The general inverse matrix polynomial index structure problem

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Outline

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1. Preliminary concepts
2. Goal of the talk and previous works
3. The Index Sum Theorem
4. The main theorem
5. Existence, possible sizes and eigenstructures of ℓ-ifications
6. Conclusions

Definition

The **Smith normal form** of $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{F}[\lambda]^{m \times n}$ is the following diagonal matrix obtained under unimodular transformations $U(\lambda)$ and $V(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \left[\begin{array}{cccc|c} p_1(\lambda) & 0 & \dots & 0 & \\ 0 & p_2(\lambda) & \ddots & \vdots & 0_{r\times(n-r)} \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \dots & 0 & p_r(\lambda) & \\ \hline & & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right].$$

- $p_1(\lambda), \dots, p_r(\lambda)$ are monic scalar polynomials in $\mathbb{F}$,
- $p_j(\lambda)$ divides $p_{j+1}(\lambda)$, for $j = 1, \dots, r-1$,
- $p_1(\lambda), \dots, p_r(\lambda)$ are unique and are the **invariant polynomials** of $P(\lambda)$.
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Finite eigenvalues and elementary divisors of $P(\lambda)$

- $\alpha \in \overline{\mathbb{F}}$ is a **finite eigenvalue** of $P(\lambda)$ if $p_j(\alpha) = 0$, for some $j = 1, \ldots, r$.
- The partial multiplicity sequence of $P(\lambda)$ at $\alpha \in \overline{\mathbb{F}}$ is the sequence

$$0 \leq \delta_1 \leq \delta_2 \leq \cdots \leq \delta_r, \qquad \text{s. t.}$$

$$p_j(\lambda) = (\lambda - \alpha)^{\delta_j} \, q_j(\lambda), \quad \text{with} \quad q_j(\alpha) \neq 0, \quad \text{for} \quad j = 1, \ldots, r$$

- The **elementary divisors** of $P(\lambda)$ **at** α are the collection of factors

$$(\lambda - \alpha)^{\delta_j} \quad \text{with} \quad \delta_j > 0$$

  This is a local definition at α that may drive us out of $\mathbb{F}$.

- For a global definition that keeps us in $\mathbb{F}$, we follow Gantmacher:

$$p_j(\lambda) = \phi_1(\lambda)^{\beta_{j1}} \, \phi_2(\lambda)^{\beta_{j2}} \cdots \phi_s(\lambda)^{\beta_{js}}, \quad \text{for} \quad j = 1, \ldots, r,$$

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Example of invariant polynomials and elementary divisors

$$P(\lambda) = \begin{bmatrix} 1 & & & \\ & \lambda - 1 & & \\ & & (\lambda-1)(\lambda^2+1) & \\ & & & 0 \end{bmatrix} \in \mathbb{R}[\lambda]^{4\times 4}$$

- **Invariant polynomials of $P(\lambda)$:**

$$p_1(\lambda) = 1, \; p_2(\lambda) = (\lambda - 1), \; p_3(\lambda) = (\lambda-1)(\lambda^2+1).$$

- Finite **elementary divisors of $P(\lambda)$ (in $\mathbb{R}$):** $(\lambda-1), \; (\lambda-1), \; (\lambda^2+1)$
- "Finite elementary divisors" $\Longleftrightarrow$ "**Nontrivial** Invariant polynomials"
- "Finite elementary divisors plus **rank**" $\Longleftrightarrow$ "Invariant polynomials"
- Eigenvalues of $P(\lambda)$: $\{1, 1, i, -i\}$
  - Elementary divisors of $P(\lambda)$ at 1: $(\lambda-1), \; (\lambda-1)$
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Definition

Let

$$P(\lambda) = P_d \lambda^d + P_{d-1}\lambda^{d-1} + \cdots + P_0, \qquad P_d \neq 0$$

be a matrix polynomial of **degree** d, the **reversal** of $P(\lambda)$ is

$$\mathrm{rev}P(\mu) := \mu^d P\left(\frac{1}{\mu}\right) = P_d + P_{d-1}\,\mu + \cdots + P_0\,\mu^d\,.$$

Remark: We take in this talk the reversal with respect the **degree**.

Definition (Elementary divisors at ∞)

$\mu = \infty$ is an eigenvalue of $P(\lambda)$ if 0 is an eigenvalue of $\mathrm{rev}P(\mu)$.

The partial multiplicity sequence of $P(\lambda)$ at ∞ is the same as that of 0 in $\mathrm{rev}P(\mu)$.

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An $m \times n$ matrix polynomial $P(\lambda)$ whose rank r is smaller than m and/or n has non-trivial left and/or right null-spaces over the field $\mathbb{F}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

$\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P)$ have bases consisting entirely of vector polynomials.

Definition (Minimal bases)

A **right minimal basis** of $P(\lambda)$ is a basis of $\mathcal{N}_r(P)$

1. consisting of vector polynomials
2. whose sum of degrees is minimal among all bases of $\mathcal{N}_r(P)$ consisting of vector polynomials.

Analogous definition for **left minimal basis**.

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$$\mathcal{N}_r(P) := \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \; : \; P(\lambda) x(\lambda) \equiv 0 \right\}.$$

$\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P)$ have bases consisting entirely of vector polynomials.

Definition (Minimal bases)

A **right minimal basis** of $P(\lambda)$ is a basis of $\mathcal{N}_r(P)$

1. consisting of vector polynomials
2. whose sum of degrees is minimal among all bases of $\mathcal{N}_r(P)$ consisting of vector polynomials.

Analogous definition for **left minimal basis**.

There are many (infinite) right minimal bases of $P(\lambda)$, but...

Theorem (Forney, 1975)

The ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{N}_r(P)$ is always the same.

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These degrees are called the **right minimal indices** of $P(\lambda)$.

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Example of right minimal basis and minimal indices

$$P(\lambda) = \begin{bmatrix} 1 & -\lambda^3 & & & \\ & & 1 & -\lambda & \\ & & & 1 & -\lambda \end{bmatrix} \in \mathbb{R}[\lambda]^{3\times 5}$$

$$\mathcal{N}_r(P) = \text{Span}\{ \underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{u_2} \} = \text{Span}\{ \underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ \lambda^3 \\ \lambda^2 \\ \lambda \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} \lambda^5 \\ \lambda^2 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{w_2} \}$$

Sum of degrees of $\{u_1, u_2\} = 3 + 2 = 5$ (right minimal bases of $P(\lambda)$)

Sum of degrees of $\{w_1, w_2\} = 3 + 5 = 8$

Right minimal indices of $P(\lambda)$ = $\{2, 3\}$

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Definition

Given an $m \times n$ matrix polynomial $P(\lambda)$ with rank r, the eigenstructure of $P(\lambda)$ consists of the following lists of scalar polynomials and non-negative integers:

(i) the invariant polynomials $p_1(\lambda), \ldots, p_r(\lambda)$, with degrees $\delta_1, \ldots, \delta_r$ (finite structure),

(ii) the partial multiplicity sequence at ∞, $\gamma_1, \ldots, \gamma_r$ (infinite structure),

(iii) the right minimal indices $\varepsilon_1, \ldots, \varepsilon_{n-r}$ (right singular structure), and

(iv) the left minimal indices $\eta_1, \ldots, \eta_{m-r}$ (left singular structure).

- This is a classical terminology.
- In some recent references
  - the eigenstructure of $P(\lambda)$ stands only for parts (i) and (ii), and
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Outline

If a **complete eigenstructure is prescribed** and **a degree d is also prescribed**, we want

- to find necessary and sufficient conditions for the existence of a matrix polynomial $P(\lambda)$ with precisely this eigenstructure and this degree,
- and to construct such $P(\lambda)$.

Remarks:

- If the degree is not prescribed, then the problem is trivial...one can realize any eigenstructure with degree one via the Kronecker canonical form of pencils.
- As far as we know, all previous results on inverse polynomial eigenvalue problems do not consider prescribed simultaneously the complete eigenstructure and an arbitrary degree.

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Theorem (Praagman 1991, De Terán & D & Mackey, 2013)

Let $P(\lambda)$ be an $m \times n$ matrix polynomial of **degree d and rank r** *having the following regular and singular eigenstructure:*

- r *invariant polynomials $p_j(\lambda)$ of degrees δ_j, for $j = 1, \ldots, r$,*
- r *infinite partial multiplicities $\gamma_1, \ldots, \gamma_r$,*
- $n - r$ *right minimal indices $\varepsilon_1, \ldots, \varepsilon_{n-r}$, and*
- $m - r$ *left minimal indices $\eta_1, \ldots, \eta_{m-r}$,*

where some of the degrees, partial multiplicities or indices can be zero, and/or one or both of the lists of minimal indices can be empty. Then

$$\sum_{j=1}^{r} \delta_j + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j = dr.$$

Definition

A list $\mathcal{L}$ of elementary divisors over a field $\mathbb{F}$ is the concatenation of two lists:

- a list $\mathcal{L}_{\text{fin}}$ of positive integer powers of monic irreducible polynomials of degree at least 1 with coefficients in $\mathbb{F}$, and
- a list $\mathcal{L}_\infty$ of elementary divisors $\mu^{\alpha_1}, \mu^{\alpha_2}, \ldots, \mu^{\alpha_{g_\infty}}$ at ∞.

Definition

- The length of the longest sublist of $\mathcal{L}_{\text{fin}}$ containing powers of the same irreducible polynomial is denoted by $g_{\text{fin}}(\mathcal{L})$.
- The length of $\mathcal{L}_\infty$ is denoted by $g_\infty(\mathcal{L})$.
- The sum of the degrees of the elements in $\mathcal{L}_{\text{fin}}$ is denoted by $\delta_{\text{fin}}(\mathcal{L})$.
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Remark.

- In $\mathbb{C}$, $\mathcal{L}_{\text{fin}} = \{(\lambda - a_1)^{\delta_1}, (\lambda - a_2)^{\delta_2}, \ldots, \}$, with $a_i \in \mathbb{C}$.
- In $\mathbb{R}$,

  $$\mathcal{L}_{\text{fin}} = \{(\lambda - a_1)^{\delta_1}, (\lambda - a_2)^{\delta_2}, \ldots, (\lambda^2 + b_1\lambda + c_1)^{\beta_1}, (\lambda^2 + b_2\lambda + c_2)^{\beta_2}, \ldots\},$$

  with $a_i, b_i, c_i \in \mathbb{R}$ and the quadratic polys having complex roots.

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Theorem

Let $\mathcal{L}$ be a list of elementary divisors over an infinite field $\mathbb{F}$,
let $\mathcal{M}_r = \{\varepsilon_1, \ldots, \varepsilon_p\}$ be a list of right minimal indices,
let $\mathcal{M}_l = \{\eta_1, \ldots, \eta_q\}$ be a list of left minimal indices, and define

$$S(\mathcal{L}) = \delta_{\text{fin}}(\mathcal{L}) + \delta_\infty(\mathcal{L}) + \sum_{i=1}^{q} \eta_i + \sum_{i=1}^{p} \varepsilon_i \,.$$

Then, there exists a matrix polynomial $P(\lambda)$ with coefficients in $\mathbb{F}$ of degree d and whose elementary divisors, right minimal indices, and left minimal indices are, respectively, those in the lists $\mathcal{L}$, $\mathcal{M}_r$, and $\mathcal{M}_l$, **if and only if**

(i) d *is a divisor of* $S(\mathcal{L})$, (ii) $\dfrac{S(\mathcal{L})}{d} \geq g_{\text{fin}}(\mathcal{L})$, *and* (iii) $\dfrac{S(\mathcal{L})}{d} > g_\infty(\mathcal{L})$.

In this case,

- *the rank of $P(\lambda)$ is $S(\mathcal{L})/d$, and*
- *the size of $P(\lambda)$ is $(q + S(\mathcal{L})/d) \times (p + S(\mathcal{L})/d)$.*

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The proof of sufficiency is long and involved, but necessity it is easy

- Necessity of (i) d is a divisor of $S(\mathcal{L})$:
  If $P(\lambda)$ exists, then it must satisfy the **Index Sum Theorem**, i.e.,

$$d\,\mathrm{rank}(P) = S(\mathcal{L}) = \delta_{\mathrm{fin}}(\mathcal{L}) + \delta_{\infty}(\mathcal{L}) + \sum_{i=1}^{q} \eta_i + \sum_{i=1}^{p} \varepsilon_i \,.$$

- Necessity of (ii) $\dfrac{S(\mathcal{L})}{d} \geq g_{\mathrm{fin}}(\mathcal{L})$: If $P(\lambda)$ exists, then

$$\mathrm{rank}(P) = \frac{S(\mathcal{L})}{d} \geq g_{\mathrm{fin}}(\mathcal{L}) = \text{largest geometric multiplicity finite e-values}$$

- Necessity of (iii) $\dfrac{S(\mathcal{L})}{d} > g_{\infty}(\mathcal{L})$

  If $P(\lambda) = P_0 + \lambda P_1 + \cdots + \lambda^d P_d$ with $P_d \neq 0$ exists, then
  $0 \neq \mathrm{rev}P(0) = P_d$ and

$$\mathrm{rank}(P) = \frac{S(\mathcal{L})}{d} > g_{\infty}(\mathcal{L}) = \text{geometric multiplicity of } \infty \text{ e-value}$$

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  If $P(\lambda)$ exists, then it must satisfy the **Index Sum Theorem**, i.e.,

$$d \operatorname{rank}(P) = S(\mathcal{L}) = \delta_{\text{fin}}(\mathcal{L}) + \delta_{\infty}(\mathcal{L}) + \sum_{i=1}^{q} \eta_i + \sum_{i=1}^{p} \varepsilon_i \,.$$

- Necessity of (ii) $\frac{S(\mathcal{L})}{d} \geq g_{\text{fin}}(\mathcal{L})$: If $P(\lambda)$ exists, then

$$\operatorname{rank}(P) = \frac{S(\mathcal{L})}{d} \geq g_{\text{fin}}(\mathcal{L}) = \text{largest geometric multiplicity finite e-values}$$

- Necessity of (iii) $\frac{S(\mathcal{L})}{d} > g_{\infty}(\mathcal{L})$

  If $P(\lambda) = P_0 + \lambda P_1 + \cdots + \lambda^d P_d$ with $P_d \neq 0$ exists, then
  $0 \neq \operatorname{rev} P(0) = P_d$ and

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Outline

Definition (De Terán & D & Mackey, 2013)

A matrix polynomial $R(\lambda)$ of degree $\ell > 0$ is said to be an ℓ-ification of a given matrix polynomial $P(\lambda)$ if for some $r, s \geq 0$ there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda) \begin{bmatrix} R(\lambda) & \\ & I_s \end{bmatrix} V(\lambda) = \begin{bmatrix} P(\lambda) & \\ & I_r \end{bmatrix}.$$

If, in addition, rev $R(\lambda)$ is an ℓ-ification of rev $P(\lambda)$, then $R(\lambda)$ is said to be a *strong* ℓ-ification of $P(\lambda)$.

Remarks:

- (strong) ℓ-ifications generalize the concept of (strong) linearizations.
- For $\ell = 1$, we (almost) recover the definition of (strong) linearization.
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Let $P(\lambda) = \lambda^d A_d + \lambda^{d-1} A_{d-1} + \cdots + \lambda A_1 + A_0 \in \mathbb{F}[\lambda]^{m\times n}$, $d = \ell s$, and define the following matrix polynomials of degree $\leq \ell$:

$$B_1(\lambda) := \lambda^\ell A_\ell + \lambda^{\ell-1} A_{\ell-1} + \cdots + \lambda A_1 + A_0 \,,$$
$$B_j(\lambda) := \lambda^\ell A_{\ell j} + \lambda^{\ell-1} A_{\ell j-1} + \cdots + \lambda A_{\ell(j-1)+1} \,, \qquad \text{for } j = 2, \ldots, s\,.$$

Then

$$C_1^\ell(\lambda) := \begin{bmatrix} B_s(\lambda) & B_{s-1}(\lambda) & B_{s-2}(\lambda) & \cdots & B_1(\lambda) \\ -I_n & \lambda^\ell I_n & 0 & \cdots & 0 \\ & -I_n & \lambda^\ell I_n & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & -I_n & \lambda^\ell I_n \end{bmatrix} \in \mathbb{F}[\lambda]^{(m+(s-1)n)\times sn}$$

is a **strong ℓ-ification** of $P(\lambda)$ (De Terán & D & Mackey, 2013).

Remark: Concrete examples easily constructed in terms of the coefficients of $P(\lambda)$ of (weak or standard) ℓ-ifications are also provided in De Terán & D & Mackey, 2013. We restrict in this talk to the strong case for brevity.

Strong ℓ-ifications indeed exist...(at least sometimes)

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Theorem (De Terán & D & Mackey, 2013)

Consider a matrix polynomial $P(\lambda)$ and another matrix polynomial $R(\lambda)$ of degree $\ell > 0$, and the following three conditions on $P(\lambda)$ and $R(\lambda)$:

(a) $P(\lambda)$ *and* $R(\lambda)$ *have the same numbers of right and left minimal indices.*

(b) $P(\lambda)$ *and* $R(\lambda)$ *have exactly the same finite elementary divisors.*

(c) $P(\lambda)$ *and* $R(\lambda)$ *have exactly the same infinite elementary divisors.*

Then:

(1) $R(\lambda)$ *is an* ℓ*-ification of* $P(\lambda)$ *if and only if conditions* (a) *and* (b) *hold.*

(2) $R(\lambda)$ *is a strong* ℓ*-ification of* $P(\lambda)$ *if and only if conditions* (a), (b), *and* (c) *hold.*

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In plain words: (strong) ℓ-ifications have the same “spectral” properties as (strong) linearizations.

Theorem

Let $P(\lambda)$ be an $m \times n$ singular matrix polynomial over an infinite field $\mathbb{F}$ with rank $r > 0$, let ℓ be a positive integer, and let

$$\widetilde{r} = \min \left\{ y \in \mathbb{Z}^+ \,:\, y \geq \frac{\delta_{\text{fin}}(P) + \delta_\infty(P)}{\ell}, \quad y \geq g_{\text{fin}}(P), \quad \textit{and} \quad y > g_\infty(P) \right\}.$$

Then, there is an $s_1 \times s_2$ strong ℓ-fication of $P(\lambda)$ **if and only if**

$$s_1 \geq (m - r) + \widetilde{r}, \quad s_2 \geq (n - r) + \widetilde{r}, \quad \textit{and} \quad s_1 - s_2 = m - n.$$

In particular, the minimum-size strong ℓ-ification of $P(\lambda)$ has sizes

$$s_1 = (m - r) + \widetilde{r} \qquad \textit{and} \qquad s_2 = (n - r) + \widetilde{r},$$

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Possible minimal indices of strong ℓ-ifications of singular polynomials

Theorem

Let $P(\lambda)$ be an $m \times n$ singular matrix polynomial over an infinite field $\mathbb{F}$ with rank $r > 0$, and let ℓ be a positive integer.

Let $0 \leq \widetilde{\eta}_1 \leq \cdots \leq \widetilde{\eta}_{m-r}$ be a list of left minimal indices, let $0 \leq \widetilde{\varepsilon}_1 \leq \cdots \leq \widetilde{\varepsilon}_{n-r}$ be a list of right minimal indices, and define

$$\widetilde{S} = \delta_{\text{fin}}(P) + \delta_\infty(P) + \sum_{i=1}^{m-r} \widetilde{\eta}_i + \sum_{i=1}^{n-r} \widetilde{\varepsilon}_i \,.$$

Then, there exists a strong ℓ-ification $R(\lambda)$ of $P(\lambda)$ having the previous lists of minimal indices ***if and only if***

$$\text{(i)} \;\; \ell \text{ is a divisor of } \widetilde{S}, \quad \text{(ii)} \;\; \frac{\widetilde{S}}{\ell} \geq g_{\text{fin}}(P), \quad \text{and} \quad \text{(iii)} \;\; \frac{\widetilde{S}}{\ell} > g_\infty(P).$$

The size of this $R(\lambda)$ is $s_1 \times s_2$, where

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Theorem

Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial over an infinite field $\mathbb{F}$ of degree d, and let ℓ be a positive integer.

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In addition, the size of any strong ℓ-ification of $P(\lambda)$ is

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In particular, if $\ell > dn$, then there are no strong ℓ-ifications of $P(\lambda)$.

This generalizes a previous result in De Terán & D & Mackey, 2013.

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Outline

- We have solved an inverse matrix polynomial eigenstructure problem
- which is more complete than other inverse results available in the literature,
- since we consider simultaneously prescribed
  1. the complete eigenstructure (including minimal indices), and
  2. an arbitrary degree.
- This inverse result has been used to determine all possible sizes and eigenstructures of the ℓ-ifications of a given poly $P(\lambda)$.
- Apart from trivial restrictions coming from the rank and geometric multiplicities, we can summarize our results in plain words as

"We can realize with a prescribed degree every eigenstructure

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allowed by the Index Sum Theorem and only these eigenstructures

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- This inverse result has been used to determine all possible sizes and eigenstructures of the ℓ-ifications of a given poly $P(\lambda)$.
- Apart from trivial restrictions coming from the rank and geometric multiplicities, we can summarize our results in plain words as

"We can realize with a prescribed degree every eigenstructure (or ℓ-ification) allowed by the Index Sum Theorem and only these eigenstructures (or ℓ-ifications)."

- We have solved an inverse matrix polynomial eigenstructure problem
- which is more complete than other inverse results available in the literature,
- since we consider simultaneously prescribed
  1. the complete eigenstructure (including minimal indices), and
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