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GENERALIZED IRREGULAR SAMPLING IN SHIFT-INVARIANT SPACES

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This article concerns the problem of stable recovering of any function in a shift-invariant space from irregular samples of some filtered versions of the function itself. These samples arise as a perturbation of regular samples. The starting point is the generalized regular sampling theory which allows any function f in a shift-invariant space to be recovered from the samples at $\{rn\}_{n\in\mathbb{Z}}$ of s filtered versions $\mathcal{L}_1 f, \mathcal{L}_2 f, \ldots, \mathcal{L}_s f$ of f, where the number of channels s is greater or equal than the sampling period r. These regular samples can be expressed as the frame coefficients of a function related to f in $L^2(0,1)$ with respect to certain frame for $L^2(0,1)$. The irregular samples are also obtained as a perturbation of the aforesaid frame. As a natural consequence, the irregular sampling results arise from the theory of perturbation of frames. The paper concludes by putting the theory to work in some spline examples where Kadec-type results are obtained.

Keywords: Shift-invariant spaces; perturbation of frames; generalized irregular sampling.

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1. Statement of the Problem

Suppose that s linear time-invariant systems (filters) $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, are defined on a shift-invariant space V_φ of $L^2(\mathbb{R})$

$$V_\varphi := \left\{ f(t) = \sum_{n\in\mathbb{Z}} a_n \, \varphi(t-n) \, : \, \{a_n\} \in \ell^2(\mathbb{Z}) \right\},$$

where the function $\varphi \in L^2(\mathbb{R})$ is a stable generator for V_φ. In Ref. 7, it has been proved that any function $f \in V_\varphi$ is recoverable by means of a stable sampling formula which involves the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$, where the sampling period $r \in \mathbb{N}$ necessarily satisfies $r \le s$. Concretely, the sampling formula for $f \in V_\varphi$ reads

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \big(\mathcal{L}_j f\big)(rn)\ S_j(t - rn)\,, \quad t \in \mathbb{R}\,,$$

where the sequence $\{S_j(t-rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for V_φ. This frame is a Riesz basis for V_φ whenever $r = s$.

From a practical point of view, it is desirable to have a similar result, but for a sequence of samples taken with a non-uniform distribution along the real line, this may not be the case. A straightforward application of this result would be the recovering of signals affected by time-jitter error, i.e., taken at points $t_n = rn + \varepsilon_n$ with ε_n some measurement uncertainty. Very often in the mathematical literature, this problem has been solved as a perturbation problem of orthonormal (Riesz) bases or frames in a Hilbert space. A classical example where this methodology applies is the irregular sampling in Paley–Wiener spaces. Indeed, any function in the classical Paley–Wiener space

$$PW_\pi := \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \ :\ \text{supp}\ \widehat{f} \subseteq [-\pi, \pi]\}\,,$$

i.e., bandlimited to $[-\pi, \pi]$, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ on the integers as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\text{sinc}(t - n)\,, \quad t \in \mathbb{R}\,, \tag{1.1}$$

where sinc denotes the cardinal sine function, $\text{sinc}(t) = \sin \pi t/\pi t$. For any $f \in PW_\pi$, one has

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(w) e^{iwt} dw = \left\langle \widehat{f}, \frac{e^{-iwt}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]}, \quad t \in \mathbb{R}\,,$$

so any value $f(t_n)$ of f is the inner product in $L^2[-\pi, \pi]$ of its Fourier transform $\widehat{f}$ and the complex exponential $e^{-it_n w}/\sqrt{2\pi}$. Thus, expanding $\widehat{f}$ with respect to the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ in $L^2[-\pi, \pi]$, and applying the inverse Fourier transform $\mathcal{F}^{-1}$, one deduces the Shannon sampling formula (1.1). An irregular sampling formula in PW_π at a sequence $\{t_n\}_{n\in\mathbb{Z}}$ of real points may be obtained by perturbating the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ in such a way that the sequence of complex exponentials $\{e^{-it_n w}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi, \pi]$. This is the case if, for instance, the sequence $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ verifies the Kadec condition: $\sup_{n\in\mathbb{Z}}|t_n - n| < 1/4$. Here, the classical Paley–Wiener criterion on stability of bases has been used (see Ref. 17, p. 38). Moreover, the Paley–Wiener-Levinson sampling theorem states that any function $f \in PW_\pi$ can be recovered from its samples $\{f(t_n)\}_{n\in\mathbb{Z}}$ by means of the Lagrange-type interpolation series

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(t)}{G'(t_n)(t - t_n)}\,, \quad t \in \mathbb{R}\,,$$

where G stands for the infinite product $G(t) := (t - t_0)\prod_{n=1}^{\infty}(1 - t/t_n)(1 - t/t_{-n})$ (see Ref. 17).

This perturbation technique has been successfully used for obtaining irregular sampling formulas in a general shift-invariant space V_φ. See, for instance, the papers of Chen *et al.*,[1,2] García *et al.*,[8] Liu and Walter,[13] Sun and Zhou,[15] and Sun.[14] See also Liu.[12]

The main aim in this paper is to recover any function $f \in V_\varphi$ from the perturbed sequence of samples $\{(\mathcal{L}_j f)(rn+\varepsilon_{j,n})\}$. It has been proved in Ref. 7 that the regular samples $\{(\mathcal{L}_j f)(rn)\}$ can be expressed as the frame coefficients of an appropriate function in $L^2(0,1)$, related to f, with respect to a particular frame in $L^2(0,1)$. Recall that a sequence $\{f_k\}$ is a frame for a separable Hilbert space $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|f\|^2 \leq \sum_k |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}\,.$$

Given a frame $\{f_k\}$ for $\mathcal{H}$ the representation property of any vector $f \in \mathcal{H}$ as a series $f = \sum_k c_k f_k$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for overcomplete frames) is sacrificed. For instance, any $f \in \mathcal{H}$ can be expanded as

$$f = \sum_k \langle f, S^{-1} f_k\rangle f_k = \sum_k \langle f, f_k\rangle S^{-1} f_k\,,$$

where S^{-1} denotes the inverse of the frame operator $Sf := \sum_k \langle f, f_k\rangle f_k$, which defines $\{S^{-1} f_k\}$, the canonical dual frame of $\{f_k\}$. For more details on the frame theory see the superb Christensen's monograph[3] and references therein. As we prove in Sec. 3, the irregular samples $\{(\mathcal{L}_j f)(rn + \varepsilon_{j,n})\}$ are obtained as a perturbation of the aforesaid frame. Thus, the theory on perturbation of frames (see Chap. 15 in Ref. 3) yields generalized irregular sampling in the shift-invariant space V_φ for suitable error sequences $\{\varepsilon_{j,n}\}$. Moreover, for some important examples, the allowed sequences $\{\varepsilon_{j,n}\}$ will be given in terms of $\sup_{j,n}|\varepsilon_{j,n}|$.

2. Preliminaries on Generalized Sampling in Shift-Invariant Spaces

In this section we introduce the preliminaries on shift-invariant spaces needed in the sequel. Also, we present the generalized regular sampling theory for these spaces as stated in Ref. 7.

2.1. *Shift-invariant spaces*

Let $\varphi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space

$$V_\varphi := \left\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot - n) \,:\, \{a_n\} \in \ell^2(\mathbb{Z}) \right\} \subset L^2(\mathbb{R})\,,$$

i.e., the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for V_φ if and only if $0 < \|\Phi\|_0 \le \|\Phi\|_\infty < \infty$, where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2$ in $(0,1)$, and $\|\Phi\|_\infty$ its essential supremum ($\widehat{\varphi}$ stands for the Fourier transform $\widehat{\varphi}(w) := \int_{-\infty}^{\infty} \varphi(t)e^{-2\pi i t w}dt$). Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds (see Ref. 3, p. 143).

We assume throughout the paper that the functions in the shift-invariant space V_φ are continuous on $\mathbb{R}$. This is equivalent to the generator φ being continuous on $\mathbb{R}$ with the function $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ uniformly bounded on $\mathbb{R}$ (see Ref. 18). Thus, any $f \in V_\varphi$ is defined on $\mathbb{R}$ as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n \varphi(t-n)$.

Besides, V_φ is a reproducing kernel Hilbert space (RKHS) since the evaluation functionals are bounded in V_φ. Indeed, for each fixed $t \in \mathbb{R}$ we have

$$|f(t)|^2 \le \frac{\|f\|^2}{\|\Phi\|_0} \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 \,, \quad f \in V_\varphi \,, \tag{2.1}$$

where we have used Cauchy–Schwarz's inequality in $f(t) = \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n)$, and the Riesz basis condition

$$\|\Phi\|_0 \sum_{n\in\mathbb{Z}} |a_n|^2 \le \|f\|^2 \le \|\Phi\|_\infty \sum_{n\in\mathbb{Z}} |a_n|^2, \quad f \in V_\varphi \,.$$

The inequality (2.1) shows that convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence which is uniform on $\mathbb{R}$.

On the other hand, the space V_φ is the image of $L^2(0,1)$ by means of the isomorphism $\mathcal{T} : L^2(0,1) \to V_\varphi$ which maps the orthonormal basis $\{e^{-2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ for V_φ (see Ref. 8), i.e.,

$$(\mathcal{T}F)(t) := \sum_{n\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi i n \cdot} \big\rangle_{L^2(0,1)} \varphi(t-n), \quad F \in L^2(0,1).$$

Notice that for each $F \in L^2(0,1)$ the function $f = \mathcal{T}F$ is given by

$$f(t) = \langle F, K_t \rangle_{L^2(0,1)} \,, \quad t \in \mathbb{R} \,.$$

The kernel transform $t \in \mathbb{R} \to K_t \in L^2(0,1)$ is defined as $K_t(x) := \overline{Z\varphi}(t,x)$ where $Z\varphi$ denotes the Zak transform of φ. Recall that the Zak transform of $f \in L^2(\mathbb{R})$ is formally defined in $\mathbb{R}^2$ as $(Zf)(t,w) := \sum_{n\in\mathbb{Z}} f(t+n)e^{-2\pi i n w}$. See Refs. 10 and 11 for properties and uses of the Zak transform.

The following shifting property of $\mathcal{T}$ will be used later: For $F \in L^2(0,1)$, $r \in \mathbb{N}$ and $n \in \mathbb{Z}$ we have

$$\mathcal{T}\big[F(\cdot)e^{-2\pi i r n \cdot}\big](t) = \mathcal{T}[F](t - rn) \,, \quad t \in \mathbb{R} \,. \tag{2.2}$$

2.2. *Generalized regular sampling*

In this section we introduce the generalized regular sampling theory in a shift-invariant space V_φ. These sampling formulas involve samples of filtered versions of the functions in V_φ. As in Ref. 7, we distinguish two types of linear time-invariant systems $\mathcal{L}_j$:

(a) The impulse response l_j of $\mathcal{L}_j$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, for any $f \in V_\varphi$ we have

$$\big(\mathcal{L}_j f\big)(t) := [f * l_j](t) = \int_{-\infty}^{\infty} f(x) l_j(t-x)dx = \langle f(\cdot), \phi_j(\cdot - t)\rangle_{L^2(\mathbb{R})}\,, \quad t \in \mathbb{R}\,,$$

where $\phi_j(t) := \overline{l_j(-t)}$. Notice that $\mathcal{L}_j f$ is a continuous and bounded function in $L^2(\mathbb{R})$.

(b) The impulse response l_j has the form $l_j = \sum_{k=0}^{N} c_k \delta^{(k)}(t+d_k)$, where $\delta^{(k)}$ denotes the kth derivative of the Dirac delta and c_k, d_k are constants for $k = 0, 1, \ldots, N$. For each $f \in V_\varphi$ we have

$$\big(\mathcal{L}_j f\big)(t) := \sum_{k=0}^{N} c_k f^{(k)}(t+d_k)\,, \quad t \in \mathbb{R}\,.$$

In this case we also assume that $\varphi^{(N)}$ exists on $\mathbb{R}$, and $\sum_{n\in\mathbb{Z}} |\varphi^{(k)}(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ for each $k = 0, 1, \ldots, N$.

Whenever $\mathcal{L}_j$ is a filter of the type (a) or (b) above, for any $t \in \mathbb{R}$ the sequence $\{(\mathcal{L}_j\varphi)(t+n)\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ (see Lemma 1 in Ref. 7). Thus, for any fixed $t \in \mathbb{R}$, the Zak transform $(Z\mathcal{L}_j\varphi)(t,w) = \sum_{n\in\mathbb{Z}} \mathcal{L}_j\varphi(t+n)e^{-2\pi i n w}$ defines a function in $L^2(0,1)$. For notational ease we choose $t = 0$ without loss of generality. For $j = 1, 2, \ldots, s$, the functions g_j in $L^2(0,1)$ defined by

$$g_j(w) := \sum_{n\in\mathbb{Z}} \mathcal{L}_j\varphi(n)e^{-2\pi i n w} = (Z\mathcal{L}_j\varphi)(0,w)\,, \quad j = 1, 2, \ldots, s\,, \tag{2.3}$$

play an important role since they allow a representation of the samples $\{(\mathcal{L}_j f)(rn)\}$. Namely: Let f be a function in V_φ such that $f = \mathcal{T}F$ where $F \in L^2(0,1)$. For every $j = 1, 2, \ldots, s$, we have

$$\big(\mathcal{L}_j f\big)(rn) = \big\langle F(\cdot), \overline{g}_j(\cdot)\, e^{-2\pi i r n \cdot} \big\rangle_{L^2(0,1)}\,, \quad n \in \mathbb{Z}\,. \tag{2.4}$$

Equation (2.4) leads us to study when a sequence $\{a_j(\cdot)\, e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ (or equivalently the sequence $\{\overline{a}_j(\cdot)e^{2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$), where $a_j \in L^2(0,1)$ for each $j = 1, 2, \ldots, s$, is a Bessel sequence, a frame, or a Riesz basis for $L^2(0,1)$. To this end, associated with the functions $a_j,\ j=1,2,\dots,s$, we introduce the $s\times r$ matrix function defined for $w\in(0,1)$ as

$$\mathbf{A}(w) := \begin{bmatrix} a_1(w) a_1\left(w+\frac{1}{r}\right) \cdots a_1\left(w+\frac{r-1}{r}\right) \\ a_2(w) a_2\left(w+\frac{1}{r}\right) \cdots a_2\left(w+\frac{r-1}{r}\right) \\ \vdots \qquad \vdots \qquad \vdots \\ a_s(w) a_s\left(w+\frac{1}{r}\right) \cdots a_s\left(w+\frac{r-1}{r}\right) \end{bmatrix} = \left[a_j\left(w+\frac{k-1}{r}\right)\right]_{\substack{j=1,2,\dots,s\\ k=1,2,\dots,r}},$$

and its related constants

$$\alpha_{\mathbf{A}} := \underset{w\in(0,1/r)}{ess\,inf}\ \lambda_{\min}[\mathbf{A}^*(w)\mathbf{A}(w)],$$
$$\beta_{\mathbf{A}} := \underset{w\in(0,1/r)}{ess\,sup}\ \lambda_{\max}[\mathbf{A}^*(w)\mathbf{A}(w)],$$

where $\mathbf{A}^*(w)$ denotes the transpose conjugate of the matrix $\mathbf{A}(w)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbf{A}^*(w)\mathbf{A}(w)$. Observe that $0\le\alpha_{\mathbf{A}}\le\beta_{\mathbf{A}}\le\infty$. Notice that in the definition of the matrix $\mathbf{A}(w)$ we are considering the 1-periodic extensions of the involved functions $a_j,\ j=1,2,\dots,s$. The next lemma gives the answer:

Lemma 2.1. *Let a_j be in $L^2(0,1)$ for $j=1,2,\dots,s$ and let $\mathbf{A}(w)$ be its associated matrix. Then,*

(i) *The sequence $\{\overline{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Bessel sequence in $L^2(0,1)$ if and only if $a_j\in L^\infty(0,1)$ for $j=1,\dots,s$. In this case, the optimal Bessel bound is $\beta_{\mathbf{A}}/r$.*

(ii) *The sequence $\{\overline{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0<\alpha_{\mathbf{A}}\le\beta_{\mathbf{A}}<\infty$. In this case, the optimal frame bounds are $\alpha_{\mathbf{A}}/r$ and $\beta_{\mathbf{A}}/r$.*

(iii) *The sequence $\{\overline{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Riesz basis for $L^2(0,1)$ if and only if it is a frame for $L^2(0,1)$ and $r=s$.*

For the proof of (i) and (ii) see Ref. 7. Concerning (iii), it can be proved by using the fact that a frame is a Riesz basis if and only if it has a biorthogonal sequence (see Ref. 3, p. 124). In this case, this biorthogonal sequence is precisely $\{sb_j(\cdot)e^{2\pi isn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ where $[b_1,b_2,\dots,b_s]$ is the first row of the matrix $\mathbf{A}^{-1}$.

Assume that the function $g_j\in L^\infty(0,1)$ for $j=1,2,\dots,s$, and that there exists a vector $[a_1(w),\dots,a_s(w)]$ with entries in $L^\infty(0,1)$ such that

$$\left[a_1(w),\dots,a_{s(w)}\right]\mathbf{G}(w) = [1,0,\dots,0] \quad \text{a.e. in } (0,1),$$

where $\mathbf{G}(w)$ denotes the corresponding $\mathbf{A}(w)$ matrix for the functions $g_j,\ j=1,2,\dots,s$, in (2.3). It has been proved in Ref. 7 that any $F\in L^2(0,1)$

can be expanded in terms of the dual frames $\{ra_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\ j=1,2,\dots,s}$ and $\{\overline{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\ j=1,2,\dots,s}$ for $L^2(0,1)$ as

$$\begin{aligned} F(w) &= r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\left\langle F(\cdot),\overline{g}_j(\cdot)\,e^{-2\pi irn\cdot}\right\rangle_{L^2(0,1)}a_j(w)e^{-2\pi inrw} \\ &= r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_jf)(rn)a_j(w)e^{-2\pi inrw} \quad \text{in } L^2(0,1)\,. \end{aligned}$$

Thus, the isomorphism $\mathcal{T}$ gives the following sampling formula in V_φ

$$f(t) = r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_jf)(rn)(\mathcal{T}a_j)(t-rn)\,, \quad t\in\mathbb{R}\,,$$

where we have used (2.2). In fact, much more can be said about the above sampling formula in V_φ (see Theorem 1, Theorem 2 and Corollary 1 in Ref. 7):

Theorem 2.1. *Assume that the functions $g_j \in L^\infty(0,1)$ for each $j=1,2,\dots,s$. Then the following statements are equivalent*:

(a) $\alpha_{\mathbf{G}} > 0$.
(b) *There exists a frame for V_φ having the form $\{S_j(t-rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ such that for any $f\in V_\varphi$,*

$$f = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_jf)(rn)\ S_j(\cdot - rn) \quad \text{in } L^2(\mathbb{R})\,. \tag{2.5}$$

(c) *There exist functions $a_j\in L^\infty(0,1)$, $j=1,2,\dots,s$, such that*

$$\big[a_1(w),\dots,a_s(w)\big]\,\mathbf{G}(w) = [1,0,\dots,0] \text{ a.e. in } (0,1)\,. \tag{2.6}$$

In the case where the equivalent conditions are satisfied, the reconstruction functions in (2.5) *are given by $S_j = r\mathcal{T}a_j$, where the functions a_j, $j=1,2,\dots,s$, satisfy* (2.6). *The convergence of the series in* (2.5) *is also absolute and uniform on $\mathbb{R}$. If $r=s$ then there exists a unique frame $\{S_j(t-sn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ for V_φ for which the sampling formula* (2.5) *holds. Moreover, this frame is a Riesz basis for V_φ, and the corresponding functions a_j, $j=1,2,\dots,s$, form the first row of the matrix $\mathbf{G}^{-1}$.*

3. Generalized Irregular Sampling

The key point for generalized regular sampling is Eq. (2.4), which allows us to represent the regular samples $\{(\mathcal{L}_jf)(rn)\}$ by means of the frame $\{\overline{(Z\mathcal{L}_j\varphi)}(0,\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ for $L^2(0,1)$. In this section, we prove that the perturbed samples $\{(\mathcal{L}_jf)(rn+\varepsilon_{j,n})\}$ are obtained from a perturbed version of the above frame. Thus, stable generalized irregular sampling may be studied from the perturbation theory of frames.

3.1. *An expression for the irregular samples*

Whenever the linear system $\mathcal{L}_j$ is of the type (a), the Minkowski inequality for integrals shows that the sequence $\{(\mathcal{L}_j\varphi)(t+n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for any fixed $t \in \mathbb{R}$ (see Lemma 1 in Ref. 7). Trivially, the same applies for $\mathcal{L}_j$ of the type (b). Therefore, for any fixed $t \in \mathbb{R}$, the function $(Z\mathcal{L}_j\varphi)(t,w) := \sum_{n\in\mathbb{Z}}(\mathcal{L}_j\varphi)(t+n)e^{-2\pi inw}$ belongs to $L^2(0,1)$ and the following expression for the perturbed samples $(\mathcal{L}_jf)(rn+\varepsilon_{j,n})$ holds:

Lemma 3.1. *Let f be a function in V_φ such that $f = \mathcal{T}F$ where $F \in L^2(0,1)$. For $n \in \mathbb{Z}$, $j = 1, 2, \ldots, s$, and $\varepsilon_{j,n} \in \mathbb{R}$ we have*

$$\big(\mathcal{L}_jf\big)(rn+\varepsilon_{j,n}) = \big\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\big\rangle_{L^2(0,1)}\,.$$

Proof. Assume that $\mathcal{L}_j$ is a system of the type (a). For each $n \in \mathbb{Z}$ we have

$$\begin{aligned}(\mathcal{L}_jf)(rn+\varepsilon_{j,n}) &= \big\langle f(\cdot), \phi_j(\cdot - rn - \varepsilon_{j,n})\big\rangle_{L^2(\mathbb{R})}\\ &= \left\langle \sum_{k\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi ik\cdot}\big\rangle_{L^2(0,1)}\varphi(\cdot - k), \phi_j(\cdot - rn - \varepsilon_{j,n})\right\rangle_{L^2(\mathbb{R})}\\ &= \sum_{k\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi ik\cdot}\big\rangle_{L^2(0,1)}\mathcal{L}_j\varphi(rn+\varepsilon_{j,n}-k)\,.\end{aligned}$$

Parseval's equality and a change in the summation index gives

$$\begin{aligned}(\mathcal{L}_jf)(rn+\varepsilon_{j,n}) &= \left\langle F(\cdot), \sum_{k\in\mathbb{Z}} \overline{\mathcal{L}_j\varphi}(rn+\varepsilon_{j,n}-k)\, e^{-2\pi ik\cdot}\right\rangle_{L^2(0,1)}\\ &= \big\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\big\rangle_{L^2(0,1)}\,.\end{aligned}$$

Assume now that $\mathcal{L}_j$ is a system of the type (b). Under our hypotheses on $\mathcal{L}_j$, for each $k = 0, 1, \ldots, N$ we have that $f^{(k)}(t) = \sum_{l\in\mathbb{Z}}\big\langle F(\cdot), e^{-2\pi il\cdot}\big\rangle_{L^2(0,1)}\varphi^{(k)}(t-l)$. Hence, for each $n \in \mathbb{Z}$, one gets

$$\begin{aligned}(\mathcal{L}_jf)(rn+\varepsilon_{j,n}) &= \sum_{k=0}^{N} c_k f^{(k)}(rn+\varepsilon_{j,n}+d_k)\\ &= \sum_{k=0}^{N} c_k \sum_{l\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi il\cdot}\big\rangle_{L^2(0,1)}\varphi^{(k)}(rn+\varepsilon_{j,n}+d_k-l)\\ &= \left\langle F(\cdot), \sum_{k=0}^{N}\overline{c}_k \sum_{l\in\mathbb{Z}} \overline{\varphi}^{(k)}(rn+\varepsilon_{j,n}+d_k-l)e^{-2\pi il\cdot}\right\rangle_{L^2(0,1)}\\ &= \big\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\big\rangle_{L^2(0,1)}\,.\end{aligned}$$

□

It is worth to point out that the same proof in Lemma 3.1 shows that

$$\mathcal{L}_jf(t) = \langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(t,\cdot)\rangle_{L^2(0,1)}\,, \quad t \in \mathbb{R}\,,$$

for any linear system $\mathcal{L}_j$ of the type (a) or (b). Lemma 3.1 leads us to study the case for which the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$. On the other hand, we know that, whenever $0<\alpha_{\mathbf{G}}\le\beta_{\mathbf{G}}<\infty$, the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(0,\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ with optimal frame bounds $\alpha_{\mathbf{G}}/r$ and $\beta_{\mathbf{G}}/r$. In the case of $r=s$, the above sequence is a Riesz basis for $L^2(0,1)$. One possibility is to use the theory of perturbation of frames in order to find the suitable error sequences for which the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$. The following result on perturbation of frames (the proof can be found in Ref. 3, p. 354) will be used later:

Lemma 3.2. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A, B, and let $\{g_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$. If there exists a constant $R<A$ such that*

$$\sum_{k=1}^{\infty}|\langle f_k-g_k,f\rangle|^2\le R\|f\|^2\quad\textit{for each } f\in\mathcal{H}\,,$$

then $\{g_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds

$$A\left(1-\sqrt{\frac{R}{A}}\right)^2,\quad\left(1+\sqrt{\frac{R}{B}}\right)^2.$$

If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis, then $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis.

3.2. *The resulting sampling theory*

Given an error sequence $\varepsilon:=\{\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ we define on $\ell^2(\mathbb{Z})$ the operator $D_\varepsilon=\big[D_{\varepsilon,1},\dots,D_{\varepsilon,s}\big]$, where

$$D_{\varepsilon,j}c:=\left\{\sum_{k\in\mathbb{Z}}\big[\mathcal{L}_j\varphi(rn-k+\varepsilon_{j,n})-\mathcal{L}_j\varphi(rn-k)\big]c_k\right\}_{n\in\mathbb{Z}}$$

for each $c=\{c_l\}_{l\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$. The operator norm is defined as usual

$$\|D_\varepsilon\|:=\sup_{c\in\ell^2(\mathbb{Z})\setminus\{0\}}\frac{\|D_\varepsilon c\|_{\ell^2_s(\mathbb{Z})}}{\|c\|_{\ell^2(\mathbb{Z})}},$$

where $\|D_\varepsilon c\|^2_{\ell^2_s(\mathbb{Z})}:=\sum_{j=1}^{s}\|D_{\varepsilon,j}c\|^2_{\ell^2(\mathbb{Z})}$ for each $c\in\ell^2(\mathbb{Z})$.

Theorem 3.1. *Assume that $g_j\in L^\infty(0,1)$ for $j=1,2,\dots,s$ with $\alpha_{\mathbf{G}}>0$. If the error sequence $\varepsilon:=\{\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,\dots,s}$ satisfies the inequality $\|D_\varepsilon\|^2<\alpha_{\mathbf{G}}/r$, then there exists a frame $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ for V_φ such that, for any $f\in V_\varphi$*

$$f(t)=\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_jf)(rn+\varepsilon_{j,n})S^\varepsilon_{j,n}(t),\quad t\in\mathbb{R}\,,\tag{3.1}$$

where the convergence of the series is in the $L^2(\mathbb{R})$-sense, absolute and uniform on $\mathbb{R}$. Moreover, when $r=s$ the sequence $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Riesz basis for V_φ, and the interpolation property $(\mathcal{L}_lS^\varepsilon_{j,n})(sm+\varepsilon_{j,m})=\delta_{j,l}\delta_{n,m}$ holds.

Proof. The sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(0,\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame (a Riesz basis if $r = s$) for $L^2(0,1)$ with frame (Riesz) bounds $\alpha_{\mathbf{G}}/r$ and $\beta_{\mathbf{G}}/r$. For any $F(w) = \sum_{l\in\mathbb{Z}} c_l e^{-2\pi ilw}$ in $L^2(0,1)$ we have

$$\begin{aligned}
&\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \left| \left\langle \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot} - \overline{(Z\mathcal{L}_j\varphi)}(0,\cdot)e^{-2\pi irn\cdot}\,,\, F(\cdot)\right\rangle_{L^2(0,1)} \right|^2 \\
&\quad = \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \left| \left\langle \sum_{k\in\mathbb{Z}} [\overline{\mathcal{L}_j\varphi}(k+\varepsilon_{j,n}) - \overline{\mathcal{L}_j\varphi}(k)]e^{-2\pi i(rn-k)\cdot}\,,\, \sum_{l\in\mathbb{Z}} c_l e^{-2\pi il\cdot} \right\rangle_{L^2(0,1)} \right|^2 \\
&\quad = \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} [\overline{\mathcal{L}_j\varphi}(k+\varepsilon_{j,n}) - \overline{\mathcal{L}_j\varphi}(k)]\,\overline{c}_{rn-k} \right|^2 \\
&\quad = \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} [\mathcal{L}_j\varphi(rn-k+\varepsilon_{j,n}) - \mathcal{L}_j\varphi(rn-k)]c_k \right|^2 \\
&\quad = \sum_{j=1}^{s} \|D_{\varepsilon,j}\{c_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})} \le \|D_\varepsilon\|^2 \|\{c_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})} = \|D_\varepsilon\|^2\|F\|^2_{L^2(0,1)}\,.
\end{aligned}$$

Hence, from Lemma 3.2, the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ (a Riesz basis if $r = s$). Let $\{h^\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ be its canonical dual frame. Hence, for any $F \in L^2(0,1)$

$$\begin{aligned}
F &= \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s} \left\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi irn\cdot}\right\rangle_{L^2(0,1)} h^\varepsilon_{j,n} \\
&= \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s} (\mathcal{L}_j f)(rn+\varepsilon_{j,n})h^\varepsilon_{j,n}.
\end{aligned}$$

Applying the isomorphism $\mathcal{T}$, one gets (3.1) in $L^2(\mathbb{R})$ where $S^\varepsilon_{j,n} = \mathcal{T}h^\varepsilon_{j,n}$. Since $\mathcal{T}$ is an isomorphism between $L^2(0,1)$ and V_φ, $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for V_φ (a Riesz basis if $r = s$).

Pointwise convergence in the sampling series is absolute due to the unconditional character of a frame. The uniform convergence on $\mathbb{R}$ is a consequence of (2.1). The interpolatory property in the case $r = s$ follows from the uniqueness of the coefficients with respect to a Riesz basis. □

The next result yields a uniform bound of the norm $\|D_\varepsilon\|$ regardless the sequence ε such that $\{\varepsilon_{j,n}\}_{n\in\mathbb{Z}}$ is in $[\alpha_j,\beta_j] \subset [-r,r]$, for each $j = 1,2,\dots,s$.

Theorem 3.2. *For any sequence* $\varepsilon := \{\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,\dots,s}$ *such that* $\{\varepsilon_{j,n}\}_{n\in\mathbb{Z}} \subset [\alpha_j,\beta_j] \subset [-r,r]$ *for each* $j = 1,2,\dots,s$ *the following inequality holds*

$$\|D_\varepsilon\|^2 \le \sum_{j=1}^{s} \Lambda_j\Gamma_j\,, \tag{3.2}$$

where, for each $j = 1, 2, \dots, s$, the constants Λ_j and Γ_j are given by

$$\Lambda_j := \sup_{\substack{l=0,1,\dots,r-1 \\ \{d_k\}\subset[\alpha_j,\beta_j]}} \sum_{k\in\mathbb{Z}} |\mathcal{L}_j\varphi(rk+l+d_k) - \mathcal{L}_j\varphi(rk+l)|\,,$$

$$\Gamma_j := \sup_{d\in[\alpha_j,\beta_j]} \sum_{k\in\mathbb{Z}} |\mathcal{L}_j\varphi(k+d) - \mathcal{L}_j\varphi(k)|\,.$$

Proof. Suppose that $\sum_{j=1}^{s} \Lambda_j\Gamma_j < \infty$; otherwise the result obviously holds. Denoting

$$d_{n,k}^{(j)} := \mathcal{L}_j\varphi(rn-k+\varepsilon_{j,n}) - \mathcal{L}_j\varphi(rn-k)\,,$$

for $k, n \in \mathbb{Z}$, the inequalities

$$\sum_{l\in\mathbb{Z}} |d_{l,k}^{(j)}| \le \Lambda_j \quad \text{and} \quad \sum_{l\in\mathbb{Z}} |d_{n,l}^{(j)}| \le \Gamma_j$$

hold. For any sequence $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have

$$\begin{aligned}
\|D_\varepsilon c\|^2_{\ell^2_s(\mathbb{Z})} &= \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\left|\sum_{k\in\mathbb{Z}} d_{n,k}^{(j)}\; c_k\right|^2 \le \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\sum_{l,k\in\mathbb{Z}} |d_{n,l}^{(j)} c_l \overline{d}_{n,k}^{(j)} \overline{c}_k| \\
&= \sum_{j=1}^{s}\sum_{l,k\in\mathbb{Z}} |c_l|\;|c_k| \sum_{n\in\mathbb{Z}} |d_{n,l}^{(j)} d_{n,k}^{(j)}| \le \sum_{j=1}^{s}\sum_{l,k\in\mathbb{Z}} \frac{|c_l|^2+|c_k|^2}{2} \sum_{n\in\mathbb{Z}} |d_{n,l}^{(j)} d_{n,k}^{(j)}| \\
&= \sum_{j=1}^{s}\sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{k,n\in\mathbb{Z}} |d_{n,l}^{(j)} d_{n,k}^{(j)}| \le \sum_{j=1}^{s}\sum_{l\in\mathbb{Z}} |c_l|^2 \Gamma_j \sum_{n\in\mathbb{Z}} |d_{n,l}^{(j)}| \\
&\le \sum_{j=1}^{s}\sum_{l\in\mathbb{Z}} |c_l|^2 \Gamma_j \Lambda_j = \left(\sum_{j=1}^{s} \Lambda_j\Gamma_j\right) \|c\|^2_{\ell^2(\mathbb{Z})}\,,
\end{aligned}$$

which concludes the proof. □

Next we give a particular example for which Theorem 3.2 works. Namely, suppose that, for each $j = 1, 2, \dots, s$, the function $\mathcal{L}_j\varphi \in \mathcal{C}^1(\mathbb{R})$, and there exists $\eta_j > 0$ such that $(\mathcal{L}_j\varphi)'(t) = O(|t|^{-(1+\eta_j)})$ whenever $|t| \to \infty$. Then, it is easy to check that, for $\delta_j > 0$,

$$M_{(\mathcal{L}_j\varphi)'}(\delta_j) := \sum_{k} \max_{t\in[k-\delta_j,k+\delta_j]} |(\mathcal{L}_j\varphi)'(t)| < \infty\,.$$

Corollary 3.1. *Suppose that, for each $j = 1, 2, \dots, s$, the function $\mathcal{L}_j\varphi \in \mathcal{C}^1(\mathbb{R})$ and $M_{(\mathcal{L}_j\varphi)'}(\delta_j) < \infty$, where $\delta_j := \sup_{n\in\mathbb{Z}}|\varepsilon_{j,n}|$. Consider*

$$N_{(\mathcal{L}_j\varphi)'}(\delta_j) := \sup_{l=0,1,\dots,r-1} \sum_{k} \max_{t\in[rk+l-\delta_j,rk+l+\delta_j]} |(\mathcal{L}_j\varphi)'(t)|\,.$$

Then, the condition

$$\sum_{j=1}^{s} \delta_j^2 \; N_{(\mathcal{L}_j\varphi)'}(\delta_j) M_{(\mathcal{L}_j\varphi)'}(\delta_j) < \frac{\alpha_{\mathbf{G}}}{r}$$

implies the existence of a frame $\{S^{\varepsilon}_{j,n}\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ *for* V_φ *such that, for any* $f \in V_\varphi$

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn + \varepsilon_{j,n}) S^{\varepsilon}_{j,n}(t)\,, \quad t \in \mathbb{R}\,,$$

uniformly on $\mathbb{R}$*. If* $r = s$*, the sequence* $\{S^{\varepsilon}_{j,n}\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ *is a Riesz basis for* V_φ*.*

Proof. For each $j = 1, 2, \ldots, s$, the mean value theorem gives

$$\sup_{d\in[-\delta_j,\delta_j]} \sum_{k\in\mathbb{Z}} |\mathcal{L}_j\varphi(k+d) - \mathcal{L}_j\varphi(k)| \le \delta_j M_{(\mathcal{L}_j\varphi)'}(\delta_j)\,,$$

and

$$\sup_{\substack{l=0,1,\ldots,r-1\\ \{d_k\}\subset[-\delta_j,\delta_j]}} \sum_{k\in\mathbb{Z}} |\mathcal{L}_j\varphi(rk+l+d_k) - \mathcal{L}_j\varphi(rk+l)| \le \delta_j N_{(\mathcal{L}_j\varphi)'}(\delta_j)\,.$$

Theorem 3.1 concludes the proof. □

Notice that $N_{(\mathcal{L}_j\varphi)'}(\delta_j) \le M_{(\mathcal{L}_j\varphi)'}(\delta_j)$. For $r = 1$ the equality holds.

3.3. *The frame algorithm*

Formula (3.1) in Theorem 3.1 is useless from a practical point of view. The involved frame $\{S^{\varepsilon}_{j,n}\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$, which depends on the error sequence $\{\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$, is impossible to determine. As a consequence, in order to recover any function $f \in V_\varphi$ from the samples $\{\mathcal{L}_j f(rn + \varepsilon_{j,n})\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ we should use the frame algorithm (see Ref. 6). We are going to implement this algorithm in the $\ell^2(\mathbb{Z})$ setting. To this end, consider the canonical isometry

$$\mathcal{U} : \ell^2(\mathbb{Z}) \to L^2(0,1)\,, \quad \mathcal{U}\,\{a_l\}_{l\in\mathbb{Z}} := \sum_{l\in\mathbb{Z}} a_l e^{-2\pi i l w}\,.$$

For $f(t) = \sum_{l\in\mathbb{Z}} a_l \varphi(t-l) \in V_\varphi$, denote by $\mathbb{F}$ the sequence

$$\mathbb{F} := \mathcal{U}^{-1} F = \mathcal{U}^{-1}\mathcal{T}^{-1} f = \{a_l\}_{l\in\mathbb{Z}}\,.$$

The samples $\{(\mathcal{L}_j f)(rn + \varepsilon_{j,n})\}$ can be written as

$$(\mathcal{L}_j f)(rn + \varepsilon_{j,n}) = \left\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi i rn\cdot}\right\rangle_{L^2(0,1)} = \left\langle \mathbb{F}, \mathbb{L}_{j,n}\right\rangle_{\ell^2(\mathbb{Z})}$$

where, for $j = 1, 2, \ldots, s$ and $n \in \mathbb{Z}$,

$$\mathbb{L}_{j,n} := \mathcal{U}^{-1}\left(\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_{j,n},\cdot)e^{-2\pi i rn\cdot}\right) = \left\{\overline{(\mathcal{L}_j\varphi)}(rn - l + \varepsilon_{j,n})\right\}_{l\in\mathbb{Z}}\,.$$

The sequence $\{\mathbb{L}_{j,n}\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ is a frame for $\ell^2(\mathbb{Z})$. Indeed, assume that $\{\varepsilon_{j,n}\}_{n\in\mathbb{Z}} \subset [\alpha_j, \beta_j] \subset [-r, r]$ for each $j = 1, 2, \ldots, s$, and that $\sum_{j=1}^{s} \Lambda_j \Gamma_j < \alpha_{\mathbf{G}}/r$. According to the proof from Theorem 3.1 and Lemma 3.2, the sequence $\{(\overline{Z\mathcal{L}_j\varphi})(\varepsilon_{j,n}, \cdot)e^{-2\pi i r n \cdot}\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ is a frame for $L^2(0,1)$ with bounds

$$A := \frac{\alpha_{\mathbf{G}}}{r}\left(1 - \sqrt{\frac{r}{\alpha_{\mathbf{G}}}\sum_{j=1}^{s}\Lambda_j\Gamma_j}\right)^2, \quad B := \frac{\beta_{\mathbf{G}}}{r}\left(1 + \sqrt{\frac{r}{\beta_{\mathbf{G}}}\sum_{j=1}^{s}\Lambda_j\Gamma_j}\right)^2. \quad (3.3)$$

Since $\mathcal{U}^{-1}$ is an isometry, the sequence $\{\mathbb{L}_{j,n}\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ is a frame for $\ell^2(\mathbb{Z})$ with the same bounds.

Hence, the recovery of the function $f = \mathcal{T}\mathcal{U}\mathbb{F} \in V_\varphi$ from the samples $\{(\mathcal{L}_j f)(rn + \varepsilon_{j,n})\}$ is reduced to recovering the sequence $\mathbb{F}$ from the sequence $\left\{\langle \mathbb{F}, \mathbb{L}_{j,n}\rangle_{\ell^2(\mathbb{Z})}\right\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$. In so doing, the classical frame algorithm reads as:

Consider

$$\begin{aligned}\mathbb{F}_0 = \mathcal{A}\mathbb{F} &:= \frac{2}{A+B}\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\langle \mathbb{F}, \mathbb{L}_{j,n}\rangle_{\ell^2(\mathbb{Z})}\mathbb{L}_{j,n} \\ &= \frac{2}{A+B}\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn + \varepsilon_{j,n})\mathbb{L}_{j,n}\end{aligned}$$

and define recursively $\mathbb{F}_{k+1} = \mathbb{F}_k + \mathcal{A}(\mathbb{F} - \mathbb{F}_k)\,,\ k \in \mathbb{N}$. Then, the sequence $\{f_k\}_{k\in\mathbb{N}}$ in V_φ given by $f_k(t) = \sum_{l\in\mathbb{Z}} a_l^{(k)}\varphi(t-l)$ where $\mathbb{F}_k = \{a_l^{(k)}\}_{l\in\mathbb{Z}}$, satisfies

$$\begin{aligned}\|f - f_k\|_{L^2(\mathbb{R})} &\leq \|\mathcal{T}\|\ \|\mathbb{F} - \mathbb{F}_k\|_{\ell^2(\mathbb{Z})} \leq \|\mathcal{T}\|\,\gamma^{k+1}\|\mathbb{F}\|_{\ell^2(\mathbb{Z})} \\ &\leq \|\mathcal{T}\|\,\|\mathcal{T}^{-1}\|\,\gamma^{k+1}\|f\|_{L^2(\mathbb{R})} = \sqrt{\frac{\|\Phi\|_\infty}{\|\Phi\|_0}}\,\gamma^{k+1}\|f\|_{L^2(\mathbb{R})}\,,\end{aligned}$$

where $\gamma := (B - A)/(B + A)$, and we have used that $\|\mathcal{T}^{-1}\|^{-2} = \|\Phi\|_0$ and $\|\mathcal{T}\|^2 = \|\Phi\|_\infty$ (see Proposition 3.6.8 in Ref. 3). It is worth to mention that, in some important examples, the value of $\sum_{j=1}^{s}\Lambda_j\Gamma_j$ in (3.3) can be explicitely computed in terms of $\delta := \sup_{j,n}|\varepsilon_{j,n}|$.

In order to improve this algorithm, especially when the ratio B/A is large, we can use the methods proposed by Gröchenig in Ref. 9 to accelere of the frame algorithm.

3.4. *Examples in spline spaces*

For each fixed $m \in \mathbb{N}$, the B-spline N_m is defined as $N_m := N_1 * N_1 * \ldots * N_1$ (m times) where N_1 denotes the characteristic function of the interval $(0,1)$. It is known[4] that $\{N_m(\cdot - k\}_{k\in\mathbb{Z}}$ is a Riesz sequence in $L^2(\mathbb{R})$. The corresponding shift-invariant space V_{N_m} is the space of splines of degree $m-1$ in $L^2(\mathbb{R})$ with nodes at the integers. For these shift-invariant spaces, it is not difficult to obtain the sum $\sum_{j=1}^{s}\Lambda_j\Gamma_j$ in terms of $\delta := \sup_{j,n}|\varepsilon_{j,n}|$. Thus, the largest δ satisfying the condition $\sum_{j=1}^{s}\Lambda_j\Gamma_j < \alpha_{\mathbf{G}}/r$ can be explicitely computed. Notice that, having in mind (3.3), the rate of convergence γ can also be computed in terms of $\delta := \sup_{j,n}|\varepsilon_{j,n}|$. We present here some illustrative examples:

3.4.1. *Recovering linear, quadratic and cubic splines from irregular samples*

For $r = s = 1$ and $(\mathcal{L}_1 f)(t) := f(t+a)$, Theorem 3.1 gives the irregular sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n+a+\varepsilon_n)\ S_n^{\varepsilon}(t)\,, \quad t\in\mathbb{R}\,, \tag{3.4}$$

where $\{S_n^{\varepsilon}\}_{n\in\mathbb{Z}}$ denotes the Riesz basis associated with the perturbation $\{\varepsilon_n\}_{n\in\mathbb{Z}}$. A challenge problem is to measure how large should be $\delta := \sup_n|\varepsilon_n|$ in order to have a sampling formula like (3.4) in the shift-invariant space spanned by the B-spline N_m.

- *Linear Splines*: Consider the linear B-spline $N_2(t) := t\mathcal{X}_{[0,1)} + (2-t)\mathcal{X}_{[1,2)}$. For $d\in[0,1]$ we have

$$\sum_{k\in\mathbb{Z}} |N_2(k+d) - N_2(k)| = |N_2(d)| + |N_2(1+d) - N_2(1)| = d + d = 2d.$$

By symmetry we also have $\sum_{k\in\mathbb{Z}} |N_2(k-d) - N_2(k)| = 2d$. Thus, for $a = 0$ and $[\alpha_1, \beta_1] = [-\delta, \delta]$, where $\delta \le 1$, we obtain $\Gamma_1 = 2\delta$. Besides,

$$\Lambda_1 = \sup_{d\in[-\delta,\delta]} |N_2(d)| + \sup_{d\in[-\delta,\delta]} |N_2(1+d) - N_2(1)| + \sup_{d\in[-\delta,\delta]} |N_2(2+d)| = 3\delta.$$

Hence, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta,\delta]$, $\delta \le 1$, we have that $\|D_\varepsilon\|^2 \le \Lambda_1\Gamma_1 = 6\delta^2$. Since

$$\alpha_{\mathbf{G}} = \inf_{w\in(0,1)} |(ZN_1)(0,w)|^2 = \inf_{w\in(0,1)} \left|e^{-2\pi i w}\right|^2 = 1\,,$$

whenever

$$\sup_{n\in\mathbb{Z}} |\varepsilon_n| < C := \frac{1}{\sqrt{6}} \approx 0.408\,,$$

a sampling formula like (3.4) with $a = 0$ holds. This bound improves upon the value of $1/3$ obtained by Chen *et al.*[2] Assuming that $\varepsilon_n \ge 0$ (or $\varepsilon_n \le 0$) for all $n\in\mathbb{Z}$, the bound $1/\sqrt{6}$ can be improved up to $1/2$ since $\Lambda_1 = 2\delta$ in this case. In fact, it has been proved in Ref. 5 that $1/2$ is the optimal bound in this example.

Since $\beta_{\mathbf{G}} = 1$ we obtain the convergence rate $\gamma = 2\sqrt{6}\ \delta/(1+6\ \delta^2)$ where $\delta := \sup_{n\in\mathbb{Z}}|\varepsilon_n|$.

- *Cubic Splines*: Consider the cubic B-spline

$$N_4(t) := \frac{t^3}{6}\mathcal{X}_{[0,1)}(t) + \left(\frac{2}{3} - 2t + 2t^2 - \frac{t^3}{2}\right)\mathcal{X}_{[1,2)}(t) + \left(-\frac{22}{3} + 10t - 4t^2 + \frac{t^3}{2}\right)\mathcal{X}_{[2,3)}(t) + \left(\frac{32}{3} - 8t + 2t^2 - \frac{t^3}{6}\right)\mathcal{X}_{[3,4)}(t).$$

Fix $a = 0$. For $d \in [0, 1)$ the symmetry of N_4 gives

$$\sum_{k=0}^{4} |N_4(k-d) - N_4(k)| = \sum_{k=0}^{4} |N_4(k+d) - N_4(k)| = d + d^2 - \frac{2d^3}{3}.$$

Hence, for $[\alpha_1, \beta_1] = [-\delta, \delta]$, where $\delta < 1$, one gets $\Gamma_1 = \delta + \delta^2 - 2\delta^3/3$. Bearing in mind the symmetry and monotony of N_4, and that $N_4(1+\delta) - N_4(1) > N_4(1) - N_4(1-\delta)$, we obtain

$$\Lambda_1 = 2N_4(\delta) + 2[N_4(1+\delta) - N_4(1)] + N_4(2) - N_4(2+\delta) = \delta + 2\delta^2 - \frac{7\delta^3}{6}.$$

Thus, for any $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta, \delta]$ we have

$$\|D_\varepsilon\|^2 \leq \Lambda_1\Gamma_1 = \frac{7\delta^6}{9} - \frac{5\delta^5}{2} + \frac{\delta^4}{6} + 3\delta^3 + \delta^2.$$

Since

$$\alpha_{\mathbf{G}} = \inf_{w\in(0,1)} |(ZN_4)(0,\cdot)|^2 = \inf_{w\in(0,1)} \left|\frac{1}{6}e^{-2\pi iw} + \frac{2}{3}e^{-4\pi iw} + \frac{1}{6}e^{-6\pi iw}\right|^2 = \frac{1}{9}\,,$$

for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta, \delta]$ such that $\sup_{n\in\mathbb{Z}}|\varepsilon_n| < C \approx 0.253$, where C is the root of the polynomial $7\delta^6/9 - 5\delta^5/2 + \delta^4/6 + 3\delta^3 + \delta^2 - 1/9$ in $(0, 1/2)$, we can recover any cubic spline in $L^2(\mathbb{R})$ from its samples at $\{n + \varepsilon_n\}_{n\in\mathbb{Z}}$ by means of a sampling expansion expansion like (3.4).

- *Quadratic Splines*: Here the generator is the quadratic B-spline

$$N_3(t) := \frac{t^2}{2}\mathcal{X}_{[0,1)}(t) + \left(3t - t^2 - \frac{3}{2}\right)\mathcal{X}_{[1,2)}(t) + \frac{(3-t)^2}{2}\mathcal{X}_{[2,3)}(t).$$

For $a = 0$ we get

$$\alpha_{\mathbf{G}} := \inf_{w\in(0,1)} |(ZN_3)(0,\cdot)|^2 = \frac{1}{4}\inf_{w\in(0,1)} |e^{-2\pi iw} + e^{-4\pi iw}|^2 = 0\,,$$

and, consequently, we cannot consider $a = 0$. However, for $a = 1/2$

$$\alpha_{\mathbf{G}} = \inf_{w\in(0,1)} |(ZN_3)(1/2,\cdot)|^2 = \inf_{w\in(0,1)} \left|\frac{1}{8} + \frac{3}{4}e^{-2\pi iw} + \frac{1}{8}e^{-4\pi iw}\right|^2 = \frac{1}{4}.$$

For $d \in [0, 1/2)$ we get

$$\sum_{k=0}^{2} |N_3(k+1/2+d) - N_3(k)| = \sum_{k=0}^{2} |N_3(k+1/2-d) - N_3(k)| = d + d^2.$$

Then, for $[\alpha_1, \beta_1] = [-\delta, \delta]$, where $\delta < 1/2$, we obtain that $\Gamma_1 = \delta + \delta^2$. Bearing in mind the symmetry and monotony of N_3 and that $N_3(1/2+\delta) - N_3(1/2) > N_3(1/2) - N_3(1/2-\delta)$, we get

$$\Lambda_1 = 2[N_3(1/2+\delta) - N_3(1/2)] + N_4(3/2) - N_4(3/2+\delta) = \delta + 2\delta^2.$$

Thus, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta, \delta]$, where $\delta < 1/2$, we get

$$\|D_\varepsilon\|^2 \le \Lambda_1\Gamma_1 = \delta^2 + 3\delta^3 + 2\delta^4.$$

As a consequence, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ satisfying $\sup_{n\in\mathbb{Z}}|\varepsilon_n| < C \approx 0.334$, where C is the root of $\delta^2 + 3\delta^3 + 2\delta^4 - 1/4$ in $(0, 1/2)$, a sampling formula like (3.4) with $a = 1/2$ holds. This bound improves upon the value of $1/4$ obtained by Chen *et al.*[2]

For irregular sampling in Spline spaces, see also the δ upper bounds obtained in Ref. 5 by using another technique.

3.4.2. *Recovering cubic Splines from the derivative sampling*

For $r = s = 2$, consider the systems $(\mathcal{L}_1 f)(t) := f(t + a)$ and $(\mathcal{L}_2 f)(t) := f'(t + a)$. For $a = 0$ we have

$$\mathbf{G}(w) = \begin{pmatrix} ZN_4(0, w) & ZN_4\left(0, w + \dfrac{1}{2}\right) \\[2ex] ZN_4'(0, w) & ZN_4'\left(0, w + \dfrac{1}{2}\right) \end{pmatrix} = \frac{1}{6}\begin{pmatrix} z + 4z + z^3 & -z + 4z^2 - z^3 \\ 3z - 3z^3 & -3z + 3z^3 \end{pmatrix},$$

where $z = e^{-2\pi i w}$. Since $\det \mathbf{G} = 2(z^5 - z^3)/3$ vanishes at $w = 0$, it follows that $\alpha_{\mathbf{G}} = 0$. Hence, Theorem 3.1 does not apply. However, taking $a = 1/2$ we obtain

$$g_1(w) = (ZN_4)(1/2, w) = \frac{1}{48} + \frac{23}{48}e^{-2\pi i w} + \frac{23}{48}e^{-4\pi i w} + \frac{1}{48}e^{-6\pi i w},$$

$$g_2(w) = (ZN_4')(1/2, w) = \frac{1}{8} + \frac{5}{8}e^{-2\pi i w} - \frac{5}{8}e^{-4\pi i w} - \frac{1}{8}e^{-6\pi i w}.$$

The eigenvalues of the matrix $\mathbf{G}^*(w)\mathbf{G}(w)$ are

$$1 + \frac{157}{288}\sin^2 2\pi w \pm \frac{7}{288}\sqrt{576\sin^2 2\pi w + 265\sin^4 2\pi w}.$$

The minimum on $(0, 1/2)$ of the smallest eigenvalue is attained at $w = \frac{1}{2\pi}\arctan\sqrt{\frac{392}{403}}$ and takes the value $\alpha_{\mathbf{G}} = \frac{216}{265}$. Besides, the maximum on $(0, 1/2)$ of the largest eigenvalue is $\beta_{\mathbf{G}} = 9/4$ attained at $w = 1/4$.

For $d \in [0, 1/2]$, we have

$$\sum_{k=0}^{3} |N_4(k + 1/2 - d) - N_4(k)| = \sum_{k=0}^{3} |N_4(k + 1/2 + d) - N_4(k)| = \frac{3}{2}d - \frac{2}{3}d^3.$$

For $d \in (0, 1/3)$ the inequality $N_4'(5/2) > N_4'(5/2 + d)$ holds. Thus, for $d \in [0, 1/3)$ we get

$$\sum_{k=0}^{3} |N_4'(k + 1/2 - d) - N_4'(k)| = \sum_{k=0}^{3} |N_4'(k + 1/2 + d) - N_4'(k)| = 2d.$$

Therefore, whenever $[\alpha_1,\beta_1]=[\alpha_2,\beta_2]=[-\delta,\delta]$, with $0<\delta<1/3$, we obtain that $\Gamma_1=(3/2)\delta-(2/3)\delta^3$ and $\Gamma_2=2\delta$.

Now, bearing in mind the symmetry of N_4 and the inequalities $N_4(1/2+\delta)-N_4(1/2)>N_4(1/2)-N_4(1/2-\delta)$ and $N_4(5/2)-N_4(5/2+d)>N_4(5/2-d)-N_4(5/2)$, we get

$$\begin{aligned}
&\sup_{d\in[-\delta,\delta]}|N_4(3/2+\delta)-N_4(3/2)|+\sup_{d\in[-\delta,\delta]}|N_4(7/2+\delta)-N_4(7/2)|\\
&\quad=\sup_{d\in[-\delta,\delta]}|N_4(1/2+\delta)-N_4(1/2)|+\sup_{d\in[-\delta,\delta]}|N_4(5/2+\delta)-N_4(5/2)|\\
&\quad=\frac{3\delta}{4}+\frac{\delta^2}{2}-\frac{\delta^3}{3}.
\end{aligned}$$

Analogously, using the symmetry of N_4', the inequality $N_4'(1/2+\delta)-N_4'(1/2)>N_4'(1/2)-N_4'(1/2-\delta)$, and that $\sup_{d\in[-\delta,\delta]}|N_4'(5/2+d)-N_4(5/2)|=N_4'(5/2-\delta)-N_4(5/2)$, we get

$$\begin{aligned}
&\sup_{d\in[-\delta,\delta]}|N_4'(3/2+\delta)-N_4'(3/2)|+\sup_{d\in[-\delta,\delta]}|N_4'(7/2+\delta)-N_4'(7/2)|\\
&\quad=\sup_{d\in[-\delta,\delta]}|N_4'(1/2+\delta)-N_4'(1/2)|+\sup_{d\in[-\delta,\delta]}|N_4'(5/2+\delta)-N_4'(5/2)|\\
&\quad=\delta+2\delta^2.
\end{aligned}$$

Hence, $\Lambda_1=(3/4)\,\delta+(1/2)\,\delta^2-(1/3)\,\delta^3$ and $\Lambda_2=\delta+2\delta^2$. Thus, for any sequence $\varepsilon=\{\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2}\subset[\delta,\delta]$, where $\delta<1/3$, we get

$$\|D_\varepsilon\|^2\le\Lambda_1\Gamma_1+\Lambda_2\Gamma_2=\frac{25\,\delta^2}{8}+\frac{19\,\delta^3}{4}-\delta^4-\frac{\delta^5}{3}+\frac{2\,\delta^6}{9}\,.$$

Thus, from Theorem 3.1, whenever $\sup_{j,n}|\varepsilon_{j,n}|<C\approx0.3022$, where C is the root of $25\delta^2/8+19\delta^3/4-\delta^4-\delta^5/3+2\delta^6/9-108/265=0$ in $(0,1/3)$, there exists a Riesz basis $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z},j=1,2}$ for V_{N_4} such that the expansion

$$f(t)=\sum_{n\in\mathbb{Z}}\left[f(2n+1/2+\varepsilon_{1,n})\;S^\varepsilon_{1,n}(t)+f'(2n+1/2+\varepsilon_{2,n})\;S^\varepsilon_{2,n}(t)\right],\quad t\in\mathbb{R}\,,$$

holds.

3.4.3. *Recovering cubic splines from average sampling*

For each $f\in V_{N_4}$ consider the system defined as $(\mathcal{L}_1f)(t):=\int_{t-1/2}^{t+1/2}f(x)dx$. For $d\in[0,1/2]$ we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}|(\mathcal{L}_1N_4)(k-d)-(\mathcal{L}_1N_4)(k)|&=\sum_{k\in\mathbb{Z}}|(\mathcal{L}_1N_4)(k+d)-(\mathcal{L}_1N_4)(k)|\\
&=\frac{23d}{24}+\frac{5d^2}{8}-\frac{d^3}{6}-\frac{d^4}{4},
\end{aligned}$$

where we have used the symmetry of $\mathcal{L}_1N_4$ with respect the line $t=2$. Thus, for $[\alpha_1,\beta_1]=[-\delta,\delta]$, where $\delta\le1/2$, we obtain $\Gamma_1=23\delta/24+5\delta^2/8-\delta^3/6-\delta^4/4$.

Besides,

$$\Lambda_1 = 2 \sup_{d\in[-\delta,\delta]} \left|(\mathcal{L}_1 N_4)(d) - (\mathcal{L}_1 N_4)(0)\right| + 2 \sup_{d\in[-\delta,\delta]} \left|(\mathcal{L}_1 N_4)(1+d) - (\mathcal{L}_1 N_4)(1)\right| + \sup_{d\in[-\delta,\delta]} \left|(\mathcal{L}_1 N_4)(2) - (\mathcal{L}_1 N_4)(2+d)\right| = \frac{5\delta^2}{4} - \frac{\delta^3}{6} - \frac{\delta^4}{2} + \frac{23\delta}{24}.$$

Hence, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta,\delta]$, $\delta \le 1/2$, we get

$$\|D_\varepsilon\|^2 \le \Lambda_1\Gamma_1 = \frac{529\delta^2}{576} + \frac{115\delta^3}{64} + \frac{133\delta^4}{288} - \frac{33\delta^5}{32} - \frac{43\delta^6}{72} + \frac{\delta^7}{8} + \frac{\delta^8}{8}.$$

Moreover

$$\begin{aligned} \alpha_{\mathbf{G}} &= \inf_{w\in(0,1)} |g_1(w)|^2 \\ &= \inf_{w\in(0,1)} \left| \frac{1 + 76e^{-2\pi i w} + 230e^{-4\pi i w} + 76e^{-6\pi i n w} + e^{-8\pi i n w}}{384} \right|^2 \\ &= \frac{25}{576}. \end{aligned}$$

Hence, from Theorem 3.1, whenever $\sup_n|\varepsilon_n| < C \approx 0.185$, where C is the root of $529\delta^2/576 + 115\delta^3/64 + 133\delta^4/288 - 33\delta^5/32 - 43\delta^6/72 + \delta^7/8 + \delta^8/8 - 25/576 = 0$ in $(0, 1/2)$, there exists a Riesz basis $\{S_n^\varepsilon\}_{n\in\mathbb{Z}}$ for V_{N_4} such that the expansion

$$f(t) = \sum_{n\in\mathbb{Z}} (\mathcal{L}_1 f)(n+\varepsilon_n)\; S_n^\varepsilon(t)\,, \quad t \in \mathbb{R}\,,$$

holds for each $f \in V_{N_4}$.

Finally, it is worth to mention that the bounds C obtained by using our general technique could be improved in some particular cases by means of a more specific tool. For instance, a similar argument as in the proof of Theorem 2.3 in Ref. 16 could give a better result in this average sampling example. See also Ref. 5 for irregular sampling in spline subspaces.

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