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A Converse of the Kramer Sampling Theorem

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Abstract. The classical Kramer sampling theorem is a universal method to obtain orthogonal sampling formulas. In the this paper a converse of this theorem is given. Concretely we assume that a pointwise sampling formula holds in the range space of a linear integral transform defined in a suitable $\mathcal{L}^2$ space. Then, under appropriate pointwise conditions on the sampling functions, we obtain a Riesz basis in the $\mathcal{L}^2$ space. Although our setup leads to a Riesz basis in general, it can further be specified so as to single out orthogonality as in Kramer's result.

Key words and phrases: Kramer Sampling Theorem, reproducing kernel Hilbert space, Riesz bases.
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1 Introduction

The classical Kramer sampling theorem provides a method for obtaining orthogonal sampling theorems [2, 6, 7, 12]. The statement of this result is as follows: *Let $K(x,t)$ be a function, defined for all t in a suitable subset D of $\mathbb{R}$ such that, as a function of x, $K(\cdot,t) \in \mathcal{L}^2(I)$ for every number $t \in D$, where I is an interval of the real line. Assume that there exists a sequence of distinct real numbers $\{t_n\}_{n\in\mathbb{Z}} \subset D$, such that $\{K(\cdot,t_n)\}_{n\in\mathbb{Z}}$ is a complete orthogonal sequence of functions of $\mathcal{L}^2(I)$. Then for any f of the form*

$$f(t) = \int_I F(x)K(x,t)\,dx\,,$$

where $F \in \mathcal{L}^2(I)$, we have

$$f(t) = \lim_{N\to\infty} \sum_{|n|\le N} f(t_n)S_n(t)\,, \tag{1}$$

with

$$S_n(t) = \frac{\int_I K(x,t)\overline{K(x,t_n)}\,dx}{\int_I |K(x,t_n)|^2\,dx}\,.$$

The series in (1) converges absolutely and uniformly wherever $\|K(\cdot,t)\|_{\mathcal{L}^2(I)}$ is bounded.

Taking $I = [-\pi,\pi]$, $K(x,t) = e^{itx}$ and $\{t_n = n\}_{n\in\mathbf{Z}}$, we get the well–known *Whittaker–Shannon–Kotel'nikov* sampling formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}$$

for functions in $\mathcal{L}^2(\mathbb{R})$ whose Fourier transform has support in $[-\pi,\pi]$, i.e., *bandlimited* to $[-\pi,\pi]$ in the classical sense.

Now, if we take $I = [0,1]$, $K(x,t) = \sqrt{xt}J_\nu(xt)$ and $\{t_n\}$, the sequence of the positive zeros of the Bessel function J_ν of ν-th order with $\nu > -1$, then

$$f(t) = \sum_n f(t_n)\frac{2\sqrt{t_n t}J_\nu(t)}{J'_\nu(t_n)(t^2-t_n^2)}$$

for every f of the form $f(t) = \int_0^1 F(x)\sqrt{xt}J\nu(xt)dx$, where $F \in \mathcal{L}^2(0,1)$.

We note that one of the richest sources of Kramer kernels is in the subject of self–adjoint boundary value problems; see, for example, [12, 3] and the references cited therein. The biorthogonal version of the Kramer sampling theorem has been stated and proved in [6, p. 84].

Having in mind the Kramer sampling theorem, a procedure has been proposed in [4] to obtain orthogonal sampling formulas in a unified way. Namely, let $\{\phi_n(x)\}_{n=0}^{\infty}$ be an orthonormal basis of an $\mathcal{L}^2(I)$ space, where I is an interval in $\mathbb{R}$. Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of functions $S_n : \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$, defined for all $t \in \Omega$, and let $\{t_n\}_{n=0}^{\infty}$ be a sequence in Ω satisfying the following two conditions:

(a) $S_n(t_k) = a_n\delta_{n,k}$ where $\delta_{n,k}$ denotes the Kronecker delta and $a_n \neq 0$,

(b) $\sum_{n=0}^{\infty} |S_n(t)|^2 < \infty$ for each $t \in \Omega$.

By defining the kernel $K(x,t) \overset{df}{=} \sum_{n=0}^{\infty} S_n(t)\overline{\phi_n}(x)$, $(x,t) \in I\times\Omega$, any function f of the form $f(t) \overset{df}{=} \int_I F(x)K(x,t)dx$, where $F \in \mathcal{L}^2(I)$, can be expanded as the sampling series

$$f(t) = \sum_{n=0}^{\infty} f(t_n)\frac{S_n(t)}{a_n}\,, \tag{2}$$

where the convergence of the series is, at least, pointwise in the set Ω.

In a similar way, one can obtain non-orthogonal sampling formulas by using a Riesz basis in $\mathcal{L}^2(I)$, instead of the orthonormal one [5].

Roughly speaking, the main purpose of this paper is to show that assuming that a sampling expansion like (2) holds for every function in the range space of a linear integral transform whose kernel is K and the sampling functions $\{S_n\}_{n=0}^{\infty}$ satisfy some appropriate conditions, then $\{a_n^{-1}K(\cdot,t_n)\}_{n=0}^{\infty}$ is a Riesz basis in $\mathcal{L}^2(I)$. The case when $\{K(\cdot,t_n)\}_{n=0}^{\infty}$ is an orthogonal basis is derived as a particular case.

Finally, notice that in [8] a reproducing kernel Hilbert space is obtained from the concept of *sampling theorem* associated with a class of continuous functions by using a completely different approach and hypotheses.

2 The result

Let I be an interval of the real line $\mathbb{R}$, and Ω a fixed subset of $\mathbb{R}$. We consider a complex-valued kernel $K(x,t)$ verifying that $K(\cdot,t)$ is in $\mathcal{L}^2(I)$ for each $t\in\Omega$. For $F\in\mathcal{L}^2(I)$ the function $f(t)\stackrel{df}{=}\int_I F(x)K(x,t)dx$ is well-defined as a function $f:\Omega\to\mathbb{C}$. We denote by $\mathcal{H}$ the set of functions obtained in this way and by T the linear integral transform

$$T:\mathcal{L}^2(I)\ni F\mapsto f\in\mathcal{H}. \tag{3}$$

If we define in $\mathcal{H}$ a norm as $\|f\|_{\mathcal{H}}=\inf\{\|F\|_{\mathcal{L}^2(I)}\}$, where the infimum is taken over all $F\in\mathcal{L}^2(I)$ such that $T(F)=f$, we obtain a reproducing kernel Hilbert space (RKHS hereafter) whose reproducing kernel is given by, cf. [9],

$$k(t,s)\stackrel{df}{=}\langle K(\cdot,t),K(\cdot,s)\rangle_{\mathcal{L}^2(I)} \tag{4}$$

(recall that the Moore-Aronszajn procedure [1] leads to the same RKHS via the *positive definite* function k). Under these circumstances it is known that the linear operator T is one-to-one if and only if T is an isometry between $\mathcal{L}^2(I)$ and $\mathcal{H}$, or, equivalently, if and only if the set of functions $\{K(\cdot,t)\}_{t\in\Omega}$ is complete in $\mathcal{L}^2(I)$ [9].

From now on we confine ourselves to the case where, *a priori*, T is one-to-one, although, at the end of the section, a remark will be made for the assumption of T being one-to-one to be dropped so as to get a similar result. It is simply a consequence of the theorem.

We have all the prerequisites done to prove the following result:

Theorem 1 *Let $\mathcal{H}$ be the range of the linear integral transform T (defined as in (3)) considered as a RKHS with the kernel k defined by (4). Let $\{S_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{H}$ such that $\sum_{n=0}^{\infty}|S_n(t)|^2<+\infty$, $t\in\Omega$ and let $\mathcal{H}_{samp}$ be a RKHS corresponding to the kernel $k_{samp}(s,t)\stackrel{df}{=}\sum_{n=0}^{\infty}S_n(s)\overline{S_n(t)}$. Then, we have the following results*

1°) *Suppose that the sequence* $\{S_n\}_{n=0}^{\infty}$ *satisfies the condition that for each sequence* $\{\alpha_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N})$ *such that* $\sum_{n=0}^{\infty} \alpha_n S_n(t) = 0$ *for all* $t \in \Omega$ *implies* $\alpha_n = 0$ *for all* n. *Then,* $\mathcal{H}_{samp} \subset \mathcal{H}$ *and* $\{S_n\}_{n=0}^{\infty}$ *is an orthonormal basis in* $\mathcal{H}_{samp}$.

2°) *Suppose in addition to* 1°) *the existence of sequences* $\{t_n\}_{n=0}^{\infty}$ *in* Ω *and* $\{a_n\}_{n=0}^{\infty}$ *in* $\mathbb{C} \setminus \{0\}$ *such that*

$$\left\{\frac{f(t_n)}{a_n}\right\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}) \text{ and } f(t) = \sum_{n=0}^{\infty} f(t_n)\frac{S_n(t)}{a_n}, \quad \text{for any } f \in \mathcal{H},$$

where the sampling series is pointwise convergent in Ω. *Then*

- $\mathcal{H}_{samp} = \mathcal{H}$.
- *The norms of* $\mathcal{H}$ *and* $\mathcal{H}_{samp}$ *are equivalent, i.e., for some constants* $0 < a \leq b$

$$a\|f\|_{samp} \leq \|f\|_{\mathcal{H}} \leq b\|f\|_{samp}, \quad f \in \mathcal{H} = \mathcal{H}_{samp}. \tag{5}$$

  Consequently, $\{S_n\}_{n=0}^{\infty}$ *is a Riesz basis for* $\mathcal{H}$.

- *The sequences* $\{\overline{a_i^{-1}K(\cdot, t_i)}\}_{i=0}^{\infty}$ *and* $\{\sum_{n=0}^{\infty} \overline{a_n^{-1}} \langle S_j, S_n\rangle_{\mathcal{H}} K(\cdot, t_n)\}_{j=0}^{\infty}$ *as well as* $\{S_i\}_{i=0}^{\infty}$ *and* $\{\overline{a_j^{-1}} \sum_{n=0}^{\infty} k_{t_j}(t_n) a_n^{-1} S_n\}_{j=0}^{\infty}$ *are biorthonormal sequences in* $\mathcal{L}^2(I)$ *and* $\mathcal{H}$ *respectively.*
- *If* $a = b$, *then* $a^2 k(s,t) = k_{samp}(s,t)$ *for all* $s, t \in \Omega$ *and the sequence* $\{K(\cdot, t_n)\}_{n=0}^{\infty}$ *is a complete and orthogonal set in* $\mathcal{L}^2(I)$.

Proof: That the sequence $\{S_n\}_{n=0}^{\infty}$ is an orthonormal basis in $\mathcal{H}_{samp}$ follows from what is in [11], but for reader's convenience we extract the proof from there. We consider $k_{samp,t}(s) = k_{samp}(s,t)$, and we prove that

$$k_{samp,t} = \sum_{n=0}^{\infty} \overline{S_n(t)} S_n \tag{6}$$

in the $\mathcal{H}_{samp}$-norm for a fixed $t \in \Omega$. Indeed, we define $f_N \stackrel{df}{=} k_{samp,t} - \sum_{n=0}^{N} \overline{S_n(t)} S_n$; taking $\xi_1, \xi_2, \cdots, \xi_M$ in $\mathbb{C}$ and $s_1, s_2, \cdots, s_M$ in Ω we have

$$\begin{aligned}\left|\sum_{i=1}^{M} \xi_i f_N(s_i)\right|^2 &= \left|\sum_{n=N+1}^{\infty} \overline{S_n(t)} \sum_{i=1}^{M} \xi_i S_n(s_i)\right|^2 \leq \\ &\leq \Big(\sum_{n=N+1}^{\infty} |S_n(t)|^2\Big)\Big(\sum_{n=0}^{\infty} |\sum_{i=1}^{M} \xi_i S_n(s_i)|^2\Big) = \\ &= \Big(\sum_{n=N+1}^{\infty} |S_n(t)|^2\Big)\Big(\sum_{i,j=1}^{M} \xi_i \overline{\xi_j} k_{samp}(s_i, s_j)\Big),\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and the definition of k_{samp}.

By using the RKHS test (see Appendix) we obtain

$\|f_N\|^2 \leq \sum_{n=N+1}^{\infty} |S_n(t)|^2 \to 0$ as $N \to \infty$. Concerning orthonormality of the sequence $\{S_n\}_{n=0}^{\infty}$ we have

$$\overline{S_m(t)} = \langle k_{\mathrm{samp},t}, S_m\rangle_{samp} = \sum_{n=1}^{\infty} \overline{S_n(t)}\langle S_n, S_m\rangle_{samp}\,,$$

where we have used the reproducing property in $\mathcal{H}_{samp}$ and (6). As a consequence, condition in 1°) implies $\langle S_n, S_m\rangle_{samp} = \delta_{n,m}$.

For the completeness of the sequence $\{S_n\}_{n=0}^{\infty}$, suppose that $\langle S_n, f\rangle_{samp} = 0$ for all $n \in \mathbb{N}$. Hence, $0 = \sum_{n=1}^{\infty} \overline{S_n(t)}\langle S_n, f\rangle_{samp} = \langle k_{\mathrm{samp},t}, f\rangle_{samp}$ for each $t \in \Omega$. By using the reproducing property in $\mathcal{H}_{samp}$ we obtain $f = 0$ in Ω. This proves 1°).

Now we prove that $\mathcal{H}_{\mathrm{samp}} = \mathcal{H}$. By the sampling property, $\{f(t_n)a_n^{-1}\}_{n=0}^{\infty}$ is in $\ell^2(\mathbb{N})$ for each $f \in \mathcal{H}$. Then, the series $\sum_{n=0}^{\infty} f(t_n)a_n^{-1}S_n$ converges in the norm of $\mathcal{H}_{samp}$. By the reproducing kernel property, we have that the series $\sum_{n=0}^{\infty} f(t_n)a_n^{-1}S_n$ is pointwise convergent. Comparing this with what we get from the sampling formula for f we deduce that

$$f = \sum_{n=0}^{\infty} f(t_n)a_n^{-1}S_n\,, \tag{7}$$

where the convergence is in $\mathcal{H}_{samp}$ and, consequently, $f \in \mathcal{H}_{samp}$.

Now we show that the identity mapping$\mathcal{H}_{samp} \hookrightarrow \mathcal{H}$ is continuous by application of the closed graph theorem. Indeed, let $\{f_n\}$ be a sequence such that $f_n \to f$ in $\mathcal{H}_{samp}$ and $f_n \to g$ in $\mathcal{H}$. Using the reproducing property in both $\mathcal{H}$ and $\mathcal{H}_{samp}$, we have for $t \in \Omega$,

$$|f_n(t) - f(t)| \leq \|f_n - f\|_{samp}\sqrt{k_{samp}(t,t)}$$
$$|f_n(t) - g(t)| \leq \|f_n - f\|_{\mathcal{H}}\sqrt{k(t,t)}\,,$$

and therefore, $\lim_{n\to\infty} f_n(t) = f(t) = g(t)$ for each $t \in \Omega$, and $f = g$.

Now, since it is also surjective, we infer that the norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathrm{samp}}$ are equivalent from the open mapping theorem. As a consequence, $\{S_n\}_{n=0}^{\infty}$ is a Riesz basis in $\mathcal{H}$ and the transform T is a linear isomorphism between $\mathcal{L}^2(I)$ and $\mathcal{H}_{samp}$. An easy calculation shows that, for each $t \in \Omega$, $K(x,t) = \sum_{n=0}^{\infty} S_n(t)\varphi_n^*(x)$ in $\mathcal{L}^2(I)$,where$\{\varphi_n^*\}_{n=0}^{\infty}$ is the biorthonormal basis associated with the Riesz basis $\{\varphi_n = T^{-1}(S_n)\}_{n=o}^{\infty}$ in $\mathcal{L}^2(I)$.

Notice that the interpolation property $S_n(t_k) = a_n\delta_{n,k}$ necessarily follows from a direct application of the sampling property to S_n. Thus, $\{a_n^{-1}K(\cdot,t_n)\}_{n=0}^{\infty}$ is a Riesz basis in $\mathcal{L}^2(I)$. Note that $\{\overline{a_n^{-1}K(\cdot,t_n)}\}_{n=0}^{\infty}$ is also a Riesz basis in $\mathcal{L}^2(I)$.

The aforesaid interpolation property immediately gives

$$a_j\delta_{j,i} = S_i(t_j) = \langle S_i, k_{t_j}\rangle_{\mathcal{H}} = \langle S_i, \sum_{n=0}^{\infty} k_{t_j}(t_n)a_n^{-1}S_n\rangle_{\mathcal{H}}$$

or

$$\delta_{j,i} = \langle S_i, \overline{a_j^{-1} \sum_{n=0}^{\infty} k_{t_j}(t_n) a_n^{-1} S_n} \rangle_{\mathcal{H}}, \tag{8}$$

which leads to biorthonormality of the second pair of sequences.

Using (7) and (4) we get from (8)

$$\begin{aligned}
\delta_{j,i} &= \langle S_i, \overline{a_j^{-1} \sum_{n=0}^{\infty} k_{t_j}(t_n) a_n^{-1} S_n} \rangle_{\mathcal{H}} \\
&= a_j^{-1} \sum_{n=0}^{\infty} \overline{k_{t_j}(t_n) \overline{a_n^{-1}}} \langle S_i, S_n \rangle_{\mathcal{H}} = a_j^{-1} \sum_{n=0}^{\infty} \overline{\overline{a_n^{-1}} \langle k_{t_j}, k_{t_n} \rangle_{\mathcal{H}}} \langle S_i, S_n \rangle_{\mathcal{H}} \\
&= a_j^{-1} \sum_{n=0}^{\infty} \overline{a_n^{-1}} \langle K(\cdot, t_n), K(\cdot, t_j) \rangle_{\mathcal{L}^2(I)} \langle S_i, S_n \rangle_{\mathcal{H}} \\
&= \langle \sum_{n=0}^{\infty} \overline{a_n^{-1}} \langle S_i, S_n \rangle_{\mathcal{H}} K(\cdot, t_n), \overline{a_j^{-1}} K(\cdot, t_j) \rangle_{\mathcal{L}^2(I)}.
\end{aligned}$$

This provides us the biorthonormality of the first pair of sequences.

The equivalence of the norms (5) can be written as $a^2 k \ll k_{samp} \ll b^2 k$ (see the corollary in the Appendix). When $a = b$, then $a^2 k = k_{samp}$ and the transform T is an isometry (up to the positive factor a) between $\mathcal{L}^2(I)$ and $\mathcal{H}_{\mathbf{samp}}$. In this case $\{\varphi_n = T^{-1}(S_n)\}_{n=0}^{\infty}$ is an orthogonal basis in $\mathcal{L}^2(I)$ and, consequently, so is the sequence $\{K(\cdot, t_n) = a_n \overline{\varphi_n}\}_{n=0}^{\infty}$. This completes the proof of the theorem. ∎

To conclude, some remarks concerning the above result are in order.

Remark 1. As to the case when, *a priori*, T is not known to be one-to-one, let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{L}^2(I)$ with $P(\phi_n) \neq 0$ for all n, where P denotes the orthogonal projection onto the closed subspace $(\operatorname{Ker} T)^{\perp}$. Consider $S_n = T(\phi_n) \in \mathcal{H}$, and suppose that these functions satisfy hypotheses in Theorem. In this case, $\{S_n\}_{n=0}^{\infty}$ is a Riesz basis in $\mathcal{H}$. Consequently, since $S_n = T[P(\phi_n)]$ and $T|_{P(\operatorname{Ker} T)} = 0$, we obtain that $\{P(\phi_n)\}_{n=1}^{\infty}$ is a Riesz basis in $P(\mathcal{L}^2(I)) = (\operatorname{Ker} T)^{\perp}$. The result comes out taking into account the orthogonal sum $\mathcal{L}^2(I) = (\operatorname{Ker} T)^{\perp} \oplus (\operatorname{Ker} T)$.

Remark 2. Theorem 1 can be stated in a more general setting. To this end, consider an abstract set Ω and a mapping $K : \Omega \longrightarrow \mathbf{H}$, where $\mathbf{H}$ denotes some separable Hilbert space. Define $f(t) := \langle h, K(t) \rangle_{\mathbf{H}}$ for $h \in \mathbf{H}$. Thus, we obtain a RKHS $\mathcal{H}$ of complex-valued functions on Ω, whose reproducing kernel is given by $k(t, s) = \langle K(s), K(t) \rangle_{\mathbf{H}}$. This allows us to include multidimensional sampling by taking Ω in $\mathbb{R}^n$, and/or dealing with $\mathcal{L}^2(\mu)$ spaces for an arbitrary measure μ, in particular for the case where μ is supported on a finite or countable set.

Remark 3. In a similar way as in the proof in 1°), we can consider two sequences $\{S_n\}_{n=0}^{\infty}$ and $\{S_n^*\}_{n=0}^{\infty}$ in $\mathcal{H}$ such that $\sum_{n=0}^{\infty} |S_n(t)|^2 < +\infty$ and

$\sum_{n=0}^{\infty} |S_n^*(t)|^2 < +\infty$, $t \in \Omega$. Defining $k_{samp}(t,s) = \sum_{n=0}^{\infty} S_n(t)\overline{S_n^*(s)}$, we can prove, following [11], that $\{S_n\}_{n=0}^{\infty}$ and $\{S_n^*\}_{n=0}^{\infty}$ are biorthogonal bases in $\mathcal{H}_{samp}$.

Remark 4. As a final remark, it is worth pointing out that the results in the theorem could be used to prove the existence of a Riesz (orthogonal) basis in $\mathcal{L}^2(I)$, starting from a sampling expansion in $\mathcal{H}$ with the conditions stated in it.

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Appendix

We state here the RKHS test used in the proof of the theorem

RKHS test [10] *Let $\mathcal{H}$ be a RKHS with reproducing kernel k on a set Ω. A function f is in $\mathcal{H}$ if and only if there is a constant $C > 0$ such that*

$$\left|\sum_{i=1}^{M} f(s_i)\xi_i\right|^2 \leq C^2 \sum_{i,j=1}^{M} k(s_i, s_j)\xi_i\overline{\xi_j}, \tag{9}$$

where $\xi_1, \xi_2, \cdots, \xi_M$ in $\mathbb{C}$ and $s_1, s_2, \cdots, s_M$ in Ω. Moreover, in this case $\|f\| = \inf\{C\}$, where the infimum is taken over all the constants C satisfying (9).

This leads to the following:

Corollary *Let K and L be two positive definite kernels on X and $\|\cdot\|_K$ and $\|\cdot\|_L$ be the norms in their RKHS's. Then*

$$c^2K \ll L \text{ if and only if } c\|\cdot\|_L \leq \|\cdot\|_K$$

with some $c > 0$.[1]

Proof: Suppose $c^2K \ll L$. Then

$$\left|\sum_{i=1}^{M} f(s_i)\xi_i\right|^2 \leq \|f\|_K^2 \sum_{i,j=1}^{M} K(s_i, s_j)\xi_i\overline{\xi_j} \leq \|f\|_K^2 \sum_{i,j=1}^{M} c^{-2}L(s_i, s_j)\xi_i\overline{\xi_j},$$

and the RKHS test gives us $c\|f\|_L \leq \|f\|_K$.

Suppose the converse. Then $B_K(1) \subset B_L(1/c)$ where B stands for a ball with its center at 0 in a corresponding space.

Thus

$$\begin{aligned}\sum_{m,n=0}^{\infty} \lambda_m\overline{\lambda_n}K(x_m, x_n) &= \|\sum_{m=0}^{\infty} \lambda_m K_{x_m}\|_K^2 = \sup\{|\langle f, \sum_{m=0}^{\infty} \lambda_m K_{x_m}\rangle_K|^2;\ f \in B_K(1)\} \\ &= \sup\{|\sum_{m=0}^{\infty} \lambda_m f(x_m)|^2;\ f \in B_K(1)\} \\ &= \sup\{|\sum_{m=0}^{\infty} \lambda_m \langle f, L_{x_m}\rangle_L|^2;\ f \in B_K(1)\}\end{aligned}$$

[1] *If K and L are two positive definite kernels, then $K \ll L$ means that the kernel $L - K$ is positive definite too*

$$\leq \sup\{|\sum_{m=0}^{\infty} \lambda_m \langle f, L_{x_m}\rangle_L|^2;\ f \in B_L(1/c)\}$$
$$= c^{-2}\|\sum_{m=0}^{\infty} \lambda_m L_{x_m}\|^2 = c^{-2} \sum_{m,n=0}^{\infty} \lambda_m \overline{\lambda_n} L(x_m, x_n),$$

and $c^2 K \ll L$. ■