) = \lambda M_6 - M_5 M_4 M_3 M_0 M_1 M_2$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

$F_\sigma(\lambda)$ has

- Consecutions at 0, 1,
- Inversions at 2, 3, 4,
- $\mathfrak{c}_0 = 2$, and
- $\mathfrak{i}_0 = 0$.

Outline

Theorem for eigenvector recovery: extracting blocks

Theorem (De Terán, D, Mackey (SIMAX, 2010))

Let $P(\lambda)$ be an $n \times n$ **regular** *matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ with permutation σ having $\mathfrak{c}_0$* **initial consecutions** *and $\mathfrak{i}_0$* **initial inversions***, and suppose that λ_0* **is a *finite* eigenvalue** *of $P(\lambda)$.*

- *If*

$$z = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_k \end{bmatrix} \in \mathbb{F}^{nk \times 1}, \qquad x_i \in \mathbb{F}^{n \times 1},$$

is a right λ_0-eigenvector of $F_\sigma(\lambda)$, **then $x_{k-\mathfrak{c}_0}$ is a right λ_0-eigenvector of $P(\lambda)$**.

- *If*

$$w^T = \left[\; w_1^T \;\middle|\; w_2^T \;\middle|\; \ldots \;\middle|\; w_k^T \;\right] \in \mathbb{F}^{1 \times nk}, \qquad w_i^T \in \mathbb{F}^{1 \times n},$$

is a left λ_0-eigenvector of $F_\sigma(\lambda)$, **then $w_{k-\mathfrak{i}_0}^T$ is a left λ_0-eigenvector of $P(\lambda)$**.

For first companion form $\mathfrak{c}_0 = 0, \mathfrak{i}_0 = k-1$, and for second $\mathfrak{c}_0 = k-1, \mathfrak{i}_0 = 0$.

For the **infinite e-value**, one has to extract the first blocks.

Theorem for eigenvector recovery: extracting blocks

Theorem (De Terán, D, Mackey (SIMAX, 2010))

Let $P(\lambda)$ be an $n \times n$ **regular** *matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ with permutation σ having $\mathfrak{c}_0$* **initial consecutions** *and $\mathfrak{i}_0$* **initial inversions**, *and suppose that λ_0* **is a *finite* eigenvalue** *of $P(\lambda)$.*

- *If*

$$z = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_k \end{bmatrix} \in \mathbb{F}^{nk \times 1}, \qquad x_i \in \mathbb{F}^{n \times 1},$$

  is a right λ_0-eigenvector of $F_\sigma(\lambda)$, **then $x_{k-\mathfrak{c}_0}$ is a right λ_0-eigenvector of $P(\lambda)$**.

- *If*

$$w^T = \left[\; w_1^T \; \middle| \; w_2^T \; \middle| \; \ldots \; \middle| \; w_k^T \; \right] \in \mathbb{F}^{1 \times nk}, \qquad w_i^T \in \mathbb{F}^{1 \times n},$$

  is a left λ_0-eigenvector of $F_\sigma(\lambda)$, **then $w_{k-\mathfrak{i}_0}^T$ is a left λ_0-eigenvector of $P(\lambda)$**.

For first companion form $\mathfrak{c}_0 = 0, \mathfrak{i}_0 = k-1$, and for second $\mathfrak{c}_0 = k-1, \mathfrak{i}_0 = 0$.

For the **infinite e-value**, one has to extract the first blocks.

Example of eigenvector recovery

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$ has size $n \times n$

$$F_\sigma(\lambda) = \lambda M_6 - M_5 M_4 M_3 M_0 M_1 M_2$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

$F_\sigma(\lambda)$ has $\mathfrak{c}_0 = 2$

$$z = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline x_3 \\ \hline x_4 \\ \hline x_5 \\ \hline x_6 \end{bmatrix}, \quad (x_i \in \mathbb{F}^{n\times 1})$$ be such that $F_\sigma(\lambda_0) z = 0 \Longrightarrow P(\lambda_0) x_4 = 0$

Example of eigenvector recovery

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$ has size $n \times n$

$$F_\sigma(\lambda) = \lambda M_6 - M_5 M_4 M_3 M_0 M_1 M_2$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

$F_\sigma(\lambda)$ has $\mathfrak{c}_0 = 2$

$z = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline x_3 \\ \hline x_4 \\ \hline x_5 \\ \hline x_6 \end{bmatrix}, \quad (x_i \in \mathbb{F}^{n\times 1})$ be such that $F_\sigma(\lambda_0)z = 0 \Longrightarrow P(\lambda_0)x_4 = 0$

Explicit expressions of e-vectors of $F_\sigma(\lambda)$ in terms e-vectors of $P(\lambda)$ (I)

- They have been developed by De Terán, D, Mackey (SIMAX, 2010).
- These expressions are useful to compare the **conditioning and backward error** of a number λ_0 as an eigenvalue of the matrix polynomial $P(\lambda)$ and as an eigenvalue of the Fiedler linearization $F_\sigma(\lambda)$:

$$\kappa_{\mathrm{rel}}(\lambda_0, P) = \frac{\sum_{i=0}^{k} |\lambda_0|^i \|A_i\|_2}{|\lambda_0|} \frac{\|y\|_2 \, \|x\|_2}{|y^* \, P'(\lambda_0) \, x|}$$

- The complete description of these expressions requires more notation, we have no time to present it here. It depends on the **consecutions and inversions** of $F_\sigma(\lambda)$.
- We simply illustrate these results with an example.
- In this problem, the **Horner shifts** of $P(\lambda) = A_k\lambda^k + \cdots + A_1\lambda + A_0$ play an important role

$$P_d(\lambda) := \lambda^d A_k + \cdots + \lambda A_{k-d+1} + A_{k-d}, \qquad 0 \le d \le k$$

- They have been developed by De Terán, D, Mackey (SIMAX, 2010).
- These expressions are useful to compare the **conditioning and backward error** of a number λ_0 as an eigenvalue of the matrix polynomial $P(\lambda)$ and as an eigenvalue of the Fiedler linearization $F_\sigma(\lambda)$:

$$\kappa_{\rm rel}(\lambda_0, P) = \frac{\sum_{i=0}^{k} |\lambda_0|^i \|A_i\|_2}{|\lambda_0|} \frac{\|y\|_2 \, \|x\|_2}{|y^* \, P'(\lambda_0) \, x|}$$

- The complete description of these expressions requires more notation, we have no time to present it here. It depends on the **consecutions and inversions** of $F_\sigma(\lambda)$.
- We simply illustrate these results with an example.
- In this problem, the **Horner shifts** of $P(\lambda) = A_k\lambda^k + \cdots + A_1\lambda + A_0$ play an important role

$$P_d(\lambda) := \lambda^d A_k + \cdots + \lambda A_{k-d+1} + A_{k-d}, \qquad 0 \le d \le k$$

Explicit expressions of e-vectors of $F_\sigma(\lambda)$ in terms e-vectors of $P(\lambda)$ (I)

- They have been developed by De Terán, D, Mackey (SIMAX, 2010).
- These expressions are useful to compare the **conditioning and backward error** of a number λ_0 as an eigenvalue of the matrix polynomial $P(\lambda)$ and as an eigenvalue of the Fiedler linearization $F_\sigma(\lambda)$:

$$\kappa_{\mathrm{rel}}(\lambda_0, P) = \frac{\sum_{i=0}^{k} |\lambda_0|^i \|A_i\|_2}{|\lambda_0|} \frac{\|y\|_2 \, \|x\|_2}{|y^* \, P'(\lambda_0) \, x|}$$

- The complete description of these expressions requires more notation, we have no time to present it here. It depends on the **consecutions and inversions** of $F_\sigma(\lambda)$.
- We simply illustrate these results with an example.
- In this problem, the **Horner shifts** of $P(\lambda) = A_k\lambda^k + \cdots + A_1\lambda + A_0$ play an important role

$$P_d(\lambda) := \lambda^d A_k + \cdots + \lambda A_{k-d+1} + A_{k-d}, \qquad 0 \le d \le k$$

Explicit expressions of e-vectors of $F_\sigma(\lambda)$ in terms e-vectors of $P(\lambda)$ (I)

- They have been developed by De Terán, D, Mackey (SIMAX, 2010).
- These expressions are useful to compare the **conditioning and backward error** of a number λ_0 as an eigenvalue of the matrix polynomial $P(\lambda)$ and as an eigenvalue of the Fiedler linearization $F_\sigma(\lambda)$:

$$\kappa_{\mathrm{rel}}(\lambda_0, P) = \frac{\sum_{i=0}^{k} |\lambda_0|^i \|A_i\|_2}{|\lambda_0|} \frac{\|y\|_2 \, \|x\|_2}{|y^* \, P'(\lambda_0) \, x|}$$

- The complete description of these expressions requires more notation, we have no time to present it here. It depends on the **consecutions and inversions** of $F_\sigma(\lambda)$.
- We simply illustrate these results with an example.
- In this problem, the **Horner shifts** of $P(\lambda) = A_k\lambda^k + \cdots + A_1\lambda + A_0$ play an important role

$$P_d(\lambda) := \lambda^d A_k + \cdots + \lambda A_{k-d+1} + A_{k-d}, \qquad 0 \le d \le k$$

Explicit expressions of e-vectors of $F_\sigma(\lambda)$ in terms e-vectors of $P(\lambda)$ (I)

- They have been developed by De Terán, D, Mackey (SIMAX, 2010).
- These expressions are useful to compare the **conditioning and backward error** of a number λ_0 as an eigenvalue of the matrix polynomial $P(\lambda)$ and as an eigenvalue of the Fiedler linearization $F_\sigma(\lambda)$:

$$\kappa_{\mathrm{rel}}(\lambda_0, P) = \frac{\sum_{i=0}^{k} |\lambda_0|^i \|A_i\|_2}{|\lambda_0|} \frac{\|y\|_2 \, \|x\|_2}{|y^* \, P'(\lambda_0) \, x|}$$

- The complete description of these expressions requires more notation, we have no time to present it here. It depends on the **consecutions and inversions** of $F_\sigma(\lambda)$.
- We simply illustrate these results with an example.
- In this problem, the **Horner shifts** of $P(\lambda) = A_k\lambda^k + \cdots + A_1\lambda + A_0$ play an important role

$$P_d(\lambda) := \lambda^d A_k + \cdots + \lambda A_{k-d+1} + A_{k-d}, \qquad 0 \leq d \leq k$$

Explicit expressions of e-vectors of $F_\sigma(\lambda)$ in terms e-vectors of $P(\lambda)$ (II)

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$F_\sigma(\lambda) = \lambda M_6 - M_0\,(M_1 M_3 M_5)\,(M_2 M_4)$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & I_n \\ & & & -A_0 & 0 & 0 \end{bmatrix}$$

If λ_0 e-val of P and $P(\lambda_0)x = 0$, $y^T P(\lambda_0) = 0$, then $F_\sigma(\lambda_0)z = 0$, $w^T F_\sigma(\lambda_0) = 0$ with

$$z = \begin{bmatrix} \lambda_0^2 x \\ \lambda_0 x \\ \lambda_0 P_2(\lambda_0) x \\ x \\ P_4(\lambda_0) x \\ P_5(\lambda_0) x \end{bmatrix}, \quad w^T = \left[\; y^T\lambda_0^3 \;\middle|\; y^T\lambda_0^3 P_1(\lambda_0) \;\middle|\; y^T\lambda_0^2 \;\middle|\; y^T\lambda_0^2 P_3(\lambda_0) \;\middle|\; y^T\lambda_0 \;\middle|\; y^T \;\right]$$

Explicit expressions of e-vectors of $F_\sigma(\lambda)$ in terms e-vectors of $P(\lambda)$ (II)

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$F_\sigma(\lambda) = \lambda M_6 - M_0\,(M_1 M_3 M_5)\,(M_2 M_4)$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & I_n \\ & & & -A_0 & 0 & 0 \end{bmatrix}$$

If λ_0 e-val of P and $P(\lambda_0)x = 0$, $y^T P(\lambda_0) = 0$, then $F_\sigma(\lambda_0)z = 0$, $w^T F_\sigma(\lambda_0) = 0$ with

$$z = \begin{bmatrix} \lambda_0^2 x \\ \lambda_0 x \\ \lambda_0 P_2(\lambda_0) x \\ x \\ P_4(\lambda_0) x \\ P_5(\lambda_0) x \end{bmatrix}, \quad w^T = \left[\; y^T\lambda_0^3 \;\middle|\; y^T\lambda_0^3 P_1(\lambda_0) \;\middle|\; y^T\lambda_0^2 \;\middle|\; y^T\lambda_0^2 P_3(\lambda_0) \;\middle|\; y^T\lambda_0 \;\middle|\; y^T \;\right]$$

Outline

Magnitudes relevant in linear system and control theory.

$\mathbb{F}(\lambda)$ denotes the field of rational functions with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)^n$ the set of n-tuples with entries in $\mathbb{F}(\lambda)$.

Definition: Minimal bases and indices (Forney, SIAM J. Control, 1975)

- Let $\mathbb{S} \subseteq \mathbb{F}(\lambda)^n$ be a subspace and $\mathcal{B} = \{\mathbf{v}_1(\lambda), \ldots, \mathbf{v}_p(\lambda)\}$ be a **polynomial basis** of $\mathbb{S}$ with $\beta_i = \deg \mathbf{v}_i(\lambda)$. We say that $\mathcal{B}$ is a **minimal basis** of $\mathbb{S}$ if $\sum_i \beta_i$ is minimal over all polynomial bases of $\mathbb{S}$.
- The ordered sequence $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_p$ of degrees is the same for all minimal bases of $\mathbb{S}$. These degrees are called **minimal indices** of $\mathbb{S}$.

Definition: Minimal bases and indices of a singular matrix poly $P(\lambda)$

A right minimal basis of P is a min. basis of $\mathcal{N}_r(P) = \{x(\lambda) : P(\lambda)x(\lambda) = 0\}$.

The right minimal indices of $P(\lambda)$ are the minimal indices of $\mathcal{N}_r(P)$.

Left-definitions analogous for left-null space.

Magnitudes relevant in linear system and control theory.

$\mathbb{F}(\lambda)$ denotes the field of rational functions with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)^n$ the set of n-tuples with entries in $\mathbb{F}(\lambda)$.

Definition: Minimal bases and indices (Forney, SIAM J. Control, 1975)

- Let $\mathbb{S} \subseteq \mathbb{F}(\lambda)^n$ be a subspace and $\mathcal{B} = \{\mathbf{v}_1(\lambda), \ldots, \mathbf{v}_p(\lambda)\}$ be a **polynomial basis** of $\mathbb{S}$ with $\beta_i = \deg \mathbf{v}_i(\lambda)$. We say that $\mathcal{B}$ is a **minimal basis** of $\mathbb{S}$ if $\sum_i \beta_i$ is minimal over all polynomial bases of $\mathbb{S}$.
- The ordered sequence $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_p$ of degrees is the same for all minimal bases of $\mathbb{S}$. These degrees are called **minimal indices** of $\mathbb{S}$.

Definition: Minimal bases and indices of a singular matrix poly $P(\lambda)$

A right minimal basis of P is a min. basis of $\mathcal{N}_r(P) = \{x(\lambda) : P(\lambda)x(\lambda) = 0\}$.

The right minimal indices of $P(\lambda)$ are the minimal indices of $\mathcal{N}_r(P)$.

Left-definitions analogous for left-null space.

Magnitudes relevant in linear system and control theory.

$\mathbb{F}(\lambda)$ denotes the field of rational functions with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)^n$ the set of n-tuples with entries in $\mathbb{F}(\lambda)$.

Definition: Minimal bases and indices (Forney, SIAM J. Control, 1975)

- Let $\mathbb{S} \subseteq \mathbb{F}(\lambda)^n$ be a subspace and $\mathcal{B} = \{\mathbf{v}_1(\lambda), \ldots, \mathbf{v}_p(\lambda)\}$ be a **polynomial basis** of $\mathbb{S}$ with $\beta_i = \deg \mathbf{v}_i(\lambda)$. We say that $\mathcal{B}$ is a **minimal basis** of $\mathbb{S}$ if $\sum_i \beta_i$ is minimal over all polynomial bases of $\mathbb{S}$.
- The ordered sequence $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_p$ of degrees is the same for all minimal bases of $\mathbb{S}$. These degrees are called **minimal indices** of $\mathbb{S}$.

Definition: Minimal bases and indices of a singular matrix poly $P(\lambda)$

A right minimal basis of P is a min. basis of $\mathcal{N}_r(P) = \{x(\lambda) : P(\lambda)x(\lambda) = 0\}$.

The right minimal indices of $P(\lambda)$ are the minimal indices of $\mathcal{N}_r(P)$.

Left-definitions analogous for left-null space.

Theorem (De Terán, D, Mackey (SIMAX, 2010))

Let $P(\lambda)$ be an $n \times n$ **singular** *matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ with permutation σ having $\mathfrak{i}(\sigma)$* **total number of inversions** *and $\mathfrak{c}(\sigma)$* **total number of consecutions**.

(a) *If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then*

$$\varepsilon_1 + \mathfrak{i}(\sigma) \leq \varepsilon_2 + \mathfrak{i}(\sigma) \leq \cdots \leq \varepsilon_p + \mathfrak{i}(\sigma),$$

are the right minimal indices of $F_\sigma(\lambda)$.

(b) *If $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ are the left minimal indices of $P(\lambda)$, then*

$$\eta_1 + \mathfrak{c}(\sigma) \leq \eta_2 + \mathfrak{c}(\sigma) \leq \cdots \leq \eta_p + \mathfrak{c}(\sigma),$$

are the left minimal indices of $F_\sigma(\lambda)$.

Theorem (Recovery of minimal bases (DT, D, M, SIMAX, 2010))

It follows **exactly the same block-extraction rule as eigenvector recovery**.

Theorem for recovery of minimal indices

Theorem (De Terán, D, Mackey (SIMAX, 2010))

Let $P(\lambda)$ be an $n \times n$ **singular** *matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ with permutation σ having $\mathfrak{i}(\sigma)$* **total number of inversions** *and $\mathfrak{c}(\sigma)$* **total number of consecutions**.

(a) *If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then*

$$\varepsilon_1 + \mathfrak{i}(\sigma) \leq \varepsilon_2 + \mathfrak{i}(\sigma) \leq \cdots \leq \varepsilon_p + \mathfrak{i}(\sigma)\,,$$

are the right minimal indices of $F_\sigma(\lambda)$.

(b) *If $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ are the left minimal indices of $P(\lambda)$, then*

$$\eta_1 + \mathfrak{c}(\sigma) \leq \eta_2 + \mathfrak{c}(\sigma) \leq \cdots \leq \eta_p + \mathfrak{c}(\sigma)\,,$$

are the left minimal indices of $F_\sigma(\lambda)$.

Theorem (Recovery of minimal bases (DT, D, M, SIMAX, 2010))

It follows **exactly the same block-extraction rule as eigenvector recovery**.

Example: recovery of minimal indices

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$F_\sigma(\lambda) = \lambda M_6 - M_0\,(M_1 M_3 M_5)\,(M_2 M_4)$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & I_n \\ & & & -A_0 & 0 & 0 \end{bmatrix}$$

Note that $\mathfrak{i}(\sigma) = 2$ and $\mathfrak{c}(\sigma) = 3$, so

(a) If $0 \le \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then

$$\varepsilon_1 + 2 \le \varepsilon_2 + 2 \le \cdots \le \varepsilon_p + 2\,,$$

are the right minimal indices of $F_\sigma(\lambda)$.

(b) If $0 \le \eta_1 \le \eta_2 \le \cdots \le \eta_p$ are the left minimal indices of $P(\lambda)$, then

$$\eta_1 + 3 \le \eta_2 + 3 \le \cdots \le \eta_p + 3\,,$$

are the left minimal indices of $F_\sigma(\lambda)$.

Example: recovery of minimal indices

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$F_\sigma(\lambda) = \lambda M_6 - M_0\,(M_1M_3M_5)\,(M_2M_4)$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & I_n \\ & & & -A_0 & 0 & 0 \end{bmatrix}$$

Note that $\mathfrak{i}(\sigma) = 2$ and $\mathfrak{c}(\sigma) = 3$, so

(a) If $0 \le \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then

$$\varepsilon_1 + 2 \le \varepsilon_2 + 2 \le \cdots \le \varepsilon_p + 2\,,$$

are the right minimal indices of $F_\sigma(\lambda)$.

(b) If $0 \le \eta_1 \le \eta_2 \le \cdots \le \eta_p$ are the left minimal indices of $P(\lambda)$, then

$$\eta_1 + 3 \le \eta_2 + 3 \le \cdots \le \eta_p + 3\,,$$

are the left minimal indices of $F_\sigma(\lambda)$.

Example: recovery of minimal indices

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$F_\sigma(\lambda) = \lambda M_6 - M_0\,(M_1 M_3 M_5)\,(M_2 M_4)$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & I_n \\ & & & -A_0 & 0 & 0 \end{bmatrix}$$

Note that $\mathfrak{i}(\sigma) = 2$ and $\mathfrak{c}(\sigma) = 3$, so

(a) If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then

$$\varepsilon_1 + 2 \leq \varepsilon_2 + 2 \leq \cdots \leq \varepsilon_p + 2\,,$$

are the right minimal indices of $F_\sigma(\lambda)$.

(b) If $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ are the left minimal indices of $P(\lambda)$, then

$$\eta_1 + 3 \leq \eta_2 + 3 \leq \cdots \leq \eta_p + 3\,,$$

are the left minimal indices of $F_\sigma(\lambda)$.

Outline

Lemma

- There are no Fiedler pencils that are symmetric whenever $P(\lambda)$ is symmetric.
- There are no Fiedler pencils that are palindromic whenever $P(\lambda)$ is palindromic.

Definition

An $n \times n$ matrix polynomial $P(\lambda)$ is

- **symmetric** if $P(\lambda) = P(\lambda)^T$.
- **palindromic** if $\operatorname{rev} P(\lambda) = P(\lambda)^T$.

Lemma

- There are no Fiedler pencils that are symmetric whenever $P(\lambda)$ is symmetric.
- There are no Fiedler pencils that are palindromic whenever $P(\lambda)$ is palindromic.

Definition

An $n \times n$ matrix polynomial $P(\lambda)$ is

- **symmetric** if $P(\lambda) = P(\lambda)^T$.
- **palindromic** if $\mathrm{rev}\, P(\lambda) = P(\lambda)^T$.

Idea: Given $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{F}^{n\times n}$, recall the matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix} \in \mathbb{F}^{nk\times nk}\,, \quad j = 1, \ldots, k-1\,,$$

$$M_0 := \begin{bmatrix} I_{n(k-1)} & \\ & -A_0 \end{bmatrix} \in \mathbb{F}^{nk\times nk}\,, \quad M_k := \begin{bmatrix} A_k & \\ & I_{n(k-1)} \end{bmatrix} \in \mathbb{F}^{nk\times nk},$$

and note that $M_1, M_2, \ldots, M_{k-1}$ **are always invertible**.

Then multiply any Fiedler pencil

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

by **some of the factors** $M_1^{-1}, M_2^{-1}, \ldots, M_{k-1}^{-1}$ in a certain order to obtain pencils **strictly equivalent to** $F_\sigma(\lambda)$ (so **strong linearizations for** $P(\lambda)$) of the type

$$\lambda M_{\sigma_1} - M_{\sigma_0} := \lambda\,(M_{p_0}^{-1} \cdots M_{p_{s_1}}^{-1}) M_k (M_{q_0}^{-1} \cdots M_{q_{s_2}}^{-1}) - M_{r_0} M_{r_1} \cdots M_{r_{s_3}}$$

Idea: Given $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{F}^{n\times n}$, recall the matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix} \in \mathbb{F}^{nk\times nk}, \quad j = 1, \ldots, k-1\,,$$

$$M_0 := \begin{bmatrix} I_{n(k-1)} & \\ & -A_0 \end{bmatrix} \in \mathbb{F}^{nk\times nk}, \quad M_k := \begin{bmatrix} A_k & \\ & I_{n(k-1)} \end{bmatrix} \in \mathbb{F}^{nk\times nk},$$

and note that $M_1, M_2, \ldots, M_{k-1}$ **are always invertible**.

Then multiply any Fiedler pencil

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

by **some of the factors** $M_1^{-1}, M_2^{-1}, \ldots, M_{k-1}^{-1}$ in a certain order to obtain pencils **strictly equivalent to** $F_\sigma(\lambda)$ (so **strong linearizations for** $P(\lambda)$) of the type

$$\lambda M_{\sigma_1} - M_{\sigma_0} := \lambda\,(M_{p_0}^{-1} \cdots M_{p_{s_1}}^{-1}) M_k (M_{q_0}^{-1} \cdots M_{q_{s_2}}^{-1}) - M_{r_0} M_{r_1} \cdots M_{r_{s_3}}$$

Idea: Given $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{F}^{n\times n}$, recall the matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix} \in \mathbb{F}^{nk\times nk}\,, \quad j = 1, \ldots, k-1\,,$$

$$M_0 := \begin{bmatrix} I_{n(k-1)} & \\ & -A_0 \end{bmatrix} \in \mathbb{F}^{nk\times nk}\,, \quad M_k := \begin{bmatrix} A_k & \\ & I_{n(k-1)} \end{bmatrix} \in \mathbb{F}^{nk\times nk},$$

and note that $M_1, M_2, ..., M_{k-1}$ **are always invertible**.

Then multiply any Fiedler pencil

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

by **some of the factors** $M_1^{-1}, M_2^{-1}, \ldots, M_{k-1}^{-1}$ in a certain order to obtain pencils **strictly equivalent to** $F_\sigma(\lambda)$ (so **strong linearizations for** $P(\lambda)$) of the type

$$\lambda M_{\sigma_1} - M_{\sigma_0} := \lambda\,(M_{p_0}^{-1} \cdots M_{p_{s_1}}^{-1}) M_k (M_{q_0}^{-1} \cdots M_{q_{s_2}}^{-1}) - M_{r_0} M_{r_1} \cdots M_{r_{s_3}}$$

Definition

$$\lambda M_{\sigma_1} - M_{\sigma_0} := \lambda\,(M_{p_0}^{-1} \cdots M_{p_{s_1}}^{-1}) M_k (M_{q_0}^{-1} \cdots M_{q_{s_2}}^{-1}) - M_{r_0} M_{r_1} \cdots M_{r_{s_3}}$$

is a **proper generalized Fiedler pencil (strong linearization)** for $P(\lambda)$ if

- $(p_0, \ldots, p_{s_1}, k, q_0, \ldots, q_{s_2}, r_0, \ldots, r_{s_3})$ is a permutation of $(0, 1, \ldots, k)$.
- $0 \in \{r_0, r_1, \ldots, r_{s_3}\}$.

Generalized Fiedler (GF) pencils that preserve symmetry for odd degree

Theorem (Antoniou & Vologiannidis, ELA, 2004)

Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be an $n \times n$ matrix poly of **odd degree**, then the proper generalized Fiedler linearization for $P(\lambda)$

$$S(\lambda) = \lambda\, M_k M_{k-2}^{-1} \cdots M_3^{-1} M_1^{-1} - M_{k-1} M_{k-3} \cdots M_2 M_0$$

is symmetric whenever $P(\lambda)$ is symmetric.

It follows easily from

$$M_j^{-1} = \begin{bmatrix} I_{n(k-j-1)} & & & \\ & 0 & I_n & \\ & I_n & A_j & \\ & & & I_{n(j-1)} \end{bmatrix} \in \mathbb{F}^{nk \times nk}, \quad j = 1, \ldots, k-1 .$$

Theorem (Antoniou & Vologiannidis, ELA, 2004)

Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be an $n \times n$ matrix poly of **odd degree**, then the proper generalized Fiedler linearization for $P(\lambda)$

$$S(\lambda) = \lambda\, M_k M_{k-2}^{-1} \cdots M_3^{-1} M_1^{-1} - M_{k-1} M_{k-3} \cdots M_2 M_0$$

is symmetric whenever $P(\lambda)$ is symmetric.

It follows easily from

$$M_j^{-1} = \begin{bmatrix} I_{n(k-j-1)} & & & \\ & 0 & I_n & \\ & I_n & A_j & \\ & & & I_{n(j-1)} \end{bmatrix} \in \mathbb{F}^{nk \times nk}, \quad j = 1, \ldots, k-1\,.$$

GF pencils that preserve symmetry for odd degree: Example

$$P(\lambda) = A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$S(\lambda) = \lambda\, M_5 M_3^{-1} M_1^{-1} - M_4 M_2 M_0$$

$$= \lambda \begin{bmatrix} A_5 & & & & \\ & 0 & I_n & & \\ & I_n & A_3 & & \\ & & & 0 & I_n \\ & & & I_n & A_1 \end{bmatrix} - \begin{bmatrix} -A_4 & I_n & & & \\ I_n & 0 & & & \\ & & -A_2 & I_n & \\ & & I_n & 0 & \\ & & & & -A_0 \end{bmatrix}$$

Theorem (De Terán, D, Mackey, JCAM, 2011)

Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be an $n \times n$ matrix poly of **odd degree**. Consider any **proper generalized Fiedler pencil** of the type

$$L(\lambda) = \lambda(\cdots M_k \cdots M_{k-i_1}^{-1} M_{k-i_0}^{-1}) - (M_{i_0} M_{i_1} \cdots M_0 \cdots),$$

and define

$$R = \begin{bmatrix} & & I_n \\ & \cdot^{\cdot^{\cdot}} & \\ I_n & & \end{bmatrix} \in \mathbb{F}^{nk\times nk} \text{ and } S = \begin{bmatrix} \pm I_n & & \\ & \ddots & \\ & & \pm I_n \end{bmatrix} \in \mathbb{F}^{nk\times nk},$$

where the signs are easily determined by the consecutions/inversions of the factors in $(M_{i_0} M_{i_1} \cdots M_0 \cdots)$. Then

$$L_{palin}(\lambda) = S\,R\,L(\lambda)$$

is a strong linearization of $P(\lambda)$ that is palindromic whenever $P(\lambda)$ is palindromic.

$$P(\lambda) = A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

There are many, let us illustrate one with lowest (anti-)bandwidth

$$L_{palin}(\lambda) = S\,R\left(\lambda\, M_1^{-1} M_3^{-1} M_5 - M_0 M_2 M_4\right)$$

$$= \lambda \begin{bmatrix} & & & I_n & A_1 \\ & & & 0 & -I_n \\ & I_n & A_3 & & \\ & 0 & -I_n & & \\ A_5 & & & & \end{bmatrix} + \begin{bmatrix} & & & & A_0 \\ & & I_n & 0 & \\ & & A_2 & -I_n & \\ I_n & 0 & & & \\ A_4 & -I_n & & & \end{bmatrix},$$

$$P(\lambda) = A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

There are many, let us illustrate one with lowest (anti-)bandwidth

$$L_{palin}(\lambda) = S\,R\,(\lambda\, M_1^{-1} M_3^{-1} M_5 - M_0 M_2 M_4)$$

$$= \lambda \begin{bmatrix} & & & I_n & A_1 \\ & & & 0 & -I_n \\ & I_n & A_3 & & \\ & 0 & -I_n & & \\ A_5 & & & & \end{bmatrix} + \begin{bmatrix} & & & & A_0 \\ & & I_n & 0 & \\ & & A_2 & -I_n & \\ I_n & 0 & & & \\ A_4 & -I_n & & & \end{bmatrix},$$

Recovery of eigenvectors, minimal indices and bases from proper GF pencils (Bueno, De Terán, D, SIMAX, 2011)

- Exactly the same recovery rules via block-extraction for
  - minimal bases, and
  - eigenvectors of finite eigenvalues of $P(\lambda)$,

  considering consecutions and inversions only for the zero degree term of the pencil.
- Different, but simple, rules for eigenvectors of the infinite eigenvalue also via block-extraction. They involve consecutions and inversions only for the first degree term of the pencil.
- Different, but simple, rules for minimal indices.
- Explicit expressions of e-vectors of GF pencils in terms of e-vectors of $P(\lambda)$ (Bueno, De Terán, D, presented last ILAS 2011).

Recovery of eigenvectors, minimal indices and bases from proper GF pencils (Bueno, De Terán, D, SIMAX, 2011)

- Exactly the same recovery rules via block-extraction for
  - minimal bases, and
  - eigenvectors of finite eigenvalues of $P(\lambda)$,

  considering consecutions and inversions only for the zero degree term of the pencil.
- Different, but simple, rules for eigenvectors of the infinite eigenvalue also via block-extraction. They involve consecutions and inversions only for the first degree term of the pencil.
- Different, but simple, rules for minimal indices.
- Explicit expressions of e-vectors of GF pencils in terms of e-vectors of $P(\lambda)$ (Bueno, De Terán, D, presented last ILAS 2011).

Recovery of eigenvectors, minimal indices and bases from proper GF pencils (Bueno, De Terán, D, SIMAX, 2011)

- Exactly the same recovery rules via block-extraction for
  - minimal bases, and
  - eigenvectors of finite eigenvalues of $P(\lambda)$,

  considering consecutions and inversions only for the zero degree term of the pencil.
- Different, but simple, rules for eigenvectors of the infinite eigenvalue also via block-extraction. They involve consecutions and inversions only for the first degree term of the pencil.
- Different, but simple, rules for minimal indices.
- Explicit expressions of e-vectors of GF pencils in terms of e-vectors of $P(\lambda)$ (Bueno, De Terán, D, presented last ILAS 2011).

Recovery of eigenvectors, minimal indices and bases from proper GF pencils (Bueno, De Terán, D, SIMAX, 2011)

- Exactly the same recovery rules via block-extraction for
  - minimal bases, and
  - eigenvectors of finite eigenvalues of $P(\lambda)$,

  considering consecutions and inversions only for the zero degree term of the pencil.

- Different, but simple, rules for eigenvectors of the infinite eigenvalue also via block-extraction. They involve consecutions and inversions only for the first degree term of the pencil.

- Different, but simple, rules for minimal indices.

- Explicit expressions of e-vectors of GF pencils in terms of e-vectors of $P(\lambda)$ (Bueno, De Terán, D, presented last ILAS 2011).

Example of eigenvector recovery in proper GF pencils

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$G(\lambda) = \lambda\, M_3^{-1} M_6 M_5^{-1} - M_4 M_0 M_2 M_1$$

$$= \lambda \begin{bmatrix} & A_6 & & & & \\ I_n & A_5 & & & & \\ & & & I_n & & \\ & & I_n & A_3 & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} I_n & & & & & \\ & -A_4 & I_n & & & \\ & I_n & & & & \\ & & & -A_2 & -A_1 & I_n \\ & & & I_n & & \\ & & & & -A_0 & \end{bmatrix}$$

$M_4M_0M_2M_1$ has $\mathfrak{c}_0 = 1$ initial consecutions (consecution at 0, inversion at 1, nothing at $2,3,4,5$

$$z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \quad (x_i \in \mathbb{F}^{n\times 1})$$ be such that $G(\lambda_0)z = 0 \Longrightarrow P(\lambda_0)x_5 = 0$

Example of eigenvector recovery in proper GF pencils

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$G(\lambda) = \lambda\, M_3^{-1} M_6 M_5^{-1} - M_4 M_0 M_2 M_1$$

$$= \lambda \begin{bmatrix} & A_6 & & & & \\ I_n & A_5 & & & & \\ & & & I_n & & \\ & & I_n & A_3 & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} I_n & & & & & \\ & -A_4 & I_n & & & \\ & I_n & & & & \\ & & & -A_2 & -A_1 & I_n \\ & & & I_n & & \\ & & & & -A_0 & \end{bmatrix}$$

$M_4M_0M_2M_1$ has $\mathfrak{c}_0 = 1$ initial consecutions (consecution at 0, inversion at 1, nothing at $2, 3, 4, 5$

$$z = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline x_3 \\ \hline x_4 \\ \hline x_5 \\ \hline x_6 \end{bmatrix}, \quad (x_i \in \mathbb{F}^{n\times 1})$$ be such that $G(\lambda_0)z = 0 \Longrightarrow P(\lambda_0)x_5 = 0$

Outline

1. Definition of Fiedler pencils. Consecutions and inversions.
2. Recovery of eigenvectors from Fiedler pencils
3. Recovery of minimal indices and bases from Fiedler pencils
4. Preservation of structures and generalized Fiedler pencils
5. **Eigenvectors of GF pencils with repeated factors**
6. Conclusions

- Example: $P(\lambda) = A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$

$$
\begin{aligned}
L(\lambda) &= \lambda\, M_2^{-1}M_4^{-1}M_1^{-1}M_3^{-1}M_5M_2^{-1}M_4^{-1} - M_2^{-1}M_4^{-1}M_0 \\
&= M_2^{-1}M_4^{-1}\ \left(\lambda\, M_1^{-1}M_3^{-1}M_5M_2^{-1}M_4^{-1} - M_0\right) \\
&= \lambda \begin{bmatrix} 0 & 0 & 0 & I_n & 0 \\ 0 & A_5 & 0 & A_4 & 0 \\ 0 & 0 & 0 & 0 & I_n \\ I_n & A_4 & 0 & A_3 & A_2 \\ 0 & 0 & I_n & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} 0 & I_n & 0 & 0 & 0 \\ I_n & A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & I_n & A_2 & 0 \\ 0 & 0 & 0 & 0 & -A_0 \end{bmatrix},
\end{aligned}
$$

- They are defined in terms of two basic ideas:
  - **(a)** They are strictly equivalent to Fiedler pencils via multiplication by M_i or M_i^{-1}.
  - **(b)** Although there are repeated factors, the pencil is made of blocks that can be either $\pm I_n$, $\pm A_i$, 0_n.
- Necessary and sufficient condition for (b) are presented by Vologiannidis and Antoniou (MCSS 2011).

GF pencils with repeated factors (Vologiannidis & Antoniou, MCSS 2011)

- Example: $P(\lambda) = A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$

$$
\begin{aligned}
L(\lambda) &= \lambda\, M_2^{-1} M_4^{-1} M_1^{-1} M_3^{-1} M_5 M_2^{-1} M_4^{-1} - M_2^{-1} M_4^{-1} M_0 \\
&= M_2^{-1} M_4^{-1} \; (\lambda\, M_1^{-1} M_3^{-1} M_5 M_2^{-1} M_4^{-1} - M_0) \\
&= \lambda \begin{bmatrix} 0 & 0 & 0 & I_n & 0 \\ 0 & A_5 & 0 & A_4 & 0 \\ 0 & 0 & 0 & 0 & I_n \\ I_n & A_4 & 0 & A_3 & A_2 \\ 0 & 0 & I_n & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} 0 & I_n & 0 & 0 & 0 \\ I_n & A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & I_n & A_2 & 0 \\ 0 & 0 & 0 & 0 & -A_0 \end{bmatrix},
\end{aligned}
$$

- They are defined in terms of two basic ideas:
  - **(a)** They are strictly equivalent to Fiedler pencils via multiplication by M_i or M_i^{-1}.
  - **(b)** Although there are repeated factors, the pencil is made of blocks that can be either $\pm I_n$, $\pm A_i$, 0_n.

- Necessary and sufficient condition for (b) are presented by Vologiannidis and Antoniou (MCSS 2011).

- Example: $P(\lambda) = A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$

$$\begin{aligned} L(\lambda) &= \lambda\, M_2^{-1} M_4^{-1} M_1^{-1} M_3^{-1} M_5 M_2^{-1} M_4^{-1} - M_2^{-1} M_4^{-1} M_0 \\ &= M_2^{-1} M_4^{-1} \;\; (\lambda\, M_1^{-1} M_3^{-1} M_5 M_2^{-1} M_4^{-1} - M_0) \\ &= \lambda \begin{bmatrix} 0 & 0 & 0 & I_n & 0 \\ 0 & A_5 & 0 & A_4 & 0 \\ 0 & 0 & 0 & 0 & I_n \\ I_n & A_4 & 0 & A_3 & A_2 \\ 0 & 0 & I_n & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} 0 & I_n & 0 & 0 & 0 \\ I_n & A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & I_n & A_2 & 0 \\ 0 & 0 & 0 & 0 & -A_0 \end{bmatrix}, \end{aligned}$$

- They are defined in terms of two basic ideas:
  - **(a)** They are strictly equivalent to Fiedler pencils via multiplication by M_i or M_i^{-1}.
  - **(b)** Although there are repeated factors, the pencil is made of blocks that can be either $\pm I_n$, $\pm A_i$, 0_n.
- Necessary and sufficient condition for (b) are presented by Vologiannidis and Antoniou (MCSS 2011).

Eigenvectors of GF pencils with repeated factors

- We have (complicated) rules to get explicit expressions of the e-vectors of these pencils in terms of the e-vectors of $P(\lambda)$.
- They do not depend only on the Horner shifts of $P(\lambda)$.
- Example for degree 8.

$$L(\lambda) = \lambda M_6^{-1} M_8 M_7^{-1} M_2 M_3 M_4 - M_1 M_2 M_3 M_4 M_5 M_0 M_2 M_3 M_4$$

If $P(\lambda)x = 0$, the $L(\lambda)z = 0$ with

$$z = \left[\begin{array}{c|c|c|c|c|c|c|c|c} \lambda^3 P_0 x & \lambda^2 x & \lambda x & \lambda P_6 x & \lambda(P_3 + A_4 P_6)x & \lambda(P_4 + A_3 P_6)x & \ldots & \lambda(P_5 + A_2 P_6)x & x \end{array} \right]^{\mathcal{B}},$$

where all Horner shifts are evaluated in λ.

- We have (complicated) rules to get explicit expressions of the e-vectors of these pencils in terms of the e-vectors of $P(\lambda)$.
- They do not depend only on the Horner shifts of $P(\lambda)$.
- Example for degree 8.

$$L(\lambda) = \lambda M_6^{-1} M_8 M_7^{-1} M_2 M_3 M_4 - M_1 M_2 M_3 M_4 M_5 M_0 M_2 M_3 M_4$$

If $P(\lambda)x = 0$, the $L(\lambda)z = 0$ with

$$z = \left[\begin{array}{c|c|c|c|c|c|c} \lambda^3 P_0 x & \lambda^2 x & \lambda x & \lambda P_6 x & \lambda(P_3 + A_4 P_6)x & \lambda(P_4 + A_3 P_6)x & \ldots \end{array} \right.$$
$$\left. \begin{array}{c|c} \ldots \ \lambda(P_5 + A_2 P_6)x & x \end{array} \right]^{\mathcal{B}},$$

where all Horner shifts are evaluated in λ.

- We have (complicated) rules to get explicit expressions of the e-vectors of these pencils in terms of the e-vectors of $P(\lambda)$.
- They do not depend only on the Horner shifts of $P(\lambda)$.
- Example for degree 8.

$$L(\lambda) = \lambda M_6^{-1} M_8 M_7^{-1} M_2 M_3 M_4 - M_1 M_2 M_3 M_4 M_5 M_0 M_2 M_3 M_4$$

If $P(\lambda)x = 0$, the $L(\lambda)z = 0$ with

$$z = \left[\begin{array}{c|c|c|c|c|c|c|c|c} \lambda^3 P_0 x & \lambda^2 x & \lambda x & \lambda P_6 x & \lambda(P_3 + A_4 P_6)x & \lambda(P_4 + A_3 P_6)x & \ldots & \lambda(P_5 + A_2 P_6)x & x \end{array} \right]^{\mathcal{B}},$$

where all Horner shifts are evaluated in λ.

Eigenvectors of GF pencils with repeated factors

- We have (complicated) rules to get explicit expressions of the e-vectors of these pencils in terms of the e-vectors of $P(\lambda)$.
- They do not depend only on the Horner shifts of $P(\lambda)$.
- Example for degree 8.

$$L(\lambda) = \lambda M_6^{-1} M_8 M_7^{-1} M_2 M_3 M_4 - M_1 M_2 M_3 M_4 M_5 M_0 M_2 M_3 M_4$$

If $P(\lambda)x = 0$, the $L(\lambda)z = 0$ with

$$z = \left[\ \lambda^3 P_0 x \ \middle| \ \lambda^2 x \ \middle| \ \lambda x \ \middle| \ \lambda P_6 x \ \middle| \ \lambda(P_3 + A_4 P_6)x \ \middle| \ \lambda(P_4 + A_3 P_6)x \ \middle| \ \ldots \right.$$
$$\left. \ldots \ \middle| \ \lambda(P_5 + A_2 P_6)x \ \middle| \ x \ \right]^{\mathcal{B}},$$

where all Horner shifts are evaluated in λ.

Outline

1. Definition of Fiedler pencils. Consecutions and inversions.
2. Recovery of eigenvectors from Fiedler pencils
3. Recovery of minimal indices and bases from Fiedler pencils
4. Preservation of structures and generalized Fiedler pencils
5. Eigenvectors of GF pencils with repeated factors
6. **Conclusions**

Conclusions and Future work

- The algebraic and the recovery properties of Fiedler pencils are well-understood.
- The **fundamental open problem to ascertain the practical relevance of Fiedler pencils** is to compare the **conditioning and backward errors** of eigenvalues in Fiedler pencils with respect conditioning and backward errors in companion forms and in the original polynomial $P(\lambda)$.
- This is a difficult problem. Two outgoing works
  - De Terán and Tisseur: cubic matrix polynomials.
  - D and Pérez-Álvaro: scalar polynomials.
- Probably, the most relevant numerical applications in eigenvalue/vector computations of (generalized) Fiedler pencils will be in
  - Symmetric and palindromic matrix polynomials of odd-degree.
  - Scalar polynomials with very large degree, where low bandwith may be important.

- The algebraic and the recovery properties of Fiedler pencils are well-understood.
- The **fundamental open problem to ascertain the practical relevance of Fiedler pencils** is to compare the **conditioning and backward errors** of eigenvalues in Fiedler pencils with respect conditioning and backward errors in companion forms and in the original polynomial $P(\lambda)$.
- This is a difficult problem. Two outgoing works
  - De Terán and Tisseur: cubic matrix polynomials.
  - D and Pérez-Álvaro: scalar polynomials.
- Probably, the most relevant numerical applications in eigenvalue/vector computations of (generalized) Fiedler pencils will be in
  - Symmetric and palindromic matrix polynomials of odd-degree.
  - Scalar polynomials with very large degree, where low bandwith may be important.

- The algebraic and the recovery properties of Fiedler pencils are well-understood.
- The **fundamental open problem to ascertain the practical relevance of Fiedler pencils** is to compare the **conditioning and backward errors** of eigenvalues in Fiedler pencils with respect conditioning and backward errors in companion forms and in the original polynomial $P(\lambda)$.
- This is a difficult problem. Two outgoing works
  - De Terán and Tisseur: cubic matrix polynomials.
  - D and Pérez-Álvaro: scalar polynomials.
- Probably, the most relevant numerical applications in eigenvalue/vector computations of (generalized) Fiedler pencils will be in
  - Symmetric and palindromic matrix polynomials of odd-degree.
  - Scalar polynomials with very large degree, where low bandwith may be important.

- The algebraic and the recovery properties of Fiedler pencils are well-understood.
- The **fundamental open problem to ascertain the practical relevance of Fiedler pencils** is to compare the **conditioning and backward errors** of eigenvalues in Fiedler pencils with respect conditioning and backward errors in companion forms and in the original polynomial $P(\lambda)$.
- This is a difficult problem. Two outgoing works
  - De Terán and Tisseur: cubic matrix polynomials.
  - D and Pérez-Álvaro: scalar polynomials.
- Probably, the most relevant numerical applications in eigenvalue/vector computations of (generalized) Fiedler pencils will be in
  - Symmetric and palindromic matrix polynomials of odd-degree.
  - Scalar polynomials with very large degree, where low bandwith may be important.