Global backward error analyses for polynomial eigenvalue problems solved via linearizations

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joint work with **Piers Lawrence** (KU Leuven, Belgium),
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The Setting

- We consider a **general** $m \times n$ **matrix polynomial**, square or rectangular, regular or singular,

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 , \qquad P_i \in \mathbb{F}^{m\times n},$$

with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$,

- and we assume that its **complete eigenstructure**
- has been computed by applying a **backward stable algorithm**
- to a **strong linearization** $\mathcal{L}(\lambda)$ of $P(\lambda)$
- that allows us to **recover the minimal indices** of $P(\lambda)$ from those of $\mathcal{L}(\lambda)$ via **uniform shifts**.
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Outline

Definition

The complete eigenstructure of an $m \times n$ matrix polynomial $P(\lambda)$ is given by:

- its finite eigenvalues, together with their elementary divisors,
- its infinite eigenvalue, together with its elementary divisors,
- $n - r$ right minimal indices $\varepsilon_1, \ldots, \varepsilon_{n-r}$, and
- $m - r$ left minimal indices $\eta_1, \ldots, \eta_{m-r}$,

where r is the rank of $P(\lambda)$.

Remark

The complete eigenstructure is composed by

- the **regular** structure $\longrightarrow$ eigenvalues,
- the **singular** structure $\longrightarrow$ minimal indices.

Minimal indices only appear in **singular polynomials**, i.e., either rectangular or square with $\det P(\lambda) = 0$. Other polynomials are called **regular**.

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- A **linearization** for $P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$ is a **linear matrix polynomial (or matrix pencil)** $\mathcal{L}(\lambda)$, such that,

$$U(\lambda)\, \mathcal{L}(\lambda)\, V(\lambda) = \begin{bmatrix} I_s & \\ & P(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

$P(\lambda)$ and $\mathcal{L}(\lambda)$ have the same finite elementary divisors.

- $\mathcal{L}(\lambda)$ is a "strong linearization" if, **in addition**, $\operatorname{rev} \mathcal{L}(\lambda)$ is a linearization for $\operatorname{rev} P(\lambda)$, where $\operatorname{rev} P(\lambda) := P_0 \lambda^d + \cdots + P_{d-1} \lambda + P_d$.

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The Frobenius companion form of the $m \times n$ matrix polynomial $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix}$$

Theorem ($C_1(\lambda)$ is much more than a strong linearization!!)

(a) *If* $0 \leq \varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $P(\lambda)$*, then the right minimal indices of* $C_1(\lambda)$ *are* $\varepsilon_1 + d - 1 \leq \cdots \leq \varepsilon_p + d - 1$.

(b) *If* $0 \leq \eta_1 \leq \cdots \leq \eta_q$ *are the left minimal indices of* $P(\lambda)$*, then the left minimal indices of* $C_1(\lambda)$ *are* $\eta_1 \leq \cdots \leq \eta_q$.

Example of strong linearization whose right (resp. left) minimal indices allow us to recover the ones of the polynomial via uniform shifts.

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Introduction: Backward stable algorithms on strong linearizations (I)

- There are **"backward stable" algorithms that compute the complete eigenstructure of any matrix pencil**:
  1. QZ algorithm for regular pencils (Moler & Stewart, 1973).
  2. Staircase or GUPTRI algorithm for singular pencils (Van Dooren, 1979; Demmel-Kågström, 1993).
- They can be applied to **strong linearizations** $\mathcal{L}(\lambda)$ **of a matrix polynomial** $P(\lambda)$ and
- if **such linearizations enjoy known uniform shifting relations for the minimal indices**, then
- **these algorithms allow us to compute the complete eigenstructure of** $P(\lambda)$, even in the singular case.
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- *The computed complete eigenstructure of $\mathcal{L}(\lambda)$ is the exact complete eigenstructure of a matrix pencil $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ such that*

$$\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F} = O(\mathbf{u}),$$

  where $\mathbf{u} \approx 10^{-16}$ is the unit roundoff and

- $\|\cdot\|_F$ is the Frobenius norm, i.e., for any matrix polynomial

$$\|Q_k\lambda^k + \cdots + Q_1\lambda + Q_0\|_F = \sqrt{\|Q_k\|_F^2 + \cdots + \|Q_1\|_F^2 + \|Q_0\|_F^2}\,.$$

- But, does this imply that (after shifting properly the minimal indices) the computed complete eigenstructure of $P(\lambda)$ is the exact complete eigenstructure of a matrix polynomial of the same degree $P(\lambda) + \Delta P(\lambda)$ such that

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- For solving this question, we pose the following **theoretical problems** of matrix perturbation theory.

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To solve the following matrix perturbation problems

- **Data:**
  1. Matrix polynomial $P(\lambda)$ of degree d.
  2. Strong linearization $\mathcal{L}(\lambda)$ of $P(\lambda)$ enjoying uniform shift-relations for the minimal indices.
  3. Perturbation pencil $\Delta\mathcal{L}(\lambda)$.
- **Problem 1:** To establish conditions on $\|\Delta\mathcal{L}(\lambda)\|_F$ such that $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization for some matrix polynomial $P(\lambda) + \Delta P(\lambda)$ of degree d, and such that
- **Problem 2:** the shift-relations between minimal indices of $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ and $P(\lambda) + \Delta P(\lambda)$ are equal to those between $\mathcal{L}(\lambda)$ and $P(\lambda)$.
- **Problem 3:** To prove a perturbation bound

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq C_{P,\mathcal{L}} \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F},$$

  with $C_{P,\mathcal{L}}$ a number depending on $P(\lambda)$ and $\mathcal{L}(\lambda)$.
- For those $P(\lambda)$ and $\mathcal{L}(\lambda)$ s.t. $C_{P,\mathcal{L}}$ is moderate, to use backward stable algorithms on $\mathcal{L}(\lambda)$ gives backward stability for $P(\lambda)$, i.e., from the polynomial point of view.

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  2. Strong linearization $\mathcal{L}(\lambda)$ of $P(\lambda)$ enjoying uniform shift-relations for the minimal indices.
  3. Perturbation pencil $\Delta\mathcal{L}(\lambda)$.
- **Problem 1:** To establish conditions on $\|\Delta\mathcal{L}(\lambda)\|_F$ such that $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization for some matrix polynomial $P(\lambda) + \Delta P(\lambda)$ of degree d, and such that
- **Problem 2:** the shift-relations between minimal indices of $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ and $P(\lambda) + \Delta P(\lambda)$ are equal to those between $\mathcal{L}(\lambda)$ and $P(\lambda)$.
- **Problem 3:** To prove a perturbation bound

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq C_{P,\mathcal{L}} \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F},$$

  with $C_{P,\mathcal{L}}$ a number depending on $P(\lambda)$ and $\mathcal{L}(\lambda)$.
- For those $P(\lambda)$ and $\mathcal{L}(\lambda)$ s.t. $C_{P,\mathcal{L}}$ is moderate, to use backward stable algorithms on $\mathcal{L}(\lambda)$ gives backward stability for $P(\lambda)$, i.e., from the polynomial point of view.

...has a number of key features which are not present in any other analyses published so far:

1. for the first time, it is NOT a first order analysis, since it is a rigorous analysis valid for perturbations $\Delta\mathcal{L}(\lambda)$ of finite norm,
2. it provides very detailed bounds, and not just vague big-O bounds as other analyses do,
3. it is valid simultaneously for all the linearizations in the very large new class of block Kronecker pencils, which includes Fiedler linearizations for which this type of backward error analyses are not yet known,
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Previous works on this type of "global" backward error analyses

There are just a few: only first order results, only for Frobenius linearizations or their counterparts in other bases, often only valid for regular polynomials, or do not pay attention to minimal indices...

- Van Dooren & De Wilde (LAA 1983).
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We mention briefly **weaker "local" backward analyses valid only** for regular matrix polynomials and for **each particular computed eigenpair**, i.e., **with a different perturbation for each eigenpair**:

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With additional assumptions and scalings, these analyses may yield coefficient-wise backward error bounds.

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Outline

Two fundamental auxiliary matrix polynomials in this talk

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1\times(k+1)},$$

and their Kronecker products by identities

$$L_k(\lambda) \otimes I_n := \begin{bmatrix} -I_n & \lambda I_n & & & \\ & -I_n & \lambda I_n & & \\ & & \ddots & \ddots & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{F}[\lambda]^{nk\times n(k+1)},$$

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Revisiting the Frobenius companion form with these matrices at hand

The Frobenius companion form of the $m \times n$ matrix polynomial $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix},$$

and can be written as

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Let $\lambda M_1 + M_0$ be an arbitrary pencil. Then any pencil of the form

$$\mathcal{L}(\lambda) = \underbrace{\left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right]}_{(\varepsilon+1)n \quad \eta m} \begin{array}{l} \}\,(\eta+1)m \\ \}\,\varepsilon n \end{array},$$

is called a block Kronecker pencil (one-block row and column cases included).

Theorem (key theorem of block Kronecker pencils)

Any block Kronecker pencil $\mathcal{L}(\lambda)$ is a strong linearization of the matrix polynomial

$$Q(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_m)(\lambda M_1 + M_0)(\Lambda_\varepsilon(\lambda) \otimes I_n) \in \mathbb{F}[\lambda]^{m\times n},$$

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- *Let* $P(\lambda) = \sum_{k=0}^{d} P_k \lambda^k \in \mathbb{F}[\lambda]^{m \times n}$*,*
- *let* $\mathcal{L}(\lambda)$ *be a block Kronecker pencil with* $\varepsilon + \eta + 1 = d$*, and*
- *let us consider* M_0 *and* M_1 *partitioned into* $(\eta + 1) \times (\varepsilon + 1)$ *blocks each of size* $m \times n$*.*

If the sum of the blocks on the $(d-k)$th block antidiagonal of M_0 plus the sum of the blocks on the $(d-k+1)$th block antidiagonal of M_1 is equal to P_k, *for* $k = 0, \ldots, d$*,*

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Any Fiedler pencil of a matrix polynomial $P(\lambda)$ (rectangular or square) can be transformed via permutations into a block Kronecker pencil with this "antidiagonal structure".

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Examples of block Kronecker pencils (I)

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{F}[\lambda]^{m\times n}$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 + P_4 & 0 & 0 & -I_m & 0 \\ 0 & \lambda P_3 + P_2 & 0 & \lambda I_m & -I_m \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right]$$

Examples of block Kronecker pencils (II)

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{F}[\lambda]^{m \times n}$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 & \lambda P_4 & \lambda P_3 & -I_m & 0 \\ 0 & 0 & \lambda P_2 & \lambda I_m & -I_m \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right]$$

Examples of block Kronecker pencils (III)

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{F}[\lambda]^{m\times n}$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 & A & P_2 & -I_m & 0 \\ \lambda P_4 & -\lambda A & \lambda B + P_1 & \lambda I_m & -I_m \\ \lambda P_3 & -\lambda B & P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right]$$

for any matrices A and B.

Outline

A "hidden" structure in block Kronecker pencils

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right]$$

- $L_\varepsilon(\lambda) \otimes I_n$ and $L_\eta(\lambda) \otimes I_m$ are particular instances of minimal bases with all their row degrees equal to 1.
- **Reminder**: $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ with $m < n$ is a minimal basis with all its row degrees equal if and only if the complete eigenstructure of $Q(\lambda)$ consists of only $n - m$ right minimal indices.
- $\Lambda_\varepsilon(\lambda)^T \otimes I_n$ and $\Lambda_\eta(\lambda)^T \otimes I_m$ are also minimal bases with all their row degrees equal to ε and η, respectively.
- Moreover, these pairs of minimal bases are "dual" each other, i.e.,

$$(L_\varepsilon(\lambda) \otimes I_n)\,(\Lambda_\varepsilon(\lambda) \otimes I_n) = 0\,, \qquad \text{(a pair "dual" min. bases)}$$
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Definition and key properties of strong block minimal bases pencils

Definition

A matrix pencil

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

is a strong block minimal bases pencil if

- $K_1(\lambda)$ and $K_2(\lambda)$ are minimal bases with all their row degrees equal to 1,
- the row degrees of a minimal basis dual to $K_1(\lambda)$ are all equal, and
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Theorem (key theorem on strong block minimal bases pencils)

Any strong block minimal bases pencil $\mathcal{L}(\lambda)$ is a strong linearization of the matrix polynomial

$$Q(\lambda) := N_2(\lambda) M(\lambda) N_1(\lambda)^T,$$

where $N_1(\lambda)$ (resp. $N_2(\lambda)$) **is a minimal basis dual to** *$K_1(\lambda)$ (resp. $K_2(\lambda)$). The right minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ shifted by deg(N_1), and the left minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ shifted by deg(N_2).*

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- **For us in this talk,** because they include “sufficiently small” (but not necessarily infinitesimal) perturbations of block Kronecker pencils that preserve the 0 block in 2×2 position, that preserve the shift relationships between minimal indices, and for which the corresponding perturbed matrix polynomial is known in terms of perturbations of dual minimal bases.
- **In general,** they unify and simplify the theory of many Fiedler-type linearizations, even for polynomials in non-monomial bases, and display structures transparently. (Not in this talk, still in development, Bueno, FMD, Fassbender, Lawrence, Noferini, Pérez, Shayanfar, Robol, Vandebril, Van Dooren ...)

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Outline

Theorem

Let $\mathcal{L}(\lambda)$ be a block Kronecker pencil for $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{F}[\lambda]^{m\times n}$, i.e.,

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

If $\Delta\mathcal{L}(\lambda)$ is any pencil with the same size as $\mathcal{L}(\lambda)$ and such that

$$\|\Delta\mathcal{L}(\lambda)\|_F < \left(\frac{\pi}{16}\right)^2 \frac{1}{d^{5/2}} \frac{1}{1 + \|\lambda M_1 + M_0\|_F},$$

then $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization of a matrix poly $P(\lambda) + \Delta P(\lambda)$ with grade d and such that

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq 68\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|\lambda M_1 + M_0\|_F + \|\lambda M_1 + M_0\|_F^2\right) \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

In addition, the right (resp. left) minimal indices of $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ are those of $P(\lambda) + \Delta P(\lambda)$ shifted by ε (resp. η), i.e., the shift relations are preserved.

Theorem

Let $\mathcal{L}(\lambda)$ be a block Kronecker pencil for $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{F}[\lambda]^{m\times n}$ with $\eta = 0$ or $\varepsilon = 0$, i.e., with the form

$$\mathcal{L}(\lambda) = \left[\frac{\lambda M_1 + M_0}{L_\varepsilon(\lambda) \otimes I_n} \right] \quad \text{or} \quad \mathcal{L}(\lambda) = \left[\; \lambda M_1 + M_0 \; \middle| \; L_\eta(\lambda)^T \otimes I_m \; \right].$$

If $\Delta\mathcal{L}(\lambda)$ is any pencil with the same size as $\mathcal{L}(\lambda)$ and such that

$$\|\Delta\mathcal{L}(\lambda)\|_F < \frac{\pi}{12\, d^{3/2}},$$

then $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization of a matrix poly $P(\lambda) + \Delta P(\lambda)$ with grade d and such that

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq 4\, d\, \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|\lambda M_1 + M_0\|_F\right) \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

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How are these theorems proved? STEP 1. Restoring the zero block.

The perturbation destroys the $(2,2)$-zero block

$$\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda) = \left[\begin{array}{c|c} \lambda M_1 + M_0 + \Delta\mathcal{L}_{11}(\lambda) & L_\eta(\lambda)^T \otimes I_m + \Delta\mathcal{L}_{12}(\lambda) \\ \hline L_\varepsilon(\lambda) \otimes I_n + \Delta\mathcal{L}_{21}(\lambda) & \Delta\mathcal{L}_{22}(\lambda) \end{array}\right].$$

Our first step restores the $(2,2)$-zero block via a strict equivalence close to the identity

$$\begin{bmatrix} I_{(\eta+1)m} & 0 \\ C & I_{\varepsilon n} \end{bmatrix} (\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)) \begin{bmatrix} I_{(\varepsilon+1)n} & D \\ 0 & I_{\eta m} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda M_1 + M_0 + \Delta\mathcal{L}_{11}(\lambda) & L_\eta(\lambda)^T \otimes I_m + \Delta\widetilde{\mathcal{L}}_{12}(\lambda) \\ L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda) & 0 \end{bmatrix} =: \mathcal{L}(\lambda) + \Delta\widetilde{\mathcal{L}}(\lambda).$$

- C and D are solutions of an **underdetermined quadratic system of two matrix equations** whose existence is proved and whose norms are properly bounded.
- Strict equivalences preserve complete eigenstructures of matrix pencils.
- This step is NOT needed for block Kronecker pencils without the zero block.

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How are these theorems proved? STEPS 2 and 3.

Step 2. We prove that

$$\mathcal{L}(\lambda) + \Delta\widetilde{\mathcal{L}}(\lambda) := \begin{bmatrix} \lambda M_1 + M_0 + \Delta\mathcal{L}_{11}(\lambda) & L_\eta(\lambda)^T \otimes I_m + \Delta\widetilde{\mathcal{L}}_{12}(\lambda) \\ L_\varepsilon(\lambda) \otimes I_n + \Delta\widetilde{\mathcal{L}}_{21}(\lambda) & 0 \end{bmatrix}$$

is a **strong block minimal bases pencil** with degrees of the dual minimal bases equal to ε and η (the unperturbed ones) and matrix polynomial

$$P(\lambda) + \Delta P(\lambda)$$
$$:= \left(\Lambda_\eta(\lambda)^T \otimes I_m + \Delta R_\eta(\lambda)^T\right)(\lambda M_1 + M_0 + \Delta\mathcal{L}_{11}(\lambda))\left(\Lambda_\varepsilon(\lambda) \otimes I_n + \Delta R_\varepsilon(\lambda)\right),$$

where $\|\Delta R_\eta(\lambda)\|_F$ and $\|\Delta R_\varepsilon(\lambda)\|_F$ are carefully bounded.

Step 3. $\|\Delta P(\lambda)\|_F$ is finally bounded.

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Notation: $L_k(\lambda) \otimes I_\ell = \lambda F_{k\ell} - E_{k\ell}$ and $\Delta\mathcal{L}_{ij}(\lambda) = \lambda \Delta B_{ij} + \Delta A_{ij}$.

The system for the unknown matrices C and D is

$$\begin{aligned} C(E_{\eta m}^T - \Delta A_{12}) + (E_{\varepsilon n} - \Delta A_{21})D &= \Delta A_{22} + C(M_0 + \Delta A_{11})D \\ C(F_{\eta m}^T + \Delta B_{12}) + (F_{\varepsilon n} + \Delta B_{21})D &= -\Delta B_{22} - C(M_1 + \Delta B_{11})D \end{aligned}$$

- It is underdetermined: more entries of C and D than scalar equations.
- We extend classical techniques used by Stewart (1972) in the study of perturbation bounds for invariant and deflating subspaces of matrices and pencils, which are valid only for determined matrix equations.
- In this way, we prove that the system is consistent and find a convenient bound for the norm of one of its solutions:

$$\sqrt{\|C\|_F^2 + \|D\|_F^2} \leq \frac{3d}{1 - 3d\|\Delta\mathcal{L}(\lambda)\|_F} \|\Delta\mathcal{L}(\lambda)\|_F .$$

A few words on the system of matrix equations in Step 1

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$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq \underbrace{68\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|\lambda M_1 + M_0\|_F + \|\lambda M_1 + M_0\|_F^2\right)}_{C_{P,\mathcal{L}}} \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

- It can be proved that if $\|P(\lambda)\|_F \ll 1$ or $\|P(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$,
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- Therefore, for getting "backward stability" from Block Kronecker linearizations, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|\lambda M_1 + M_0\|_F \approx \|P(\lambda)\|_F$, then

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

For Fiedler and Frobenius linearizations $\|\lambda M_1 + M_0\|_F = \|P(\lambda)\|_F$

Discussion of the perturbation bounds for block Kronecker pencils

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- and that, if $\|\lambda M_1 + M_0\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$.
- Therefore, for getting "backward stability" from Block Kronecker linearizations, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|\lambda M_1 + M_0\|_F \approx \|P(\lambda)\|_F$, then

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

For Fiedler and Frobenius linearizations $\|\lambda M_1 + M_0\|_F = \|P(\lambda)\|_F$.

Outline

Conclusions

- We have proved that the computation of the complete eigenstructure of a matrix polynomial $P(\lambda)$ (regular or singular, square or rectangular)
- via block Kronecker pencils **is backward stable from the polynomial point of view**
- **if** $\|P(\lambda)\|_F = 1$ **and** $\|\lambda M_1 + M_0\|_F \approx \|P(\lambda)\|_F$.
- This proves, in particular, for the first time "global backward stability" for all Fiedler pencils.
- The new perturbation analysis presents a number of novel features and establishes a framework that can be generalized to other linearizations.
- In particular, we are currently advancing in several "global structured backward error analyses" for structured complete polynomial eigenproblems.

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