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An Estimation of the Truncation Error for the Two-Channel Sampling Formulas

A.G. García, M.A. Hernández-Medina, and A. Portal

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ABSTRACT. The aim of the article is to obtain an estimation for the truncation error in the two-channel sampling formulas. Since these formulas are expansions with respect to suitable Riesz bases in Paley-Wiener spaces, the truncation error will be estimated by using the hypercircle inequality in the Riesz bases setting. In so doing, the norm of an involved operator is calculated, and the remainder of the series of the absolute square sampling functions is estimated.

1. Introduction

The two-channel sampling theory in Paley-Wiener spaces has been well established in mathematical literature. See, for instance, [5, Chapter 12] where the so-called Riesz basis method is developed, or the former references [4, 8]. Roughly speaking, a bandlimited signal is filtered by using appropriate Fourier multipliers and then, the two filtered signals are sampled at suitable sampling rates. Finally, the original signal is recovered from the samples thus obtained, by means of a sampling series. A challenging problem is to obtain a good approximation of the initial signal by using only a finite number of samples. This problem is intimately related to the problem of obtaining an estimation for the truncation error in the corresponding sampling series. As far as we know, there is not a general framework to obtain an estimation of the truncation error in the two-channel sampling theory. In this article we propose a general framework by using the hypercircle inequality in the Riesz bases setting.

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Recall that the hypercircle inequality [1, 3, 9] estimates the error when we are evaluating a bounded linear functional on a hypercircle C_r of a Hilbert space $\mathbb{H}$, i.e., the intersection of a hyperplane P of finite co-dimension and the closed ball of radius r, B_r. The approximate value is just the evaluation of the functional in the nearest point to the origin in the hyperplane P. To be precise,
Let P be a hyperplane of co-dimension N in a Hilbert space $\mathbb{H}$, and let w_N be the element of P nearest to the origin. Then, for any x in the hypercircle $C_r = P \cap B_r$ and any bounded linear functional L in $\mathbb{H}$ we have

$$|L(x) - L(w_N)|^2 \leq \big(r^2 - \|w_N\|^2\big) \sum_{k=N+1}^{\infty} |L(x_k)|^2 \,,$$

where $\{x_k\}_{k=1}^{\infty}$ denotes an orthonormal basis in $\mathbb{H}$ such that P is given by the equations: $\langle x, x_i \rangle = a_i$, $i = 1, \dots, N$ and $w_N = \sum_{i=1}^{N} a_i\, x_i$.

Besides, the hypercircle inequality has been proven to be very useful in numerical analysis (see, for instance, [3]). Also, the hypercircle inequality has been used to estimate the truncation error in the Whittaker-Shannon-Kotel'nikov sampling formula [6, 9]. This well-known sampling result reads as follows:
Any function f in the Paley-Wiener space $PW_{\pi\sigma} := \{f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R}) \,:\, \operatorname{supp} \widehat{f} \subseteq [-\pi\sigma, \pi\sigma]\}$, where $\widehat{f}$ stands for the Fourier transform of f, can be expanded as the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f\Big(\frac{n}{\sigma}\Big) \frac{\sin \pi(\sigma t - n)}{\pi(\sigma t - n)} = \sum_{n=-\infty}^{\infty} f(n/\sigma)\, \operatorname{sinc}(\sigma t - n)\,, \quad t \in \mathbb{R}\,.$$

Assuming that $\|f\| \leq r$, the hypercircle inequality shows that the truncation error in the WSK sampling formula satisfies the inequality

$$\big|f(t) - f_N(t)\big|^2 \leq \sigma r^2 - \sum_{n=-N}^{N} |f(n/\sigma)|^2 \,,$$

where $f_N(t) := \sum_{n=-N}^{N} f(n/\sigma)\, \operatorname{sinc}(\sigma t - n)$.

Extending the hypercircle inequality in the Riesz bases setting allow us to estimate the truncation error for non-orthogonal sampling expansions [2]. Such an estimation uses only the samples and the norm of the function together with other quantities, intrinsically related with the sampling formula. As pointed out in [3], the use of a non-optimal approximation including known data (the samples here) can be sometimes preferred to the best approximation, which may have rather cumbersome coefficients. The two-channel sampling formulas are expansions with respect to suitable Riesz bases in Paley-Wiener spaces [5]. Consequently, the hypercircle inequality in the Riesz bases setting is an appropriate tool to estimate their truncation error. Briefly, this result can be introduced as follows: Let $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ be a pair of dual Riesz bases in a separable Hilbert space $\mathbb{H}$. It is known that there exist an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in a suitable Hilbert space $\widehat{\mathbb{H}}$ (maybe $\mathbb{H}$ itself) and an invertible bounded operator $\mathcal{T} : \widehat{\mathbb{H}} \longrightarrow \mathbb{H}$ such that $\mathcal{T}(e_n) = x_n$, for each $n \in \mathbb{N}$. Moreover, $(\mathcal{T}^*)^{-1}(e_n) = x_n^*$ for each $n \in \mathbb{N}$, where $\mathcal{T}^*$ denotes the adjoint operator of $\mathcal{T}$. Under these circumstances the hypercircle inequality becomes (see [2]):
Given the vector $w_M = \sum_{k=1}^{M} \alpha_k x_k$ and the subspace $K = \{y \in \mathbb{H} : \langle y, x_k^* \rangle_{\mathbb{H}} = 0, \quad 1 \leq k \leq M\}$, consider the hyperplane $P := w_M + K$ and the hypercircle $C_r = P \cap B_r$. Then,

for any $x \in C_r$ *and any bounded linear functional* L *in* $\mathbb{H}$ *we have*

$$|L(x) - L(w_M)|^2 \leq \left(\left\| \mathcal{T}^{-1} \right\|^2 r^2 - \sum_{k=1}^{M} |\alpha_k|^2 \right) \sum_{k=M+1}^{\infty} |L(x_k)|^2 . \tag{1.1}$$

The article is organized as follows: First, we introduce the two-channel sampling theory in the Paley-Wiener space PW_π, stressing that the obtained sampling formulas are expansions with respect to suitable Riesz bases in PW_π. Next, we estimate the truncation error by using the hypercircle inequality in the Riesz bases setting. In so doing, we calculate the norm of an involved operator with the sampling functions, and we estimate the remainder of the series of the absolute square sampling functions. Finally, we particularize the obtained estimation for the derivative sampling formula.

2. Preliminaries on Two-Channel Sampling Formulas

In this section we introduce the Riesz basis method for obtaining Riesz bases in $L^2[-\pi, \pi]$, which gives the two-channel sampling theory in PW_π. We follow, with a slightly different notation, the Higgins' approach in [5, Chapter 12]. Throughout the article, $\mathcal{H}_\pi$ will denote the Hilbert space $L^2[0, \pi] \oplus L^2[0, \pi]$ with its usual norm, i.e., for $(\phi, \varphi) \in \mathcal{H}_\pi$,

$$\|(\phi, \varphi)\|^2_{\mathcal{H}_\pi} = \|\phi\|^2_{L^2[0,\pi]} + \|\varphi\|^2_{L^2[0,\pi]} .$$

Since $\{e_n(w) = e^{-2inw}/\sqrt{\pi}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0, \pi]$, we obtain an orthonormal basis $\{\alpha_n\}_{n\in\mathbb{Z}} \cup \{\beta_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}_\pi$ by setting

$$\alpha_n(w) := (e_n(w), \mathbf{0}) ; \quad \beta_n(w) := (\mathbf{0}, e_n(w)) , \quad n \in \mathbb{Z} .$$

Let M_1, M_2 be two bounded measurable functions (multipliers) on $[-\pi, \pi]$. For $\phi \in L^2[0, \pi]$, we denote its π-periodic extension by ϕ^p. On the other hand, if $\phi \in L^2[-\pi, \pi]$, it can be expressed as $\phi = \phi^{①} + \phi^{②}$, where $\phi^{①} = \phi\chi_{[-\pi,0]}$ and $\phi^{②} = \phi\chi_{[0,\pi]}$.

Next, we define the operator

$$\begin{array}{rccl} \mathcal{S}: & \mathcal{H}_\pi & \longrightarrow & L^2[-\pi, \pi] \\ & (\phi_1, \phi_2) & \longmapsto & M_1(w)\phi_1^p(w) + M_2(w)\phi_2^p(w) , \end{array}$$

which is bounded because $\|\mathcal{S}(\phi_1, \phi_2)\|^2 \leq 4 \max\{\|M_1\|^2_\infty , \|M_2\|^2_\infty\}\|(\phi_1, \phi_2)\|^2$. It maps the orthonormal basis $\{\alpha_n\}_{n\in\mathbb{Z}} \cup \{\beta_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}_\pi$ onto the sequence $\{x_n\}_{n\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$ in $L^2[-\pi, \pi]$ given by

$$x_n(w) := \mathcal{S}(\alpha_n)(w) = \frac{1}{\sqrt{\pi}} M_1(w)e^{-i2nw} ;$$
$$y_n(w) := \mathcal{S}(\beta_n)(w) = \frac{1}{\sqrt{\pi}} M_2(w)e^{-i2nw} , \; n \in \mathbb{Z} .$$

If the operator $\mathcal{S}$ is invertible, then the sequence $\{x_n\}_{n\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$ will be a Riesz basis in $L^2[-\pi, \pi]$. This will be the case whenever the matrix

$$\mathcal{M}(w) := \begin{pmatrix} M_1^{①}(w - \pi) & M_2^{①}(w - \pi) \\ M_1^{②}(w) & M_2^{②}(w) , \end{pmatrix} , \quad w \in [0, \pi] , \tag{2.1}$$

satisfies the condition $0 < A \le |\det(\mathcal{M}(w))|$ a.e. in $[0,\pi]$, for some positive constant A. Indeed, if we denote $\Phi = (\phi_1, \phi_2)^T \in \mathcal{H}_\pi$, then

$$\mathcal{M}(w)\Phi(w) = \begin{pmatrix} M_1^{①}(w-\pi)\phi_1^p(w) + M_2^{①}(w-\pi)\phi_2^p(w) \\ M_1^{②}(w)\phi_1^p(w) + M_2^{②}(w)\phi_2^p(w) \end{pmatrix}, \; w \in [0,\pi].$$

On the other hand, for $w \in [-\pi,\pi]$, we get

$$\begin{aligned} \mathcal{S}(\phi_1,\phi_2)(w) &= M_1(w)\phi_1^p(w) + M_2(w)\phi_2^p(w) \\ &= \big[M_1^{①}(w) + M_1^{②}(w)\big]\phi_1^p(w) + \big[M_2^{①}(w) + M_2^{②}(w)\big]\phi_2^p(w) \\ &= \big[M_1^{①}(w)\phi_1^p(w-\pi) + M_2^{①}(w)\phi_2^p(w-\pi)\big] + \big[M_1^{②}(w)\phi_1^p(w) + M_2^{②}(w)\phi_2^p(w)\big], \end{aligned}$$

from which we deduce the relationship between $\mathcal{M}(w)\Phi(w)$ and $\mathcal{S}(\phi_1,\phi_2)(w)$. Namely: for $w \in [-\pi, 0]$, $\mathcal{S}(\phi_1,\phi_2)(w)$ is given by the first row of $\mathcal{M}(\widetilde{w})\Phi(\widetilde{w})$, $\widetilde{w} = w+\pi \in [0,\pi]$, whereas $\mathcal{S}(\phi_1,\phi_2)(w)$, for $w \in [0,\pi]$, is given by the second row of $\mathcal{M}(w)\Phi(w)$.

Taking M_1^* and M_2^* such that

$$\mathcal{M}^{-1}(w) := \begin{pmatrix} M_1^{*①}(w-\pi) & M_1^{*②}(w) \\ M_2^{*①}(w-\pi) & M_2^{*②}(w) \end{pmatrix}, \quad w \in [0,\pi]$$

we obtain an explicit formula for evaluating $\mathcal{S}^{-1}(f)$ for $f \in L^2[-\pi,\pi]$. Namely,

$$\begin{aligned} \big[\mathcal{S}^{-1}(f)\big](w) &= \mathcal{M}^{-1}(w) \begin{pmatrix} f^{①}(w-\pi) \\ f^{②}(w) \end{pmatrix} \\ &= \begin{pmatrix} M_1^{*①}(w-\pi) f^{①}(w-\pi) + M_1^{*②}(w) f^{②}(w) \\ M_2^{*①}(w-\pi) f^{①}(w-\pi) + M_2^{*②}(w) f^{②}(w) \end{pmatrix}, \quad w \in [0,\pi]. \end{aligned}$$

Observe that, as a consequence, we obtain that the sequence $\{x_n^*\}_{n\in\mathbb{Z}} \cup \{y_n^*\}_{n\in\mathbb{Z}}$ where

$$x_n^*(w) := \frac{1}{\sqrt{\pi}}\,\overline{M_1^*}(w)e^{-i2nw}; \quad y_n^*(w) := \frac{1}{\sqrt{\pi}}\,\overline{M_2^*}(w)e^{-i2nw}, \; n \in \mathbb{Z},$$

is the dual Riesz basis of $\{x_n\}_{n\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$.

By using Fourier duality we obtain the two-channel sampling formulas in PW_π. Indeed, given $f \in PW_\pi$ we expand its Fourier transform $\widehat{f} := \mathcal{F}(f)$ (throughout the article the Fourier transform is defined as $\widehat{f}(w) := (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(t)\, e^{-itw}\, dt$) with respect to the Riesz basis $\{x_n\}_{n\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$ in $L^2[-\pi,\pi]$. Thus, we have

$$\widehat{f} = \sum_{n\in\mathbb{Z}} \big[\langle \widehat{f}, x_n^*\rangle x_n + \langle \widehat{f}, y_n^*\rangle y_n\big].$$

Taking the inverse Fourier transform $\mathcal{F}^{-1}$ we get in the reproducing kernel Hilbert space PW_π

$$f(t) = \sum_{n\in\mathbb{Z}} \big[\langle \widehat{f}, x_n^*\rangle \big(\mathcal{F}^{-1}x_n\big)(t) + \langle \widehat{f}, y_n^*\rangle \big(\mathcal{F}^{-1}y_n\big)(t)\big], \quad t \in \mathbb{R},$$

where

$$\big[\mathcal{F}^{-1}x_n\big](t) = \frac{1}{\sqrt{\pi}}\big(\mathcal{F}^{-1}M_1\big)(t-2n) \;\text{ and }\; \big[\mathcal{F}^{-1}y_n\big](t) = \frac{1}{\sqrt{\pi}}\big(\mathcal{F}^{-1}M_2\big)(t-2n), \; t \in \mathbb{R}.$$

For the coefficients we have

$$\langle \widehat{f}, x_n^* \rangle = \sqrt{2} f_1(2n) \text{ and } \langle \widehat{f}, y_n^* \rangle = \sqrt{2} f_2(2n) \; n \in \mathbb{Z},$$

where f_j, $j = 1, 2$, are the filtered versions of f defined by their Fourier transforms $\widehat{f_j} = M_j^* \widehat{f}$, $j = 1, 2$. Summarizing, we can state the two-channel sampling result:

Theorem 1. *Any function $f \in PW_\pi$ can be recovered from the samples $\{f_1(2n)\}_{n\in\mathbb{Z}}$ and $\{f_2(2n)\}_{n\in\mathbb{Z}}$ of f_1 and f_2, respectively, by means of the sampling series*

$$f(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} \big[f_1(2n)\, S(t-2n) + f_2(2n)\, T(t-2n) \big], \quad t \in \mathbb{R}, \tag{2.2}$$

where $S(t) = (\mathcal{F}^{-1}M_1)(t)/\sqrt{\pi}$ and $T(t) = (\mathcal{F}^{-1}M_2)(t)/\sqrt{\pi}$. The convergence of the series is absolute and uniform in $\mathbb{R}$.

3. The Truncation Error for the Two-Channel Sampling Formulas

In this section we are concerned with the estimation of the truncation error for a two-channel sampling formula as in (2.2). Our goal is to get an estimation for the truncation error $T_N(t) := |f(t) - f_N(t)|$ where f_N denotes the sum in (2.2) from $-N$ to N. Since the Paley-Wiener space PW_π is a reproducing kernel Hilbert space, the point-evaluation functional $\mathcal{E}_t(f) := f(t)$, $f \in PW_\pi$, is bounded. As a consequence,

$$T_N(t) := |f(t) - f_N(t)| = |\mathcal{E}_t(f) - \mathcal{E}_t(f_N)|, \quad t \in \mathbb{R}, \tag{3.1}$$

can be considered as the left term in inequality (1.1). In addition, the sampling expansion (2.2) was obtained by expanding $f \in PW_\pi$ with respect to the Riesz basis $\{S(\cdot - 2n)\}_{n\in\mathbb{Z}} \cup \{T(\cdot - 2n)\}_{n\in\mathbb{Z}}$. This Riesz basis is the image, via $\mathcal{F}^{-1}$, of the Riesz basis $\{x_n\}_{n\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$ in $L^2[-\pi, \pi]$. As we have seen in Section 2, $\{x_n\}_{n\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$ is the image, via the bounded invertible operator $\mathcal{S}$, of the orthonormal basis $\{\alpha_n\}_{n\in\mathbb{Z}} \cup \{\beta_n\}_{n\in\mathbb{Z}}$ in the Hilbert space $\mathbb{H}$. Hence, the invertible bounded operator $\mathcal{T} := \mathcal{F}^{-1}\,\mathcal{S}$ satisfies $\mathcal{T}(\alpha_n) = S(\cdot - 2n)$ and $\mathcal{T}(\beta_n) = T(\cdot - 2n)$ for each $n \in \mathbb{Z}$. Thus, we have obtained:

Theorem 2. *Let $f \in PW_\pi$ with $\|f\| \le r$. Then, the truncation error for the sampling formula* (2.2) *satisfies the inequality*

$$\begin{aligned} &|f(t) - f_N(t)|^2 \\ &\le \Big[\|\mathcal{T}^{-1}\|^2 r^2 - 2 \sum_{n=-N}^{N} \Big(|f_1(2n)|^2 + |f_2(2n)|^2 \Big) \Big] \sum_{|n|>N} \big[|S(t-2n)|^2 + |T(t-2n)|^2 \big], \end{aligned}$$

where $t \in \mathbb{R}$ and $f_N(t) = \sqrt{2} \sum_{n=-N}^{N} \big[f_1(2n)\, S_n(t) + f_2(2n)\, T_n(t) \big]$.

So, we must estimate the series $\sum_{|n|>N} \big[|S(t-2n)|^2 + |T(t-2n)|^2 \big]$ and calculate the norm $\|\mathcal{T}^{-1}\|$.

3.1 The Computation of $\|\mathcal{T}^{-1}\|$

In order to calculate $\|\mathcal{T}^{-1}\|$, t he following equalities hold:

(i) $\|\mathcal{T}\| = \|\mathcal{S}\| = \|\mathcal{T}^*\| = \|\mathcal{S}^*\|$ where $\mathcal{T}^*$ and $\mathcal{S}^*$ denote the adjoint operators of $\mathcal{T}$ and $\mathcal{S}$, respectively.

(ii) $\|\mathcal{T}^{-1}\| = \|\mathcal{S}^{-1}\|$.

The equality $\|\mathcal{T}\| = \|\mathcal{S}\|$ is a straightforward consequence of the fact that the inverse Fourier transform $\mathcal{F}^{-1}$ is a unitary operator. The other equalities follow from the general theory of adjoint operators. For (ii), the following chain of equalities holds

$$\|\mathcal{T}^{-1}\| = \|(\mathcal{T}^{-1})^*\| = \|(\mathcal{S}^{-1}\,\mathcal{F})^*\| = \|\mathcal{F}\,(\mathcal{S}^{-1})^*\| = \|(\mathcal{S}^{-1})^*\| = \|\mathcal{S}^{-1}\|\,.$$

From now on, we assume that the multipliers M_1 and M_2 are real, or purely imaginary valued. In this case, a straightforward calculation shows that $\|\mathcal{S}^{-1}f\|^2 = \|\mathcal{S}^{-1}(\Re f)\|^2 + \|i\mathcal{S}^{-1}(\Im f)\|^2$ for any $f \in L^2[-\pi,\pi]$. Thus, we have

$$\begin{aligned}\frac{\|\mathcal{S}^{-1}f\|^2}{\|f\|^2} &= \frac{\|\mathcal{S}^{-1}(\Re f)\|^2 + \|i\mathcal{S}^{-1}(\Im f)\|^2}{\|f\|^2}\\ &= \frac{\|\Re f\|^2}{\|f\|^2}\frac{\|\mathcal{S}^{-1}(\Re f)\|^2}{\|\Re f\|^2} + \frac{\|\Im f\|^2}{\|f\|^2}\frac{\|\mathcal{S}^{-1}(\Im f)\|^2}{\|\Im f\|^2}\,.\end{aligned}$$

Taking into account that $\|f\|^2 = \|\Re f\|^2 + \|\Im f\|^2$, we can consider a real valued function f without loss of generality.

If $f \in L^2[-\pi,\pi]$, then $\|f\|^2 = \int_0^\pi \left[|f^{①}(w-\pi)|^2 + |f^{②}(w)|^2\right] dw$ and

$$\begin{aligned}&\|\mathcal{S}^{-1}f\|^2\\ &= \left\|M_1^{*①}(w-\pi)f^{①}(w-\pi) + M_1^{*②}(w)f^{②}(w)\right\|^2\\ &\quad + \left\|M_2^{*①}(w-\pi)f^{①}(w-\pi) + M_2^{*②}(w)f^{②}(w)\right\|^2\\ &= \left\|M_1^* f\right\|^2 + \left\|M_2^* f\right\|^2\\ &\quad + 2\int_0^\pi \left[M_1^{*①}(w-\pi)\overline{M_1^{*②}(w)} + M_2^{*①}(w-\pi)\overline{M_2^{*②}(w)}\right] f^{①}(w-\pi)f^{②}(w)\,dw.\end{aligned}$$

Notice that under the assumption on the multipliers M_1 and M_2, the function $M_1^{*①}(w-\pi)\overline{M_1^{*②}(w)} + M_2^{*①}(w-\pi)\overline{M_2^{*②}(w)}$ is real valued on $[0,\pi]$. Now, denoting by

$$\begin{aligned}F(w) &:= M_1^{*①}(w-\pi)\overline{M_1^{*②}(w)} + M_2^{*①}(w-\pi)\overline{M_2^{*②}(w)}\,,\\ G_1(w-\pi) &:= \left|M_1^{*①}(w-\pi)\right|^2 + \left|M_2^{*①}(w-\pi)\right|^2,\\ G_2(w) &:= \left|M_1^{*②}(w)\right|^2 + \left|M_2^{*②}(w)\right|^2, \quad w \in [0,\pi]\,,\end{aligned}$$

we obtain that

$$\begin{aligned}\|\mathcal{S}^{-1}f\|^2 &= \int_0^\pi F(w)\left|f^{①}(w-\pi) + f^{②}(w)\right|^2 dw\\ &\quad + \int_0^\pi [G_1(w-\pi) - F(w)]\left|f^{①}(w-\pi)\right|^2 dw\\ &\quad + \int_0^\pi [G_2(w) - F(w)]\left|f^{②}(w)\right|^2 dw\,.\end{aligned}$$

Now, defining the set $A(f) := \{w \in [0,\pi] \,:\, |f^{①}(w-\pi)|^2 + |f^{②}(w)|^2 \neq 0\}$, we get $\|f\|^2 = \int_{A(f)} \big[|f^{①}(w-\pi)|^2 + |f^{②}(w)|^2\big]\, dw$ and

$$\big\|\mathcal{S}^{-1} f\big\|^2 = \int_{A(f)} F(w)\big|f^{①}(w-\pi) + f^{②}(w)\big|^2\, dw + \int_{A(f)} [G_1(w-\pi) - F(w)]\big|f^{①}(w-\pi)\big|^2\, dw + \int_{A(f)} [G_2(w) - F(w)]\big|f^{②}(w)\big|^2\, dw\,.$$

Hence,

$$\frac{\big\|\mathcal{S}^{-1} f\big\|^2}{\|f\|^2} = \int_{A(f)} \frac{\big|f^{①}(w-\pi)\big|^2 + \big|f^{②}(w)\big|^2}{\|f\|^2}\, H(w)\, dw\,,$$

where the function H is given in $A(f)$ by

$$H(w) := G_1(w-\pi) + \frac{2 f^{①}(w-\pi) f^{②}(w) F(w) + \big[G_2(w) - G_1(w-\pi)\big]\big|f^{②}(w)\big|^2}{\big|f^{①}(w-\pi)\big|^2 + \big|f^{②}(w)\big|^2}\,.$$

Consider the sets $C(f) := \{w \in [0,\pi] \,:\, |f^{②}(w)| = 0\}$, and $B(f) := A(f) \setminus C(f)$.

For $w \in C(f)$, we have that $H(w) = G_1(w-\pi) =: \alpha(w)$. For $w \in B(f)$ we consider the function

$$J(x) := K + \frac{2\lambda x + L}{x^2+1}\,, \quad x \in \mathbb{R}\,,$$

whose maximum is achieved at $x = \frac{-L}{2\lambda} + \sqrt{\frac{L^2}{4\lambda^2} + 1}$ when $\lambda \neq 0$. This value is precisely $K + \frac{L}{2} + \sqrt{\frac{L^2}{4} + \lambda^2}$. On the other hand, when $\lambda = 0$, its maximum is reached at $x = 0$ and takes the value $K + L$.

Writing $H(w)$ as

$$H(w) = G_1(w-\pi) + \frac{2\frac{f^{①}(w-\pi)}{f^{②}(w)} F(w) + G_2(w) - G_1(w-\pi)}{\left(\frac{f^{①}(w-\pi)}{f^{②}(w)}\right)^2 + 1}\,,$$

we obtain that, whenever $F(w) \neq 0$

$$H(w) \leq \frac{G_1(w-\pi) + G_2(w)}{2} + \sqrt{\frac{\big[G_2(w) - G_1(w-\pi)\big]^2}{4} + F(w)^2} =: \beta(w)\,.$$

Whenever $F(w) = 0$, we have that $H(w) \leq G_2(w) \leq \beta(w)$. In any case, for $w \in B(f)$ we have $H(w) \leq \beta(w)$.

Finally, for any $w \in A(f)$, we have

$$H(w) \leq \max\big\{\alpha(w), \beta(w)\big\} \leq \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} =: m\,,$$

where $\|\cdot\|_\infty$ denotes the essential supremum in the interval $[0,\pi]$. Thus,

$$\frac{\big\|\mathcal{S}^{-1} f\big\|^2}{\|f\|^2} \leq \int_{A(f)} \frac{\big|f^{①}(w-\pi)\big|^2 + \big|f^{②}(w)\big|^2}{\|f\|^2}\, m\, dw\,,$$

from which we deduce that

$$\left\|\mathcal{S}^{-1}\right\|^2 \leq m := \max\{\|\alpha\|_\infty\,, \|\beta\|_\infty\}\,.$$

Finally, this bound is saturated. Suppose that $m = \|\beta\|_\infty$. For each $n \in \mathbb{N}$, define the set

$$D_n := \left\{ w \in [0,\pi] \,:\, \beta(w) > m - \frac{1}{n} \right\}$$

and consider the sequence of functions $\{\phi_n\}_{n\in\mathbb{N}}$ given by

$$\phi_n(w) := \gamma(w)\chi_{D_n}(w+\pi) + \chi_{D_n}(w)\,,$$

where χ_{D_n} denotes the characteristic function of the set D_n and γ the function defined in $[0,\pi]$ by

$$\gamma(w) = \begin{cases} \frac{G_1(w-\pi)-G_2(w)}{2F(w)} + \sqrt{\frac{[G_2(w)-G_1(w-\pi)]^2}{4F^2(w)} + 1} & \text{if} \quad F(w) \neq 0 \\ 0 & \text{if} \quad F(w) = 0\,. \end{cases}$$

We obtain that

$$\frac{\left\|\mathcal{S}^{-1}\phi_n\right\|^2}{\|\phi_n\|^2} \geq \int_{A(\phi_n)} \frac{\left|\phi_n^{①}(w-\pi)\right|^2 + \left|\phi_n^{②}(w)\right|^2}{\|\phi_n\|^2} \left(m - \frac{1}{n}\right) dw = m - \frac{1}{n} \to m \text{ as } n \to \infty\,.$$

When $m = \|\alpha\|_\infty$, taking the functions $\phi_n(w) := \chi_{D_n}(w+\pi)$ we achieve the result in a similar way.

Therefore,

$$\left\|\mathcal{S}^{-1}\right\|^2 = \max\{\|\alpha\|_\infty\,, \|\beta\|_\infty\}\,,$$

where the functions α and β have been defined above.

After some tedious calculations, one can find the same value for $\|\mathcal{S}^{-1}\|^2$ whenever the complex valued multipliers satisfy the condition:

$$\Im\left[M_1^{*①}(w-\pi)\overline{M_1^{*②}(w)} + M_2^{*①}(w-\pi)\overline{M_2^{*②}(w)}\right] = 0 \quad \text{a.e. in} \quad [0,\pi]\,.$$

3.2 Estimating the Sum $\sum_{|n|>N}\left[|S(t-2n)|^2 + |T(t-2n)|^2\right]$

The knowledge of the asymptotic behavior of the functions S and T allow us to give an estimation for the series $\sum_{|n|>N}\left[|S(t-2n)|^2+|T(t-2n)|^2\right]$. In fact, the following result holds:

Lemma 1. *Let F be a function such that, for $L, b > 0$ and $\nu > 1/2$, it satisfies*

$$|F(t)| \leq \frac{L}{|t|^\nu}\,, \ \textit{for} \quad |t| \geq b\,.$$

Then, for $N > (b+1)/2$, we obtain

$$\sum_{|n|>N} |F(t-2n)|^2 \leq \frac{2L^2}{2(2\nu-1)(2N-|t|)^{2\nu-1}}\,, \quad |t| < 2N - b - 1\,.$$

Proof. For $|t| < 2N - b - 1$ and $|n| > N$ we have $|t - 2n| > b$. As a consequence, for $|t| < 2N - b - 1$ and $2N > b + 1$, we get

$$\begin{aligned}\sum_{|n|>N} |F(t-2n)|^2 &\le \sum_{|n|>N} \frac{L^2}{|t-2n|^{2\nu}} \le \sum_{|n|>N} \frac{L^2}{(|2n|-|t|)^{2\nu}} \\ &= 2L^2 \sum_{n=N+1}^{\infty} \frac{1}{(2n-|t|)^{2\nu}} \le \frac{2L^2}{2} \sum_{n=N}^{\infty} \int_{2N-|t|}^{2(N+1)-|t|} \frac{1}{x^{2\nu}}\, dx \\ &= \frac{2L^2}{2} \int_{2N-|t|}^{\infty} \frac{1}{x^{2\nu}}\, dx = \frac{2L^2}{2(2\nu-1)(2N-|t|)^{2\nu-1}}\,. \end{aligned}$$

□

Whenever N is small we could calculate the whole series $\sum_{n=-\infty}^{\infty}\big[|S(t-2n)|^2 + |T(t-2n)|^2\big]$. Having in mind that

$$\begin{aligned} S(t-2n) &= \left\langle x_n, \frac{e^{-it\cdot}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]} = \left\langle \mathcal{S}(\alpha_n), \frac{e^{-it\cdot}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]} = \left\langle \alpha_n, \mathcal{S}^*\left(\frac{e^{-it\cdot}}{\sqrt{2\pi}}\right) \right\rangle_{\mathcal{H}_\pi} \\ T(t-2n) &= \left\langle y_n, \frac{e^{-it\cdot}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]} = \left\langle \mathcal{S}(\beta_n), \frac{e^{-it\cdot}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]} = \left\langle \beta_n, \mathcal{S}^*\left(\frac{e^{-it\cdot}}{\sqrt{2\pi}}\right) \right\rangle_{\mathcal{H}_\pi}, \end{aligned}$$

the Parseval equality gives

$$\sum_{n=-\infty}^{\infty} \big[|S(t-2n)|^2 + |T(t-2n)|^2\big] = \big\|\mathcal{S}^*\big(e^{-it\cdot}/\sqrt{2\pi}\big)\big\|^2_{\mathcal{H}_\pi} \le \|\mathcal{T}\|^2\,.$$

To compute $\|\mathcal{S}^*(e^{-it\cdot}/\sqrt{2\pi})\|_{\mathcal{H}_\pi}$, we express the adjoint operator $\mathcal{S}^*$ in terms of the matrix $\mathcal{M}(w)$ given by (2.1), which represent, the operator $\mathcal{S}$. Indeed, for $(\phi_1, \phi_2) \in \mathcal{H}_\pi$, we have that

$$\mathcal{M}(w)\begin{pmatrix}\phi_1(w)\\ \phi_2(w)\end{pmatrix} = \begin{pmatrix}\mathcal{S}(\phi_1,\phi_2)^{①}(w-\pi)\\ \mathcal{S}(\phi_1,\phi_2)^{②}(w)\end{pmatrix}, \qquad \text{a.e. in} \quad [0,\pi]\,.$$

Consequently, for $(\phi_1, \phi_2) \in \mathcal{H}_\pi$ and $g \in L^2[-\pi, \pi]$ we have

$$\begin{aligned}\big\langle \mathcal{S}(\phi_1,\phi_2), g\big\rangle &= \big\langle \mathcal{S}(\phi_1,\phi_2)^{①}, g^{①}\big\rangle + \big\langle \mathcal{S}(\phi_1,\phi_2)^{②}, g^{②}\big\rangle \\ &= \int_0^\pi \big(\mathcal{S}(\phi_1,\phi_2)^{①}(w-\pi), \mathcal{S}(\phi_1,\phi_2)^{②}(w)\big)\overline{\begin{pmatrix} g^{①}(w-\pi)\\ g^{②}(w)\end{pmatrix}}\, dw \\ &= \int_0^\pi \big(\phi_1(w), \phi_2(w)\big)\mathcal{M}^T(w)\overline{\begin{pmatrix} g^{①}(w-\pi)\\ g^{②}(w)\end{pmatrix}}\, dw\,, \end{aligned}$$

where $\mathcal{M}^T(w)$ stands for the transpose matrix of $\mathcal{M}(w)$. Therefore, $\mathcal{S}^* g = (\varphi_1, \varphi_2)$ is given by

$$\begin{pmatrix}\varphi_1(w)\\ \varphi_2(w)\end{pmatrix} = \overline{\mathcal{M}^T(w)}\begin{pmatrix} g^{①}(w-\pi)\\ g^{②}(w)\end{pmatrix}, \qquad \text{a.e. in} \quad [0,\pi]\,.$$

Hence,

$$\big\|\mathcal{S}^*\big(e^{-it\cdot}/\sqrt{2\pi}\big)\big\|_{\mathcal{H}_\pi} = \frac{1}{\sqrt{2\pi}}\left\|\overline{\mathcal{M}^T(w)}\begin{pmatrix} e^{-it(w-\pi)}\\ e^{-itw}\end{pmatrix}\right\|_{\mathcal{H}_\pi}.$$

3.3 An Example: Truncation Error in the Derivative Sampling Formula

The derivative sampling formula in the Paley-Wiener space PW_π can be derived from Theorem 1 by using the multipliers: $M_1(w) = 1 - \frac{|w|}{\pi}$ and $M_2(w) = \frac{i \operatorname{sgn} w}{\pi}$, $w \in [-\pi, \pi]$. In this case,

$$\mathcal{M}(w) = \frac{1}{\pi}\begin{pmatrix} w & -i \\ \pi - w & i \end{pmatrix} \quad \text{and} \quad \mathcal{M}^{-1}(w) = \begin{pmatrix} 1 & 1 \\ -i(w-\pi) & -iw \end{pmatrix}, \quad w \in [0, \pi],$$

and hence, $M_1^*(w) = 1$ and $M_2^*(w) = -iw$. The corresponding pair of dual Riesz bases in PW_π are

$$\left\{x_n = \frac{1}{\sqrt{\pi}}\left(1 - \frac{|w|}{\pi}\right)e^{-2inw}\right\}_{n\in\mathbb{Z}} \cup \left\{y_n = \frac{i \operatorname{sgn} w}{\pi\sqrt{\pi}}\, e^{-2inw}\right\}_{n\in\mathbb{Z}}$$

and

$$\left\{x_n^* = \frac{1}{\sqrt{\pi}}\, e^{-2inw}\right\}_{n\in\mathbb{Z}} \cup \left\{y_n^* = \frac{1}{\sqrt{\pi}}\, iwe^{-2inw}\right\}_{n\in\mathbb{Z}}.$$

The sampling functions in Theorem 1 are

$$S(t-2n) = \frac{1}{\sqrt{2}} \operatorname{sinc}^2\left(\frac{t-2n}{2}\right) \quad \text{and} \quad T(t-2n) = \frac{-1}{\sqrt{2}}(t-2n) \operatorname{sinc}^2\left(\frac{t-2n}{2}\right),$$

the sequences of samples $\{f(2n)\}_{n\in\mathbb{Z}}$ and $\{-f'(2n)\}_{n\in\mathbb{Z}}$, and the corresponding sampling formula (2.2) reads:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[f(2n) + (t-2n)f'(2n)\right] \operatorname{sinc}^2\left(\frac{t-2n}{2}\right), \quad t \in \mathbb{R}. \tag{3.2}$$

Concerning the truncation error for (3.2) we have that

$$F(w) = 1 - \pi w + w^2; \;\; G_1(w-\pi) = 1 + (w-\pi)^2; \;\; G_2(w) = 1 + w^2, \quad w \in [0, \pi].$$

Hence, $\alpha(w) = 1 + (w-\pi)^2$ whose maximum in $[0, \pi]$ is $\|\alpha\|_\infty = 1 + \pi^2$, and

$$\beta(w) = w^2 - \pi w + 1 + \frac{\pi^2}{2} + \sqrt{\frac{\left(\pi^2 - 2\pi w\right)^2}{4} + w^2 - \pi w + 1},$$

whose maximum in $[0, \pi]$ is $\|\beta\|_\infty = 1 + \pi^2/2 + \sqrt{\pi^4/4 + 1}$. Therefore,

$$\left\|\mathcal{S}^{-1}\right\|^2 = 1 + \pi^2/2 + \sqrt{\pi^4/4 + 1}.$$

In this example the sampling functions are generated by the functions $S(t) = (1/\sqrt{2}) \operatorname{sinc}^2(t/2)$ and $T(t) = (-1/\sqrt{2})\, t \operatorname{sinc}^2(t/2)$ which satisfy

$$|S(t)| \le \frac{4}{\pi^2\sqrt{2}}\frac{1}{|t|^2} \quad \text{and} \quad |T(t)| \le \frac{4}{\pi^2\sqrt{2}}\frac{1}{|t|}, \quad |t| \ge \delta > 0.$$

Thus, for $N > 1$ Lemma 1 gives

$$\sum_{|n|>N} |S(t-2n)|^2 + |T(t-2n)|^2 \le \frac{8}{3\pi^4(2N-|t|)^3} + \frac{8}{\pi^4(2N-|t|)},$$

whenever $|t| < 2N - 1 - \delta$. Therefore, the truncation error for the sampling formula (3.2) reads:

$$|f(t) - f_N(t)|^2 \leq \left[\left(1 + \pi^2/2 + \sqrt{\pi^4/4 + 1}\right)r^2 - 2 \sum_{n=-N}^{N} \left(|f(2n)|^2 + \left|f'(2n)\right|^2\right)\right] \times \left[\frac{8}{3\pi^4(2N - |t|)^3} + \frac{8}{\pi^4(2N - |t|)}\right], \qquad |t| < 2N - 1 - \delta\,,$$

where $f_N(t) = \sum_{n=-N}^{N} \left[f(2n) + (t - 2n) f'(2n)\right] \operatorname{sinc}^2\left(\frac{t - 2n}{2}\right)$.

For more details on the derivative sampling topic we address the interested reader to the superb survey in [7, Chapter 3].

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Departamento de Matemáticas, Universidad Carlos III de Madrid,
Avda. de la Universidad 30, 28911 Leganés-Madrid, Spain
e-mail: agarcia@math.uc3m.es

Departamento de Matemática Aplicada, E.T.S.I.T., U.P.M., Avda. Complutense s/n , 28040 Madrid, Spain
e-mail: mahm@mat.upm.es

Departamento de Matemáticas, Universidad Carlos III de Madrid,
Avda. de la Universidad 30, 28911 Leganés-Madrid, Spain
e-mail: aportal@math.uc3m.es