An approximation property of the functions defined through a resolvent sampling kernel

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Abstract

In this article, we introduce the σ-resolvent sampling kernels associated with an unbounded symmetric operator with compact resolvent defined on a Hilbert space $\mathbb{H}$, where σ denotes an entire $\mathbb{H}$-valued function. Related to a σ-resolvent sampling kernel we construct by duality a reproducing kernel Hilbert space of entire functions $\mathcal{H}_\sigma$ where a sampling formula holds. We prove that any function obtained by duality through a generalized Lagrange-Kramer sampling kernel is uniformly approximated in compact sets of $\mathbb{C}$ by functions defined through σ-resolvent sampling kernels. In particular, we study the obtained spaces for σ constant from an algebraic and topological point of view.

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1 Statement of the problem

For the past few years a significant mathematical literature on the topic of sampling theorems associated with differential or difference problems has flourished [2, 5, 6, 8, 9, 10]. See also [16] and the references therein. In its turn, we might consider the Weiss-Kramer sampling theorem as the *leitmotiv* of all these results [12, 15]. Roughly speaking, the common situation for these sampling problems is the following:

Let f be a function defined on $\mathbb{C}$ by $f(z) = \int_I F(x)\,K(x,z)\,dx$, $F \in L^2(I)$, (or $f(z) = \sum_n F(n)\,K(n,z)$, $F \in \ell^2$). The kernel K, which belongs to $L^2(I)$ (or ℓ^2) for each fixed $z \in \mathbb{C}$, satisfies the differential (difference) equation appearing in a differential (difference) problem (P) which has the sequence of eigenvalues $\{z_n\}$. Moreover, whenever we substitute in K the spectral parameter z by $\{z_n\}$ we obtain the sequence of orthogonal eigenfunctions associated with (P) which constitutes an orthogonal basis for $L^2(I)$ (ℓ^2). Under these circumstances, f is an entire function which can be recovered from its samples $\{f(z_n)\}$ by means of a sampling formula $f(z) = \sum_n f(z_n)\,S_n(z)$, where the sampling functions $\{S_n\}$ are given by $S_n(z) = \|K(\cdot, z_n)\|^{-2}\,\langle K(\cdot, z), K(\cdot, z_n)\rangle$ (the inner product in $L^2(I)$ or ℓ^2).

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Many often, the sampling functions can be written as Lagrange-type interpolation functions $S_n(z) = G(z)/((z - z_n)G'(z_n))$, where G is an entire function having simple zeros at $\{z_n\}$. In this case, for a fixed $z \in \mathbb{C}$, the expansion of the kernel $K(x,z)$ $(K(n,z))$ in the orthogonal basis $\{K(x,z_n)\}$ $(\{K(m,z_n)\}_n)$ has the form

$$K(x,z) = \sum_n \frac{A_n\, G(z)}{z - z_n} \frac{K(x,z_n)}{\|K(\cdot,z_n)\|} \quad \left(K(m,z) = \sum_n \frac{A_n\, G(z)}{z - z_n} \frac{K(m,z_n)}{\|K(\cdot,z_n)\|} \right),$$

where the coefficients are entire functions and $\sum_n \left|\frac{A_n\, G(z)}{z-z_n}\right|^2 = \|K(\cdot,z)\|^2$ is bounded in compact sets of $\mathbb{C}$ (the A_n are constants). Roughly speaking, the form assumed for the coefficients comes out, in general, by applying the Green's formula (or Lagrange's formula) associated with the differential or difference problem. In these circumstances we say that K is a *Lagrange-Kramer sampling kernel*. For a characterization of the Kramer sampling kernels giving a sampling formula written as a Lagrange-type interpolation series see Ref. [7].

In this work, we introduce the σ-resolvent sampling kernels associated with a symmetric operator with compact resolvent defined on a Hilbert space $\mathbb{H}$, where σ denotes an entire $\mathbb{H}$-valued function. Related to a σ-resolvent sampling kernel we construct by duality a reproducing kernel Hilbert space of entire functions $\mathcal{H}_\sigma$ where a sampling formula holds: The σ-sampling theorem. In this setting, we prove that any function obtained through a generalized Lagrange-Kramer sampling kernel, a generalization of a Lagrange-Kramer sampling kernel, is uniformly approximated in compact sets of $\mathbb{C}$ by functions defined through σ-resolvent sampling kernels. Finally, we confine ourselves to the particular case where σ is a constant function. We study some algebraic properties of the associated spaces $\mathcal{H}_\sigma$, and also some topological properties for the topology of uniform convergence in compact subsets of $\mathbb{C}$. The corresponding results for the classical Paley-Wiener spaces are exhibited.

2 Sampling theory associated with a resolvent kernel

In this Section we introduce the sampling resolvent kernels associated with a symmetric operator with compact resolvent, and the corresponding sampling theory.

2.1 Preliminaries on symmetric operators with compact resolvent

Let $\mathbb{H}$ be a complex Hilbert space and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ be a symmetric (formally self–adjoint) linear operator, densely defined on $\mathbb{H}$. Assume that there exists an inverse operator $\mathcal{T} = \mathcal{A}^{-1}$, compact and defined on $\mathbb{H}$. We know from the spectral theorem for symmetric compact operators defined on a Hilbert space that $\mathcal{T}$ has discrete spectrum [14]. Moreover, if $\{\mu_n\}_{n=1}^\infty$ is the sequence of eigenvalues of $\mathcal{T}$, then $\lim_{n\to\infty} |\mu_n| = 0$. We may assume that $|\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_n| \geq \ldots$ The eigenspace associated with each eigenvalue μ_n is finite dimensional. Set $k_n = \dim\ker(\mu_n I - \mathcal{T}) < \infty$. Note that 0 is not an eigenvalue of $\mathcal{T}$, so the sequence $\{e_n\}_{n=1}^\infty$ of eigenvectors of $\mathcal{T}$ is a complete orthonormal system (applying the Gram–Schmidt method in each eigenspace) of $\mathbb{H}$. The sequences $\{z_n = \mu_n^{-1}\}_{n=1}^\infty$ and $\{e_n\}_{n=1}^\infty$ are, respectively, the sequence of eigenvalues and the sequence of associated eigenvectors of the operator $\mathcal{A}$. Since $\lim_{n\to\infty} |\mu_n| = 0$, we have $0 < |z_1| \leq |z_2| \leq \ldots \leq |z_n| \leq \ldots$ and $\lim_{n\to\infty} |z_n| = \infty$. We can arrange the sequence of eigenvectors of $\mathcal{A}$ as $\{(e_{n,i})_{i=1}^{k_n}\}_{n=1}^\infty$, where $n \in \mathbb{N}$ and $i = 1, 2, \ldots, k_n$.

The resolvent operator $R_z := (z\,I - \mathcal{A})^{-1}$ is meromorphic in $\mathbb{C}$ with simple poles at

$\{z_n\}_{n=1}^\infty$. For each $x \in \mathbb{H}$ the following expansion holds in $\mathbb{H}$ [14]:

$$R_z(x) = \sum_{n=1}^{\infty} \left(\frac{1}{z - z_n} \sum_{i=1}^{k_n} \langle x, e_{n,i} \rangle_{\mathbb{H}} \, e_{n,i} \right). \tag{1}$$

From now on, P will denote an entire function having simple and real zeros at $\{z_n\}_{n=1}^\infty$, and taking real values on $\mathbb{R}$. In the particular case that the exponent of convergence of the sequence $\{z_n\}_{n=1}^\infty$ is finite, i.e.,

$$\eta = \inf \left\{ \alpha > 0 \mid \sum_{k=1}^{\infty} \frac{1}{|z_k|^\alpha} < +\infty \right\} < \infty,$$

we can take P to be the canonical product associated with the sequence of eigenvalues $\{z_n\}_{n=1}^\infty$ given by

$$P(z) = \begin{cases} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp(\sum_{i=0}^{p} \frac{1}{i} (\frac{z}{z_n})^i) & \text{if } p \geq 1 \\ \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) & \text{if } p = 0 \end{cases} \tag{2}$$

where p is the smallest non negative integer larger than $\eta - 1$.

More information about the function P associated with a particular differential or difference problem, can be found, for instance, in [2, 6, 8, 10, 16].

Given an entire $\mathbb{H}$-valued function $\sigma : \mathbb{C} \longrightarrow \mathbb{H}$, we define the following $\mathbb{H}$-valued function:

$$\begin{array}{rrcl} K_\sigma : & \mathbb{C} & \longrightarrow & \mathbb{H} \\ & z & \longrightarrow & K_\sigma(z) := P(z) R_z[\sigma(z)] \,. \end{array} \tag{3}$$

Since σ is an entire function, and the resolvent operator is meromorphic with simple poles at $\{z_n\}_{n=1}^\infty$, the next Lemma, whose proof is an easy exercise, allows us to assert that $R_z[\sigma(z)]$ is a meromorphic $\mathbb{H}$-valued function with simple poles at $\{z_n\}_{n=1}^\infty$.

Lemma 1 *Let $\mathbb{H}$ be a Hilbert space and let $\mathcal{B}(\mathbb{H})$ be the space of the bounded operators on $\mathbb{H}$. Assume that the maps $A : \mathcal{U} \longrightarrow \mathcal{B}(\mathbb{H})$ and $\sigma : \mathcal{U} \longrightarrow \mathbb{H}$ are holomorphic on a domain $\mathcal{U}$ of the complex plane $\mathbb{C}$. Then, the mapping $F : \mathcal{U} \longrightarrow \mathbb{H}$ defined by $F(z) = [A(z)](\sigma(z))$ is holomorphic in $\mathcal{U}$, and $F'(z) = [A'(z)](\sigma(z)) + [A(z)](\sigma'(z))$ for each $z \in \mathcal{U}$.*

Moreover, since P has simple zeros at $\{z_n\}_{n=1}^\infty$ we finally obtain that K_σ is an entire $\mathbb{H}$-valued function.

Definition 1 *The entire $\mathbb{H}$-valued function K_σ associated with an entire $\mathbb{H}$-valued function σ will be called a σ-resolvent sampling kernel.*

Using the expansion (1), one obtains the following expansion for $K_\sigma(z)$, $z \in \mathbb{C}$,

$$K_\sigma(z) = \sum_{n=1}^{\infty} \frac{P(z)}{z - z_n} \sum_{i=1}^{k_n} \langle \sigma(z), e_{n,i} \rangle_{\mathbb{H}} e_{n,i} \,. \tag{4}$$

In particular, $K_\sigma(z_m) = P'(z_m) \sum_{i=1}^{k_m} \langle \sigma(z_m), e_{m,i} \rangle_{\mathbb{H}} \, e_{m,i}$.

2.2 The resulting sampling theory

Let K_σ be the σ-resolvent sampling kernel associated with an entire $\mathbb{H}$-valued function σ. Define the mapping T_σ by

$$\begin{aligned} T_\sigma : \ \mathbb{H} &\longrightarrow \ \mathbb{C}^{\mathbb{C}} \\ x &\longrightarrow \ T_\sigma(x)\,, \end{aligned} \tag{5}$$

where $[T_\sigma(x)](z) := \langle K_\sigma(z), x\rangle_{\mathbb{H}}$, $z \in \mathbb{C}$. Note that, for each $x \in \mathbb{H}$, the function $T_\sigma(x)$ is an entire function. The mapping T_σ is anti-linear, i.e.,

$$T_\sigma(\alpha x + \beta y) = \overline{\alpha} T_\sigma(x) + \overline{\beta} T_\sigma(y) \quad \text{for all } x, y \in \mathbb{H} \quad \text{and } \alpha, \beta \in \mathbb{C}\,.$$

We denote by $\mathcal{H}_\sigma$ the range space of T_σ, i.e., $\mathcal{H}_\sigma := T_\sigma(\mathbb{H})$. Endowing $\mathcal{H}_\sigma$ with the norm

$$\|f\|_{\mathcal{H}_\sigma} := \inf\{\|x\|_{\mathbb{H}} : f = T_\sigma(x)\}\,,$$

we obtain a Hilbert space of entire functions [11, 13]. In fact, the infimum is actually reached: There exists $\widetilde{x} = P_{(\ker T_\sigma)^\perp}(x)$, the orthogonal projection onto $(\ker T_\sigma)^\perp$ of any $x \in \mathbb{H}$ such that $T_\sigma(x) = f$, satisfying $\|f\|_{\mathcal{H}_\sigma} = \|\widetilde{x}\|_{\mathbb{H}}$. Thus, the anti-linear mapping T_σ is continuous and satisfies $\|T_\sigma\| \le 1$.

Moreover, the space $\mathcal{H}_\sigma$ is a reproducing kernel Hilbert space (RKHS hereafter), since the point-evaluation functional $E_z(f) := f(z)$ is continuous for each $z \in \mathbb{C}$. Its reproducing kernel k_σ is given by [11]

$$k_\sigma(z, \omega) = \langle K_\sigma(z), K_\sigma(\omega)\rangle_{\mathbb{H}} = P(z)\,\overline{P(\omega)}\langle R_z(\sigma(z)), R_\omega(\sigma(\omega))\rangle_{\mathbb{H}}\,.$$

Recall that for each $\omega \in \mathbb{C}$ the function l_ω defined as $l_\omega(z) := k_\sigma(z, \omega)$ belongs to $\mathcal{H}_\sigma$, and the reproducing property holds

$$f(\omega) = \langle f, l_\omega\rangle_{\mathcal{H}_\sigma} = \langle f, k_\sigma(\cdot, \omega)\rangle_{\mathcal{H}_\sigma} \ \text{ for } \omega \in \mathbb{C} \text{ and } f \in \mathcal{H}_\sigma\,.$$

For a function $f \in \mathcal{H}_\sigma$ such that $f(z_n) \ne 0$ for any eigenvalue z_n, the following expansion holds:

Theorem 1 *Let $f = T_\sigma(x)$ be in $\mathcal{H}_\sigma$ such that $f(z_n) \ne 0$ for all $n \in \mathbb{N}$. Then, f admits the expansion as the series:*

$$f(z) = \sum_{n=1}^{\infty} f(z_n)\,\frac{F_n[f](z)}{F_n[f](z_n)}\,\frac{P(z)}{(z - z_n)P'(z_n)}\,, \quad z \in \mathbb{C}\,, \tag{6}$$

where $F_n[f](z) := \sum_{i=1}^{k_n} \langle \sigma(z), e_{n,i}\rangle_{\mathbb{H}} \overline{\langle x, e_{n,i}\rangle}_{\mathbb{H}}$ depends on f. The convergence of the series in (6) *is absolute and uniform in compact subsets of $\mathbb{C}$. Furthermore, it converges in the $\mathcal{H}_\sigma$-norm sense.*

Proof: Given $f \in \mathcal{H}_\sigma$, consider $x \in \mathbb{H}$ such that $f = T_\sigma(x)$. Expanding x with respect to the orthonormal basis $\{(e_{n,i})_{i=1}^{k_n}\}_{n=1}^{\infty}$ we have

$$x = \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} \langle x, e_{n,i}\rangle_{\mathbb{H}}\, e_{n,i} \quad \text{in } \mathbb{H}\,.$$

Applying the continuous anti-linear mapping T_σ we obtain

$$f = T_\sigma(x) = \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} \overline{\langle x, e_{n,i}\rangle}_{\mathbb{H}}\, T_\sigma(e_{n,i}) \quad \text{in } \mathcal{H}_\sigma\,. \tag{7}$$

As a consequence of the RKHS theory, the convergence in the $\mathcal{H}_\sigma$-sense implies pointwise convergence for any $z \in \mathbb{C}$. Now, for each $n \in \mathbb{N}$, $T_\sigma(e_{n,i})$ is given by

$$[T_\sigma(e_{n,i})](z) = \langle K_\sigma(z), e_{n,i}\rangle_{\mathbb{H}} = \frac{P(z)\langle \sigma(z), e_{n,i}\rangle_{\mathbb{H}}}{z - z_n}, \quad z \in \mathbb{C} \setminus \{z_n\},$$

and $[T_\sigma(e_{n,i})](z_n) = P'(z_n)\langle \sigma(z_n), e_{n,i}\rangle_{\mathbb{H}}$. Consequently, (7) can be written as

$$f(z) = \sum_{n=1}^{\infty} \frac{P(z)}{z - z_n} F_n[f](z), \quad z \in \mathbb{C} \setminus \{z_n\}. \tag{8}$$

Since $K_\sigma(z_n) = P'(z_n) \sum_{i=1}^{k_n} \langle \sigma(z_n), e_{n,i}\rangle_{\mathbb{H}}\, e_{n,i}$, we obtain

$$f(z_n) = \langle K_\sigma(z_n), x\rangle_{\mathbb{H}} = P'(z_n)\, F_n[f](z_n). \tag{9}$$

Substituting (9) in (8) we obtain (6) since $F_n[f](z_n) \neq 0$ for all $n \in \mathbb{N}$.

Notice that the orthonormal basis $\{(e_{n,i})_{i=1}^{k_n}\}_{n=1}^{\infty}$ is, in particular, an unconditional basis. Therefore, we deduce that the series in (6) is pointwise unconditionally convergent and hence, absolutely convergent. The uniform convergence is a standard result in the setting of the RKHS theory [13] since $\|K_\sigma(\cdot)\|_{\mathbb{H}}$ is bounded (actually it is continuous) in compact subsets of $\mathbb{C}$. ■

The mapping T_σ is injective if and only if T_σ an isometry, or equivalently, if and only if the set $\{K_\sigma(z)\}_{z\in\mathbb{C}}$ is complete in $\mathbb{H}$ [11, 13]. In particular, whenever $k_n = 1$ for all $n \in \mathbb{N}$ and $\langle \sigma(z_n), e_n\rangle_{\mathbb{H}} \neq 0$ for all $n \in \mathbb{N}$, we have that the anti-linear mapping T_σ is a bijective isometry between $\mathbb{H}$ and $\mathcal{H}_\sigma$ since $\{K_\sigma(z_n) = P'(z_n)\langle \sigma(z_n), e_n\rangle_{\mathbb{H}} e_n\}_{n=1}^{\infty}$ is a complete sequence in $\mathbb{H}$. In this case, $\{T_\sigma(e_n)\}_{n=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}_\sigma$, and (6) is an orthonormal expansion in $\mathcal{H}_\sigma$. More specifically, the following sampling theorem which will be referred throughout the paper as the *σ-sampling theorem*, holds:

Corollary 1 (σ-sampling theorem) *Assume that $\langle \sigma(z_n), e_n\rangle_{\mathbb{H}} \neq 0$ for all $n \in \mathbb{N}$. Any function $f \in \mathcal{H}_\sigma$ can be expanded as the sampling series*

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{\langle \sigma(z), e_n\rangle_{\mathbb{H}}}{\langle \sigma(z_n), e_n\rangle_{\mathbb{H}}} \frac{P(z)}{(z - z_n)P'(z_n)}, \quad z \in \mathbb{C}. \tag{10}$$

The convergence of the series in (10) *is absolute and uniform in compact subsets of* $\mathbb{C}$.

Notice that whenever $\langle \sigma(z_m), e_m\rangle_{\mathbb{H}} = 0$ for all $m \in \mathbb{I}(\sigma) \subset \mathbb{N}$, the RKHS $\mathcal{H}_\sigma$ will be isometrically isomorphic to the closed subspace in $\mathbb{H}$ spanned by $\{e_n\}_{n\notin\mathbb{I}(\sigma)}$. If for any $m \in \mathbb{I}(\sigma)$ we suppose that $\langle \sigma(z), e_m\rangle_{\mathbb{H}} = 0$ for all $z \in \mathbb{C}$, formula (10) still remains valid for all $n \notin \mathbb{I}(\sigma)$.

Definition 2 *We denote by Σ the set of all entire $\mathbb{H}$-valued functions σ such that $\langle \sigma(z_n), e_n\rangle_{\mathbb{H}} \neq 0$ for all $n \in \mathbb{N}$, or such that $\langle \sigma(z), e_m\rangle_{\mathbb{H}} = 0$ for all $z \in \mathbb{C}$ whenever $m \in \mathbb{I}(\sigma)$. We define $\mathcal{H}_\Sigma$ as the set $\cup_{\sigma\in\Sigma}\mathcal{H}_\sigma$*

Notice that if we take the function $\sigma(z) := a \in \mathbb{H}$ constant, independently of the multiplicity of the eigenvalues of the operator $\mathcal{A}$, we have

$$\frac{F_n[f](z)}{F_n[f](z_n)} = 1, \qquad \text{for any } f \in \mathcal{H}_\sigma.$$

Therefore, the following sampling theorem holds:

Corollary 2 *Consider the entire $\mathbb{H}$-valued function defined by $K_a(z) := P(z)R_z(a)$. For a fixed $x \in \mathbb{H}$, the function f given by $f(z) = \langle K_a(z), x\rangle_{\mathbb{H}}$, $z \in \mathbb{C}$, can be recovered from its samples $\{f(z_n)\}_{n=1}^{\infty}$ through the Lagrange-type interpolation series*

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{P(z)}{(z-z_n)P'(z_n)}\,, \quad z \in \mathbb{C}\,.$$

The convergence of the series is absolute and uniform in compact subsets of $\mathbb{C}$.

Sampling theorems associated with a sampling kernel defined through an expression like (1) have been obtained in [17].

3 Approximation property derived from the σ-sampling theorem

In this Section we enlarge the class of sampling kernels by introducing the generalized Lagrange-Kramer sampling kernels. These kernels generalize most of the sampling kernels associated with differential or difference problems. Throughout this Section we assume that $k_n = 1$ for each $n \in \mathbb{N}$.

Let $\Psi : \mathbb{C} \setminus \{z_n\}_{n=1}^{\infty} \longrightarrow \mathbb{H}$ be a function whose expansion in the orthonormal basis $\{e_n\}_{n=1}^{\infty}$

$$\Psi(z) = \sum_{n=1}^{\infty} \frac{w_n(z)}{z-z_n}\, e_n\,,$$

verifies the two following conditions:

i) For each $n \in \mathbb{N}$, $w_n(z)$ is an entire function such that $w_n(z_n) \neq 0$.

ii) For each compact set $\mathcal{K} \subset \mathbb{C}$ there exists a constant $C_{\mathcal{K}}$ such that

$$\sup_{z\in\mathcal{K}} \sum_{n=1}^{\infty} \left| \frac{P(z)\, w_n(z)}{z-z_n} \right|^2 \leq C_{\mathcal{K}}\,,$$

for some entire function P having simple zeros at $\{z_n\}_{n=1}^{\infty}$.

Notice that condition *ii)* is independent of the chosen entire function P. Furthermore, the choice of $\{w_n\}_{n=1}^{\infty}$ will allow a variety of asymptotic behaviours for the sequence $\{z_n\}_{n=1}^{\infty}$.

Definition 3 *Let Ψ be a function as above. The kernel K_Ψ defined as*

$$\begin{array}{rccl} K_\Psi : & \mathbb{C} & \longrightarrow & \mathbb{H} \\ & z & \longrightarrow & K_\Psi(z) := P(z)\ \Psi(z)\,, \end{array}$$

is said to be a generalized Lagrange-Kramer sampling kernel.

Lemma 2 *Any generalized Lagrange-Kramer sampling kernel K_Ψ defines an entire $\mathbb{H}$-valued function.*

Proof: Using Theorem 1.1 in [14, p. 267] this is equivalent to proving that, for each $x \in \mathbb{H}$, the function defined as $g(z) := \langle K_\Psi(z), x\rangle_{\mathbb{H}}$, which reads as

$$g(z) = \sum_{n=1}^{\infty} \frac{P(z)\, w_n(z)}{z-z_n} \langle e_n, x\rangle_{\mathbb{H}}\,,$$

is an entire function. From the Cauchy-Schwarz inequality and condition *ii)* we obtain that the above series is uniformly bounded in compact subsets of $\mathbb{C}$. By using the classical Montel theorem [4] we obtain a subsequence that converges uniformly in compact subsets to an entire function which necessarily coincides with g. ■

For a fixed generalized Lagrange-Kramer sampling kernel K_Ψ we consider the set of functions

$$\mathcal{H}_{K_\Psi} := \Big\{ f : \mathbb{C} \longrightarrow \mathbb{C} \mid f(z) := \langle K_\Psi(z), x \rangle_{\mathbb{H}} : x \in \mathbb{H} \Big\} .$$

Proceeding as in Section 3, we can prove that $\mathcal{H}_{K_\Psi}$ is a RKHS of entire functions. It is worth remarking that, by using the same technique as in Theorem 6, the sampling formula

$$f(z) = \sum_{m=1}^{\infty} f(z_m) \frac{w_m(z)}{w_m(z_m)} \frac{P(z)}{(z - z_m)P'(z_m)} , \quad z \in \mathbb{C} ,$$

holds in $\mathcal{H}_{K_\Psi}$. This sampling formula justifies the name of generalized Lagrange-Kramer sampling kernel for K_Ψ. The main goal in this Section is to prove, using the σ-sampling theorem (see Corollary 1), that any function $f \in \mathcal{H}_{K_\Psi}$ can be approximated, uniformly in compact subsets of $\mathbb{C}$, by a sequence of entire functions in $\mathcal{H}_\Sigma$ (see Definition 2). For the sake of completeness we include the statement of the Moore-Smith theorem which will be needed later. Its proof can be found in [3, p. 236]:

Lemma 3 *Let M be a complete metric space with metric ρ, and let $\{x_{n,m}\}$, $n, m \in \mathbb{N}$, be given. Assume there are sequences $\{y_n\}$, $\{z_m\}$ in M such that*

1. $\lim_{n\to\infty} \rho(x_{n,m}, z_m) = 0$ *uniformly in m, and*
2. *For each $n \in \mathbb{N}$,* $\lim_{m\to\infty} \rho(x_{n,m}, y_n) = 0$.

Then there is $x \in M$ such that

$$\lim_{m\to\infty} \lim_{n\to\infty} \rho(x_{n,m}, x) = \lim_{n\to\infty} \lim_{m\to\infty} \rho(x_{n,m}, x) = \lim_{m,n\to\infty} \rho(x_{n,m}, x) = 0 .$$

The aforesaid approximation property reads as follows:

Theorem 2 *Any function f in $\mathcal{H}_{K_\Psi}$ is the uniform limit in compact subsets of $\mathbb{C}$ of a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}_\Sigma$.*

Proof: Let $f(z) = \langle K_\Psi(z), x \rangle_{\mathbb{H}}$ be a function belonging to $\mathcal{H}_{K_\Psi}$ obtained from a fixed $x \in \mathbb{H}$. For each $n \in \mathbb{N}$, consider $\sigma_n : \mathbb{C} \longrightarrow \mathbb{H}$ which satisfies

$$\langle \sigma_n(z), e_k \rangle = \begin{cases} w_k(z) & \text{si } k \le n \\ 0 & \text{si } k > n . \end{cases}$$

Taking $K_n(z) := K_{\sigma_n}(z)$, and having in mind that

$$K_\Psi(z) = \sum_{k=1}^{\infty} \frac{P(z)\, w_k(z)}{z - z_k} e_k ,$$

we have that $K_n(z) \longrightarrow K_\Psi(z)$ as $n \to \infty$ in $\mathbb{H}$. Now we consider the sequence in $\mathcal{H}_\Sigma$ given by $f_n(z) := \langle K_n(z), x \rangle_{\mathbb{H}}$. The pointwise convergence $f_n(z) \to f(z)$ as $n \to \infty$ in $\mathbb{C}$ is a straightforward consequence of the strong convergence of the kernels in $\mathbb{H}$. The σ-sampling theorem gives the sampling expansion

$$f_n(z) = \sum_{k=1}^{n} f_n(z_k) \frac{\langle \sigma_n(z), e_k \rangle_{\mathbb{H}}}{\langle \sigma_n(z_k), e_k \rangle_{\mathbb{H}}} \frac{P(z)}{(z-z_k)P'(z_k)} \,.$$

Next we prove that

$$\lim_{n\to\infty} \sum_{m=1}^{n} f_n(z_m) \left[\frac{\langle \sigma_n(z), e_m \rangle_{\mathbb{H}}}{\langle \sigma_n(z_m), e_m \rangle_{\mathbb{H}}} \frac{P(z)}{(z-z_m)P'(z_m)} \right] = \sum_{m=1}^{\infty} f(z_m) \left[\frac{w_m(z)}{w_m(z_m)} \frac{P(z)}{(z-z_m)P'(z_m)} \right]$$

uniformly in compact subsets of $\mathbb{C}$. To interchange the limit and the series we use the Moore-Smith theorem. To this end, consider

$$x_{m,n}(z) = \sum_{k=1}^{n} f_m(z_k) \left[\frac{\langle \sigma_m(z), e_k \rangle_{\mathbb{H}}}{\langle \sigma_m(z_k), e_k \rangle_{\mathbb{H}}} \frac{P(z)}{(z-z_k)P'(z_k)} \right] ,$$

where $z \in \mathbb{C}$. We have to prove that $x_{m,n}(z) \to f_m(z)$ as $n \to \infty$, uniformly in compact subsets of $\mathbb{C}$ and uniformly in m. Indeed,

$$\left| x_{m,n}(z) - f_m(z) \right|^2 \leq \|K_\Psi(z)\|_{\mathbb{H}}^2 \sum_{k=n+1}^{\infty} |\langle x, e_k \rangle_{\mathbb{H}}|^2 \,,$$

and $\|K_\Psi(z)\|_{\mathbb{H}}$ is uniformly bounded in compact subsets of $\mathbb{C}$.

On the other hand,

$$x_{m,n}(z) \longrightarrow \sum_{k=1}^{n} f(z_k) \left[\frac{w_k(z)}{w_k(z_k)} \frac{P(z)}{(z-z_k)P'(z_k)} \right] , \quad \text{as } m \to \infty \,, \tag{11}$$

uniformly in compact subsets of $\mathbb{C}$. To prove the uniform convergence, take $m > n$. We can write

$$\sum_{k=1}^{n} \frac{P(z)}{(z-z_k)P'(z_k)} \left[f_m(z_k) \frac{\langle \sigma_m(z), e_k \rangle_{\mathbb{H}}}{\langle \sigma_m(z_k), e_k \rangle_{\mathbb{H}}} - f(z_k) \frac{w_k(z)}{w_k(z_k)} \right] = \sum_{k=1}^{n} [f_m(z_k) - f(z_k)] \frac{w_k(z)}{w_k(z_k)} \frac{P(z)}{(z-z_k)P'(z_k)} \tag{12}$$

Now, given a compact $\mathcal{K}$ and $\epsilon > 0$, there exist $m_1, m_2, \ldots, m_n \in \mathbb{N}$ such that

$$|f_m(z_k) - f(z_k)| < \frac{\epsilon}{nR_{\mathcal{K}}} \,, \quad k = 1, \ldots, n \,,$$

for each $m \geq \max\{m_1, m_2, \ldots, m_n\}$, where

$$R_{\mathcal{K}} = \sup_{1\leq k\leq n} \sup_{z\in\mathcal{K}} \left\{ \left| \frac{P(z)}{(z-z_k)P'(z_k)} \frac{w_k(z)}{w_k(z_k)} \right| \right\} ,$$

and, as a consequence of (12), we obtain the uniform convergence in (11). Finally,the Moore-Smith theorem gives

$$f(z)=\sum_{m=1}^{\infty} f(z_m)\frac{w_m(z)}{w_m(z_m)}\frac{P(z)}{(z-z_m)P'(z_m)}\,,$$

and that $f_n \to f$ uniformly in compact subsets of $\mathbb{C}$ which concludes the proof. ■

The generalized Lagrange-Krmaer sampling kernel class enlarges the σ-resolvent sampling kernel class. Indeed, let $\mathcal{A}$ be a Hilbert-Schmidt operator. The sequence $\{1/z_n\}_{n=1}^{\infty}$ of its eigenvalues is in $\ell^2(\mathbb{N})$. As a consequence, the $\mathbb{H}$-valued function given by

$$K(z):=\sum_{n=1}^{\infty}\frac{P(z)}{z-z_n}\,e_n\,,$$

defines a Lagrange-Kramer sampling kernel since the series

$$\sum_{n=1}^{\infty}\frac{|P(z)|^2}{|z-z_n|^2}$$

is uniformly bounded in compact subsets of $\mathbb{C}$. Notice that this kernel K does not coincide with K_σ for any $\sigma\in\Sigma$.

Our next goal is to prove an inverse result for Theorem 2 in the following sense: If every function f defined by means of a kernel K can be uniformly approximated in compact subsets of $\mathbb{C}$ through a sequence of functions $\{f_m\}_{m=1}^{\infty}$, associated with a sequence of kernels $\{K_{\sigma_m}\}_{m=1}^{\infty}$ where $\sigma_m\in\Sigma$, then K is a generalized Lagrange-Kramer sampling kernel.

First, we need the following Lemma which proof is similar to those of Lemma 2:

Lemma 4 *A function $K:\mathbb{C}\longrightarrow\mathbb{H}$ defines an entire $\mathbb{H}$-valued function if and only if, for each $z\in\mathbb{C}$ the expansion of $K(z)$ in the orthonormal basis $\{e_n\}_{n=1}^{\infty}$*

$$K(z)=\sum_{n=1}^{\infty}G_n(z)\,e_n$$

satisfies the following conditions:

(a) For each $n\in\mathbb{N}$, G_n is an entire function, and

(b) $\|K(z)\|^2=\sum_{n=1}^{\infty}|G_n(z)|^2$ is uniformly bounded on compact subsets of $\mathbb{C}$.

Theorem 3 *Let $K:\mathbb{C}\longrightarrow\mathbb{H}$ be a kernel satisfying $K(z_n)\neq 0$ for all $n\in\mathbb{N}$. Assume that there exists a sequence of kernels $\{K_{\sigma_m}\}_{m=1}^{\infty}$, where $\sigma_m\in\Sigma$, such that for any function f defined in $\mathbb{C}$ as $f(z):=\langle K(z),x\rangle_{\mathbb{H}}$ for some $x\in\mathbb{H}$, the sequence of functions $\{f_m\}_{m=1}^{\infty}$ defined by $f_m(z)=\langle K_{\sigma_m}(z),x\rangle_{\mathbb{H}}$ verifies that $f_m\longrightarrow f$ as $m\to\infty$, uniformly in compact subsets of $\mathbb{C}$. Then, K is a generalized Lagrange-Kramer sampling kernel.*

Proof: The analytic convergence theorem assures that K is an entire function. As a consequence of Lemma 4, the coefficients G_n of its expansion $K(z)=\sum_{n=1}^{\infty}G_n(z)\,e_n$ are entire functions, and the series $\sum_{n=1}^{\infty}|G_n(z)|^2$ is uniformly bounded in compact subsets of $\mathbb{C}$.

On the other hand, for each $z\in\mathbb{C}$, $K_{\sigma_m}(z)\longrightarrow K(z)$ as $m\to\infty$, weakly in $\mathbb{H}$. In particular, taking a vector e_n in the orthonormal basis we obtain

$$\langle K_{\sigma_m}(z)-K(z),e_n\rangle_{\mathbb{H}}=\frac{\langle\sigma_m(z),e_n\rangle_{\mathbb{H}}}{z-z_n}P(z)-G_n(z)\longrightarrow 0 \quad \text{as } m\to\infty\,.$$

Hence, there exists $w_n(z) = \lim_{m\to\infty}\langle \sigma_m(z), e_n\rangle_{\mathbb{H}}$ defining an entire function for each $n \in \mathbb{N}$ via the analytic convergence theorem. Besides, $G_n(z) = \frac{w_n(z)}{z-z_n}P(z)$ and $K(z_n) = w_n(z_n)P'(z_n) \neq 0$ which implies that $w_n(z_n) \neq 0$ for all $n \in \mathbb{N}$. As a consequence, K is a generalized Lagrange-Kramer sampling kernel. ■

Having in mind the proof of the above theorem, its hypotheses can be relaxed in the following way:

Corollary 3 *Let $K : \mathbb{C} \longrightarrow \mathbb{H}$ be an entire kernel satisfying $K(z_n) \neq 0$ for all $n \in \mathbb{N}$. Assume that there exists a sequence of kernels $\{K_{\sigma_m}\}_{m=1}^{\infty}$, where $\sigma_m \in \Sigma$, such that, for each $z \in \mathbb{C}$, $K_{\sigma_m}(z) \longrightarrow K(z)$ weakly in $\mathbb{H}$ as $m \to \infty$. Then, K is a generalized Lagrange-Kramer sampling kernel.*

4 The case of constant σ: The spaces $\mathcal{H}_a$

From now on we confine ourselves to the case where $\sigma(z) := a \in \mathbb{H}$ is constant. We denote the corresponding RKHS of entire functions as $\mathcal{H}_a$. We assume that the multiplicity of each eigenvalue z_n of $\mathcal{A}$ is $k_n = 1$, denoting by $\{e_n\}_{n=1}^{\infty}$ the corresponding orthonormal basis of eigenfunctions. Our goal is to study the relationship between spaces $\mathcal{H}_a$ obtained from different choices of $a \in \mathbb{H}$. Recall that, for $f := T_a(x)$, where $x \in \mathbb{H}$, the sampling expansion reads

$$f(z) = \sum_{n=1}^{\infty} f(z_n)\frac{P(z)}{(z-z_n)P'(z_n)}\,, \quad z \in \mathbb{C}\,, \tag{13}$$

and $f(z_n) = P'(z_n)\overline{\langle x, e_n\rangle}_{\mathbb{H}}\langle a, e_n\rangle_{\mathbb{H}}$ for each $n \in \mathbb{N}$.

Consider the spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ associated with a, $b \in \mathbb{H}$ and set $f \in \mathcal{H}_a \cap \mathcal{H}_b$. There exist $x, y \in \mathbb{H}$ such that $f = T_b(x) = T_a(y)$. For each $k \in \mathbb{N}$ we have

$$f(z_k) = P'(z_k)\langle e_k, x\rangle_{\mathbb{H}}\langle b, e_k\rangle_{\mathbb{H}} = P'(z_k)\langle e_k, y\rangle_{\mathbb{H}}\langle a, e_k\rangle_{\mathbb{H}}\,.$$

Hence, $\langle e_k, x\rangle_{\mathbb{H}}\langle b, e_k\rangle_{\mathbb{H}} = \langle e_k, y\rangle_{\mathbb{H}}\langle a, e_k\rangle_{\mathbb{H}}$ for each $k \in \mathbb{N}$.

Lemma 5 *Let a, b in $\mathbb{H}$. Consider the sequence $\{\xi_k := \langle b, e_k\rangle_{\mathbb{H}}/\langle a, e_k\rangle_{\mathbb{H}}\}_{k=1}^{\infty}$, where we assume that $\langle b, e_m\rangle_{\mathbb{H}} = 0$ whenever $\langle a, e_m\rangle_{\mathbb{H}} = 0$. In this case, ξ_m is taken to be 1. Then,*

$$\mathcal{H}_b \subseteq \mathcal{H}_a \Leftrightarrow \{\xi_k\}_{k=1}^{\infty} \in \ell^{\infty}(\mathbb{N})\,.$$

Moreover, the inclusion $\mathcal{H}_b \hookrightarrow \mathcal{H}_a$ is continuous.

Proof: First of all, notice that whenever $f(z_k) = g(z_k)$ for all $k \in \mathbb{N}$, where $f \in \mathcal{H}_a$ and $g \in \mathcal{H}_b$, the sampling theorem (13) implies that $f = g$. Therefore, a function $g = T_b(x) \in \mathcal{H}_b$, where $x \in \mathbb{H}$, belongs to $\mathcal{H}_a$ if and only if there exists $y \in \mathbb{H}$ such that

$$\overline{\langle x, e_k\rangle}_{\mathbb{H}}\langle b, e_k\rangle_{\mathbb{H}} = \overline{\langle y, e_k\rangle}_{\mathbb{H}}\langle a, e_k\rangle_{\mathbb{H}}\,, \quad \text{for all } k \in \mathbb{N}\,. \tag{14}$$

Because of (14), the inclusion $\mathcal{H}_b \subseteq \mathcal{H}_a$ is equivalent to $\{\langle x, e_k\rangle_{\mathbb{H}}\,\xi_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ for each $x \in \mathbb{H}$. The last assertion is equivalent to the fact that the sequence $\{\xi_k\}_{k=1}^{\infty}$ belongs to $\ell^{\infty}(\mathbb{N})$.

To prove the continuity, consider $f \in \mathcal{H}_b$. There exists $x \in \mathbb{H}$ such that $T_b(x) = f$ and $\|f\|_b^2 = \|x\|^2 = \sum_{k=1}^{\infty}|\langle x, e_k\rangle_{\mathbb{H}}|^2$. On the other hand, there exists $y \in \mathbb{H}$ such that $f = T_a(y)$, $\|f\|_a^2 = \|y\|^2$, and $\langle y, e_k\rangle_{\mathbb{H}} = \langle x, e_k\rangle_{\mathbb{H}}\,\xi_k$ for all $k \in \mathbb{N}$. Finally,

$$\|f\|_b^2 = \sum_{k=1}^{\infty}|\xi_k|^2\,|\langle x, e_k\rangle_{\mathbb{H}}|^2 \leq \|\{\xi_k\}\|_{\infty}^2\,\|f\|_a^2\,,$$

from which we obtain the continuity for the inclusion $\mathcal{H}_b \subseteq \mathcal{H}_a$. ■

Next, in the Hilbert space $\mathbb{H}$ we can define a binary relation " $\approx$ " as follows: For $a,\ b \in \mathbb{H}$,

$$a \approx b \Leftrightarrow \exists \alpha,\ \beta > 0 \text{ such that } 0 < \alpha \leq |\langle b, e_k\rangle_{\mathbb{H}}/\langle a, e_k\rangle_{\mathbb{H}}| \leq \beta \text{ for all } k \in \mathbb{N}\,.$$

Lemma 6 *The binary relation " $\approx$ " defines an equivalence relation in $\mathbb{H}$.*

Proof: The reflexivity is trivial. Now, assume $a \approx b$ with $0 < \alpha \leq |\langle b, e_k\rangle_{\mathbb{H}}/\langle a, e_k\rangle_{\mathbb{H}}| \leq \beta$. Then, $0 < 1/\beta \leq |\langle a, e_k\rangle_{\mathbb{H}}/\langle b, e_k\rangle_{\mathbb{H}}| \leq 1/\alpha$. Finally, consider $a \approx b$ and $b \approx c$, i.e.,

$$0 < \alpha \leq |\langle b, e_k\rangle_{\mathbb{H}}/\langle a, e_k\rangle_{\mathbb{H}}| \leq \beta\,, \quad 0 < \alpha' \leq |\langle c, e_k\rangle_{\mathbb{H}}/\langle b, e_k\rangle_{\mathbb{H}}| \leq \beta'\,.$$

Hence, $0 < \alpha\alpha' \leq |\langle c, e_k\rangle_{\mathbb{H}}/\langle a, e_k\rangle_{\mathbb{H}}| \leq \beta\beta'$ which implies $a \approx c$. ■

As a consequence of the above Lemma the following result holds:

Theorem 4 *The spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ coincide (as sets of functions) if and only if $a \approx b$. Moreover, the associated norms in $\mathcal{H}_a$ and $\mathcal{H}_b$ are equivalents.*

Consider the set of entire functions $\mathcal{H}_{res} := \cup_{a\in\mathbb{H}}\mathcal{H}_a$. As a consequence of Lemma 5, $\mathcal{H}_{res}$ is a linear space of entire functions. Indeed, given two spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ there exists another $\mathcal{H}_c$ which contains them (take for instance $c \in \mathbb{H}$ having $\{|\langle a, e_k\rangle_{\mathbb{H}}| + |\langle b, e_k\rangle_{\mathbb{H}}|\}_{k=1}^{\infty}$ as Fourier coefficients).

In the present context of constant σ, a Lagrange-Kramer sampling kernel K_A will be associated with a sequence $A := \{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{C}\setminus\{0\}$ such that the series $\sum_{n=1}^{\infty}\frac{|P(z)|^2|\alpha_n|^2}{|z-z_n|^2}$ is uniformly bounded in compact subsets of $\mathbb{C}$, and

$$K_A(z) = \sum_{n=1}^{\infty}\frac{P(z)\,\alpha_n}{z-z_n}\,e_n\,, \quad z \in \mathbb{C}\,.$$

Notice that it is easy to prove the existence of such a sequence. Indeed, given the closed disk $\overline{D}(0;m)$, there exists a constant C_m such that $\big|P(z)/(z-z_n)\big| \leq C_m$ for all $z \in \overline{D}(0;m)$ and for all $n \in \mathbb{N}$. Choosing, for instance, $\alpha_n = 1/(2^n C_n)$, the conditions for a Lagrange-Kramer sampling kernel are satisfied.

The set $\mathcal{H}_A := \{f : \mathbb{C} \longrightarrow \mathbb{C} \mid f(z) := \langle K_A(z), x\rangle_{\mathbb{H}},\ x \in \mathbb{H}\}$ is a RKHS of entire functions. In this case, $\mathcal{H}_A \subseteq \mathcal{H}_B$ if and only if the sequence $\{\alpha_n/\beta_n\}_{n=1}^{\infty}$ belongs to $\ell^{\infty}(\mathbb{N})$. Moreover, the inclusion $\mathcal{H}_A \hookrightarrow \mathcal{H}_B$ is continuous. As a consequence, the space $\mathcal{H}_{wt} := \cup\mathcal{H}_A$, where the union is taken over all sequences A defining Lagrange-Kramer sampling kernels, is a linear space of entire functions.

Lemma 7 *Given $a \in \mathbb{H}$, assume that there exists a sequence $A := \{\alpha_n\}_{n=1}^{\infty}$ defining a Lagrange-Kramer sampling kernel such that:*

$$|\langle a, e_k\rangle_{\mathbb{H}}| \leq |\alpha_k|\,,\ \textit{for each } k \in \mathbb{N}\,.$$

Then, $\mathcal{H}_a \subseteq \mathcal{H}_A$ with continuous inclusion.

Proof: It is enough to take into account that the sequence $\{\langle a, e_k\rangle_{\mathbb{H}}/\alpha_k\}_{k=1}^{\infty}$ belongs to $\ell^{\infty}(\mathbb{N})$. ■

Whenever the operator $\mathcal{A}$ is Hilbert-Schmidt, the sequence $\widetilde{A} := \{\alpha_n = 1\}_{n=1}^{\infty}$ defines a Lagrange-Kramer sampling kernel. As a consequence of Lemma 7 we have that $\mathcal{H}_a \subseteq \mathcal{H}_{\widetilde{A}}$ for all $a \in \mathbb{H}$, and therefore $\mathcal{H}_{res} \subseteq \mathcal{H}_{\widetilde{A}}$ in this case.

4.1 The topology of the uniform convergence in $\mathcal{H}_{res}$ and $\mathcal{H}_{wt}$

Consider the space $\mathcal{H}(\mathbb{C})$ of all entire functions endowed with the topology of uniform convergence in compact sets. Recall that this topology is associated with the metric

$$d(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\,\frac{\|f-g\|_n}{1+\|f-g\|_n}\,,\quad f,g\in\mathcal{H}(\mathbb{C})\,,$$

where $\|f-g\|_m:=\sup_{z\in\overline{D}(0;m)}|f(z)-g(z)|$. Thus, $\mathcal{H}(\mathbb{C})$ becomes a Fréchet space [4].

We endow $\mathcal{H}_{res}$, a linear subspace of $\mathcal{H}(\mathbb{C})$, with the induced topology of the uniform convergence in compact sets of $\mathbb{C}$. In Theorem 2 we have proved that $\mathcal{H}_{wt}\subset\overline{\mathcal{H}}_{res}$, the closure taken with respect to the above topology. In fact, any $f\in\mathcal{H}_{wt}$, i.e., $f(z)=\langle K_A(z),x\rangle_{\mathbb{H}}$, where $A:=\{\alpha_n\}_{n=1}^{\infty}$ defines the Lagrange-Kramer sampling kernel K_A, can be approximated, uniformly in compact sets of $\mathbb{C}$, by means of the sequence $\{f_m\}_{m=1}^{\infty}\subset\mathcal{H}_{res}$ given by $f_m(z)=\langle K_{a_m}(z),x\rangle_{\mathbb{H}}$ where a_m is the vector in $\mathbb{H}$ whose Fourier coefficients in the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ are given by

$$\langle a_m,e_i\rangle_{\mathbb{H}}=\begin{cases}1 & \text{if } i\le m\\ 0 & \text{if } i>m.\end{cases}$$

For each $m\in\mathbb{N}$, define $\mathcal{H}_m:=\mathcal{H}_{a_m}$. Notice that $\mathcal{H}_m$ is the same set of functions for any element in $\mathbb{H}$ having only the first m Fourier coefficients different from 0. Thus, we may also consider the linear space $\mathcal{H}_{fin}:=\cup_{m\in\mathbb{N}}\mathcal{H}_m$. As linear subspaces of $\mathcal{H}(\mathbb{C})$, the closures of $\mathcal{H}_{res}$ and $\mathcal{H}_{fin}$ coincide. Obviously we have the inclusion $\overline{\mathcal{H}}_{fin}\subseteq\overline{\mathcal{H}}_{res}$. Concerning the other inclusion, Theorem 2 gives $\mathcal{H}_{res}\subseteq\overline{\mathcal{H}}_{fin}$, and consequently $\overline{\mathcal{H}}_{fin}=\overline{\mathcal{H}}_{res}$.

In general, as we will see in Section 4.2, $\overline{\mathcal{H}}_{fin}=\overline{\mathcal{H}}_{res}\not\subset\mathcal{H}_{wt}$. Next, we prove a sufficient condition ensuring that the uniform limit in compact subsets of a sequence $\{f_m\}_{m=1}^{\infty}$ in $\mathcal{H}_{fin}$ belongs to $\mathcal{H}_{wt}$. To this end, let $\{f_m\}_{m=1}^{\infty}$ be a sequence in $\mathcal{H}_{fin}$ such that $f_m\in\mathcal{H}_m$, for each $m\in\mathbb{N}$, and $f_m\longrightarrow f$ as $m\to\infty$, uniformly in compact sets of $\mathbb{C}$. Consider $A:=\{\alpha_n\}_{n=1}^{\infty}$ a sequence defining a Lagrange-Kramer sampling kernel K_A. For each $m\in\mathbb{N}$, let b_m be the element in $\mathbb{H}$ whose Fourier coefficientes with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ are given by

$$\langle b_m,x_i\rangle_{\mathbb{H}}=\begin{cases}\alpha_i & \text{if } i\le m\\ 0 & \text{if } i>m,\end{cases}$$

and denote by K_{b_m} the associated kernel. Since $f_m\in\mathcal{H}_m=\mathcal{H}_{b_m}$, there exists $h_m\in\mathbb{H}$ such that $f_m(z)=\langle K_{b_m}(z),h_m\rangle_{\mathbb{H}}$, for $z\in\mathbb{C}$. Under the above circumstances, the following result holds:

Theorem 5 *Assume that $f_m\longrightarrow f$ as $m\to\infty$, uniformly in compact subsets of $\mathbb{C}$, and that the sequence $\{h_m\}_{m=1}^{\infty}$ is bounded in $\mathbb{H}$. Then, there exists $x\in\mathbb{H}$ such that $f(z)=\langle K_A(z),x\rangle_{\mathbb{H}}$ for all $z\in\mathbb{C}$.*

Proof: For $m\in\mathbb{N}$ with $m\ge i$ we have

$$|f_m(z_i)-f_n(z_i)|=|\alpha_i P'(z_i)||\langle e_i,h_m-h_n\rangle_{\mathbb{H}}|\,,$$

from which we conclude that

$$\langle h_n-h_m,\sum_{k=1}^{M}\lambda_k\,e_k\rangle_{\mathbb{H}}\longrightarrow 0\quad\text{ as } n,m\to\infty \tag{15}$$

in the dense subspace of all the finite linear combinations of vectors in the orthonormal basis $\{e_n\}_{n=1}^{\infty}$. As a consequence of the boundedness of the sequence $\{h_n\}_{n=1}^{\infty}$ we obtain that it is a weakly Cauchy sequence in $\mathbb{H}$. Indeed, let $x \in \mathbb{H}$. Given $\varepsilon > 0$, there exists x_ε, a finite linear combination of elements in the basis $\{e_n\}_{n=1}^{\infty}$, such that $\|x - x_\varepsilon\|_{\mathbb{H}} \le \varepsilon$. Hence,

$$|\langle h_n - h_m, x\rangle_{\mathbb{H}}| \le |\langle h_n - h_m, x - x_\varepsilon\rangle_{\mathbb{H}}| + |\langle h_n - h_m, x_\varepsilon\rangle_{\mathbb{H}}| \le 2M\varepsilon + |\langle h_n - h_m, x_\varepsilon\rangle_{\mathbb{H}}|\,,$$

where M denotes a bound for $\{h_m\}_{m=1}^{\infty}$. Taking (15) into account, we obtain that $\langle h_n - h_m, x\rangle_{\mathbb{H}} \longrightarrow 0$ as $n, m \to \infty$. Since every Hilbert space is weakly complete [1, p. 45], there exists $x \in \mathbb{H}$ such that $h_n \longrightarrow x$ weakly in $\mathbb{H}$ as $n \to \infty$. Set $h(z) = \langle K_A(z), x\rangle_{\mathbb{H}}$, $z \in \mathbb{C}$. Our goal is to prove that $h(z) = f(z)$ for all $z \in \mathbb{C}$. First, notice that $f(z_n) = h(z_n)$ for all $n \in \mathbb{N}$. Indeed,

$$f_m(z_n) = \langle b_m, e_n\rangle_{\mathbb{H}} P'(z_n)\langle e_n, h_m\rangle_{\mathbb{H}} \longrightarrow \alpha_n P'(z_n)\langle e_n, x\rangle_{\mathbb{H}} = h(z_n)$$

as $m \to \infty$. The proof will be concluded if we prove that f satisfies the same sampling formula than h does, namely, formula (13). To this end we use an argument similar to that used in Theorem 2. Define

$$x_{m,n}(z) = \sum_{k=1}^{n} f_m(z_k)\frac{P(z)}{P'(z_k)(z - z_k)}\,.$$

For a fixed $m \in \mathbb{N}$, we have that $x_{m,n}(z) \longrightarrow f_m(z)$ as $n \to \infty$ and $z \in \mathbb{C}$. Furthermore, this convergence is uniform in m. Indeed,

$$\begin{aligned}
\Big|x_{m,n}(z) - f_m(z)\Big|^2 &= \Big|\sum_{k=n+1}^{\infty} \frac{P(z)\langle b_m, e_k\rangle_{\mathbb{H}}}{z - z_k}\langle e_k, h_m\rangle_{\mathbb{H}}\Big|^2 \\
&\le \sum_{k=n+1}^{\infty}\left|\frac{P(z)\langle b_m, e_k\rangle_{\mathbb{H}}}{z - z_n}\right|^2 \sum_{k=n+1}^{\infty} |\langle h_m, e_k\rangle_{\mathbb{H}}|^2 \le \sum_{k=n+1}^{\infty}\left|\frac{P(z)\alpha_k}{z - z_k}\right|^2 \|h_m\|^2 \\
&\le C\sum_{k=n+1}^{\infty}\left|\frac{P(z)\alpha_k}{z - z_k}\right|^2 \longrightarrow 0
\end{aligned}$$

as $n \to \infty$, regardless $m \in \mathbb{N}$.

On the other hand, as $m \to \infty$ we have

$$x_{m,n}(z) \longrightarrow \sum_{k=1}^{n} f(z_k)\frac{P(z)}{(z - z_k)P'(z_k)} = \sum_{k=1}^{n} h(z_k)\frac{P(z)}{(z - z_k)P'(z_k)}$$

Finally, using the Moore-Smith theorem we obtain

$$\lim_{m\to\infty} f_m(z) = \lim_{m\to\infty}\sum_{k=1}^{\infty} f(z_k)\frac{P(z)}{(z - z_k)P'(z_k)} = \sum_{k=1}^{\infty} f(z_k)\frac{P(z)}{(z - z_k)P'(z_k)} = f(z)\,,$$

that is, $f(z) = h(z) = \langle K_A(z), x\rangle_{\mathbb{H}}$ for all $z \in \mathbb{C}$. ■

4.2 The Paley-Wiener case

Consider the boundary value problem

$$\begin{aligned}
-iy'(t) &= zy(t)\,, \quad t \in (-\pi, \pi)\,, \\
y(-\pi) &= y(\pi)\,.
\end{aligned}$$

Consider the associated operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow L^2(-\pi,\pi)$ defined by $\mathcal{A}f := -if'$, where

$$\mathcal{D}(\mathcal{A}) := \big\{ f : [-\pi,\pi] \to \mathbb{C} \mid f \in AC[-\pi,\pi], f' \in L^2(-\pi,\pi) \text{ and } f(\pi) = f(-\pi) \big\} .$$

$\mathcal{A}$ is self-adjoint, its spectrum is given by $\sigma(\mathcal{A}) = \{z_n = n : n \in \mathbb{Z}\}$ and the corresponding sequence of orthogonal eigenfunctions is $\{e_n(t) = e^{int}\}_{n\in\mathbb{Z}}$,

The resolvent operator of $\mathcal{A}$, $R_z = (zI - \mathcal{A})^{-1}$, is given by

$$[R_z g](t) = \frac{e^{iz(t+\pi)}}{e^{iz\pi} - e^{-iz\pi}} \int_{-\pi}^{\pi} e^{-izs} g(s)\, ds - e^{izt} \int_{-\pi}^{t} e^{-izs} g(s)\, ds\,, \quad t \in (-\pi,\pi)\,,$$

where $g \in \mathbb{H} := L^2(-\pi,\pi)$ and $z \in \mathbb{C} \setminus \mathbb{Z}$. Therefore, in this case, if we take $P(z) = e^{iz\pi} - e^{-iz\pi} = 2i \sin \pi z$, the associated resolvent sampling kernel is given by

$$[K_g(z)](t) = e^{iz(t+\pi)} \int_{-\pi}^{\pi} e^{-izs} g(s)\, ds - 2ie^{izt} \sin \pi z \int_{-\pi}^{t} e^{-izs} g(s)\, ds\,, \quad t \in (-\pi,\pi)\,,$$

for $g \in \mathbb{H}$. The associated sampling result reads as follows:

For a fixed $f \in \mathbb{H}$, the function $F(z) := \langle K_g(z), f\rangle_{\mathbb{H}}$, $z \in \mathbb{C}$ can be expanded as the cardinal sampling series:

$$F(z) = \sum_{n=-\infty}^{\infty} F(n)\, \operatorname{sinc}(z-n)\,.$$

Notice that the classical Fourier kernel is a Lagrange-Kramer sampling kernel:

$$[K(z)](t) := \frac{e^{izt}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} \operatorname{sinc}(z-n) \frac{e^{int}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sin \pi z}{\pi(z-n)} \frac{e^{int}}{\sqrt{2\pi}}\,,$$

for the sequence $A := \{(-1)^n/(2\pi i)\}_{n=-\infty}^{\infty}$ and $P(z) = 2i \sin \pi z$. The corresponding space $\mathcal{H}_A$ coincides with the classical Paley-Wiener space PW_π.

We denote $\mathcal{H}_{PW} := \mathcal{H}_{res}$. Notice that, as a consequence of Lemma 7, the inclusion $\mathcal{H}_{PW} \subset PW_\pi$ holds. Concerning the closure of $\mathcal{H}_{PW}$ with respect to the topology of the uniform convergence in compact sets we have the following result:

Theorem 6 *The subspace $\mathcal{H}_{PW}$ is dense in $\mathcal{H}(\mathbb{C})$.*

Proof: First we prove that $PW_\pi \subseteq \overline{\mathcal{H}}_{PW}$. Let G be in PW_π and consider $g := \mathcal{F}^{-1}(G)$ in $L^2(-\pi,\pi)$. For fixed $z \in \mathbb{C}$, the function $t \mapsto [K_g(z)](t)$ is continuous in $[-\pi,\pi]$. Moreover, $[K_g(z)](t_n) \to [K_g(z)](t)$, as $n \to \infty$, uniformly on compact subsets of $\mathbb{C}$ whenever $t_n \to t$, as $n \to \infty$.

Let $\{h_n\}_{n=1}^{\infty}$ be the sequence of functions defined by

$$h_n(t) = \begin{cases} n & \text{if } -\pi \le t \le -\pi + \frac{1}{n} \\ \\ 0 & \text{if } -\pi + \frac{1}{n} < t \le \pi \end{cases}$$

For each $n \in \mathbb{N}$, define $f_n(z) = \langle K_g(z), h_n\rangle \in \mathcal{H}_g$. The sequence of entire functions $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}_g$ converges to the given function $G \in PW_\pi$ uniformly in compact sets of $\mathbb{C}$. Indeed,

$$f_n(z) = \int_{-\pi}^{-\pi+\frac{1}{n}} [K_g(z)](s)\, ds = [K_g(z)](\xi_n)\ , \text{ with } \xi_n \in [-\pi, -\pi + \frac{1}{n}]\,,$$

where we have used the mean value theorem for integrals. Hence,

$$f_n(z) \to [K_g(z)](-\pi) = G(z) \in PW_\pi \text{ as } n \to \infty\,,$$

uniformly in compact subsets of $\mathbb{C}$.

We finish the proof by showing that any polynomial can be approximated, uniformly in compact subsets, by functions in PW_π. Consider a sequence $\{g_n\}_{n=1}^\infty$ in $L^2(-\pi,\pi)$ satisfying:

1. $g_n \in C^\infty$, $g_n(x) \geq 0$ and $\operatorname{supp} g_n \subset [-\frac{1}{n}, \frac{1}{n}]$, for each $n \in \mathbb{N}$.

2. $\displaystyle\int_{-\frac{1}{n}}^{\frac{1}{n}} g_n(s)\, ds = 1$.

The sequence $\{g_n\}_{n=1}^\infty$ converges to Dirac's delta in $\mathcal{E}'$, the space of compact supported distributions. By the Paley-Wiener-Schwartz-Ehrenpreis theorem, $\{\widehat{g_n}\}_{n=1}^\infty$ converges uniformly in compact subsets of $\mathbb{C}$ to the constant function 1. Let $p(z)$ be a polynomial in the complex variable z. The sequence of entire functions $\{p_n(z) := p(z)\widehat{g_n}(z)\}_{n=1}^\infty$ converges to $p(z)$ as $n \to \infty$, uniformly in compact subsets of $\mathbb{C}$. Moreover, for each $n \in \mathbb{N}$, the function p_n belongs to PW_π. Indeed, p_n is the Fourier transform of $p(D)g_n$, where $Dh = -ih'$ and $\widehat{h}(z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-izs}h(s)\, ds$. Since the set of polynomials is dense in $\mathcal{H}(\mathbb{C})$, we conclude that PW_π is dense in $\mathcal{H}(\mathbb{C})$ with respect to the compact uniform convergence. ■

Consequently, in the Paley-Wiener case we have the inclusions (as subspaces of $\mathcal{H}(\mathbb{C})$ with the topology of uniform convergence in compact sets):

$$\mathcal{H}_{PW} \subset PW_\pi \subset \overline{\mathcal{H}}_{PW} = \mathcal{H}(\mathbb{C})\,.$$

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