Structured linearizations that preserve the sign characteristic of Hermitian matrix polynomials

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ICIAM 2015. Minisymposium on Nonlinear Eigenvalue Problems.
Beijing, China, August 10-14, 2015

Basic concepts: matrix polynomials and their linearizations

- We consider matrix polynomials of degree k

$$P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{C}^{n\times n}\,, \quad A_k \neq 0.$$

- A linearization for $P(\lambda)$ is an $nk \times nk$ **linear matrix polynomial (or matrix pencil)** $L(\lambda)$, such that,

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{n(k-1)} & \\ & P(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

Property: $P(\lambda)$ and $L(\lambda)$ have the same finite spectral structure.

- $L(\lambda)$ is a "strong linearization" if, **in addition**, $\operatorname{rev} L(\lambda)$ is a linearization for $\operatorname{rev} P(\lambda)$, where

$$\operatorname{rev} P(\lambda) := \lambda^k A_0 + \cdots + \lambda A_{k-1} + A_k.$$

Property: $P(\lambda)$ and $L(\lambda)$ have the same finite and infinite spectral structures.

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Importance and ideal properties of linearizations

- Most algorithms for computing eigenvalues of matrix polynomials are based on solving the eigenvalue problem for a linearization via well established methods as QZ, (memory efficient) Arnoldi...
- Most matrix polynomials appearing in applications have particular structures
- and in order to preserve numerically via structured algorithms the symmetries imposed in the spectrum by these structures:
- *"It would be preferable if the structural properties of the polynomial were faithfully reflected in the linearization..."*

  Mackey, Mackey, Mehl, and Mehrmann, "Vector spaces of linearizations for matrix polynomials", SIMAX, 2006.
- Also "good" linearizations in applications should be easily constructible without performing operations, should allow us to recover easily eigenvectors, minimal indices/bases of $P(\lambda)$, should lead to polynomial backward errors, should not increase eigenvalue condition numbers...

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- Some unexpected surprises have emerged. In particular "There are structured matrix polynomials arising in applications (Hermitian, symmetric, alternating, palindromic,...) of even degree which do not have a linearization with the same structure." (DTDM, MMMM)
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- Suggested by F. Tisseur at ICIAM 2011-Minisymposium on "Nonlinear Eigenvalue Problems".

  "Many Hermitian linearizations of (the very important in applications) Hermitian matrix polynomials have been devised in the literature, but only one is guaranteed to have the same sign-characteristic as the original matrix polynomial (Al-Ammari, Tisseur, LAA 2012).

  Can we identify more interesting Hermitian linearizations of Hermitian matrix polynomials preserving the sign-characteristic?"

- The "one" is the last pencil in the canonical basis of $\mathbb{DL}(P)$ (MMMM 2006; Higham, M, M, Tisseur, 2006) and Al-Ammari & Tisseur's proof is only valid for Hermitian matrix polys with semisimple real eigenvalues.

- In this talk, we find infinitely many of such linearizations (without restricting to semisimple real eigenvalues) which can be constructed very easily: some of them previously known (without guarantee that they preserve Sign Characteristic), others new.

Sign-characteristic problem for linearizations of Hermitian matrix polys

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Outline

- We consider Hermitian matrix polynomials of degree k, i.e.,

$$P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0 \,, \quad A_i = A_i^* \in \mathbb{C}^{n\times n} \,,$$

- with A_k **nonsingular**.
- This restriction is required in Gohberg, Lancaster, Rodman's SC-Theory (Annals of Mathematics, 1980; book, 2005) and it implies:
  1. Only regular matrix polynomials without infinite eigenvalues are considered, and
  2. linearizations $\equiv$ strong linearizations.
- We are looking forward the still unpublished unifying extended SC-Theory by Al-Ammari, Mehrmann, Nakatsukasa, Noferini, Tisseur, Xu including infinite eigenvalues, singular polynomials, and other structured matrix polynomials.

General assumption of the talk

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- The elementary divisors (Jordan blocks) of a Hermitian matrix polynomial corresponding to its nonreal eigenvalues are paired up:

$$(\lambda-\lambda_0)^{m_1}, (\lambda-\overline{\lambda_0})^{m_1}, \ldots, (\lambda-\lambda_0)^{m_g}, (\lambda-\overline{\lambda_0})^{m_g}$$

- The SC of a Hermitian matrix polynomial $P(\lambda)$ is a set of signs attached to the elementary divisors of $P(\lambda)$ associated to its real eigenvalues.
- If $\lambda_0 \in \mathbb{R}$ is a simple eigenvalue of $P(\lambda)$ (i.e., λ_0 has only one elementary divisor of degree one) with eigenvector $\mathbf{x}$, then the corresponding sign is

$$\text{sign}\,(\,\mathbf{x}^* P'(\lambda_0)\mathbf{x}\,)$$

- Parenthesis: this allows us to construct interesting examples...

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Parenthesis: Hermitian linearizations may NOT preserve the SC

Example: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$ is 1×1 Hermitian.

- **Eigenvalues of $p(\lambda)$ and SC:**

$$\text{eigs} = \{1, 2\} \longrightarrow \text{SC}(p) = (\text{sign}(p'(1)), \text{sign}(p'(2)) = (-1, +1).$$

- **Two Hermitian linearizations of $p(\lambda)$:**

$$L(\lambda) = \begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 2 \end{bmatrix} \quad \text{and} \quad \widetilde{L}(\lambda) = \begin{bmatrix} -\lambda + 1 & 0 \\ 0 & \lambda - 2 \end{bmatrix}.$$

It is tempting to say that $L(\lambda)$ is the most natural one, but

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- The definitions and results used in this part can be found in Gohberg, Lancaster, Rodman, *Indefinite Linear Algebra and Applications* (2005).
- We need to deal first with **pairs of matrices** and in particular with

Definition (Selfadjoint pair)

A pair of matrices (T, N), where $T, N \in \mathbb{C}^{n \times n}$ and $N = N^*$ is nonsingular Hermitian, is said to be selfadjoint if

$$T^* = NTN^{-1}.$$

In other words,

- if T is selfadjoint with respect to the (possibly indefinite) inner product defined by N ($\langle x, y \rangle_N := x^* N y$).

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Definition (Unitary Similarity of Selfadjoint Matrix Pairs)

Two selfadjoint pairs of the same size (T_1, N_1) and (T_2, N_2) are unitarily similar if there exists a nonsingular matrix H such that

$$T_1 = H^{-1} T_2 H, \quad N_1 = H^* N_2 H.$$

- It is an equivalence relation in the set of selfadjoint matrix pairs.
- There is a canonical form of selfadjoint matrix pairs under unitary similarity.
- For describing this canonical form we need

$$\mathcal{R}_m := \begin{bmatrix} 0 & \dots & 1 \\ \vdots & .\cdot^{\cdot} & \vdots \\ 1 & \dots & 0 \end{bmatrix} \quad \text{and} \quad J_k(\mu) := \begin{bmatrix} \mu & 1 & \dots & 0 \\ & \mu & \ddots & \vdots \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}.$$

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Theorem (Canonical form under Unitary Similarity of Selfadjoint Pairs)

Any selfadjoint pair (T, N) is unitarily similar to $(J, P_{\epsilon,J})$, where

$$J = J_{l_1}(\lambda_1) \oplus \cdots \oplus J_{l_r}(\lambda_r) \oplus (J_{k_1}(\mu_1) \oplus J_{k_1}(\overline{\mu_1})) \oplus \cdots \oplus (J_{k_c}(\mu_c) \oplus J_{k_c}(\overline{\mu_c}))$$

is a Jordan normal form for T, $\lambda_1, \ldots, \lambda_r$ are the real eigenvalues of T, and $\mu_1, \overline{\mu_1}, \ldots, \mu_c, \overline{\mu_c}$, are the nonreal eigenvalues of T; and

$$P_{\epsilon,J} = \epsilon_1 \mathcal{R}_{l_1} \oplus \cdots \oplus \epsilon_r \mathcal{R}_{l_r} \oplus \mathcal{R}_{2k_1} \oplus \cdots \oplus \mathcal{R}_{2k_c},$$

where

$$\epsilon = \{\epsilon_1, \ldots, \epsilon_r\}$$

is an ordered set of signs ± 1 uniquely determined by (T, N) up to permutation of signs corresponding to equal Jordan blocks.

Definition (SC of a Selfadjoint Pair)

The set of signs $\epsilon = \{\epsilon_1, \ldots, \epsilon_r\}$ is the SC of the selfadjoint pair (T, N).

Theorem (Standard triples of Hermitian matrix polynomials)

Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ be an $n \times n$ Hermitian matrix polynomial with A_k nonsingular. Then:

- *For any standard triple (X, T, Y) of $P(\lambda)$, there exists a unique nonsingular Hermitian matrix N such that the matrix pair (T, N) is selfadjoint.*
- *All these infinitely many selfadjoint pairs associated to $P(\lambda)$ have the same SC.*

Definition (Sign characteristic of Hermitian matrix polynomials)

The SC of the Hermitian matrix polynomial $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ with A_k nonsingular is the SC of any of its associated selfadjoint pairs (T, N).

The Sign Characteristic (SC) of Hermitian matrix polynomials (V)

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Corollary (Explicit characterization of the SC of Hermitian polynomials)

Let $P(\lambda) = A_k\lambda^k + \cdots + A_1\lambda + A_0$ *be a Hermitian matrix polynomial with* A_k *nonsingular and let us define*

$$C_P := \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & I_n \\ -A_k^{-1}A_0 & -A_k^{-1}A_1 & \cdots & \cdots & -A_k^{-1}A_{k-1} \end{bmatrix},$$

$$B_P := \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_2 & \vdots & \iddots & \\ \vdots & A_k & & \\ A_k & & & 0 \end{bmatrix}.$$

Then the sign characteristic of $P(\lambda)$ is the sign characteristic of the selfadjoint pair (C_P, B_P).

Why is the SC important?

- It determines important features of the structured eigenvalue perturbation theory of Hermitian matrix polynomials. For example:
  - Two equal (or extremely close) real simple eigenvalues can become nonreal under structured perturbations only if they have opposite signs in the SC.
- So, structured and unstructured eigenvalue perturbation theories are very different for “colliding” eigenvalues,
- and SC would explain different behaviours of backward stable structured and unstructured algorithms.
- SC is useful to classify different families of Hermitian matrix polynomials appearing in applications: hyperbolic, quasihyperbolic, gyroscopically stabilized, overdamped quadratics,....

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Outline

- We consider matrix polynomials of degree k

$$P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{C}^{n\times n}\,, \quad A_k \neq 0.$$

- A linearization for $P(\lambda)$ is an $nk \times nk$ **linear matrix polynomial** (or matrix pencil) $L(\lambda)$, such that,

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{n(k-1)} & \\ & P(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

Property: $P(\lambda)$ and $L(\lambda)$ have the same finite spectral structure.

Theorem (Odd degree case)

*Let $P(\lambda) = A_k \lambda^k + \cdots + A_1 \lambda + A_0$ be a Hermitian $n \times n$ matrix polynomial with A_k nonsingular and k **odd**, and $L(\lambda)$ be a Hermitian $nk \times nk$ matrix pencil. Then,*

$L(\lambda)$ is a linearization of $P(\lambda)$ with the same SC

if and only if

there exists a unimodular matrix polynomial $V(\lambda)$ such that

$$V(\lambda)L(\lambda)V(\lambda)^* = \begin{bmatrix} I_{n(k-1)/2} & & \\ & -I_{n(k-1)/2} & \\ & & P(\lambda) \end{bmatrix}.$$

Remark

$$(V_q \lambda^q + \cdots + V_1 \lambda + V_0)^* := V_q^* \lambda^q + \cdots + V_1^* \lambda + V_0^*.$$

SC-preserving linearizations and unimodular congruences (I)

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$$V(\lambda)L(\lambda)V(\lambda)^* = \begin{bmatrix} I_{n(k-2)/2} & & & & \\ & -I_{n(k-2)/2} & & & \\ & & I_{neg} & & \\ & & & -I_{pos} & \\ & & & & P(\lambda) \end{bmatrix},$$

where A_k is strictly congruent to $\operatorname{diag}(-I_{neg}, I_{pos})$.

Lemma

Let $P(\lambda) = A_k\lambda^k + \cdots + A_1\lambda + A_0$ be a Hermitian matrix polynomial with A_k nonsingular and let

$$D_k(\lambda, P) := \lambda \begin{bmatrix} 0 & \cdots & \cdots & 0 & A_k \\ \vdots & & & \iddots & A_{k-1} \\ \vdots & & \iddots & \iddots & A_{k-2} \\ 0 & \iddots & \iddots & \iddots & \vdots \\ A_k & A_{k-1} & A_{k-2} & \cdots & A_1 \end{bmatrix} - \left[\begin{array}{cccc|c} 0 & \cdots & 0 & A_k & \\ \vdots & & \iddots & A_{k-1} & \\ 0 & \iddots & \iddots & \vdots & \\ A_k & A_{k-1} & \ldots & A_2 & \\ \hline & & & & -A_0 \end{array}\right].$$

Then $D_k(\lambda, P)$ is a Hermitian linearization of $P(\lambda)$ that preserves the sign characteristic of $P(\lambda)$.

- $D_k(\lambda, P)$ is the "one" considered by Al-Ammari & Tisseur (2012) for semisimple eigenvalues.
- This lemma follows from proving that the selfadjoint pairs (C_P, B_P) and (C_{D_k}, B_{D_k}) of $P(\lambda)$ and $D_k(\lambda, P)$, respectively, are unitarily similar.

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Theorem

Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ be an $n \times n$ Hermitian matrix polynomial with A_k nonsingular, $D_k(\lambda, P)$ be the pencil defined in the previous slide, and $L(\lambda)$ be an $nk \times nk$ pencil. Then,

$L(\lambda)$ is a Hermitian linearization of $P(\lambda)$ that preserves the SC of $P(\lambda)$

if and only if

*$L(\lambda)$ is *congruent to $D_k(\lambda, P)$,*

i.e., if and only if

$$L(\lambda) = S \; D_k(\lambda, P) \, S^*,$$

for a nonsingular constant matrix S.

- Any nonsingular S provides a SC-preserving linearization of $P(\lambda)$,
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Outline

- Given a regular Hermitian matrix polynomial of degree k, then
- the pencil $D_k(\lambda, P)$ is the last pencil in the canonical basis of the vector space of Hermitian pencils $\mathbb{DL}(P)$ (Mackey[2], Mehl, Mehrmann, SIMAX 2006 & Higham, Mackey[2], Tisseur, SIMAX 2006).
- $\mathbb{DL}(P)$ has dimension k and "almost all" their elements are Hermitian strong linearizations of $P(\lambda)$.
- The canonical basis of $\mathbb{DL}(P)$ is denoted by

$$\{D_1(\lambda, P), D_2(\lambda, P), \ldots, D_k(\lambda, P)\},$$

and all its pencils are immediately constructible from the coefficients of $P(\lambda)$.
- They are also particular examples of Fiedler Pencils with Repetition (Vologiannidis & Antoniou, MCSS, 2011).
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The canonical basis of $\mathbb{DL}(P)$ (I)

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$$P(\lambda) = A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$D_1(\lambda, P) = \lambda \left[\begin{array}{c|ccc} A_4 & & & \\ \hline & -A_2 & -A_1 & -A_0 \\ & -A_1 & -A_0 & \\ & -A_0 & & \end{array}\right] - \left[\begin{array}{cccc} -A_3 & -A_2 & -A_1 & -A_0 \\ -A_2 & -A_1 & -A_0 & \\ -A_1 & -A_0 & & \\ -A_0 & & & \end{array}\right]$$

$$D_2(\lambda, P) = \lambda \left[\begin{array}{cc|cc} & A_4 & & \\ A_4 & A_3 & & \\ \hline & & -A_1 & -A_0 \\ & & -A_0 & \end{array}\right] - \left[\begin{array}{c|ccc} A_4 & & & \\ \hline & -A_2 & -A_1 & -A_0 \\ & -A_1 & -A_0 & \\ & -A_0 & & \end{array}\right]$$

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Theorem

Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ be a Hermitian matrix polynomial of degree k with A_k nonsingular. Let $D_m(\lambda, P)$ be the mth pencil in the canonical basis of $\mathbb{DL}(P)$ for $m = 1, \ldots, k$ and suppose A_0 is nonsingular if $m \neq k$.

If $k - m$ is even, then $D_m(\lambda, P)$ is a Hermitian linearization of $P(\lambda)$ with the same sign characteristic as $P(\lambda)$.

- **Remark:** If $k - m$ is odd, the preservation of SC depends on $P(\lambda)$ in a nontrivial way: for some $P(\lambda)$ we have preservation, but not for others.

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Preservation of SC for arbitrary pencils in $\mathbb{DL}(P)$

- Ongoing work by Breen, Bueno, Ford & Furtado (not including me).
- Given the Hermitian matrix polynomial $P(\lambda) = A_k\lambda^k + \cdots + A_1\lambda + A_0$, then we define the space of associated pencils

$$\mathbb{DL}(P) := \{v_1 D_1(\lambda, P) + v_2 D_2(\lambda, P) + \cdots + v_k D_k(\lambda, P) \, : \, v_i \in \mathbb{R}\}$$

- Arbitrary pencils in $\mathbb{DL}(P)$ are not as nice as $D_j(\lambda, P)$ pencils, because they are constructed via operations that may be affected by errors.

Theorem (Breen, Bueno, Ford & Furtado, 2015...in progress)

Let $L(\lambda) := v_1 D_1(\lambda, P) + v_2 D_2(\lambda, P) + \cdots + v_k D_k(\lambda, P)$. Assume that A_k is nonsingular and the roots of

$$q(x; v) = v_1 x^{k-1} + v_2 x^{k-2} + \cdots + v_{k-1} x + v_k$$

are not eigenvalues of $P(\lambda)$. If $q(\lambda_i; v) > 0$ for any λ_i real eigenvalue of $P(\lambda)$, then $L(\lambda)$ is a Hermitian linearization of $P(\lambda)$ preserving its SC.

- An "if and only if" condition has been proved. It requires to know in advance detailed spectral information of $P(\lambda)$.

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Outline

The famous block tridiagonal linearization for odd degree preserves SC

Theorem

Let $P(\lambda) = A_k\lambda^k + \cdots + A_1\lambda + A_0$ *be any matrix polynomial of* **odd degree** *and define (the* *generalized Fiedler pencil*)

$$L(\lambda) := \begin{bmatrix} \lambda A_1 + A_0 & \lambda I & & & & & & \\ \lambda I & 0 & I & & & & & \\ & I & \lambda A_3 + A_2 & \lambda I & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & I & \lambda A_{k-2} + A_{k-3} & \lambda I & & \\ & & & & \lambda I & 0 & I \\ & & & & & I & \lambda A_k + A_{k-1} \end{bmatrix}.$$

Then (Antoniou-Vologiannidis, 04; Mackey[2]*, Mehl, Mehrmann, 10):*

- *$L(\lambda)$ is always a strong linearization of $P(\lambda)$.*
- *If $P(\lambda)$ is Hermitian, then $L(\lambda)$ is also Hermitian,*
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Outline

- We have characterized many families of **Fiedler Pencils with Repetition (FPR)** and
- Hermitian Generalized Fiedler Pencils with Repetition **(HGFPR)**
- which are Hermitian linearizations that preserve the SC of Hermitian matrix polynomials.
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$P(\lambda) = A_6\lambda^6 + \cdots + A_1\lambda + A_0$, with $A_i = A_i^* \in \mathbb{C}^{n\times n}$ and A_6 nonsingular.

Let $X_1, X_2, X_3, X_4 \in \mathbb{C}^{n\times n}$ be arbitrary Hermitian matrices and define the family of pencils $F_2(P)$ as

$$\lambda \begin{bmatrix} 0 & A_6 & 0 & 0 & 0 & 0 \\ A_6 & A_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_3 & X_3 & X_2 & X_1 \\ 0 & 0 & X_3 & -A_1 & X_4 & 0 \\ 0 & 0 & X_2 & X_4 & 0 & 0 \\ 0 & 0 & X_1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_4 & -A_3 & X_3 & X_2 & X_1 \\ 0 & -A_3 & -A_2 & -A_1 & X_4 & 0 \\ 0 & X_3 & -A_1 & -A_0 & 0 & 0 \\ 0 & X_2 & X_4 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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Any pencil in $F_2(P)$ such that X_1 and X_4 are nonsingular Hermitian matrices is a Hermitian linearization of $P(\lambda)$ with the same SC as $P(\lambda)$.

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Example 2: one pencil in the family of simple FPR

$P(\lambda) = A_{10}\lambda^{10} + \cdots + A_1\lambda + A_0$, with $A_i = A_i^* \in \mathbb{C}^{n\times n}$ and A_{10} nonsingular.

Define the pencil $\lambda L_1 - L_0$ with

$$L_1 = \left[\begin{array}{cccccc|cccc}
0 & 0 & A_{10} & & & & & & & \\
0 & 0 & 0 & 0 & I_n & & & & & \\
A_{10} & 0 & A_9 & 0 & A_8 & & & & & \\
 & 0 & 0 & 0 & 0 & I_n & & & & \\
 & I_n & A_8 & 0 & A_7 & A_6 & & & & \\
 & & & I_n & A_6 & A_5 & & & & \\
\hline
 & & & & & & -A_3 & I_n & & \\
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Then $\lambda L_1 - L_0$ is a Hermitian linearization of $P(\lambda)$ with the same SC as $P(\lambda)$.

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Conclusions and ongoing developments

- We have now available infinitely many linearizations of Hermitian matrix polynomials which reflect faithfully all the structural properties of the original polynomials including their sign characteristics.
- According to preliminary outgoing work, among these linearizations, the "famous block tridiagonal one" seems to be the best from the point of view of coefficientwise eigenvalue backward errors and condition numbers,
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