Polynomial Zigzag Matrices, Dual Minimal Bases, and Applications

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joint work with

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Outline

Theorem (Kronecker Canonical Form (KCF))

For any matrix pencil $A - \lambda B$, $A, B \in \mathbb{C}^{m \times n}$, there exist nonsingular matrices U and V such that

$$U(A-\lambda B)V = L_{\varepsilon_1} \oplus \cdots \oplus L_{\varepsilon_p} \oplus L_{\eta_1}^T \oplus \cdots \oplus L_{\eta_q}^T \oplus J_{k_1}(\lambda-\lambda_1) \oplus \cdots \oplus J_{k_f}(\lambda-\lambda_f) \oplus N_{\ell_1}(\lambda) \oplus \cdots \oplus N_{\ell_s}(\lambda),$$

where

$$L_\varepsilon = \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}_{\varepsilon \times (\varepsilon+1)}, \qquad L_\eta^T = \begin{bmatrix} 1 & & \\ \lambda & \ddots & \\ & \ddots & 1 \\ & & \lambda \end{bmatrix}_{(\eta+1)\times\eta},$$

$$J_k(\lambda-\lambda_i) = \begin{bmatrix} \lambda-\lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda-\lambda_i \end{bmatrix}_{k\times k}, \qquad N_\ell(\lambda) = \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix}_{\ell\times\ell}.$$

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Minimal indices of pencils (II)

Definition (Minimal Indices of a Matrix Pencil)

Let $A - \lambda B$ be a matrix pencil with KCF

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Then

- the numbers $\varepsilon_1, \ldots, \varepsilon_p$ are the **right minimal indices** of $A - \lambda B$,
- the numbers $\eta_1, \ldots, \eta_q$ are the **left minimal indices** of $A - \lambda B$.

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In this talk:

- $\mathbb{F}$ is an arbitrary field.
- $\mathbb{F}[\lambda]$ is the ring of polynomials with coefficients in $\mathbb{F}$.
- $\mathbb{F}(\lambda)$ is the field of rational functions over $\mathbb{F}$.
- $\mathbb{F}(\lambda)^n$ is the vector space over the field $\mathbb{F}(\lambda)$ of n-tuples with entries in $\mathbb{F}(\lambda)$.
- **Example:**

$$\begin{bmatrix} \dfrac{\lambda+2}{\lambda^2} \\ \dfrac{1}{(\lambda+1)^3} \end{bmatrix} \in \mathbb{R}(\lambda)^2$$

- $\mathbb{F}(\lambda)^n$ **is known as a rational vector space and its subspaces as rational vector subspaces.** (Wolovich-1974, Forney-1975)

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Minimal bases of rational vector subspaces

- Any rational subspace $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$ has bases consisting entirely of vector polynomials.
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$$\begin{bmatrix} \dfrac{\lambda+2}{\lambda^2} \\ \dfrac{1}{(\lambda+1)^3} \end{bmatrix} \in \mathcal{V} \Longrightarrow \lambda^2 (\lambda+1)^3 \begin{bmatrix} \dfrac{\lambda+2}{\lambda^2} \\ \dfrac{1}{(\lambda+1)^3} \end{bmatrix} = \begin{bmatrix} (\lambda+2)(\lambda+1)^3 \\ \lambda^2 \end{bmatrix} \in \mathcal{V}$$

Definition (Minimal basis)

A **minimal basis** of the rational subspace $\mathcal{V} \in \mathbb{F}(\lambda)^n$ is a basis

1. consisting of vector polynomials
2. whose sum of degrees is minimal among all bases of $\mathcal{V}$ consisting of vector polynomials.

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There are many (infinite) minimal bases of a rational subspace $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$, but...

Theorem (Forney, 1975...probably known before)

The ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$ is always the same.

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These degrees are called the **minimal indices** of $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$.

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Connection to minimal indices of pencils (I)

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$$U(A-\lambda B)V = L_{\varepsilon_1} \oplus \cdots \oplus L_{\varepsilon_p} \oplus L_{\eta_1}^T \oplus \cdots \oplus L_{\eta_q}^T \oplus \text{regular blocks}$$

where

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Connection to minimal indices of pencils (II)

Proposition

Let $A - \lambda B$ be an $m \times n$ matrix pencil with entries in $\mathbb{F}$. Then:

- The **right minimal indices** of $A - \lambda B$ are the **minimal indices of the rational right NULL space** of $A - \lambda B$, i.e.,

$$\mathcal{N}_r(A - \lambda B) := \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n \times 1} \; : \; (A - \lambda B)x(\lambda) \equiv 0 \right\}.$$

- The **left minimal indices** of $A - \lambda B$ are the **minimal indices of the rational left NULL space** of $A - \lambda B$, i.e.,

$$\mathcal{N}_\ell(A - \lambda B) := \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1 \times m} \; : \; y(\lambda)^T (A - \lambda B) \equiv 0^T \right\}.$$

We use for brevity the following definition.

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Example of minimal basis and minimal indices of a pencil

$$A - \lambda B = \left[\begin{array}{cc|ccc} 1 & \lambda & & & \\ \hline & & 1 & \lambda & \\ & & & 1 & \lambda \end{array}\right] \in \mathbb{F}[\lambda]^{3\times 5}$$

Right minimal indices of $A - \lambda B$ = $\{1, 2\}$ and no left minimal indices.

$$\mathcal{N}_r(A-\lambda B) = \text{Span}\{ \underbrace{\begin{bmatrix} -\lambda \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ \lambda^2 \\ -\lambda \\ 1 \end{bmatrix}}_{u_2} \} = \text{Span}\{ \underbrace{\begin{bmatrix} -\lambda^3 \\ \lambda^2 \\ \lambda^3 \\ -\lambda^2 \\ \lambda \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} \lambda^5 \\ -\lambda^4 \\ \lambda^2 \\ -\lambda \\ 1 \end{bmatrix}}_{w_2} \}$$

Sum of degrees of $\{u_1, u_2\} = 1 + 2 = 3$ (right minimal bases of $A - \lambda B$)

Sum of degrees of $\{w_1, w_2\} = 3 + 5 = 8$

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Sum of degrees of $\{w_1, w_2\} = 3 + 5 = 8$

Remark

They are defined through the null spaces of the polynomial matrix, since for polynomial matrices of degree larger than 1, there is NO a “KCF”

Example:

$$P(\lambda) = \left[\begin{array}{cc|ccc} 1 & \lambda^3 & & & \\ \hline & & 1 & \lambda & \\ & & & 1 & \lambda \end{array}\right] \in \mathbb{F}[\lambda]^{3\times 5}$$

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Every polynomial matrix has left and right minimal bases and indices

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Sum of degrees of $\{u_1, u_2\} = 3 + 2 = 5$ (right minimal bases of $P(\lambda)$)

Sum of degrees of $\{w_1, w_2\} = 3 + 5 = 8$

Right minimal indices of $P(\lambda)$ = $\{2, 3\}$

REMARK: In the rest of the talk, we arrange all (minimal) bases as the rows of matrices and often call "basis" to the matrix.

Theorem (Forney 1975...probably known before)

The rows of a polynomial matrix $N(\lambda)$ over a field $\mathbb{F}$ are a minimal basis of the subspace they span if and only if

(a) $N(\lambda_0)$ *has full row rank for all* $\lambda_0 \in \overline{\mathbb{F}}$*, and*

(b) *the highest-row-degree coefficient matrix of* $N(\lambda)$ *has also full row rank.*

Example (of minimal basis)

$$N(\lambda) = \begin{bmatrix} -\lambda^3 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & -\lambda & 1 \end{bmatrix}$$

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- $N(\lambda)$ satisfies (a) by the 1's.
- $N(\lambda)$ satisfies (b) since its highest-row-degree coefficient matrix is

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One more example

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Example (NOT minimal basis)

$$N(\lambda) = \begin{bmatrix} -\lambda^3 & 1 & \lambda^3 & -\lambda^2 & \lambda \\ -\lambda^5 & \lambda^2 & \lambda^2 & -\lambda & 1 \end{bmatrix}$$

- This $N(\lambda)$ DOES NOT satisfy (a) because

$$N(1) = \begin{bmatrix} -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

is rank deficient.

Outline

1. Preliminary concepts: Minimal Indices and Bases
2. **Forney's theorem on dual minimal bases: Goal of the talk**
3. Polynomial Zigzag matrices
4. Solving the inverse problem for dual minimal bases
5. Conclusions and Applications

Definition (Dual Minimal Bases)

Polynomial matrices $M(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ are said to be **dual minimal bases** if

(a) both are minimal bases,

(b) $m + k = n$,

(c) and $M(\lambda)\, N(\lambda)^T = 0$.

Remark

- The "name" is not standard. Forney (1975) uses the "rational subspaces spanned by the rows of $M(\lambda)$ and $N(\lambda)$ are dual subpaces of $F(\lambda)^n$ ".
- Dual minimal bases have classical applications in Linear System Theory for constructing left and right coprime factorizations of transfer functions,
- also for solving certain matrix polynomial equations,
- and, modern applications, for constructing strong linearizations and ℓ-ifications of polynomials matrices, solving inverse problems...

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Example ($M(\lambda)N(\lambda)^T = 0$)

$$M(\lambda) = \begin{bmatrix} 1 & \lambda & & \\ & 1 & \lambda & \\ & & 1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{3\times 4}$$

$$N(\lambda) = \begin{bmatrix} \lambda^3 & -\lambda^2 & \lambda & -1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1\times 4}$$

Remarks

In general, for dual minimal bases $M(\lambda)N(\lambda)^T = 0$ viewed as polynomial matrices:

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Forney's theorem and main goal of the talk

Theorem (Forney 1975...probably known before)

Let $M(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ *and* $N(\lambda) \in \mathbb{F}[\lambda]^{k \times n}$ *be dual minimal bases with row degrees* $(\eta_1, \ldots, \eta_m)$ *and* $(\varepsilon_1, \ldots, \varepsilon_k)$*, respectively. Then*

$$\sum_{i=1}^{m} \eta_i = \sum_{j=1}^{k} \varepsilon_j \, .$$

GOAL: Solve the corresponding INVERSE PROBLEM

Given two lists of nonnegative integers $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ that have the same sum:

- do there exist dual minimal bases having these numbers as their row degrees?
- can we explicitly construct dual minimal bases having any lists of prescribed row degrees with the same sum?

Our main new tool will be...

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Outline

Example of forward Zigzag matrix:

$$Z(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & \\ & & & 1 & \lambda^3 & & & & & & \\ & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & \\ & & & & & & & & 1 & \lambda^2 & \lambda^3 \end{bmatrix}$$

Definition

$Z(\lambda) \in \mathbb{F}^{m\times n}$ with $m < n$ is a forward-zigzag matrix, if

(a) each row of $Z(\lambda)$ is of the form

$$\Big[\, \underbrace{0 \;\; \dots \;\; 0}_{\text{Maybe none}} \;\; 1 \;\; \lambda^{p_1} \;\; \lambda^{p_2} \;\; \dots \;\; \lambda^{p_k} \;\; \underbrace{0 \;\; \dots \;\; 0}_{\text{Maybe none}} \,\Big],$$

with $0 < p_1 < p_2 < \cdots < p_k$ and $k \geq 1$.

(b) $Z(\lambda)$ is in a double-echelon form: the last nonzero entry of each row and the first nonzero entry of the row just below are in the same column.

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Basic properties of Forward-Zigzag Matrices (I)

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- **Every zigzag matrix is a minimal basis.**
- I like to say that zigzag matrices generalize to degrees larger than 1 right singular blocks of the KCF of pencils, which are simple examples of forward zigzag matrices.
- For any two adjacent columns of a zigzag matrix, there is a unique row having two nonzero entries in those columns.

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- **Unit column sequence of a forward-zigzag matrix.** In the example

  U,N,N,U,U,N,N,N,U,N,N

  number of Us = number of rows

- **Degree-gap sequence of a forward-zigzag matrix.** In the example

  $$2, 5, 1, 3, 1, 3, 4, 7, 2, 1$$

- **Structure sequence of a forward-zigzag matrix.** In the example

$$\left[\, \mathrm{U} \;\; 2 \;\; \mathrm{N} \;\; 5 \;\; \mathrm{N} \;\; 1 \;\; \mathrm{U} \;\; 3 \;\; \mathrm{U} \;\; 1 \;\; \mathrm{N} \;\; 3 \;\; \mathrm{N} \;\; 4 \;\; \mathrm{N} \;\; 7 \;\; \mathrm{U} \;\; 2 \;\; \mathrm{N} \;\; 1 \;\; \mathrm{N} \,\right]$$

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Backward-Zigzag Polynomial Matrices

Example of backward Zigzag matrix:

$$\widehat{Z}(\lambda) = \begin{bmatrix} \lambda^2 & 1 & & & & & & & & & \\ & \lambda^5 & 1 & & & & & & & & \\ & & \lambda^5 & \lambda^4 & \lambda & 1 & & & & & \\ & & & & & \lambda^3 & 1 & & & & \\ & & & & & & \lambda^4 & 1 & & & \\ & & & & & & & \lambda^9 & \lambda^2 & 1 & \\ & & & & & & & & & \lambda & 1 \end{bmatrix}$$

- Backward Zigzag matrices are obtained by reversing the order of rows and columns of forward Zigzag matrices.
- Similar properties and tools as for forward Zigzag matrices.
- In the example, the structure sequence is

$$\left[\ _{\mathrm{N}}\ 2\ _{\mathrm{U}}\ 5\ _{\mathrm{U}}\ 1\ _{\mathrm{N}}\ 3\ _{\mathrm{N}}\ 1\ _{\mathrm{U}}\ 3\ _{\mathrm{U}}\ 4\ _{\mathrm{U}}\ 7\ _{\mathrm{N}}\ 2\ _{\mathrm{U}}\ 1\ _{\mathrm{U}}\ \right].$$

Example of backward Zigzag matrix:

$$\widehat{Z}(\lambda) = \begin{bmatrix} \lambda^2 & 1 & & & & & & & & & \\ & \lambda^5 & 1 & & & & & & & & \\ & & \lambda^5 & \lambda^4 & \lambda & 1 & & & & & \\ & & & & & \lambda^3 & 1 & & & & \\ & & & & & & \lambda^4 & 1 & & & \\ & & & & & & & \lambda^9 & \lambda^2 & 1 & \\ & & & & & & & & & \lambda & 1 \end{bmatrix}$$

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Definition

Suppose

1. $Z(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is a forward-zigzag matrix and
2. $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ is a backward-zigzag matrix

with the same number of columns. Then $Z(\lambda)$ and $Z^{\diamondsuit}(\lambda)$ are said to be dual zigzag matrices, if they have

(a) the same degree-gap sequence, but

(b) complementary unit column sequences, where U and N are each other's complement.

Corollary

#rows of $Z(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ plus #rows of $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ is equal to the number of columns, i.e.,

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Example of Dual Zigzag Matrices

$$Z(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & \\ & & & 1 & \lambda^3 & & & & & & \\ & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & \\ & & & & & & & & 1 & \lambda^2 & \lambda^3 \end{bmatrix}$$

$$Z^{\diamondsuit}(\lambda) = \begin{bmatrix} \lambda^2 & 1 & & & & & & & & & \\ & \lambda^5 & 1 & & & & & & & & \\ & & \lambda^5 & \lambda^4 & \lambda & 1 & & & & & \\ & & & & & \lambda^3 & 1 & & & & \\ & & & & & & \lambda^4 & 1 & & & \\ & & & & & & & \lambda^9 & \lambda^2 & 1 & \\ & & & & & & & & & \lambda & 1 \end{bmatrix}$$

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$$Z(\lambda) = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c} 1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & \\ & & & 1 & \lambda^3 & & & & & & \\ & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & \\ & & & & & & & & 1 & \lambda^2 & \lambda^3 \end{array}\right]$$

$$Z^{\diamondsuit}(\lambda) = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c} \lambda^2 & 1 & & & & & & & & & \\ & \lambda^5 & 1 & & & & & & & & \\ & & \lambda^5 & \lambda^4 & \lambda & 1 & & & & & \\ & & & & & \lambda^3 & 1 & & & & \\ & & & & & & \lambda^4 & 1 & & & \\ & & & & & & & \lambda^9 & \lambda^2 & 1 & \\ & & & & & & & & & \lambda & 1 \end{array}\right]$$

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Theorem (from dual Zigzag to dual minimal bases)

Suppose

- $Z(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *is any forward-zigzag matrix,*
- $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{(n-m)\times n}$ *is its dual backward zigzag matrix, and*
- $\Sigma_n := \operatorname{diag}(1, -1, 1, -1, \ldots, (-1)^{n-1})$.

Then

$$Z(\lambda) \cdot \left(Z^{\diamondsuit}(\lambda) \cdot \Sigma_n\right)^T = 0\,,$$

i.e., $Z(\lambda)$ **and** $(Z^{\diamondsuit}(\lambda) \cdot \Sigma_n)$ **are dual minimal bases.**

From dual Zigzag matrices to dual minimal bases

$$Z(\lambda) = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & \\
 & & & 1 & \lambda^3 & & & & & & \\
 & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & \\
 & & & & & & & & 1 & \lambda^2 & \lambda^3
\end{array}\right]$$

$$Z^{\diamondsuit}(\lambda) \cdot \Sigma_n = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\lambda^2 & -1 & & & & & & & & & \\
 & -\lambda^5 & 1 & & & & & & & & \\
 & & \lambda^5 & -\lambda^4 & \lambda & -1 & & & & & \\
 & & & & & -\lambda^3 & 1 & & & & \\
 & & & & & & \lambda^4 & -1 & & & \\
 & & & & & & & -\lambda^9 & \lambda^2 & -1 & \\
 & & & & & & & & & -\lambda & 1
\end{array}\right]$$

Both have the same degree-gap sequence:

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Our strategy

- Dual Zigzag matrices are Dual minimal bases (up to signs of columns),
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Our inverse problem

Theorem (Forney 1975...probably known before)

Let $M(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *and* $N(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ *be dual minimal bases with row degrees* $(\eta_1, \ldots, \eta_m)$ *and* $(\varepsilon_1, \ldots, \varepsilon_k)$*, respectively. Then*

$$\sum_{i=1}^{m} \eta_i = \sum_{j=1}^{k} \varepsilon_j \,.$$

GOAL: Solve the corresponding INVERSE PROBLEM

Given two lists of nonnegative integers $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ that have the same sum:

- do there exist dual minimal bases having these numbers as their row degrees?
- can we explicitly construct dual minimal bases having any lists of prescribed row degrees with the same sum?

Outline

Row degrees of a zigzag matrix and its dual

Lemma

Suppose $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ is a forward-zigzag matrix with structure sequence

$$\mathcal{S} = \left[\begin{array}{ccccccccc} \mathrm{s}_1 & \delta_1 & \mathrm{s}_2 & \delta_2 & \ldots & \mathrm{s}_{n-1} & \delta_{n-1} & \mathrm{s}_n \end{array} \right].$$

Then:

(a) *$Z(\lambda)$ has row degrees equal to the partial sums of degree gaps between any two consecutive U's and after the last U.*

(b) *$Z^{\diamondsuit}(\lambda)$ has row degrees equal to the partial sums of degree gaps before the first N and between any two consecutive N's.*

(The row degrees are ordered from top to bottom.)

Example: Row degrees of a zigzag matrix and its dual

Example of forward Zigzag matrix:

$$Z(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & & \\ & & & 1 & \lambda^3 & & & & & & & \\ & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & & \\ & & & & & & & & 1 & \lambda^2 & \lambda^3 \end{bmatrix}$$

$$\mathcal{S} = [\ \text{U} \quad 2 \quad \text{N} \quad 5 \quad \text{N} \quad 1 \quad \text{U} \quad 3 \quad \text{U} \quad 1 \quad \text{N} \quad 3 \quad \text{N} \quad 4 \quad \text{N} \quad 7 \quad \text{U} \quad 2 \quad \text{N} \quad 1 \quad \text{N}\ ]$$

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Row degrees $Z[\lambda]$

$$(2+5+1\,,\,3\,,\,1+3+4+7\,,\,2+1) = (8,3,15,3)\,.$$

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$$(2+5+1\,,\,3\,,\,1+3+4+7\,,\,2+1) = (8,3,15,3)\,.$$

Row degrees $Z^{\diamondsuit}[\lambda]$

$$(2,5,1+3+1,3,4,7+2,1) = (2,5,5,3,4,9,1)\,.$$

The partial sums of the row degrees of $Z(\lambda)$ and $Z^{\diamondsuit}(\lambda)$ can also be computed easily from the previous recipe and we get...

Corollary (Partial row degree sums of dual zigzag matrices)

Suppose $Z(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ and $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ are dual zigzag matrices with row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, respectively. Then:

$$\sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \varepsilon_i$$

whenever $(\alpha, \beta) \neq (m, k)$, $1 \leq \alpha \leq m$ and $1 \leq \beta \leq k$; that is, a leading partial sum of row degrees of a zigzag matrix is never equal to a leading partial sum of row degrees of its dual.

- But once this “extra” constraint is assumed,
- it is easy to solve the inverse problem for dual Zigzag matrices.

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Solving the inverse problem for dual Zigzag matrices: Existence

Theorem

Let $(\eta_1, \ldots, \eta_m)$ *and* $(\varepsilon_1, \ldots, \varepsilon_k)$ *be two lists of positive integers such that*

$$\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \varepsilon_i \quad \text{and} \quad \sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \varepsilon_i,$$

for $(\alpha, \beta) \neq (m, k)$, $1 \leq \alpha \leq m$ *and* $1 \leq \beta \leq k$.
Then

- *there exists a unique forward-zigzag matrix* $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+k)}$ *with row degrees* $(\eta_1, \ldots, \eta_m)$ *such that,*
- *its dual backward-zigzag matrix* $Z^{\diamond}(\lambda) \in \mathbb{F}[\lambda]^{k \times (m+k)}$ *has row degrees* $(\varepsilon_1, \ldots, \varepsilon_k)$.

In addition, **the structure sequence of** $Z(\lambda)$ **is constructed via the following simple procedure:**

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In addition, **the structure sequence of $Z(\lambda)$ is constructed via the following simple procedure:**

Example: $(\eta_1, \eta_2, \eta_3, \eta_4) = (8, 3, 15, 3)$, $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_7) = (2, 5, 5, 3, 4, 9, 1)$.

(1) Define the partial sums $\ell_0 := 0$,

$$\ell_\alpha := \sum_{i=1}^{\alpha} \eta_i, \quad \alpha = 1, 2, 3, \quad \text{and} \quad r_\beta := \sum_{i=1}^{\beta} \varepsilon_i, \quad \beta = 1, \dots, 7.$$

(2) Order them in two lists

$$\begin{bmatrix} \ell_0 & \ell_1 & \ell_2 & \ell_3 \\ 0 & 8 & 11 & 26 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 \\ 2 & 7 & 12 & 15 & 19 & 28 & 29 \end{bmatrix}.$$

(3) Merge both lists in one ordered list

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(4) Replacements $\ell_i \to$ U, $r_j \to$ N gives unit column sequence of $Z(\lambda)$:

U,N,N,U,U,N,N,N,U,N,N

(5) Differences of consecutive terms gives the degree gap sequence of $Z(\lambda)$:

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Example: $(\eta_1, \eta_2, \eta_3, \eta_4) = (8, 3, 15, 3)$, $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_7) = (2, 5, 5, 3, 4, 9, 1)$.

(1) Define the partial sums $\ell_0 := 0$,

$$\ell_\alpha := \sum_{i=1}^{\alpha} \eta_i, \quad \alpha = 1, 2, 3, \quad \text{and} \quad r_\beta := \sum_{i=1}^{\beta} \varepsilon_i, \quad \beta = 1, \dots, 7.$$

(2) Order them in two lists

$$\begin{bmatrix} \ell_0 & \ell_1 & \ell_2 & \ell_3 \\ 0 & 8 & 11 & 26 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 \\ 2 & 7 & 12 & 15 & 19 & 28 & 29 \end{bmatrix}.$$

(3) Merge both lists in one ordered list

$$\begin{bmatrix} \ell_0 & r_1 & r_2 & \ell_1 & \ell_2 & r_3 & r_4 & r_5 & \ell_3 & r_6 & r_7 \\ 0 & 2 & 7 & 8 & 11 & 12 & 15 & 19 & 26 & 28 & 29 \end{bmatrix}.$$

(4) Replacements $\ell_i \to$ U, $r_j \to$ N gives unit column sequence of $Z(\lambda)$:

U,N,N,U,U,N,N,N,U,N,N

(5) Differences of consecutive terms gives the degree gap sequence of $Z(\lambda)$:

$$2, 5, 1, 3, 1, 3, 4, 7, 2, 1$$

The general inverse problem for dual minimal bases

- If the prescribed lists $(\eta_1, \ldots, \eta_m)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$ of degrees satisfy
- $$\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \varepsilon_i \quad \text{and} \quad \sum_{i=1}^{\alpha} \eta_i = \sum_{i=1}^{\beta} \varepsilon_i,$$

  for at least one $(\alpha, \beta) \neq (m, k)$,
- then the problem has to be solved by splitting it into smaller subproblems,
- such that each of them is solved via dual Zigzag matrices,
- and the general solution is the direct sum of the smaller Zigzag solutions.
- Zero row degrees (if any) are easily incorporated by placing them in the first positions of the lists and by using $[I\,0]$ and $[0\,I]$ matrices in the direct sums.

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Outline

Conclusions

- We have found an explicit simple solution of the inverse row degree problem for dual minimal bases via the new class of Zigzag matrices.
- This solution has been used (or is being used) by us and others (Lawrence, Van Barel, ...) for:
- constructing strong linearizations and ℓ-ifications of matrix polynomials with certain desired properties,
- in backward error analyses of numerical algorithms for solving polynomial eigenvalue problems via linearizations, and
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