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Riesz bases in $L^2(0,1)$ related to sampling in shift-invariant spaces

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Abstract

The Fourier duality is an elegant technique to obtain sampling formulas in Paley–Wiener spaces. In this paper it is proved that there exists an analogue of the Fourier duality technique in the setting of shift-invariant spaces. In fact, any shift-invariant space V_φ with a stable generator φ is the range space of a bounded one-to-one linear operator T between $L^2(0,1)$ and $L^2(\mathbb{R})$. Thus, regular and irregular sampling formulas in V_φ are obtained by transforming, via T, expansions in $L^2(0,1)$ with respect to some appropriate Riesz bases.

Keywords: Shift-invariant spaces; Zak transform; Riesz bases; Sampling expansions

1. Introduction

The Whittaker–Shannon–Kotel'nikov sampling theorem states that any function f in the classical Paley–Wiener space PW_π,

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$$PW_\pi := \{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \colon \operatorname{supp} \hat{f} \subseteq [-\pi, \pi] \},$$

i.e., bandlimited to $[-\pi, \pi]$, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ on the integers as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\operatorname{sinc}(t-n), \tag{1}$$

where sinc denotes the cardinal sine function, $\operatorname{sinc}(t) = \sin \pi t / \pi t$.

This theorem and its numerous offspring have been proved in many different ways, e.g., using Fourier expansions, the Poisson summation formula, contour integrals, etc. (see, for instance, [11,19]). But the most elegant proof is probably the one due to Hardy [10], using that the Fourier transform $\mathcal{F}$ is an isometry between PW_π and $L^2[-\pi, \pi]$. For any $f \in PW_\pi$ one has

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(w) e^{iwt}\, dw = \left\langle \hat{f}, \frac{e^{-iwt}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]}, \quad t \in \mathbb{R},$$

so any value $f(t_n)$ of f is the inner product in $L^2[-\pi, \pi]$ of $\hat{f}$ and the complex exponential $e^{-it_n w}/\sqrt{2\pi}$. The key point in Hardy's proof is that an expansion converging in $L^2[-\pi, \pi]$ is transformed by $\mathcal{F}^{-1}$ into another expansion which converges in the topology of PW_π. This implies, in particular, that it converges absolutely and uniformly on $\mathbb{R}$. Recall that the Paley–Wiener space PW_π is a reproducing kernel Hilbert space (RKHS) whose reproducing kernel is $k(t,s) = \operatorname{sinc}(t-s)$. This technique has been coined in [11, p. 56] as the *Fourier duality* in Paley–Wiener spaces. Thus, expanding $\hat{f}$ with respect to the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ and transforming by $\mathcal{F}^{-1}$ we obtain the Shannon sampling formula (1). An irregular sampling formula in PW_π at a sequence $\{t_n\}_{n\in\mathbb{Z}}$ of real points may be obtained by perturbating the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ in such a way that the sequence of complex exponentials $\{e^{-it_n w}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for PW_π. This is the case if, for instance, the sequence $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ verifies the Kadec condition: $\sup_{n\in\mathbb{Z}} |t_n - n| < 1/4$. Moreover, the Paley–Wiener–Levinson sampling theorem states that any function $f \in PW_\pi$ can be recovered from its samples $\{f(t_n)\}_{n\in\mathbb{Z}}$ by means of the Lagrange-type interpolation series

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(t)}{G'(t_n)(t-t_n)},$$

where G stands for the infinite product $G(t) := (t - t_0) \prod_{n=1}^{\infty} (1 - t/t_n)(1 - t/t_{-n})$ [18]. On the other hand, the Paley–Wiener space PW_π is a particular case of a shift-invariant space, i.e., a closed subspace in $L^2(\mathbb{R})$ generated by the integer shifts of a single function $\varphi \in L^2(\mathbb{R})$. Whenever the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ forms, at least, a frame sequence in $L^2(\mathbb{R})$ (i.e., it is a frame for its closed linear span), the corresponding shift-invariant space can be described as

$$V_\varphi := \left\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot - n) \colon \{a_n\} \in \ell^2(\mathbb{Z}) \right\}.$$

The generator φ is stable if the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. For PW_π, a stable generator is $\varphi = \text{sinc}$. Wavelet subspaces are important examples of shift-invariant spaces generated by the scaling function of the corresponding multiresolution analysis. See [3,4,15] for the general theory of shift-invariant spaces and their applications. In addition, sampling theory in shift-invariant spaces and, in particular, in wavelet subspaces has been largely studied in the recent years. Let us cite, for instance, the works of Aldroubi and Gröchenig [1], Aldroubi and Unser [2], Chen, Itoh and Shiki [6,7], Janssen [13], Sun and Zhou [16,20], or Walter [14,17] among others.

The main aim in this paper is to show that the Fourier duality for Paley–Wiener spaces can be generalized to the case of a shift-invariant space V_φ with a stable generator φ. To this end, we define a bounded one-to-one linear operator T between $L^2(0,1)$ and $L^2(\mathbb{R})$ as

$$\begin{aligned} T : L^2(0,1) &\longrightarrow L^2(\mathbb{R}) \\ F &\longrightarrow f \quad \text{such that } f(t) := \langle F, K_t\rangle_{L^2(0,1)}, \end{aligned}$$

where the kernel transform $t \in \mathbb{R} \mapsto K_t \in L^2(0,1)$ is given by the Zak transform of $\bar{\varphi}$ namely, $K_t(x) := Z\bar{\varphi}(t,x)$, a.e. $x \in (0,1)$. Recall that the Zak transform of $f \in L^2(\mathbb{R})$ is formally defined as $(Zf)(t,w) := \sum_{n\in\mathbb{Z}} f(t+n)e^{-2\pi i n w}$, $t, w \in \mathbb{R}$. The shift-invariant space V_φ coincides with the range space of T. Thus, sampling expansions in V_φ can be seen as transformed expansions via T of expansions in $L^2(0,1)$ with respect to appropriate Riesz bases. Taking into account the definition of T, these bases should have the particular form $\{K_{t_n}\}_{n\in\mathbb{Z}}$. Taking the sampling points $\{t_n = a+n\}_{n\in\mathbb{Z}}$, we obtain the regular sampling in V_φ, whereas perturbing this sequence as $\{t_n = a + n + \delta_n\}_{n\in\mathbb{Z}}$, we obtain the irregular sampling. These steps will be carried out throughout the remaining sections.

2. Preliminaries on shift-invariant spaces

Let $\varphi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space

$$V_\varphi := \left\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot - n) \colon \{a_n\} \in \ell^2(\mathbb{Z}) \right\} \subset L^2(\mathbb{R}),$$

i.e., the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for V_φ if and only if

$$0 < \|\Phi\|_0 \leqslant \|\Phi\|_\infty < \infty,$$

where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\hat{\varphi}(w+k)|^2$ in $[0,1]$, and $\|\Phi\|_\infty$ its essential supremum. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds [8, p. 143].

We assume along the paper that, for each $t \in \mathbb{R}$, the series $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ converges. Thus, by using the Riesz' subsequence theorem [8, p. 39] we can choose the pointwise limit $f(t) := \sum_{n\in\mathbb{Z}} a_n\ \varphi(t-n)$ for each $t \in \mathbb{R}$, as the representative element of the class $\sum_{n\in\mathbb{Z}} a_n\ \varphi(\cdot\ - n)$ in $L^2(\mathbb{R})$. Moreover, if φ is a continuous function and the series $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ converges uniformly in compact subsets of $\mathbb{R}$, we can take any $f \in V_\varphi$ as a continuous function in $\mathbb{R}$.

Besides, V_φ is a RKHS since the evaluation functionals are bounded in V_φ. Indeed, for each fixed $t \in \mathbb{R}$ we have

$$\left|f(t)\right|^2 \leqslant \frac{1}{\|\Phi\|_0} \sum_{n\in\mathbb{Z}} \left|\varphi(t-n)\right|^2 \|f\|^2, \quad f \in V_\varphi, \tag{2}$$

where we have used Cauchy–Schwartz's inequality in $f(t) = \sum_{n\in\mathbb{Z}} a_n\ \varphi(t-n)$, and the Riesz basis condition

$$\|\Phi\|_0 \sum_{n\in\mathbb{Z}} |a_n|^2 \leqslant \|f\|^2 \leqslant \|\Phi\|_\infty \sum_{n\in\mathbb{Z}} |a_n|^2, \quad f \in V_\varphi.$$

Inequality (2) shows that convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence in $\mathbb{R}$. The convergence is uniform in subsets of the real line where $\|K_t\|^2_{L^2(0,1)} = \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is bounded.

The reproducing kernel of V_φ is given by $k(t,s) = \sum_{n\in\mathbb{Z}} \varphi(t-n)\,\overline{\varphi^*(s-n)}$ where the sequence $\{\varphi^*(\cdot\ - n)\}_{n\in\mathbb{Z}}$ denotes the dual Riesz basis of $\{\varphi(\cdot\ - n)\}_{n\in\mathbb{Z}}$. Recall that the function φ^* has Fourier transform $\widehat{\varphi^*} = \hat{\varphi}/\Phi$ [2].

3. A linear transform defining a shift-invariant space

For each $t \in \mathbb{R}$, consider the function $K_t \in L^2(0,1)$ defined by the Fourier series

$$K_t := \sum_{n\in\mathbb{Z}} \overline{\varphi(t+n)} e^{-2\pi i n x}.$$

Notice that $K_t(x) = Z\bar{\varphi}(t,x)$ a.e. $x \in (0,1)$, where Z denotes the Zak transform of $\bar{\varphi}$. See [9,12] for properties and uses of the Zak transform.

Thus, for each $F \in L^2(0,1)$ we can define the function

$$\begin{aligned} f:\mathbb{R} &\longrightarrow \mathbb{C}, \\ t &\longrightarrow f(t) := \langle F, K_t\rangle_{L^2(0,1)}. \end{aligned}$$

If we denote by T the linear transform which maps $F \in L^2(0,1)$ into f, i.e., $T(F) = f$, then we can identify the range space of T as the shift-invariant V_φ, i.e., $T(L^2(0,1)) = V_\varphi$. Indeed, for $F \in L^2(0,1)$ we have that

$$\left[T(F)\right](t) = \langle F, K_t\rangle_{L^2(0,1)} = \sum_{n\in\mathbb{Z}} \left\langle F, e^{-2\pi i n x}\right\rangle_{L^2(0,1)} \varphi(t+n), \quad t \in \mathbb{R},$$

which belongs to V_φ. Furthermore, for each $f \in V_\varphi$ there exists a sequence $\{a_n\} \in \ell^2(\mathbb{Z})$ such that $f = \sum_{n\in\mathbb{Z}} a_n \varphi(\cdot + n)$ in $L^2(\mathbb{R})$. Since $\{e^{-2\pi i n x}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(0,1)$, there exists a function $F \in L^2(0,1)$ such that $\langle F, e^{-2\pi i n x}\rangle_{L^2(0,1)} = a_n$ for every $n \in \mathbb{Z}$. Hence, $T(F) = f$. Moreover, the following result holds:

Theorem 1. *The mapping T is an invertible bounded operator between $L^2(0,1)$ and V_φ.*

Proof. The operator T is bijective since it maps the orthonormal basis $\{e^{-2\pi inx}\}_{n\in\mathbb{Z}}$ in $L^2(0,1)$ into the Riesz basis $\{\varphi(t+n)\}_{n\in\mathbb{Z}}$ in V_φ. Concerning the continuity, for $F \in L^2(0,1)$, we have

$$\|T(F)\|^2_{L^2(\mathbb{R})} = \Big\| \sum_{n\in\mathbb{Z}} \langle F, e^{-2\pi inx}\rangle_{L^2(0,1)} \varphi(t+n)\Big\|^2_{L^2(\mathbb{R})} \leqslant \|\Phi\|_\infty \sum_{n\in\mathbb{Z}} \big|\langle F, e^{-2\pi inx}\rangle\big|^2$$
$$= \|\Phi\|_\infty \|F\|^2_{L^2(0,1)},$$

where we have used the upper Riesz basis condition for $\{\varphi(\cdot+n)\}_{n\in\mathbb{Z}}$. □

Having in mind the periodicity relations of the Zak transform, the function K_t satisfies $K_{t+m}(x) = e^{2\pi imx}K_t(x)$ in $L^2(0,1)$, where $t \in \mathbb{R}$ and $m \in \mathbb{Z}$.

Now, for $f \in V_\varphi$ consider $F = T^{-1}(f) \in L^2(0,1)$. For each $n \in \mathbb{Z}$ we have

$$T\big[F(x)e^{2\pi inx}\big](t) = \big\langle F(\cdot)e^{2\pi in\cdot}, K_t(\cdot)\big\rangle_{L^2(0,1)} = \langle F, K_{t-n}\rangle_{L^2(0,1)} = f(t-n).$$

Since T is a bounded invertible operator, the sequence $\{f(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ if and only if $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$. The following theorem which can be found in [5, Theorem 2.2] gives a characterization of Bessel sequences, Riesz bases and frames in $L^2(0,1)$ having the form $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$. From now on, $\|F\|_\infty$ (respectively $\|F\|_0$) will denote the essential supremum (respectively infimum) of $|F|$ in $(0,1)$.

Theorem 2. *Given a function $F \in L^2(0,1)$, the following results hold*:

(a) *The sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(0,1)$ if and only if the function F satisfies $\|F\|_\infty < \infty$.*
(b) *The sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$ if and only if the function F satisfies $0 < \|F\|_0 \leqslant \|F\|_\infty < \infty$. In this case, the optimal Riesz bounds of $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ are $\|F\|_0^2$ and $\|F\|_\infty^2$.*
(c) *The sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a frame in $L^2(0,1)$ if and only if is a Riesz basis for $L^2(0,1)$.*

Thus we have the following corollary in V_φ.

Corollary 1. *Given a function $g \in V_\varphi$, consider $G = T^{-1}(g) \in L^2(0,1)$. Then, the sequence $\{g(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ if and only if $0 < \|G\|_0 \leqslant \|G\|_\infty < \infty$.*

4. Regular sampling in shift-invariant spaces

Regular sampling in V_φ arises by considering appropriate Riesz bases in $L^2(0,1)$. Namely, for a fixed $a \in [0,1)$, the regular samples at $\{a+n\}_{n\in\mathbb{Z}}$ of $f \in V_\varphi$ are given by

$$f(a+n) = \langle F, K_{a+n}\rangle_{L^2(0,1)} = \langle F, K_a e^{2\pi inx}\rangle_{L^2(0,1)}, \quad n \in \mathbb{Z},$$

where $F = T^{-1}(f)$. The sequence $\{K_a(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ in $L^2(0,1)$ has the biorthonormal sequence $\{e^{2\pi inx}/\bar{K}_a(x)\}_{n\in\mathbb{Z}}$ provided $1/K_a \in L^2(0,1)$. Hence, stable regular sampling in V_φ reduces to studying whenever the sequence $\{K_a(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$, and this depends on the function K_a as stated in Theorem 2. Expanding $F = T^{-1}(f)$ with respect to the Riesz basis $\{e^{2\pi inx}/\bar{K}_a(x)\}_{n\in\mathbb{Z}}$, via the invertible bounded operator T, we obtain a regular sampling formula for f.

Lemma 1. *Given $a \in [0,1)$, there exists a function $S_a \in V_\varphi$ satisfying the interpolation condition $S_a(a+n) = \delta_{n,0}$, where $n \in \mathbb{Z}$, if and only if the function $1/K_a$ belongs to $L^2(0,1)$. In this case $S_a = T(1/\bar{K}_a)$.*

Proof. Assume that there exists a function $S_a \in V_\varphi$ satisfying the interpolation condition $S_a(a+n) = \delta_{n,0}$, where $n \in \mathbb{Z}$. For $F_a = T^{-1}(S_a)$ we have

$$\begin{aligned} S_a(a+n) &= \langle F_a, K_{a+n}\rangle_{L^2(0,1)} = \langle F_a, e^{2\pi inx}K_a\rangle_{L^2(0,1)} \\ &= \int_0^1 F_a(x)\overline{K_a(x)}e^{-2\pi inx}\,dx = \delta_{n,0}, \end{aligned}$$

which implies that $F_a(x)\overline{K_a(x)} = 1$ a.e. in $(0,1)$, and consequently the function $1/K_a$ belongs to $L^2(0,1)$.

Conversely, if $1/K_a$ is in $L^2(0,1)$, we define $S_a = T(1/\bar{K}_a)$. For $n \in \mathbb{Z}$ it satisfies

$$S_a(a+n) = \left\langle \frac{1}{\bar{K}_a}, K_{a+n}\right\rangle_{L^2(0,1)} = \langle 1, e^{2\pi inx}\rangle_{L^2(0,1)} = \delta_{n,0}. \qquad \square$$

Thus we can characterize stable regular sampling in V_φ.

Theorem 3. *Consider $a \in [0,1)$ such that the function $1/K_a \in L^2(0,1)$. The following conditions are equivalent*:

(a) $0 < \|K_a\|_0 \leqslant \|K_a\|_\infty < \infty$.
(b) *There exists a Riesz basis $\{S_n\}_{n\in\mathbb{Z}}$ for V_φ such that, for each $f \in V_\varphi$, we have the pointwise expansion*

$$f(t) = \sum_{n\in\mathbb{Z}} f(a+n)S_n(t), \quad t \in \mathbb{R}.$$

Furthermore, in this case the sampling functions are $S_n(t) = S_a(t-n)$, where $S_a = T(1/\bar{K}_a)$. The sampling series converges in the $L^2(\mathbb{R})$-norm sense, absolutely and uniformly in subsets of $\mathbb{R}$ where $\|K_t\|$ is bounded.

Proof. First we prove that (a) implies (b). Consider $S_a = T(1/\bar{K}_a)$. Condition (a) implies that $0 < \|1/\bar{K}_a\|_0 \leqslant \|1/\bar{K}_a\|_\infty < \infty$ and, as a consequence, Corollary 1 gives that

$\{S_a(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. For each $f \in V_\varphi$, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ such that $f(t) = \sum_{n\in\mathbb{Z}} a_n S_a(t-n)$ where the convergence is also pointwise for each $t \in \mathbb{R}$ since V_φ is a RKHS. Taking $t = a+m$, and using the interpolatory condition $S_a(a+n) = \delta_{n,0}$, we obtain that $a_m = f(a+m)$ for any $m \in \mathbb{Z}$.

Conversely, assume that the condition (b) holds. Taking $f(t) = S_a(t-m)$, $m \in \mathbb{Z}$, we obtain that $S_m(t) = S_a(t-m)$ and , as a consequence, $\{S_a(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. Since $S_a = T(1/\bar{K}_a)$, Corollary 1 gives condition (a).

Absolute convergence comes from the inconditional character of a Riesz basis. The uniform convergence is a standard result in the setting of the RKHS theory. □

A straightforward calculation gives the Fourier transform of S_a. Indeed,

$$\widehat{S_a}(w) = \widehat{T(1/\bar{K}_a)}(w) = \frac{\hat{\varphi}(w)}{Z\varphi(a, w/2\pi)} \quad \text{a.e. in } \mathbb{R}.$$

5. Irregular sampling in shift-invariant spaces

Usually, one may consider irregular sampling as a perturbation of the regular sampling. In the present setting, we can try to recover any function $f \in V_\varphi$ from its perturbed samples $\{f(a+n+\delta_n)\}_{n\in\mathbb{Z}}$, where $a \in [0,1)$ and $\{\delta_n\}_{n\in\mathbb{Z}}$ is a sequence in $(-1,1)$. Since

$$f(a+n+\delta_n) = \langle F, K_{a+n+\delta_n}\rangle_{L^2(0,1)}, \quad n \in \mathbb{Z}, \text{ where } F = T^{-1}(f) \in L^2(0,1),$$

a challenge problem is to prove that $\{K_{a+n+\delta_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.

One possibility is to use a perturbation technique on the Riesz basis $\{K_{a+n}\}_{n\in\mathbb{Z}} = \{K_a e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ which gives the sequence of regular samples $\{f(a+n)\}_{n\in\mathbb{Z}}$. As a consequence, we need a perturbation result for those Riesz bases in $L^2(0,1)$ appearing in Theorem 2.

For an infinite matrix $M = \{m_{n,k}\}_{n,k\in\mathbb{Z}}$ defining a bounded operator in $\ell^2(\mathbb{Z})$ we denote its operator norm as $\|M\|_2 := \sup_{\|c\|_{\ell^2(\mathbb{Z})}=1} \|Mc\|_{\ell^2(\mathbb{Z})}$.

Theorem 4. *Let $F = \sum_{k\in\mathbb{Z}} a_k e^{-2\pi i k x}$ be in $L^2(0,1)$ such that $0 < \|F\|_0 \leqslant \|F\|_\infty < \infty$. Let $\{F_n\}_{n\in\mathbb{Z}}$ be a sequence of functions in $L^2(0,1)$ with Fourier expansions $F_n = \sum_{k\in\mathbb{Z}} a_k(n) e^{-2\pi i k x}$, $n \in \mathbb{Z}$. Suppose that the infinite matrix $D = \{d_{n,k}\}_{n,k\in\mathbb{Z}}$ with entries $d_{n,k} := a_{n-k}(n) - a_{n-k}$, $n, k \in \mathbb{Z}$, satisfies the condition $\|D\|_2 < \|F\|_0$. Then, the sequence $\{F_n(x)e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.*

Proof. To this end we use the following result on perturbation of Riesz bases in a Hilbert space $\mathcal{H}$ which can be found in [8, p. 354]: let $\{f_k\}_{k=1}^\infty$ be a Riesz basis for $\mathcal{H}$ with Riesz bounds A, B, and let $\{g_k\}_{k=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R < A$ such that

$$\sum_{k=1}^{\infty} \left|\langle f_k - g_k, f\rangle\right|^2 \leqslant R\,\|f\|^2, \quad \text{for each } f \in \mathcal{H},$$

then $\{g_k\}_{k=1}^\infty$ is a Riesz basis for $\mathcal{H}$.

For any $f = \sum_{j\in\mathbb{Z}} \overline{c_j} e^{2\pi ijx}$ in $L^2(0,1)$ we have

$$\sum_{n\in\mathbb{Z}} \left|\left\langle F_n(x)e^{2\pi inx} - F(x)e^{2\pi inx}, f\right\rangle\right|^2$$
$$= \sum_{n\in\mathbb{Z}} \left|\left\langle \sum_{k\in\mathbb{Z}} \bigl(a_k(n) - a_k\bigr) e^{2\pi i(n-k)x}, \sum_{j\in\mathbb{Z}} \overline{c_j} e^{2\pi ijx} \right\rangle\right|^2$$
$$= \sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} \bigl(a_{n-k}(n) - a_{n-k}\bigr) c_k \right|^2 = \sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} d_{n,k} c_k \right|^2 = \|D\, c\|^2_{\ell^2(\mathbb{Z})} \leqslant \|D\|_2^2\, \|f\|^2.$$

Taking into account that in our case $A = \|F\|_0^2$, we obtain that $\{F_n(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$. □

As a consequence of the above perturbation theorem in $L^2(0,1)$, we obtain an irregular sampling result in V_φ.

Theorem 5. *Given $a \in [0,1)$ such that $0 < \|K_a\|_0 \leqslant \|K_a\|_\infty < \infty$. Let $\Delta = \{\delta_n\}_{n\in\mathbb{Z}}$ be a sequence in $(-1,1)$ such that the infinite matrix $D_\Delta = \{d_{n,k}\}_{n,k\in\mathbb{Z}}$ whose entries are given by*

$$d_{n,k} := \overline{\varphi(a+n-k+\delta_n)} - \overline{\varphi(a+n-k)}, \quad n,k \in \mathbb{Z},$$

satisfies $\|D_\Delta\|_2 < \|K_a\|_0$. Then, there exists a Riesz basis $\{S_n\}_{n\in\mathbb{Z}}$ for V_φ such that any function $f \in V_\varphi$ can be expanded as

$$f(t) = \sum_{n\in\mathbb{Z}} f(a+n+\delta_n) S_n(t), \quad t \in \mathbb{R}.$$

The convergence of the series is absolute and uniform in subsets of $\mathbb{R}$ where $\|K_t\|$ is bounded. Also, it converges in the $L^2(\mathbb{R})$-norm sense.

Proof. Applying Theorem 4 to

$$K_a(x) = \sum_{k\in\mathbb{Z}} \overline{\varphi(a+k)} e^{-2\pi ikx} \quad \text{and}$$
$$K_{a+\delta_n}(x) = \sum_{k\in\mathbb{Z}} \overline{\varphi(a+k+\delta_n)} e^{-2\pi ikx}, \quad n \in \mathbb{Z},$$

we obtain that $\{K_{a+\delta_n} e^{2\pi inx}\}_{n\in\mathbb{Z}} = \{K_{a+n+\delta_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$. Denote by $\{G_n\}_{n\in\mathbb{Z}}$ its dual Riesz basis. Now, given $f \in V_\varphi$, we expand the function $F = T^{-1}(f) \in L^2(0,1)$ with respect to $\{G_n\}_{n\in\mathbb{Z}}$. Thus,

$$F = \sum_{n\in\mathbb{Z}} \langle F, K_{a+n+\delta_n}\rangle_{L^2(0,1)}\, G_n = \sum_{n\in\mathbb{Z}} f(a+n+\delta_n)\, G_n \quad \text{in } L^2(0,1).$$

Applying the operator T, we get

$$f = \sum_{n\in\mathbb{Z}} f(a+n+\delta_n) T(G_n) \quad \text{in } L^2(\mathbb{R}).$$

Furthermore, since T is an invertible bounded operator, the sequence $\{S_n := T(G_n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. The pointwise convergence properties of the series come out as in Theorem 3. □

The next result yields a uniform bound of the norm $\|D_\Delta\|_2$ regardless the sequence $\Delta = \{\delta_n\}_{n\in\mathbb{Z}}$ in $[\alpha, \beta] \subset [-1, 1]$.

Theorem 6. *For any sequence* $\Delta = \{\delta_n\}_{n\in\mathbb{Z}}$ *in* $[\alpha, \beta]$ *the following inequality holds*:

$$\|D_\Delta\|_2 \leqslant \sup_{\{d_n\}\subset[\alpha,\beta]} \sum_{n\in\mathbb{Z}} \left|\varphi(a+n+d_n) - \varphi(a+n)\right|. \tag{3}$$

Proof. Assume that the second term in the above inequality is finite. Otherwise, the inequality trivially holds. For any $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have

$$\begin{aligned}
\|D_\Delta c\|^2_{\ell^2(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} d_{n,k}\, c_k \right|^2 \leqslant \sum_{n\in\mathbb{Z}} \sum_{l,j\in\mathbb{Z}} |d_{n,l}|\, |c_l|\, |\overline{d_{n,j}}|\, |\overline{c_j}| \\
&= \sum_{l,j\in\mathbb{Z}} |c_l|\, |c_j| \sum_{n\in\mathbb{Z}} |d_{n,l}|\, |d_{n,j}| \leqslant \sum_{l,j\in\mathbb{Z}} \frac{|c_l|^2 + |c_j|^2}{2} \sum_{n\in\mathbb{Z}} |d_{n,l}|\, |d_{n,j}| \\
&= \sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{j,n\in\mathbb{Z}} |d_{n,l}|\, |d_{n,j}| \leqslant \sup_{l\in\mathbb{Z}} \left(\sum_{j,n\in\mathbb{Z}} |d_{n,l}|\, |d_{n,j}| \right) \|c\|^2_{\ell^2(\mathbb{Z})} \\
&\leqslant \sup_{l\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} |d_{n,l}| \right) \sum_{j\in\mathbb{Z}} |d_{n,j}|\, \|c\|^2_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

Having in mind that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |d_{n,j}| &= \sum_{j\in\mathbb{Z}} \left|\varphi(a+j-k+\delta_j) - \varphi(a+j-k)\right| \\
&= \sum_{n\in\mathbb{Z}} \left|\varphi(a+n+\delta_{n+k}) - \varphi(a+n)\right|,
\end{aligned}$$

we obtain the desired result. □

A comment about the second term in (3) is in order. Namely, looking for an estimation of the ratio between $\sum_{n\in\mathbb{Z}} |\varphi(a+n+d_n) - \varphi(a+n)|$ and $(\sup_n |d_n|)^\lambda$ for a fixed $\lambda > 0$, led Chen et al. to introduce in [6] the classes of functions $L_a^\lambda[\alpha, \beta]$.

Next we give a particular example when Theorem 6 works. Namely, suppose that the stable generator $\varphi \in \mathcal{C}^1(\mathbb{R})$ and for some $\varepsilon > 0$ it satisfies $\varphi'(t) = O(|t|^{-(1+\varepsilon)})$ as $|t| \to \infty$. Then, it is easy to prove that, for $\delta \in (0, 1]$,

$$M_{\varphi'}(\delta) := \sum_k \max_{I_k(\delta)} \left|\varphi'(t)\right| \leqslant M_{\varphi'}(1) < \infty,$$

where $I_k(\delta)$ denotes the interval $[a+k-\delta, a+k+\delta]$.

Corollary 2. *Let $\varphi \in \mathcal{C}^1(\mathbb{R})$ be a stable generator such that $M_{\varphi'}(\delta) < \infty$, where $\delta := \sup_{n\in\mathbb{Z}} |\delta_n|$. Then, the condition $\delta\, M_{\varphi'}(\delta) < \|K_a\|_0$ implies the existence of a Riesz basis $\{S_n\}_{n\in\mathbb{Z}}$ for V_φ such that any function in this space can be expanded as*

$$f(t) = \sum_{n\in\mathbb{Z}} f(a+n+\delta_n) S_n(t), \quad t \in \mathbb{R}.$$

The convergence in the series is absolute and uniform in subsets of $\mathbb{R}$ where $\|K_t\|$ is bounded. It converges also in the $L^2(\mathbb{R})$-norm sense.

Proof. The mean value theorem gives

$$\sup_{\{d_n\}\subset[-\delta,\delta]} \sum_{n\in\mathbb{Z}} \bigl|\varphi(a+n+d_n) - \varphi(a+n)\bigr| \leqslant \delta\, M_{\varphi'}(\delta).$$

Theorem 5 concludes the proof. □

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