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Orthogonal Sampling Formulas: A Unified Approach*

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Abstract. This paper intends to serve as an educational introduction to sampling theory. Basically, sampling theory deals with the reconstruction of functions (signals) through their values (samples) on an appropriate sequence of points by means of sampling expansions involving these values.

In order to obtain such sampling expansions in a unified way, we propose an inductive procedure leading to various orthogonal formulas. This procedure, which we illustrate with a number of examples, closely parallels the theory of orthonormal bases in a Hilbert space. All intermediate steps will be described in detail, so that the presentation is self-contained. The required mathematical background is a basic knowledge of Hilbert space theory.

Finally, despite the introductory level, some hints are given on more advanced problems in sampling theory, which we motivate through the examples.

Key words. orthonormal bases, sampling expansions, reproducing kernel Hilbert spaces

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1. Introduction.

In 1949 Claude Shannon [23] published a remarkable result:

If a signal $f(t)$ contains no frequencies higher than w cycles per second, then $f(t)$ is completely determined by its values $f(n/2w)$ at a discrete set of points with spacing $1/2w$ and can be reconstructed from these values by the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2w}\right) \frac{\sin \pi(2wt - n)}{\pi(2wt - n)}. \tag{1.1}$$

In engineering and mathematical terminology, the signal f is *bandlimited to* $[-2\pi w, 2\pi w]$, meaning that $f(t)$ contains no frequencies beyond w cycles per second. Equivalently, its Fourier transform F is zero outside this interval:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi w}^{2\pi w} F(x) e^{ixt} dx. \tag{1.2}$$

The engineering principle underlying (1.1) is that all the information contained in $f(t)$ is stored in its samples $f(n/2w)$. The cut-off frequency determines the so-called *Nyquist rate*,[1] the minimum rate at which the signal needs to be sampled in order to

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[1]This rate is named after the engineer H. Nyquist, who was the first to point out its importance in connection with telegraph transmission.

recover it at all intermediate times t. In the case above, $2w = 4\pi w/2\pi$ is the sampling frequency and $1/2w$ is the sampling period.

The sampling functions used in the reconstruction (1.1) are

$$S_n(t) = \frac{\sin \pi(2wt - n)}{\pi(2wt - n)}.$$

They satisfy the interpolatory property $S_n(t_k) = \delta_{n,k}$: $\delta_{n,k}$ equals 1 if $n = k$ and 0 if $n \neq k$. A series as in (1.1) is known as a *cardinal series* because the sampling functions involve the *cardinal sine* function (or *sinc function*)

$$\text{sinc}\,(t) = \begin{cases} \frac{\sin \pi t}{\pi t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

These series owe their name to J. M. Whittaker [26], a reference cited by Shannon in [23]. To be precise, J. M. Whittaker's work was a refinement of his father's, the eminent British mathematician E. T. Whittaker [25]. However, it is not clear whether they were the first mathematicians to introduce these kinds of expansions. Some interesting historical notes concerning this controversy can be found in [6, 11, 12, 28].

The Shannon sampling theorem provides the theoretical foundation for modern pulse code modulation communications systems, which were introduced, independently, by V. Kotel'nikov [14] in 1933 (in Russian) and by Shannon in 1949. This sampling theorem is presently known in the mathematical literature as the *Whittaker–Shannon–Kotel'nikov theorem* or WSK theorem.

What started as a theorem for reconstructing bandlimited signals from uniform samples has now become a whole branch of applied mathematics, known as *sampling theory*. The efforts in extending Shannon's fundamental result point in various directions: nonuniform samples, other discrete data taken from the signal, multidimensional signals, and more. Some of these extensions will appear in section 3.

In general, the problem of *sampling and reconstruction* can be stated as follows: Given a set H of functions defined on a common domain Ω, is there a discrete set $D = \{t_n\} \subset \Omega$ such that every $f \in H$ is uniquely determined by its values on D? And if this is the case, how can we recover such a function? Moreover, is there a sampling series of the form

$$f(t) = \sum_n f(t_n) S_n(t) \tag{1.3}$$

valid for every f in H, where the convergence of the series is at least absolute and uniform on closed bounded intervals?[2]

In many cases of practical interest, the set H is related to some integral transform as in (1.2), and the sampling functions satisfy an interpolatory property. All this leads us to propose a general method to obtain some sampling theorems in a unified way. Section 2 obtains orthogonal sampling theorems by the following steps:

1. Take a set of functions $\{S_n(t)\}$ interpolating at a sequence of points $\{t_n\}$.
2. Choose an orthonormal basis for an L^2 space.
3. Define an integral kernel involving $\{S_n(t)\}$ and the orthonormal basis. Consider the corresponding integral transform in the L^2 space.

[2] A sampling series may also contain samples from a transformed version of f, as the derivative, for instance. Here we confine ourselves to sampling formulas like (1.3).

4. Endow the range space of this integral transform with a norm that provides an isometric isomorphism between the range space and the L^2 space via the integral transform.

5. Thus, any Fourier expansion in the L^2 space is transformed into a Fourier expansion in the range space whose coefficients are the samples of the corresponding function, computed at the sequence $\{t_n\}$.

6. Convergence in this norm of the range space implies pointwise convergence and, as a consequence, we obtain a *sampling expansion* that holds for all functions in the range space.[3]

This methodology is used in section 3, where several well-known sampling formulas are derived in this way. Thus the main features of our approach are the following:

I. The fact of placing the problem in a functional framework, common to many diverse situations, allows us to introduce sampling theory through the well-developed theory of orthonormal bases in a Hilbert space. A number of well-known sampling formulas are obtained in a unified way.

II. The functional setting we have chosen only permits us, in principle, to derive *orthogonal* sampling expansions. Some remarks, motivated by the examples in section 3, will be made concerning other more general settings.

2. The Main Theory. Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be an orthonormal basis of an $L^2(I)$ space, where I is an interval in $\mathbb{R}$, bounded or unbounded. As usual, the inner product in $L^2(I)$ is given by $\langle F, G\rangle_{L^2(I)} = \int_I F(x)\overline{G(x)}dx$.

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of functions $S_n : \Omega \subset \mathbb{R} \longrightarrow \mathbb{R}$ (or $\mathbb{C}$), defined for all $t \in \Omega$, and let $\{t_n\}_{n=1}^{\infty}$ be a sequence in Ω satisfying conditions C1 and C2.

C1. $S_n(t_k) = a_n\delta_{n,k}$, where $\delta_{n,k}$ denotes the Kronecker delta and $a_n \neq 0$.

C2. $\sum_{n=1}^{\infty} |S_n(t)|^2 < \infty$ for each $t \in \Omega$.

Define the function $K(x,t)$ as

$$K(x,t) = \sum_{n=1}^{\infty} S_n(t)\overline{\phi_n}(x)\,, \qquad (x,t) \in I \times \Omega. \tag{2.1}$$

As a function of x, $K(\cdot,t)$ belongs to $L^2(I)$ since $\{\overline{\phi_n}\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(I)$ as well.

Now, consider $K(x,t)$ as a kernel and define on $L^2(I)$ the linear integral transformation which maps F to

$$f(t) := \int_I F(x)K(x,t)dx. \tag{2.2}$$

Remark 1. Given an integral kernel $K(x,t)$, conditions C1 and C2 can be read as the existence of a sequence $\{t_n\}_{n=1}^{\infty} \subset \Omega$ such that $\{K(x,t_n)\}_{n=1}^{\infty}$ is an orthogonal basis for $L^2(I)$. Kramer [15] originally suggested this method of obtaining sampling theorems. From a pedagogical point of view, we find it more instructive to follow an inductive construction.

The integral transform (2.2) is well defined because F and $K(\cdot,t)$ belong to $L^2(I)$ and the Cauchy–Schwarz inequality implies that $f(t)$ is defined for each $t \in \Omega$. This transformation is one to one, since $\{K(x,t_k) = a_k\overline{\phi_k}(x)\}_{k=1}^{\infty}$ is a complete sequence for $L^2(I)$; i.e., the only function orthogonal to every $\{K(x,t_k)\}_{k=1}^{\infty}$ is the zero function.

[3]The idea underlying the whole procedure is borrowed from Hardy [10], who first noticed that (1.1) is an orthogonal expansion.

In fact, if two functions f and g are equal in the sequence $\{t_k\}_{k=1}^{\infty}$ they coincide necessarily on the whole set Ω.

Now, define $\mathcal{H}$ as the range of the integral transform (2.2)

$$\mathcal{H} = \left\{ f : \Omega \longrightarrow \mathbb{C} \text{ such that } f(t) = \int_I F(x)K(x,t)dx,\ F \in L^2(I) \right\}$$

endowed with the norm $\|f\|_{\mathcal{H}} = \|F\|_{L^2(I)}$. Using the polarization identity [20, p. 276], we obtain the following fact.

• $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ *is a Hilbert space isometrically isomorphic to* $L^2(I)$*, with inner product*

$$\langle f, g \rangle_{\mathcal{H}} = \langle F, G \rangle_{L^2(I)}, \tag{2.3}$$

where $f(t) = \int_I F(x)K(x,t)dx$ *and* $g(t) = \int_I G(x)K(x,t)dx$.

Since an isometric isomorphism transforms orthonormal bases to orthonormal bases, we derive an important property for $\mathcal{H}$ by applying the integral transform (2.2) to the orthonormal basis $\{\phi_n(x)\}$.

• $\{S_n(t)\}_{n=1}^{\infty}$ *is an orthonormal basis for* $\mathcal{H}$.

Now we see that $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a reproducing kernel Hilbert space of functions defined on Ω, a crucial step for our sampling purposes. In a *reproducing kernel Hilbert space* (RKHS) H, all the evaluation functionals $E_t(f) := f(t)$, $f \in H$, are continuous for each fixed $t \in \Omega$ (or equivalently bounded since they are linear) [27, pp. 15–19]. By the Riesz representation theorem [20, p. 345], for each $t \in \Omega$ there exists a unique element $k_t \in H$ such that $f(t) = \langle f, k_t \rangle_H$ for all $f \in H$. Let $k(t,s) = \langle k_s, k_t \rangle = k_s(t)$ for $s, t \in \Omega$. Then

$$\langle f(\cdot), k(\cdot, s) \rangle = \langle f, k_s \rangle = f(s). \tag{2.4}$$

The function $k(t,s)$ is called the *reproducing kernel*[4] of H. One can easily prove that the reproducing kernel in an RKHS is unique. Indeed, let $k'(t,s)$ be another reproducing kernel for H. For a fixed $s \in \Omega$, consider $k'_s(t) = k'(t,s)$. Then, for $t \in \Omega$ we have

$$\begin{aligned} k'_s(t) &= \langle k'_s, k_t \rangle = \overline{\langle k_t, k'_s \rangle} \\ &= \overline{k_t(s)} = \overline{\langle k_t, k_s \rangle} = k_s(t). \end{aligned}$$

Hence $k(s,t) = k'(s,t)$ for all $t, s \in \Omega$.

Finally, if $\{e_n(t)\}_{n=1}^{\infty}$ is an orthonormal basis for H, then the reproducing kernel can be expressed as $k(t,s) = \sum_{n=1}^{\infty} e_n(t)\overline{e_n(s)}$. Indeed, expanding k_t in the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ we have

$$k_t = \sum_{n=1}^{\infty} \langle k_t, e_n \rangle e_n = \sum_{n=1}^{\infty} \overline{e_n(t)} e_n,$$

and therefore,

$$k(t,s) = \langle k_s, k_t \rangle = \sum_{n=1}^{\infty} \overline{e_n(s)} e_n(t). \tag{2.5}$$

[4] Note that, equivalently, an RKHS can be defined through (2.4) instead of the continuity of the evaluation functionals. In this case, the continuity of E_t follows from the Cauchy–Schwarz inequality.

- $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ *is an RKHS whose reproducing kernel is given by*

$$k(t,s) = \sum_{n=1}^{\infty} \overline{S_n(s)} S_n(t) = \langle K(\cdot,t), K(\cdot,s)\rangle_{L^2(I)}. \tag{2.6}$$

To prove it, we use the Cauchy–Schwarz inequality in (2.2), obtaining for a fixed $t \in \Omega$

$$|E_t(f)| = |f(t)| \le \|F\|_{L^2(I)} \|K(\cdot,t)\|_{L^2(I)} = \|f\|_{\mathcal{H}} \|K(\cdot,t)\|_{L^2(I)} \tag{2.7}$$

for every $f \in \mathcal{H}$.

As to the reproducing kernel formula (2.6), due to (2.5), we only need to prove the second equality. To this end, consider

$$k'(t,s) = \langle K(\cdot,t), K(\cdot,s)\rangle_{L^2(I)} = \int_I K(x,t)\overline{K(x,s)}dx.$$

Then, for a fixed $s \in \Omega$, $k'(t,s)$ is the transformed $\overline{K(x,s)}$ by (2.2). Using the isometry (2.3) we have

$$\langle f, k'(\cdot,s)\rangle_{\mathcal{H}} = \langle F, \overline{K(x,s)}\rangle_{L^2(I)} = \int_I F(x)K(x,s)dx = f(s).$$

The uniqueness of the reproducing kernel leads to the desired result.

It is worth pointing out that inequality (2.7) has important consequences for the convergence in $\mathcal{H}$. More precisely, we have the following fact.

- *Convergence in the norm* $\|\cdot\|_{\mathcal{H}}$ *implies pointwise convergence and uniform convergence on subsets of* Ω*, where* $\|K(\cdot,t)\|_{L^2(I)} = \sqrt{k(t,t)}$ *is bounded.*

At this point, we have all the ingredients to obtain a sampling formula for all the functions in $\mathcal{H}$. Indeed, expanding an arbitrary function $f \in \mathcal{H}$ in the orthonormal basis $\{S_n(t)\}_{n=1}^{\infty}$, we have

$$f(t) = \sum_{n=1}^{\infty} \langle f, S_n\rangle_{\mathcal{H}} S_n(t),$$

where the convergence is in the $\mathcal{H}$-norm sense and hence pointwise in Ω. Taking into account the isometry between $\mathcal{H}$ and $L^2(I)$, we have that

$$\langle f, S_n\rangle_{\mathcal{H}} = \langle F, \phi_n\rangle_{L^2(I)} = \frac{f(t_n)}{a_n}$$

for each $n \in \mathbb{N}$. Hence, we obtain the following *sampling formula* for $\mathcal{H}$.

- *Each function* f *in* $\mathcal{H}$ *can be recovered from its samples at the sequence* $\{t_n\}_{n=1}^{\infty}$ *through the formula*

$$f(t) = \sum_{n=1}^{\infty} f(t_n) \frac{S_n(t)}{a_n}. \tag{2.8}$$

The convergence of the series in (2.8) *is absolute and is uniform on subsets of* Ω*, where* $\|K(\cdot,t)\|_{L^2(I)} = \sqrt{k(t,t)}$ *is bounded.*

Note that an orthonormal basis is an unconditional basis. By Parseval's identity [20, p. 307], any of its reorderings is again an orthonormal basis. Therefore, the sampling series (2.8) is pointwise unconditionally convergent for each $t \in \Omega$ and hence pointwise absolutely convergent. The uniform convergence follows from inequality (2.7).

Remark 2. We also could have obtained the formula (2.8) by applying the integral transform (2.2) to the Fourier series expansion $F(x) = \sum_{n=1}^{\infty} \langle F, \phi_n \rangle_{L^2(I)} \phi_n(x)$ of a function F in $L^2(I)$.

A final comment about the functional space $\mathcal{H}$ is in order. Any $f \in \mathcal{H}$ can be described using the sequence of its values $\{f(t_n)\}_{n=1}^{\infty}$ by means of formula (2.8). In particular, the inner product and the norm in $\mathcal{H}$ can be expressed as

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \frac{f(t_n)\overline{g(t_n)}}{|a_n|^2}, \qquad \|f\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \frac{|f(t_n)|^2}{|a_n|^2}.$$

3. Some Examples. Generally speaking, one can easily construct spaces $\mathcal{H}$ as in section 2 having a sampling property at a sequence $\{t_n\}_{n=1}^{\infty}$ as in formula (2.8). To this end, let $t_1, t_2, \ldots$ be distinct real numbers such that $\sum_{n=1}^{\infty} 1/|t_n|^2 < \infty$. There exists an analytic function $P(t)$ with simple zeros at the sequence $\{t_n\}$ [18, p. 457]. Taking $S_n(t) = \frac{P(t)}{t-t_n}$ and any orthonormal basis $\{\phi_n(x)\}_{n=1}^{\infty}$ for an $L^2(I)$-space, we can follow the steps of section 2 in order to construct an RKHS $\mathcal{H}$ with the sampling property at the given sequence $\{t_n\}$. Thus, taking into account the fact that $S_n(t_k) = P'(t_n)\delta_{n,k}$, formula (2.8) ensures that any function of the form (2.2) can be expanded as the Lagrange-type interpolation series

$$f(t) = \sum_{n=1}^{\infty} f(t_n) \frac{P(t)}{(t-t_n)P'(t_n)}.$$

However, our main aim is to derive some of the well-known sampling theorems by following the method exposed in the previous section. All the examples in this section are based on the knowledge of specific orthonormal bases for known L^2-spaces. See [20, pp. 322–329] for the bases and also [30] for the integral transforms.

One of the richest sources of Kramer kernels is in the subject of self-adjoint boundary value problems [8, 28].

3.1. Classical Bandlimited Functions. The set of functions $\{e^{-inx}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi, \pi]$. We consider the Fourier integral kernel $K(x,t) = e^{itx}/\sqrt{2\pi}$. For a fixed $t \in \mathbb{R}$, we have

$$\begin{aligned} \frac{e^{itx}}{\sqrt{2\pi}} &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \langle e^{itx}, e^{inx} \rangle_{L^2[-\pi,\pi]} \frac{e^{inx}}{\sqrt{2\pi}} \\ &= \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n)}{\pi(t-n)} \frac{e^{inx}}{\sqrt{2\pi}} \quad \text{in} \quad L^2[-\pi,\pi]. \end{aligned}$$

Therefore, taking $S_n(t) = \frac{\sin \pi(t-n)}{\pi(t-n)}$ and $t_n = n$, $n \in \mathbb{Z}$, we obtain that any function of the form

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x) e^{itx} dx \quad \text{ with } F \in L^2[-\pi, \pi],$$

i.e., *bandlimited to* $[-\pi, \pi]$ in the classical sense, can be recovered from its samples at integers by means of the *cardinal series*

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}. \tag{3.1}$$

The series converges absolutely, and uniformly on $\mathbb{R}$ because in this case

$$\|K(\cdot, t)\|^2_{L^2[-\pi,\pi]} = 1 \quad \text{for all} \quad t \in \mathbb{R}.$$

The choice of the interval $[-\pi, \pi]$ is arbitrary. The same result applies to any compact interval $[-\pi\sigma, \pi\sigma]$ taking the samples $\{f(n/\sigma)\}_{n\in\mathbb{Z}}$ and replacing t with σt in the cardinal series (3.1).

The reproducing kernel of the corresponding $\mathcal{H}$ space is given by

$$\begin{aligned} k(t,s) &= \frac{1}{2\pi} \langle e^{itx}, e^{isx} \rangle_{L^2[-\pi,\pi]} = \frac{\sin \pi(t-s)}{\pi(t-s)} \\ &= \sum_{n=-\infty}^{\infty} \frac{\sin \pi(t-n)}{\pi(t-n)} \frac{\sin \pi(s-n)}{\pi(s-n)}, \end{aligned}$$

where we have used (2.6) and (2.5), respectively.

Remark 3. We can provide further information about the space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ in this particular case. Namely, since the Fourier transform is a unitary operator in $L^2(\mathbb{R})$ [20, p. 335], the space $\mathcal{H}$ coincides with the classical Paley–Wiener space PW_π, the closed subspace of $L^2(\mathbb{R})$ given by

$$PW_\pi = \{f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R}), \quad \text{supp}\, \widehat{f} \subseteq [-\pi, \pi]\},$$

where $\widehat{f}$ is the Fourier transform of f and supp $\widehat{f}$ denotes the support of $\widehat{f}$. Hence, $\widehat{f}$ is zero outside $[-\pi, \pi]$ for any $f \in PW_\pi$. Furthermore, the classical Paley–Wiener theorem [27, p. 100] shows that PW_π coincides with the space of entire functions of exponential type at most π with square integrable restriction to the real axis; i.e.,

$$PW_\pi = \{f \in \mathcal{H}(\mathbb{C}) \ : \ |f(z)| \le Ae^{\pi|z|}, \quad f|_{\mathbb{R}} \in L^2(\mathbb{R})\}.$$

Remark 4. The actual computation of the cardinal series (3.1) presents some numerical difficulties since the cardinal sine function behaves like $1/t$ as $|t| \to \infty$. One way to overcome this difficulty is the so-called *oversampling technique*, i.e., sampling the signal at a frequency higher than that given by its bandwidth. In this way we obtain sampling functions converging to zero at infinity faster than the cardinal sine functions. Indeed, consider the bandlimited function

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\delta}^{\pi\delta} F(x) e^{ixt} dx \quad \text{with } F \in L^2[-\pi\delta, \pi\delta] \ \text{ and } \delta < 1.$$

Extending F to be zero in $[-\pi, \pi] \setminus [-\pi\delta, \pi\delta]$, we have

$$F(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{e^{-inx}}{\sqrt{2\pi}} \quad \text{in } L^2[-\pi, \pi].$$

Let $\theta(x)$ be a smooth function taking the value 1 on $[-\pi\delta, \pi\delta]$ and the value 0 outside $[-\pi, \pi]$. As a consequence,

$$F(x) = \theta(x)F(x) = \sum_{n=-\infty}^{\infty} f(n)\theta(x)\frac{e^{-inx}}{\sqrt{2\pi}} \quad \text{in } L^2[-\pi, \pi],$$

and the sampling expansion

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)S(t-n)$$

holds, where S is the inverse Fourier transform $\mathcal{F}^{-1}$ of $\theta/\sqrt{2\pi}$ and, consequently, $S(t-n) = \mathcal{F}^{-1}(\theta(x)e^{-inx}/\sqrt{2\pi})(t)$. Furthermore, using the properties of the Fourier transform [24, p. 317] we see that the smoother θ is, the faster the decay of S is. However, the new sampling functions $\{S(\cdot - n)\}$ are no longer orthogonal.

Remark 5. Note that, using the orthonormal basis $\{e^{-inx}/\sqrt{2\pi}\}$, we can only obtain sampling formulas with uniformly separated sampling points. However, in many cases uniform sampling is not the most efficient way. As a rule, the sampling frequency should be higher in regions where the function is expected to undergo large variations and lower in intervals where the function is known to be almost constant. To obtain sampling formulas with irregularly spaced sampling points t_n, one can use Riesz bases or frames of complex exponentials $e^{it_n x}$. The theory obtained by including either Riesz bases or frames in the definition of the kernel (2.1) goes much in the same way as in section 2, but the mathematical requirements are more sophisticated. We omit the details to keep this presentation at an introductory level (see [2, 9] for more details of this more general framework). Finally, some iterative methods for the practical implementation of irregular sampling theory have been developed. A comprehensive account can be found in [9].

3.2. Bandlimited Functions in the Fractional Fourier Transform Sense. The sequence $\{\frac{1}{\sqrt{2\sigma}}e^{-i\pi nx/\sigma}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\sigma, \sigma]$. It is easy to prove that $\{\frac{1}{\sqrt{2\sigma}}e^{-i\pi nx/\sigma}e^{iax^2}\}_{n\in\mathbb{Z}}$, with $a \in \mathbb{R}$, is also a new orthonormal basis for $L^2[-\sigma, \sigma]$. Let a and b be two nonzero real constants. For notational ease we denote $2ab = \frac{1}{c}$. We will see the meaning of these constants later. Direct calculations show that the expansion

$$\begin{aligned} e^{-ia(t^2+x^2-2bxt)} &= \sum_{n=-\infty}^{\infty} \left\langle e^{-ia(t^2+x^2-2bxt)}, \frac{e^{i\pi nx/\sigma}}{\sqrt{2\sigma}}e^{-iax^2} \right\rangle_{L^2[-\sigma,\sigma]} \frac{e^{i\pi nx/\sigma}}{\sqrt{2\sigma}}e^{-iax^2} \\ &= \sum_{n=-\infty}^{\infty} \sqrt{2\sigma}e^{-iat^2}\frac{\sin\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})}{\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})}\frac{e^{i\pi nx/\sigma}}{\sqrt{2\sigma}}e^{-iax^2} \end{aligned}$$

holds in the $L^2[-\sigma, \sigma]$ sense. Set

$$S_n(t) = \sqrt{2\sigma}e^{-iat^2}\frac{\sin\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})}{\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})} \quad \text{and} \quad t_n = \frac{n\pi c}{\sigma}, n \in \mathbb{Z}.$$

Since $S_n(t_k) = \sqrt{2\sigma}e^{-iat_n^2}\delta_{n,k}$, we have the sampling formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n)e^{-ia(t^2-t_n^2)}\frac{\sin\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})}{\frac{\sigma}{c}(t-\frac{n\pi c}{\sigma})} \tag{3.2}$$

for any function f of the form

$$f(t)=\int_{-\sigma}^{\sigma}F(x)e^{-ia(t^2+x^2-2bxt)}dx \quad \text{with } F\in L^2[-\sigma,\sigma]. \tag{3.3}$$

Here, the reproducing kernel obtained from (2.6) is

$$k(t,s)=2\sigma e^{-ia(t^2-s^2)}\frac{\sin\frac{\sigma}{c}(t-s)}{\frac{\sigma}{c}(t-s)}.$$

Since $k(t,t)=2\sigma$, the series in (3.2) converges uniformly in $\mathbb{R}$.

Our next purpose is to see how formula (3.3) and the fractional Fourier transform (FRFT) are related. Recall that the FRFT with angle $\alpha\notin\{0,\pi\}$ of a function $f(t)$ is defined as

$$\mathcal{F}_\alpha[f](x)=\int_{-\infty}^{\infty}f(t)K_\alpha(x,t)\,dt,$$

where, apart from a normalization constant, the integral kernel $K_\alpha(x,t)$ is given by

$$e^{i\frac{\cot\alpha}{2}(t^2+x^2)-i\frac{xt}{\sin\alpha}}. \tag{3.4}$$

For $\alpha=0$ the FRFT is defined by $\mathcal{F}_0[f](x)=f(x)$, and for $\alpha=\pi$, by $\mathcal{F}_\pi[f](x)=f(-x)$. Whenever $\alpha=\pi/2$, the kernel (3.4) coincides with the Fourier kernel. Otherwise, (3.4) can be rewritten as

$$e^{ia(\alpha)[t^2+x^2-2b(\alpha)xt]},$$

where $a(\alpha)=\frac{\cot\alpha}{2}$ and $b(\alpha)=\sec\alpha$. The inversion formula of the FRFT (see [29]) is given by

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F_\alpha(x)K_{-\alpha}(x,t)dx.$$

Consequently, formula (3.2) is just the sampling expansion for a function bandlimited to $[-\sigma,\sigma]$ in the FRFT sense (3.3). Note that $2a(\alpha)b(\alpha)=\frac{1}{\sin\alpha}$, and $c=\sin\alpha$ in the sampling expansion (3.2).

3.3. Finite Cosine Transform. Let us consider the orthogonal basis $\{\cos nx\}$, $n\in\mathbb{N}\cup\{0\}$, in $L^2[0,\pi]$: note that $\|\cos nx\|^2_{L^2[0,\pi]}$ equals $\pi/2$ for $n\geq 1$ and π for $n=0$. For $t\in\mathbb{R}$ fixed, we expand the function $\cos tx$ in this basis, obtaining

$$\begin{aligned}\cos tx&=\sum_{n=0}^{\infty}\left\langle \cos tx,\frac{\cos nx}{\|\cos nx\|}\right\rangle_{L^2[0,\pi]}\frac{\cos nx}{\|\cos nx\|}\\&=\frac{\sin\pi t}{\pi t}+\sum_{n=1}^{\infty}\frac{(-1)^n 2t\sin\pi t}{\pi(t^2-n^2)}\cos nx \quad \text{in } L^2[0,\pi].\end{aligned}$$

Therefore, choosing

$$S_0(t)=\frac{\sin\pi t}{\pi t},\quad S_n(t)=\frac{(-1)^n2t\sin\pi t}{\pi(t^2-n^2)},\quad\text{and}\quad t_n=n, n\in\mathbb{N}\cup\{0\},$$

we have that any function of the form

$$f(t) = \int_0^\pi F(x)\cos tx dx \qquad \text{with } F \in L^2[0,\pi]$$

can be expanded as

$$f(t) = f(0)\frac{\sin \pi t}{\pi t} + \frac{2}{\pi}\sum_{n=1}^{\infty} f(n)\frac{(-1)^n t \sin \pi t}{t^2 - n^2}.$$

The convergence of the series is absolute and uniform on $\mathbb{R}$ since

$$\|K(\cdot,t)\|^2_{L^2[0,\pi]} = \frac{\pi}{2} + \frac{\sin 2t\pi}{4t}$$

is bounded for all $t \in \mathbb{R}$. The reproducing kernel is given by

$$k(t,s) = \frac{1}{s-t}[t \sin t\pi \cos s\pi - s \cos t\pi \sin s\pi].$$

The cardinal series (3.1) is absolutely convergent and hence unconditionally convergent. Therefore, it can be written, gathering terms, in the equivalent form

$$f(t) = \frac{\sin \pi t}{\pi}\left\{\frac{f(0)}{t} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{f(n)}{t-n} + \frac{f(-n)}{t+n}\right)\right\}.$$

As a consequence, the sampling expansion associated with the finite cosine transform is nothing more than the cardinal series (3.1) for an even function.

3.4. The Paley–Wiener Space Revisited. Consider the product Hilbert space $\boldsymbol{H} = L^2[0,\pi] \times L^2[0,\pi]$ endowed with the norm $\|F\|^2_{\boldsymbol{H}} = \|F_1\|^2_{L^2[0,\pi]} + \|F_2\|^2_{L^2[0,\pi]}$ for every $F = (F_1, F_2) \in \boldsymbol{H}$. The system of functions $\{\frac{1}{\sqrt{\pi}}(\cos nx, \sin nx)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\boldsymbol{H}$. For a fixed $t \in \mathbb{R}$ we have

$$\begin{aligned}(\cos tx, \sin tx) &= \sum_{n=-\infty}^{\infty}\left\langle(\cos tx, \sin tx), \frac{1}{\sqrt{\pi}}(\cos nx, \sin nx)\right\rangle_{\boldsymbol{H}}\frac{1}{\sqrt{\pi}}(\cos nx, \sin nx)\\ &= \sum_{n=-\infty}^{\infty}\frac{\sin \pi(t-n)}{\sqrt{\pi}(t-n)}\frac{1}{\sqrt{\pi}}(\cos nx, \sin nx)\end{aligned}$$

in the $\boldsymbol{H}$ sense. Taking $S_n(t) = \frac{\sin \pi(t-n)}{\sqrt{\pi}(t-n)}$ and $t_n = n \in \mathbb{Z}$, we have that $S_n(t_k) = \sqrt{\pi}\delta_{n,k}$. As a consequence, any function of the form

$$f(t) = \int_0^\pi \{F_1(x)\cos tx + F_2(x)\sin tx\}dx \qquad \text{with } F_1, F_2 \in L^2[0,\pi]$$

can be expanded as the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(t-n)}{\pi(t-n)}.$$

The corresponding $\mathcal{H}$ space is again the Paley–Wiener space PW_π. Indeed, for $f \in PW_\pi$ we have

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x)e^{itx}dx = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\pi}^{0} F(x)e^{itx}dx + \int_{0}^{\pi} F(x)e^{itx}dx \right\} \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\pi}^{0} F(x)(\cos tx + i\sin tx)dx + \int_{0}^{\pi} F(x)(\cos tx + i\sin tx)dx \right\} \\ &= \int_0^{\pi} \left\{ \frac{1}{\sqrt{2\pi}}[F(x)+F(-x)]\cos tx + \frac{i}{\sqrt{2\pi}}[F(x)-F(-x)]\sin tx \right\} dx \\ &= \int_0^{\pi} \{F_1(x)\cos tx + F_2(x)\sin tx\}dx, \end{aligned}$$

where $F_1(x) = \frac{1}{\sqrt{2\pi}}[F(x)+F(-x)]$ and $F_2(x) = \frac{i}{\sqrt{2\pi}}[F(x)-F(-x)]$ belong to $L^2[0,\pi]$.

In particular, taking $F_1 = F_2 = F \in L^2[0,\pi]$ we obtain the sampling expansion for a function f bandlimited to $[0,\pi]$ in the sense of the *Hartley transform*. To be more precise, any function of the form

$$f(t) = \int_0^{\pi} F(x)[\cos tx + \sin tx]dx \quad \text{ with } F \in L^2[0,\pi]$$

can be expanded as a cardinal series (3.1). Recall that the Hartley transform of a function F, defined as

$$f(t) = \int_0^{\infty} F(x)[\cos tx + \sin tx]dx,$$

was introduced by R. V. L. Hartley, an electrical engineer, as a way to overcome what he considered a drawback of the Fourier transform, namely, representing a real-valued function $F(x)$ by a complex-valued one

$$g(t) = \int_{-\infty}^{\infty} F(x)[\cos tx - i\sin tx]dx.$$

3.5. The ν-Bessel–Hankel Space. The Fourier–Bessel set $\{\sqrt{x}J_\nu(x\lambda_n)\}_{n=1}^{\infty}$ is known to be an orthogonal basis for $L^2(0,1)$, where λ_n is the nth positive zero of the Bessel function $J_\nu(t)$, $\nu > -1$. The Bessel function of order ν is given by

$$J_\nu(t) = \frac{t^\nu}{2^\nu \Gamma(\nu+1)} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(1+\nu)\cdots(n+\nu)} \left(\frac{t}{2}\right)^{2n} \right].$$

Using special function formulas (see [1, 11.3.29]), for a fixed $t > 0$, we have

$$\sqrt{xt}J_\nu(xt) = \sum_{n=1}^{\infty} \frac{2\sqrt{t\lambda_n}J_\nu(t)}{J'_\nu(\lambda_n)(t^2-\lambda_n^2)} \sqrt{x}J_\nu(x\lambda_n) \quad \text{ in } L^2(0,1).$$

Therefore, the range of the integral transform

$$f(t) = \int_0^1 F(x)\sqrt{xt}J_\nu(xt)dx, \quad F \in L^2(0,1), \tag{3.5}$$

is an RKHS $\mathcal{H}_\nu$ with reproducing kernel

$$k(s,t) = \frac{\sqrt{st}}{t^2 - s^2}\{tJ_{\nu+1}(t)J_\nu(s) - sJ_{\nu+1}(s)J_\nu(t)\},$$

and the sampling expansion

$$f(t) = \sum_{n=1}^{\infty} f(\lambda_n)\frac{2\sqrt{t\lambda_n}J_\nu(t)}{J'_\nu(\lambda_n)(t^2 - \lambda_n^2)}$$

holds for $f \in \mathcal{H}_\nu$. Note that the integral kernel in (3.5) is the kernel of the Hankel transform.

3.6. The Continuous Laguerre Transform. The sequence $\{e^{-x/2}L_n(x)\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2[0,\infty)$, where $L_n(x) = \sum_{k=0}^{n}(-1)^k\frac{1}{k!}\binom{n}{k}x^k$ is the nth Laguerre polynomial. A continuous extension $L_t(x)$ of the Laguerre polynomials can be found in [28, p. 144]. It is given by

$$L_t(x) = \sum_{n=0}^{\infty} L_n(x)\frac{\sin\pi(t-n)}{\pi(t-n)}.$$

$L_t(x)$ is a $\mathcal{C}^\infty$-function that satisfies the Laguerre differential equation

$$xy'' + (1-x)y' + ty = 0,$$

which is the same differential equation satisfied by $L_n(x)$ when t is replaced by n. For our sampling purposes, the most important feature is that the expansion

$$e^{-x/2}L_t(x) = \sum_{n=0}^{\infty}\frac{\sin\pi(t-n)}{\pi(t-n)}e^{-x/2}L_n(x)$$

holds in $L^2[0,\infty)$. Therefore, any function of the form

$$f(t) = \int_0^\infty F(x)e^{-x/2}L_t(x)dx \quad \text{with } F \in L^2[0,\infty)$$

can be expanded as the sampling series

$$f(t) = \sum_{n=0}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}.$$

3.7. The Multidimensional WSK Theorem. The general theory of section 2 can be easily adapted to higher dimensions. For simplicity we consider the bidimensional case.

The sequence $\{e^{-inx}e^{-imy}/2\pi\}$, where $n, m \in \mathbb{Z}$, is an orthonormal basis for $L^2(\boldsymbol{R})$, where $\boldsymbol{R}$ denotes the square $[-\pi,\pi]\times[-\pi,\pi]$. For a fixed $(t,s) \in \mathbb{R}^2$, we have

$$\frac{1}{2\pi}e^{itx}e^{isy} = \sum_{n,m}\frac{\sin\pi(t-n)}{\pi(t-n)}\frac{\sin\pi(s-m)}{\pi(s-m)}\frac{1}{2\pi}e^{inx}e^{imy} \quad \text{in } L^2(\boldsymbol{R}).$$

The functions

$$S_{nm}(t,s) = \frac{\sin\pi(t-n)}{\pi(t-n)}\frac{\sin\pi(s-m)}{\pi(s-m)}$$

and the sequence $\{t_{nm} = (n, m)\}$, $n, m \in \mathbb{Z}$, satisfy conditions C1 and C2 in section 2. Therefore, any function of the form

$$f(t,s) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(x,y)e^{itx}e^{isy}dxdy \quad \text{with } F \in L^2(\boldsymbol{R})$$

can be recovered by means of the double series

$$f(t,s) = \sum_{n,m} f(n,m)\frac{\sin \pi(t-n)}{\pi(t-n)}\frac{\sin \pi(s-m)}{\pi(s-m)}.$$

The series converges absolutely, and uniformly on $\mathbb{R}^2$.

Remark 6. Naturally, one can always find a rectangle enclosing the bounded support B of the bidimensional Fourier transform of a bidimensional function f. In that case, a function bandlimited to B can be reconstructed through the bidimensional WSK formula. However, this is clearly inefficient from a practical point of view, since we are using more information than strictly needed. In general, the support B of the Fourier transform is an irregularly shaped set. So, obtaining more efficient reconstruction procedures depends largely on the particular geometry of B (see [12, Chap. 14] for a more specific account).

We conclude this section by directing the interested reader to [17], a reference describing various practical applications of the sampling theorems discussed above.

Acknowledgments and Further Reading. The author is indebted to all those who, with their books, papers, and surveys, have contributed to the revitalization of this beautiful and relevant topic in applied mathematics. For a *sampling of references*, of widely varying depth of coverage, it may help to suggest four groups:

- Surveys on sampling theory [4, 7, 11, 13];
- Advanced papers on sampling theory [2, 5, 6, 9, 19];
- Specific books on sampling theory [12, 16, 28];
- General books making reference to sampling theory [3, 21, 22, 24, 27].

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