MULTI-CHANNEL SAMPLING ON SHIFT-INVARIANT SPACES WITH FRAME GENERATORS

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ABSTRACT. Let φ be a continuous function in $L^2(\mathbb{R})$ such that the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a frame sequence in $L^2(\mathbb{R})$ and assume that the shift-invariant space $V(\varphi)$ generated by φ has a multi-banded spectrum $\sigma(V)$. The main aim in this paper is to derive a multi-channel sampling theory for the shift-invariant space $V(\varphi)$. By using a type of Fourier duality between the spaces $V(\varphi)$ and $L^2[0,2\pi]$ we find necessary and sufficient conditions allowing us to obtain stable multi-channel sampling expansions in $V(\varphi)$.

KEY WORDS : SHIFT-INVARIANT SPACES, MULTI-CHANNEL SAMPLING, FRAMES
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1. INTRODUCTION

As a natural extension of the classical Shannon sampling theorem, Papoulis introduced in [17] generalized sampling for arbitrary multi-channel sampling in Paley-Wiener spaces $PW_{\pi\sigma}$ of band-limited signals: In many common situations the available data are samples of some filtered versions of the signal itself. Following [17], there have been many generalizations and applications of the multi-channel sampling. See, for example, [6, 7, 16, 19, 20] and references therein.

Although Shannon's sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in [18]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite duration signal; the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has a very slow decay, which makes computation in the signal domain very inefficient.

Moreover, many applied problems impose different a priori constraints on the type of functions. For this reason, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces. Indeed, in many practical applications, signals are assumed to belong to some shift-invariant space of the form: $V(\varphi) := \overline{\text{span}}_{L^2(\mathbb{R})}\{\varphi(t-n) : n \in \mathbb{Z}\}$ where the function φ in $L^2(\mathbb{R})$ is called the generator of $V(\varphi)$. In most of cases in the mathematical literature, it is supposed that the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $V(\varphi)$. See, for instance, [1, 2, 3, 4, 12, 15, 18, 21, 22] and the references therein. Throughout this paper we assume that the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a frame for $V(\varphi)$ and that the spectrum of $V(\varphi)$ is multi-banded in $[0,2\pi]$ (see Section 3 infra).

On the other hand, suppose that N linear time-invariant systems (filters) $\mathcal{L}_j$, $j=1,2,\dots,N$, are defined on the shift-invariant subspace $V(\varphi)$ of $L^2(\mathbb{R})$. In mathematical terms we are dealing with continuous operators which commute with shifts. The recovery of any function $f \in V(\varphi)$ from samples of the functions $\mathcal{L}_j f$, $j=1,2,\dots,N$, leads to a generalized sampling in $V(\varphi)$.

Our challenge problem is the following: Given r, N positive integers and N real numbers $0 \le a_j < r$ for $1 \le j \le N$, find multi-channel sampling expansions like

$$f(t) = \sum_{j=1}^{N} \sum_{n\in\mathbb{Z}} (\mathcal{L}_j f)(a_j + rn) S_{j,n}(t)\,, \quad t \in \mathbb{R}\,, \tag{1.1}$$

valid for any $f \in V(\varphi)$, where the sequence of sampling functions $\{S_{j,n}(t) : 1 \le j \le N, n \in \mathbb{Z}\}$ forms a frame or a Riesz basis for $V(\varphi)$.

Recently, García et al. ([8, 9, 10]) introduced a novel idea for developing a sampling theory on a shift-invariant space $V(\varphi)$ by using an analogous of the Fourier duality between the spaces $V(\varphi)$ and $L^2[0, 2\pi]$. In particular, García and Pérez-Villalón [9] (see also [14]) developed a multi-channel sampling procedure on a shift-invariant space $V(\varphi)$, where φ is a continuous Riesz generator. Unlike the author's claim (see section 4.1 in [9]), the arguments used in [9] for the case of Riesz generator cannot be directly extended to the case of a frame generator.

In the present paper, by assuming that the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a frame for $V(\varphi)$ and that the spectrum of $V(\varphi)$ is multi-banded in $[0, 2\pi]$, and allowing more general filters than those used in [9], we obtain necessary and sufficient conditions under which there exists a stable multi-channel sampling expansion on $V(\varphi)$ like that in (1.1). We also provide some illustrating examples. All these tasks will be carried out throughout the remaining sections.

2. Shift-invariant spaces and Fourier duality type

We start this section by introducing some notation and preliminaries used in the sequel. Let $\{\varphi_n\}_{n\in\mathbb{Z}}$ be a sequence of elements in a separable Hilbert space $\mathcal{H}$. We say that

- the sequence $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a Bessel sequence (with Bessel bound B) in $\mathcal{H}$ if there exists a constant $B > 0$ such that
$$\sum_{n\in\mathbb{Z}} |\langle \varphi, \varphi_n\rangle|^2 \le B\|\varphi\|^2 \text{ for all } \varphi \in \mathcal{H}\,;$$
- the sequence $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}$ (with frame bounds A and B) if there exist constants $0 < A \le B$ such that
$$A\|\varphi\|^2 \le \sum_{n\in\mathbb{Z}} |\langle \varphi, \varphi_n\rangle|^2 \le B\|\varphi\|^2 \text{ for all } \varphi \in \mathcal{H}\,;$$
- the sequence $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$ (with Riesz bounds A and B) if it is a complete set in $\mathcal{H}$ and there exist constants $0 < A \le B$ such that
$$A\|\mathbf{c}\|^2 \le \|\sum_{n\in\mathbb{Z}} c(n)\varphi_n\|^2 \le B\|\mathbf{c}\|^2 \text{ for all } \mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\,,$$
where $\|\mathbf{c}\|^2 := \sum_{n\in\mathbb{Z}} |c(n)|^2$.

For $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we take its Fourier transform to be normalized as

$$\mathcal{F}[\varphi](\xi) = \widehat{\varphi}(\xi) := \int_{-\infty}^{\infty} \varphi(t) e^{-it\xi} dt\,,$$

so that $\frac{1}{\sqrt{2\pi}}\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ becomes a unitary operator. For any $\varphi \in L^2(\mathbb{R})$ consider its related functions

$$C_\varphi(t) := \sum_{n\in\mathbb{Z}} |\varphi(t+n)|^2 \quad \text{and} \quad G_\varphi(\xi) := \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\xi+2n\pi)|^2.$$

It is known that the 1-periodic function C_φ belongs to $L^1[0,1]$ and the 2π-periodic function G_φ belongs to $L^1[0,2\pi]$; moreover,

$$\|\varphi\|^2_{L^2(\mathbb{R})} = \|C_\varphi\|_{L^1[0,1]} = \frac{1}{2\pi}\|G_\varphi\|_{L^1[0,2\pi]}.$$

Let $V(\varphi) := \overline{\text{span}}_{L^2(\mathbb{R})}\{\varphi(t-n) : n \in \mathbb{Z}\}$ be the shift-invariant space generated by φ, that is, the closed subspace of $L^2(\mathbb{R})$ spanned by $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ and supp G_φ the support of the locally integrable function G_φ as a distribution on $\mathbb{R}$. Let $\sigma(V) :=$ supp $G_\varphi \cap [0,2\pi]$ be the spectrum of $V(\varphi)$ and $\tau(V) := [0,2\pi] \setminus \sigma(V)$. For any $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, let

$$\widehat{\mathbf{c}}(\xi) := \sum_{n\in\mathbb{Z}} c(n)e^{-in\xi}$$

be the discrete Fourier transform of the sequence $\mathbf{c}$. In [5] we find the following result:

Proposition 2.1. *Let $\varphi \in L^2(\mathbb{R})$ and $0 < A \le B$. The following statements hold:*

(a) *The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Bessel sequence with a Bessel bound B for $V(\varphi)$ if and only if*

$$G_\varphi(\xi) \le B \text{ a.e. on } [0,2\pi]\,.$$

(b) *The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a frame for $V(\varphi)$ with frame bounds A, B if and only if*

$$A \le G_\varphi(\xi) \le B \text{ a.e. on } \sigma(V)\,.$$

(c) *The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for $V(\varphi)$ with Riesz bounds A, B if and only if*

$$A \le G_\varphi(\xi) \le B \text{ a.e. on } [0,2\pi]\,.$$

For any $\varphi \in L^2(\mathbb{R})$ and $\mathbf{c} = \{c(k)\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, let $T(\mathbf{c}) := (\mathbf{c} * \varphi)(t) = \sum_{k\in\mathbb{Z}} c(k)\varphi(t-k)$ be the pre-frame operator of $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$. Proposition 2.1 can be restated as (cf. [5]):

- The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Bessel sequence with a Bessel bound B if and only if T is a bounded linear operator from $\ell^2(\mathbb{Z})$ into $V(\varphi)$ with $\|T\| \le \sqrt{B}$.
- The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a frame for $V(\varphi)$ with frame bounds A, B if and only if T is a bounded linear operator from $\ell^2(\mathbb{Z})$ onto $V(\varphi)$ and

$$A\|\mathbf{c}\|^2 \le \|T(\mathbf{c})\|^2_{L^2(\mathbb{R})} \le B\|\mathbf{c}\|^2,\ \mathbf{c} \in N(T)^\perp,$$

  where $N(T) := \{\mathbf{c} \in \ell^2(\mathbb{Z}) : T(\mathbf{c}) = 0\}$ and $N(T)^\perp$ is the orthogonal complement of $N(T)$ in $\ell^2(\mathbb{Z})$.
- The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for $V(\varphi)$ with Riesz bounds A, B if and only if T is an isomorphism from $\ell^2(\mathbb{Z})$ onto $V(\varphi)$ and

$$A\|\mathbf{c}\|^2 \le \|T(\mathbf{c})\|^2_{L^2(\mathbb{R})} \le B\|\mathbf{c}\|^2,\ \mathbf{c} \in \ell^2(\mathbb{Z})\,.$$

Lemma 2.2. *Let $\varphi \in L^2(\mathbb{R})$ such that the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$. Then for any $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$,*

$$\widehat{T(\mathbf{c})}(\xi) = \widehat{\mathbf{c}}(\xi)\widehat{\varphi}(\xi)\,,$$

so that

$$\|T(\mathbf{c})\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{\mathbf{c}}(\xi)\widehat{\varphi}(\xi)|^2 d\xi = \frac{1}{2\pi}\int_0^{2\pi} |\widehat{\mathbf{c}}(\xi)|^2 G_\varphi(\xi) d\xi\,. \tag{2.1}$$

Proof. See Lemma 7.2.1 in [5] and Lemma 2.2 in [11]. □

In what follows, we always assume that the function $\varphi \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ is a continuous frame generator (i.e., the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a frame for $V(\varphi)$), and satisfying the condition: $\sup_{\mathbb{R}} C_\varphi(t) < \infty$. Thus $V(\varphi) = \{(\mathbf{c} * \varphi)(t) : \mathbf{c} \in \ell^2(\mathbb{Z})\}$ is a reproducing kernel Hilbert space (RKHS in short) and any $f(t) = (\mathbf{c}*\varphi)(t)$ in $V(\varphi)$ converges both in the $L^2(\mathbb{R})$ sense, and absolutely and uniformly on $\mathbb{R}$ to a continuous function on $\mathbb{R}$ (see [15, 22]).

By using (2.1), we have that $N(T) = \{\mathbf{c} \in \ell^2(\mathbb{Z}) : \widehat{\mathbf{c}}(\xi) = 0 \text{ a.e. on } \sigma(V)\}$ and consequently

$$N(T)^\perp = \{\mathbf{c} \in \ell^2(\mathbb{Z}) : \widehat{\mathbf{c}}(\xi) = 0 \text{ a.e. on } \tau(V)\}. \tag{2.2}$$

Now, we introduce a Fourier duality for $V(\varphi)$ useful for sampling purposes as we will see in the next section.

Let $\mathcal{T}_\varphi : L^2[0,2\pi] \longrightarrow V(\varphi)$ be the linear operator defined by

$$(\mathcal{T}_\varphi F)(t) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \langle F(\xi), e^{-ik\xi}\rangle_{L^2[0,2\pi]}\varphi(t-k) = \langle F(\xi), \frac{1}{2\pi}\overline{Z_\varphi(t,\xi)}\rangle_{L^2[0,2\pi]}\,,$$

where Z_φ denotes the Zak transform of φ given as $Z_\varphi(t,\xi) := \sum_{k\in\mathbb{Z}} \varphi(t+k)e^{-ik\xi}$ (see [12]). Notice that $\{\varphi(t-n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for each t in $\mathbb{R}$. By using (2.2), $\mathcal{T}_\varphi$ is a bounded linear operator from $L^2[0,2\pi]$ onto $V(\varphi)$ with kernel

$$N(\mathcal{T}_\varphi) = \{F(\xi) \in L^2[0,2\pi] : F(\xi) = 0 \text{ a.e. on } \sigma(V)\}\,.$$

Thus, the operator $\mathcal{T}_\varphi : L^2[\sigma(V)] \longrightarrow V(\varphi)$ becomes an isomorphism. We also note the following useful properties of $\mathcal{T}_\varphi$:

- $\widehat{\mathcal{T}_\varphi F}(\xi) = F(\xi)\widehat{\varphi}(\xi)$;
- $\mathcal{T}_\varphi[F(\xi)e^{-in\xi}](t) = (\mathcal{T}_\varphi F)(t-n),\ n \in \mathbb{Z}$.

3. Multi-channel sampling theory

For $1 \le j \le N$, let $\mathcal{L}_j$ be an LTI (linear time-invariant) system with impulse response h_j, that is,

$$\mathcal{L}_j[f](t) := (f * h_j)(t) = \int_{\mathbb{R}} f(s)h_j(t-s)ds\,.$$

Here, we assume that each system $\mathcal{L}_j$ belongs to one of the following three types:

(i) Its impulse response $h_j(t) = \delta(t+a_j),\ a_j \in \mathbb{R}$, or
(ii) $h_j \in L^2(\mathbb{R})$, or
(iii) $\widehat{h}_j \in L^\infty(\mathbb{R})$ whenever $H_\varphi(\xi) := \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\xi+2n\pi)| \in L^2[0,2\pi]$.

For type (i), $\mathcal{L}[f](t) = f(t+a)$ for any $f \in L^2(\mathbb{R})$, so that $\mathcal{L} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ becomes a unitary operator. In particular, consider $\psi(t) := \mathcal{L}[\varphi](t) = \varphi(t+a)$; for any $f(t) = (\mathbf{c} * \varphi)(t) \in V(\varphi)$ we have that $\mathcal{L}[f](t) = (\mathbf{c} * \psi)(t)$ converges absolutely and uniformly on $\mathbb{R}$ since $\sup_{\mathbb{R}} C_\psi(t) = \sum_{n\in\mathbb{Z}} |\psi(t+n)|^2 < \infty$. For types (ii) and (iii) the following result holds:

Lemma 3.1. *Let $\mathcal{L}$ be an LTI system with impulse response h of type (ii) or (iii) as above and consider the function $\psi(t) := \mathcal{L}[\varphi](t) = (\varphi * h)(t)$. Then we have:*

(a) *The function ψ belongs to the space $C_\infty(\mathbb{R}) := \{u(t) \in C(\mathbb{R}) : \lim_{|t|\to\infty} u(t) = 0\}$.*
(b) $\sup_{\mathbb{R}} C_\psi(t) < \infty$.
(c) *For any $f(t) = (\mathbf{c} * \varphi)(t) \in V(\varphi)$ with $\mathbf{c} \in \ell^2(\mathbb{Z})$, $\mathcal{L}[f](t) = (\mathbf{c} * \psi)(t)$ converges absolutely and uniformly on $\mathbb{R}$.*
(d) *For each fixed $t \in \mathbb{R}$, $\operatorname{supp} Z_\psi(t,\cdot) \cap [0,2\pi] \subset \sigma(V)$.*

Proof. First assume $h \in L^2(\mathbb{R})$. Since $\widehat{\psi}(\xi) = \widehat{\varphi}(\xi)\widehat{h}(\xi) \in L^1(\mathbb{R})$, the function $\psi \in C_\infty(\mathbb{R})$ by using the Riemann-Lebesgue Lemma. The Poisson summation formula (cf. Lemma 5.1 in [15]) gives:

$$\begin{aligned} C_\psi(t) &= \sum_{n\in\mathbb{Z}} |\psi(t+n)|^2 = \frac{1}{2\pi}\Big\| \sum_{n\in\mathbb{Z}} \psi(t+n)e^{-in\xi}\Big\|^2_{L^2[0,2\pi]} \\ &= \frac{1}{2\pi}\Big\| \sum_{n\in\mathbb{Z}} \hat{\psi}(\xi+2n\pi)e^{it(\xi+2n\pi)}\Big\|^2_{L^2[0,2\pi]} \\ &\le \frac{1}{2\pi}\|G_\varphi^{\frac12} G_h^{\frac12}\|^2_{L^2[0,2\pi]} = \|G_\varphi\|_{L^\infty(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence $\sup_{\mathbb{R}} C_\psi(t) < \infty$. Since $f(t) = (\mathbf{c} * \varphi)(t)$ converges in $L^2(\mathbb{R})$ for any $\mathbf{c} \in \ell^2(\mathbb{Z})$ and the operator $\mathcal{L} : L^2(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$ is bounded by using Young's inequality on the convolution product, we have that $\mathcal{L}[f](t) = \sum_{k\in\mathbb{Z}} c(k)\mathcal{L}[\varphi(t-k)] = \sum_{k\in\mathbb{Z}} c(k)\psi(t-k) = (\mathbf{c} * \psi)(t)$ converges absolutely and uniformly on $\mathbb{R}$ by using (b).

Now assume that $H_\varphi \in L^2[0,2\pi]$ and let $\widehat{h} \in L^\infty(\mathbb{R})$. Since $\widehat{\varphi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we obtain that $\widehat{\psi} = \widehat{\varphi}\,\widehat{h} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and consequently, $\psi \in C_\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Since $\sum_{n\in\mathbb{Z}} |\widehat{\psi}(\xi + 2n\pi)| \le \|\widehat{h}\|_{L^\infty(\mathbb{R})} H_\varphi(\xi)$, using again the Poisson summation formula we have that

$$C_\psi(t) = \frac{1}{2\pi}\Big\| \sum_{n\in\mathbb{Z}} \widehat{\psi}(\xi+2n\pi)e^{it(\xi+2n\pi)}\Big\|^2_{L^2[0,2\pi]} \le \frac{1}{2\pi}\|\widehat{h}\|^2_{L^\infty(\mathbb{R})}\|H_\varphi\|^2_{L^2[0,2\pi]}\,,$$

so that $\sup_{\mathbb{R}} C_\psi(t) < \infty$. For any $f \in L^2(\mathbb{R})$,

$$\|\mathcal{L}[f]\|_{L^2(\mathbb{R})} = \|f * h\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}}\|\widehat{f}(\xi)\widehat{h}(\xi)\|_{L^2(\mathbb{R})} \le \|\widehat{h}\|_{L^\infty(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}.$$

Hence, $\mathcal{L} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is a bounded linear operator so that, for any $f(t) = (\mathbf{c} * \varphi)(t) \in V(\varphi)$, $\mathcal{L}[f](t) = (\mathbf{c} * \psi)(t)$ converges in $L^2(\mathbb{R})$. Condition (b) implies that $(\mathbf{c} * \psi)(t)$ also converges absolutely and uniformly on $\mathbb{R}$ which proves (c).

Finally to prove (d), consider any $F \in L^2[0,2\pi]$ with $\operatorname{supp} F \subseteq \tau(V)$ and let

$$F(\xi) = \sum_{k\in\mathbb{Z}} c(k)e^{-ik\xi} \quad \text{where} \quad c(k) = \frac{1}{2\pi}\langle F(\xi), e^{-ik\xi}\rangle_{L^2[0,2\pi]}\,,\ k \in \mathbb{Z}\,.$$

The sequence $\mathbf{c} \in N(T)$ so that $T(\mathbf{c}) = (\mathbf{c} * \varphi)(t) = 0$. Since

$$\langle F(\xi), \overline{Z_\psi(t,\xi)}\rangle_{L^2[0,2\pi]} = 2\pi(\mathbf{c} * \psi)(t) = 2\pi\mathcal{L}[\mathbf{c} * \varphi](t) = 0\,,$$

we finally obtain that supp $Z_\psi(t,\cdot)\cap[0,2\pi]\subset\sigma(V)$. □

In particular, given an LTI system $\mathcal{L}$ of type (i), (ii) or (iii), for any $f=(\mathcal{T}_\varphi F)\in V(\varphi)$, where $F\in L^2[\sigma(V)]$, we have

$$\mathcal{L}[f](t)=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\langle F(\xi)\chi_{\sigma(V)}(\xi),e^{ik\xi}\rangle_{L^2[0,2\pi]}\psi(t-k)=\langle F(\xi),\frac{1}{2\pi}\overline{Z_\psi(t,\xi)}\rangle_{L^2[\sigma(V)]}. \tag{3.1}$$

Here $\chi_E(\xi)$ denotes the characteristic function of a measurable set E in $\mathbb{R}$.

As it was said before, in this work we are involved in the following problem: Given r,N positive integers and N real numbers $0\le a_j<r$ for $1\le j\le N$, find multi-channel sampling formulas in $V(\varphi)$ such that, for any $f\in V(\varphi)$,

$$f(t)=\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}}(\mathcal{L}_jf)(a_j+rn)S_{j,n}(t)\,,\quad t\in\mathbb{R}\,, \tag{3.2}$$

where the sequence of sampling functions $\{S_{j,n}(t):1\le j\le N,n\in\mathbb{Z}\}$ forms a frame or a Riesz basis for $V(\varphi)$.

First of all, notice that convergence in the $L^2(\mathbb{R})$-sense in the sampling series (3.2) implies pointwise convergence since $V(\varphi)$ is a RKHS, which is absolute and uniform on $\mathbb{R}$. Indeed, let $\{\widetilde{\varphi}(t-n)\}_{n\in\mathbb{Z}}$ be the canonical dual frame of $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$. Then the reproducing kernel of $V(\varphi)$ is

$$q(s,t):=\sum_{n\in\mathbb{Z}}\widetilde{\varphi}(s-n)\overline{\varphi(t-n)}.$$

Since $\sup_{\mathbb{R}}C_\varphi(t)<\infty$ the function $q(t,t)$ is uniformly bounded on $\mathbb{R}$. Hence, the convergence in the $L^2(\mathbb{R})$-sense implies uniform convergence on $\mathbb{R}$. The pointwise convergence is also absolute due to the unconditional convergence of a frame or Riesz basis expansion.

In this work we solve this problem for the case where $V(\varphi)$ is a shift-invariant space having a continuous frame generator φ and the spectrum $\sigma(V)$ of $V(\varphi)$ is a multi-banded region such that

$$\sigma(V)=\bigcup_{k=1}^{M}[\alpha_k,\beta_k],\ \text{ where } 0\le\alpha_1<\beta_1<\alpha_2<\beta_2<\cdots<\alpha_M<\beta_M\le2\pi\,.$$

Notice that through (3.1) and the isomorphism $\mathcal{T}_\varphi:L^2[\sigma(V)]\longrightarrow V(\varphi)$, the sampling expansion (3.2) on $V(\varphi)$ is equivalent to the expansion in $L^2[\sigma(V)]$:

$$F(\xi)=\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}}\langle F(\xi),\frac{1}{2\pi}\overline{Z_{\psi_j}(a_j,\xi)e^{-irn\xi}}\rangle_{L^2[\sigma(V)]}\ s_{j,n}(\xi)\,,\quad F\in L^2[\sigma(V)]\,,$$

where $\{s_{j,n}(\xi):1\le j\le N,n\in\mathbb{Z}\}$ is a frame or a Riesz basis for $L^2[\sigma(V)]$.

From now on we assume that $\sigma(V)=\bigcup_{k=1}^{M}[\alpha_k,\beta_k]$ and we set

$$s_k:=\alpha_k-\Big[\alpha_k\frac{r}{2\pi}\Big]\frac{2\pi}{r}\quad\text{and}\quad r_k:=\beta_k-\Big[\beta_k\frac{r}{2\pi}\Big]\frac{2\pi}{r}\,,$$

so that $0\le s_k,r_k<\frac{2\pi}{r}$, $1\le k\le M$ ($[x]$ denotes the integer part of $x\ge0$).
Next consider the set of points $\{t_k\}_{k=0}^{m}$ such that $0=t_0<t_1<\cdots<t_m=\frac{2\pi}{r}$ where

$\{t_k\}_{k=1}^{m-1} = \{s_k, r_k : 1 \le k \le M\} \setminus \{0\}$. Then,

$$I := [0, \frac{2\pi}{r}] = \bigcup_{k=1}^{m} \overline{B}_k\,, \quad B_k = (t_{k-1}, t_k).$$

Lemma 3.2. *For each $1 \le k \le m$ and each $1 \le n \le r$, we have that*

$$\text{either } \Big(B_k + (n-1)\frac{2\pi}{r}\Big) \cap \sigma(V) = \emptyset \quad \text{or} \quad \Big(B_k + (n-1)\frac{2\pi}{r}\Big) \subset \sigma(V).$$

Proof. See Lemma 1 in [20]. □

For each $1 \le k \le m$ we consider $L(k)$, the subset of $\{1, 2, \dots, r\}$ defined by

$$L(k) := \Big\{1 \le n \le r : B_k + (n-1)\frac{2\pi}{r} \subset \sigma(V)\Big\}\,,$$

and $l(k) := \#L(k)$, i.e., its number of elements. Let $\mathcal{P} := \{1 \le k \le m : l(k) > 0\}$; for each $k \in \mathcal{P}$, there are $l(k)$ positive integers $\{n_{k,j}\}_{j=1}^{l(k)}$ such that $1 \le n_{k,1} < n_{k,2} < \cdots < n_{k,l(k)} \le r$ and

$$B_k + (n_{k,j} - 1)\frac{2\pi}{r} \subset \sigma(V)\,, \quad 1 \le j \le l(k).$$

For $k \in \mathcal{P}$, let $\widetilde{B}_k := \cup_{j=1}^{l(k)} \big(B_k + (n_{k,j} - 1)\frac{2\pi}{r}\big)$. These sets $\widetilde{B}_k$ are disjoint and $\sigma(V) = \cup_{k \in \mathcal{P}} \overline{\widetilde{B}}_k$; hence, $|\sigma(V)| = \sum_{k \in \mathcal{P}} l(k)|B_k|$, where $|E|$ denotes the Lebesgue measure of E.

For each $k \in \mathcal{P}$, consider the unitary operator $D_k : L^2(\widetilde{B}_k) \longrightarrow L^2_{l(k)}(B_k)$ defined by

$$D_k(F)(\xi) := \Big[F\Big(\xi + (n_{k,1} - 1)\frac{2\pi}{r}\Big), \cdots, F\Big(\xi + (n_{k,l(k)} - 1)\frac{2\pi}{r}\Big)\Big]^\top, \; F \in L^2(\widetilde{B}_k)\,,$$

where $L^2_{l(k)}(B_k)$ denotes the Hilbert product space $L^2(B_k) \times \cdots \times L^2(B_k)$ ($l(k)$ times).

Now, for each $k \in \mathcal{P}$ we consider the $N \times l(k)$ matrix with entries in $L^2(B_k)$

$$\mathbf{G}_k(\xi) := [D_k(g_1)(\xi), \cdots, D_k(g_N)(\xi)]^T = \Big[g_i(\xi + (n_{k,j} - 1)\frac{2\pi}{r})\Big]_{1 \le i \le N,\; 1 \le j \le l(k)}\,,$$

and the $l(k) \times l(k)$ matrix with entries in $L^1(B_k)$

$$\mathbf{H}_k(\xi) := \mathbf{G}_k^*(\xi)\mathbf{G}_k(\xi)\,,$$

where $\mathbf{G}_k^*(\xi)$ denotes the adjoint of the matrix $\mathbf{G}_k(\xi)$, being

$$g_i(\xi) := \frac{1}{2\pi} Z_{\psi_i}(a_i, \xi) \in L^2[\sigma(V)]\,, \quad 1 \le i \le N\,.$$

Let $\lambda_{\min,k}(\xi)$ (respectively $\lambda_{\max,k}(\xi)$) be the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbf{H}_k(\xi)$ and the constants

$$\alpha_G := \min_{k \in \mathcal{P}} \|\lambda_{\min,k}\|_{L^0(B_k)} \quad \text{and} \quad \beta_G := \max_{k \in \mathcal{P}} \|\lambda_{\max,k}\|_{L^\infty(B_k)}. \tag{3.3}$$

Here $\|u\|_{L^0(E)}$ and $\|u\|_{L^\infty(E)}$ denote the essential infimum and the essential supremum of a measurable function u on E. We are now ready to state and prove our main sampling results.

Theorem 3.3. *Assume that the function $Z_{\psi_j}(a_j, \xi) \in L^\infty[\sigma(V)]$ for $1 \le j \le N$. Then the following statements are equivalent:*

(i) *There is a frame* $\big\{S_j(t-rn) : 1 \le j \le N, n \in \mathbb{Z}\big\}$ *for* $V(\varphi)$ *such that for each* $f \in V(\varphi)$ *the sampling formula*

$$f(t) = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} (\mathcal{L}_j f)(a_j + rn) S_j(t-rn)\,, \quad t \in \mathbb{R} \tag{3.4}$$

holds.

(ii) *There is a frame* $\big\{S_{j,n}(t) : 1 \le j \le N, n \in \mathbb{Z}\big\}$ *for* $V(\varphi)$ *such that for each* $f \in V(\varphi)$ *the sampling formula*

$$f(t) = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} (\mathcal{L}_j f)(a_j + rn) S_{j,n}(t)\,, \quad t \in \mathbb{R} \tag{3.5}$$

holds.

(iii) $\alpha_G > 0\,.$

Proof. Condition (i) implies condition (ii) trivially. Assume condition (ii); applying the isomorphism $\mathcal{T}_\varphi^{-1} : V(\varphi) \longrightarrow L^2[\sigma(V)]$ to (3.5) gives:

$$F(\xi) = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_j(\xi)} e^{-irn\xi}\rangle_{L^2[\sigma(V)]}\; s_{j,n}(\xi), \quad F \in L^2[\sigma(V)], \tag{3.6}$$

where $\{s_{j,n}(\xi) : 1 \le j \le N, n \in \mathbb{Z}\}$ is a frame for $L^2[\sigma(V)]$. By using Lemma 3.5 (i) below, the sequence $\big\{\overline{g_j(\xi)} e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\big\}$ is a Bessel sequence in $L^2[\sigma(V)]$. The expansion (3.6) on $L^2[\sigma(V)]$ implies that the sequence $\big\{\overline{g_j(\xi)} e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\big\}$ must be a frame for $L^2[\sigma(V)]$ (see Lemma 5.6.2 in [5]). Hence, condition (iii) holds by using Lemma 3.5 (ii) below.

Finally assume condition (iii); the sequence $\big\{\overline{g_j(\xi)} e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\big\}$ with $g_j(\xi) = \frac{1}{2\pi} Z_{\psi_j}(a_j, \xi)$ is a frame for $L^2[\sigma(V)]$ by Lemma 3.5 (ii) below. Let $\big\{s_j(\xi) e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\big\}$ be a dual frame of $\big\{\overline{g_j(\xi)} e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\big\}$ (cf. Lemma 3.6 below). Thus we have the following frame expansion in $L^2[\sigma(V)]$:

$$F(\xi) = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_j(\xi)} e^{-irn\xi}\rangle_{L^2[\sigma(V)]} s_j(\xi) e^{-irn\xi}, \quad F \in L^2[\sigma(V)]\,. \tag{3.7}$$

Applying the isomorphism $\mathcal{T}_\varphi : L^2[\sigma(V)] \longrightarrow V(\varphi)$ to (3.7) gives (3.4) with $S_j = \mathcal{T}_\varphi(s_j)$, $1 \le j \le N$, which proves condition (i).

□

For later use, notice that $\alpha_G > 0$ implies $l(k) \le N$ for all $k \in \mathcal{P}$. For $N = r = 1$ in Theorem 3.3, we obtain:

Corollary 3.4. *Let* $\mathcal{L}$ *be an LTI system of type (i), (ii) or (iii). There is a frame* $\big\{S(t-n) : n \in \mathbb{Z}\big\}$ *for* $V(\varphi)$ *such that for each* $f \in V(\varphi)$

$$f(t) = \sum_{n\in\mathbb{Z}} (\mathcal{L} f)(a + n) S(t-n)\,, \quad t \in \mathbb{R} \tag{3.8}$$

if and only if

$$0 < \|Z_\psi(a,\xi)\|_{L^0[\sigma(V)]} \le \|Z_\psi(a,\xi)\|_{L^\infty[\sigma(V)]} < \infty. \tag{3.9}$$

Moreover, in this case,

$$\widehat{S}(\xi) = \frac{\widehat{\varphi}(\xi)}{Z_\psi(a,\xi)} \chi_{\operatorname{supp} G_\varphi}(\xi). \tag{3.10}$$

Proof. Whenever $r = 1$, $L(k) = \{1\}$ and $\widetilde{B}_k = B_k$ for all $k \in \mathcal{P}$; thus D_k becomes the identity operator. Therefore, $\mathbf{G}_k(\xi) = g(\xi) = \frac{1}{2\pi} Z_\psi(a,\xi)$ and $\mathbf{H}_k(\xi) = \frac{1}{(2\pi)^2}|Z_\psi(a,\xi)|^2$ for $k \in \mathcal{P}$ and $\xi \in B_k$. Hence $0 < \alpha_G \le \beta_G < \infty$ if and only if condition (3.9) holds. As a consequence, (3.9) implies (3.8) by Theorem 3.3. Conversely, assume that (3.8) holds. Then $\varphi(t) = \sum_{n\in\mathbb{Z}} \psi(a+n) S(t-n)$ so that $\widehat{\varphi}(\xi) = Z_\psi(a,\xi)\widehat{S}(\xi)$ and $G_\varphi(\xi) = |Z_\psi(a,\xi)|^2 G_S(\xi)$ from which (3.9) and (3.10) follow. □

When the impulse response h is the Dirac delta distribution $\delta(t)$, the system $\mathcal{L}$ is the identity operator, and Corollary 3.4 reduces to a regular shifted sampling in $V(\varphi)$ (see Theorem 1 in [22] and Theorem 3.4 in [15]). The next technical lemma used in the proof of Theorem 3.3 enlarges the results of Lemma 3 in [9]:

Lemma 3.5. *Let g_j be in $L^2[\sigma(V)]$ for $1 \le j \le N$ and let α_G, β_G be the constants given by (3.3). Then we have:*

(i) *The sequence $\{\overline{g_j(\xi)} e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ is a Bessel sequence in $L^2[\sigma(V)]$ if and only if $\beta_G < \infty$, that is, $g_j(\xi) \in L^\infty[\sigma(V)]$ for each $1 \le j \le N$.*

(ii) *The sequence $\{\overline{g_j(\xi)} e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ is a frame for $L^2[\sigma(V)]$ if and only if*

$$0 < \alpha_G \le \beta_G < \infty. \tag{3.11}$$

Proof. First note that for any $F \in L^2[\sigma(V)]$ we have

$$\begin{aligned} \langle F(\xi), \overline{g_j(\xi)} e^{-irn\xi} \rangle_{L^2[\sigma(V)]} &= \int_{\sigma(V)} F(\xi) g_j(\xi) e^{irn\xi} d\xi \\ &= \sum_{k\in\mathcal{P}} \int_{B_k} [D_k(g_j)]^T(\xi) D_k(F_k)(\xi) e^{irn\xi} d\xi \\ &= \langle \sum_{k\in\mathcal{P}} [D_k(g_j)]^T D_k(F_k) \chi_{B_k}, e^{-irn\xi} \rangle_{L^2(I)}, \end{aligned}$$

where $F_k(\xi) := F(\xi)\chi_{\widetilde{B}_k}(\xi)$. Since $\{\sqrt{\frac{r}{2\pi}} e^{-irn\xi}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(I)$ and the sets B_k are disjoint, we have

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \left|\langle F(\xi), \overline{g_j(\xi)} e^{-irn\xi} \rangle_{L^2[\sigma(V)]}\right|^2 &= \frac{2\pi}{r} \Big\| \sum_{k\in\mathcal{P}} D_k(g_j)^T D_k(F_k) \chi_{B_k}(\xi) \Big\|^2_{L^2(I)} \\ &= \frac{2\pi}{r} \sum_{k\in\mathcal{P}} \left\| D_k(g_j)^T D_k(F_k) \right\|^2_{L^2(B_k)} \\ &= \frac{2\pi}{r} \sum_{k\in\mathcal{P}} \langle \overline{D_k(g_j)} D_k(g_j)^T D_k(F_k), D_k(F_k) \rangle_{L^2_{l(k)}(B_k)}. \end{aligned}$$

Hence,

$$\begin{aligned}(3.12)\qquad &\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}}|\langle F(\xi),\overline{g_j(\xi)}e^{-irn\xi}\rangle|^2\\ &= \frac{2\pi}{r}\sum_{k\in\mathcal{P}}\langle\sum_{j=1}^{N}\overline{D_k(g_j)}D_k(g_j)^T D_k(F_k),D_k(F_k)\rangle_{L^2_{l(k)}(B_k)}\\ &= \frac{2\pi}{r}\sum_{k\in\mathcal{P}}\langle\mathbf{H}_k(\xi)D_k(F_k),D_k(F_k)\rangle_{L^2_{l(k)}(B_k)}.\end{aligned}$$

For (i), assume that $\beta_G<\infty$. By using (3.12), for any $F\in L^2[\sigma(V)]$ we have

$$\begin{aligned}\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}}\big|\langle F(\xi),\overline{g_j(\xi)}e^{-irn\xi}\rangle_{L^2[\sigma(V)]}\big|^2 &\le \frac{2\pi}{r}\beta_G\sum_{k\in\mathcal{P}}\langle D_k(F_k),D_k(F_k)\rangle_{L^2_{l(k)}(B_k)}\\ &= \frac{2\pi}{r}\beta_G\|F\|^2_{L^2[\sigma(V)]}\end{aligned}$$

so that $\big\{\overline{g_j(\xi)}e^{-irn\xi}:1\le j\le N,n\in\mathbb{Z}\big\}$ is a Bessel sequence with bound $\frac{2\pi}{r}\beta_G$.
On the other hand, for any constant K with $0\le K<\beta_G$ we have that $K<\|\lambda_{\max,k}(\xi)\|_\infty$ for some $k\in\mathcal{P}$. Then there is a measurable set $E\subset B_k$ of positive measure such that $\lambda_{\max,k}(\xi)\ge K$ on E. Choose a measurable vector-valued function $\mathbf{F}_k(\xi):=\{F_{k,j}(\xi)\}_{j=1}^{l(k)}$ on E such that $\sum_{j=1}^{l(k)}|F_{k,j}(\xi)|^2=1$ on E and $\mathbf{H}_k(\xi)\mathbf{F}_k(\xi)=\lambda_{\max,k}(\xi)\mathbf{F}_k(\xi)$ on E. This function can be constructed as in [13, Lemma 2.4]. Extend $\mathbf{F}_k(\xi)$ over B_k by setting $\mathbf{F}_k(\xi)=0$ on $B_k\setminus E$. Thus $\mathbf{F}_k\in L^\infty_{l(k)}(B_k)$ and $\mathbf{H}_k(\xi)\mathbf{F}_k(\xi)=\lambda_{\max,k}(\xi)\mathbf{F}_k(\xi)$ on B_k. Let F be such that $F=D_k^{-1}(\mathbf{F}_k)$ on $\widetilde{B}_k$ and $F(\xi)=0$ on $\sigma(V)\setminus\widetilde{B}_k$. This function F belongs to $L^\infty[\sigma(V)]$ and satisfies

$$\begin{aligned}\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}}\big|\langle F(\xi),\overline{g_j(\xi)}e^{-irn\xi}\rangle_{L^2[\sigma(V)]}\big|^2 &= \frac{2\pi}{r}\langle\mathbf{H}_k(\xi)\mathbf{F}_k(\xi),\mathbf{F}_k(\xi)\rangle_{L^2_{l(k)}(B_k)}\\ &\ge \frac{2\pi}{r}K\langle\mathbf{F}_k,\mathbf{F}_k\rangle_{L^2_{l(k)}(E)}=\frac{2\pi}{r}K\|F\|^2_{L^2[\sigma(V)]}.\end{aligned}$$

As a consequence, $\frac{2\pi}{r}\beta_G$ is the optimal Bessel bound for $\big\{\overline{g_j(\xi)}e^{-irn\xi}:1\le j\le N,n\in\mathbb{Z}\big\}$. Moreover, if $\beta_G=\infty$, the sequence $\big\{\overline{g_j(\xi)}e^{-irn\xi}:1\le j\le N,n\in\mathbb{Z}\big\}$ cannot be a Bessel sequence. Finally, note that the spectral norm of a matrix is equivalent to its Frobenius norm. Hence $\beta_G<\infty$ if and only if all entries of $\mathbf{H}_k(\xi)$ for $k\in\mathcal{P}$ are essentially bounded which is also equivalent to $g_j\in L^\infty[\sigma(V)]$ for $1\le j\le N$.

For (ii), assume that $0<\alpha_G\le\beta_G<\infty$. A similar reasoning as the one in (i) gives that $\big\{\overline{g_j(\xi)}e^{-irn\xi}:1\le j\le N,n\in\mathbb{Z}\big\}$ is a frame for $L^2[\sigma(V)]$, where $\frac{2\pi}{r}\beta_G\ge\frac{2\pi}{r}\alpha_G$ are the optimal upper and lower bounds. In particular, if either $\alpha_G=0$ or $\beta_G=\infty$, then $\big\{\overline{g_j(\xi)}e^{-irn\xi}:1\le j\le N,n\in\mathbb{Z}\big\}$ cannot be a frame for $L^2[\sigma(V)]$. □

It is useful to note that condition (3.11) is equivalent to $g_j\in L^\infty[\sigma(V)]$ for $1\le j\le N$ and $\min_{k\in\mathcal{P}}\|\det\mathbf{H}_k(\xi)\|_{L^0(B_k)}>0$.

Lemma 3.6. *Let g_j be in $L^2[\sigma(V)]$ for $1\le j\le N$ such that $\big\{\overline{g_j(\xi)}e^{-irn\xi}:1\le j\le N,n\in\mathbb{Z}\big\}$ is a frame for $L^2[\sigma(V)]$. Then any dual frame of $\big\{\overline{g_j(\xi)}e^{-irn\xi}:1\le j\le N,n\in\mathbb{Z}\big\}$ having* *the form* $\{s_j(\xi)e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ *is obtained from the equation*

$$\frac{2\pi}{r}\mathbf{S}_k(\xi)^\top = \mathbf{G}_k(\xi)^\dagger + \mathbf{E}_k(\xi)\big(\mathbf{I}_N - \mathbf{G}_k(\xi)\mathbf{G}_k(\xi)^\dagger\big)\,, \quad k \in \mathcal{P}\,, \tag{3.13}$$

where $\mathbf{I}_N$ *is the* $N \times N$ *identity matrix,* $\mathbf{E}_k(\xi)$ *is any arbitrary* $l(k) \times N$ *matrix with entries in* $L^\infty(B_k)$*,* $\mathbf{G}_k(\xi)^\dagger := \big[\mathbf{G}_k(\xi)^*\mathbf{G}_k(\xi)\big]^{-1}\mathbf{G}_k(\xi)$ *is the pseudo-inverse matrix of* $\mathbf{G}_k(\xi)$*,*

$$\mathbf{S}_k(\xi) := [D_k(s_{1,k})(\xi), \cdots, D_k(s_{N,k})(\xi)]^\top \tag{3.14}$$

and $s_{j,k}(\xi) = s_j(\xi)\chi_{\widetilde{B}_k}(\xi)$ *for* $1 \le j \le N$*.*

Proof. Assume that the sequence $\{s_j(\xi)e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ is a dual frame of the sequence $\{\overline{g_j(\xi)}e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$. Then $s_j \in L^\infty[\sigma(V)]$ for $1 \le j \le N$. For any F_1 and F_2 in $L^2[\sigma(V)]$ we also have (cf. Lemma 5.6.2 in [5]):

$$\begin{aligned}\langle F_1, F_2\rangle_{L^2[\sigma(V)]} &= \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}}\langle F_1, s_j e^{-irn\xi}\rangle_{L^2[\sigma(V)]}\langle \overline{g_j} e^{-irn\xi}, F_2\rangle_{L^2[\sigma(V)]} \\ &= \frac{2\pi}{r}\sum_{k\in\mathcal{P}}\langle D_k(F_{1,k}), \mathbf{S}_k(\xi)\mathbf{G}_k(\xi)D_k(F_{2,k})\rangle_{L^2_{l(k)}(B_k)} \end{aligned} \tag{3.15}$$

with $\mathbf{S}_k(\xi)$ as in (3.14). Since

$$\langle F_1, F_2\rangle_{L^2[\sigma(V)]} = \sum_{k\in\mathcal{P}}\langle D_k(F_{1,k}), D_k(F_{2,k})\rangle_{L^2_{l(k)}(B_k)}\,,$$

(3.15) implies that $\frac{2\pi}{r}\mathbf{S}_k(\xi)$ must be a left inverse of the matrix $\mathbf{G}_k(\xi)$. Finally, the right hand side of (3.13) is a left inverse of $\mathbf{G}_k(\xi)$ and any left inverse $\frac{2\pi}{r}\mathbf{S}_k(\xi)^\top$ of $\mathbf{G}_k(\xi)$ is obtained from (3.13) by choosing $\mathbf{E}_k(\xi) = \frac{2\pi}{r}\mathbf{S}_k(\xi)^\top$. □

One can easily check that the canonical dual frame of $\{\overline{g_j(\xi)}e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ is obtained from (3.13) by choosing $\mathbf{E}_k(\xi) = 0$ for each $k \in \mathcal{P}$.

Next we give the Riesz basis counterpart to Theorem 3.3:

Theorem 3.7. *There exists a Riesz basis* $\{S_{j,n}(t) : 1 \le j \le N, n \in \mathbb{Z}\}$ *for* $V(\varphi)$ *for which the sampling expansion (3.5) holds on* $V(\varphi)$ *if and only if*

$$0 < \alpha_G \le \beta_G < \infty \quad \text{and} \quad l(k) = N \text{ for all } 1 \le k \le m. \tag{3.16}$$

Moreover, in this case,

$$S_{j,n}(t) = S_j(t - rn),\ 1 \le j \le N \text{ and } n \in \mathbb{Z}; \tag{3.17}$$

$$(\mathcal{L}_j S_k)(a_j + rn) = \delta_{j,k}\delta_{n,0},\ 1 \le j, k \le N \text{ and } n \in \mathbb{Z}; \tag{3.18}$$

$$|\sigma(V)| = 2\pi\frac{N}{r} \text{ (which implies } N \le r). \tag{3.19}$$

Proof. Assuming (3.16), Lemma 3.8 below proves that the sequence $\{\overline{g_j(\xi)}e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ is a Riesz basis for $L^2[\sigma(V)]$. Thus we have the Riesz basis expansion (3.7), where $\{s_j(\xi)e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ is the dual Riesz basis of $\{\overline{g_j(\xi)}e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$. The isomorphism $\mathcal{T}_\varphi$ gives the sampling expansion (3.5), where $S_{j,n}(t) = S_j(t - rn)$ and $S_j(t) = \mathcal{T}_\varphi(s_j(\xi))(t)$. Conversely assume that the Riesz basis expansion (3.5) holds on $V(\varphi)$. Applying the isomorphism $\mathcal{T}_\varphi^{-1}$ to (3.5) gives the Riesz basis expansion (3.6) on $L^2[\sigma(V)]$.

Then $\{\overline{g_j(\xi)}e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ must be the dual Riesz basis of $\{s_{j,n}(\xi) : 1 \le j \le N, n \in \mathbb{Z}\}$ so that (3.16) holds by Lemma 3.8 below. Since $\{s_{j,n}(\xi) : 1 \le j \le N, n \in \mathbb{Z}\}$ is the dual Riesz basis of $\{\overline{g_j(\xi)}e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$, $s_{j,n}(\xi) = s_j(\xi)e^{-irn\xi}$ where $s_j \in L^\infty[\sigma(V)]$ (cf. Lemma 3.6). Therefore, $S_{j,n}(t) = \mathcal{T}_\varphi(s_j(\xi)e^{-irn\xi})(t) = S_j(t-rn)$, where $S_j = \mathcal{T}_\varphi(s_j)$, $1 \le j \le N$, so that (3.17) holds. Applying the sampling formula (3.4) to S_k gives

$$S_k(t) = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} (\mathcal{L}_j S_k)(a_j + rn) S_j(t-rn)\,, \quad t \in \mathbb{R}\,,$$

from which (3.18) follows. Finally (3.19) follows immediately from (3.16) having in mind that $|\sigma(V)| = \sum_{k\in\mathcal{P}} l(k)|B_k|$. □

Whenever $\sigma(V) = [0, 2\pi]$, φ becomes a Riesz generator for $V(\varphi)$. As a consequence, Theorems 3.3 and 3.7 are the extended frame versions of Theorem 2 and Corollary 1 in [9]; there φ is a Riesz generator and the LTI system $\mathcal{L}_j$ has impulse response h_j in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for $1 \le j \le N$.

Lemma 3.8. *Let g_j be a function in $L^2[\sigma(V)]$ for $1 \le j \le N$ and let α_G, β_G be the constants given by (3.3). Then, the sequence $\{\overline{g_j(\xi)}e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ is a Riesz basis for $L^2[\sigma(V)]$ if and only if*

$$0 < \alpha_G \le \beta_G < \infty \quad \text{and} \quad l(k) = N \quad \text{for all} \quad 1 \le k \le m. \tag{3.20}$$

Proof. Note that $\{\overline{g_j(\xi)}e^{-irn\xi} : 1 \le j \le N, n \in \mathbb{Z}\}$ is a Riesz basis for $L^2[\sigma(V)]$ if and only if it is complete set in $L^2[\sigma(V)]$ and there are constants $0 < A \le B$ such that

$$A\|\mathbf{c}\|^2 \le \Big\| \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} c_j(n)\overline{g_j(\xi)}e^{-irn\xi}\Big\|^2_{L^2[\sigma(V)]} \le B\|\mathbf{c}\|^2, \tag{3.21}$$

where $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_N) \in \ell^2_N(\mathbb{Z})$ and $\|\mathbf{c}\|^2 := \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} |c_j(n)|^2$. For the middle term in (3.21) we have

$$\begin{aligned}
&\Big\| \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} c_j(n)\overline{g_j(\xi)}e^{-irn\xi}\Big\|^2_{L^2[\sigma(V)]} = \int_{\sigma(V)} \Big| \sum_{j=1}^{N} \overline{g_j(\xi)}\widehat{c}_j(r\xi)\Big|^2 d\xi \\
&= \sum_{k\in\mathcal{P}}\sum_{j=1}^{l(k)} \int_{B_k} |\mathbf{g}^*(\xi + (n_{k,j}-1)\frac{2\pi}{r})\widehat{\mathbf{c}}(r\xi)|^2 d\xi \\
&= \sum_{k\in\mathcal{P}} \langle \sum_{j=1}^{l(k)} \mathbf{g}(\xi + (n_{k,j}-1)\frac{2\pi}{r})\mathbf{g}^*(\xi + (n_{k,j}-1)\frac{2\pi}{r})\widehat{\mathbf{c}}(r\xi), \widehat{\mathbf{c}}(r\xi)\rangle_{L^2_N(B_k)} \\
&= \sum_{k\in\mathcal{P}} \langle \widetilde{\mathbf{H}}_k(\xi)\widehat{\mathbf{c}}(r\xi), \widehat{\mathbf{c}}(r\xi)\rangle_{L^2_N(B_k)},
\end{aligned}$$

where $\mathbf{g}(\xi) := [g_1(\xi), \cdots, g_N(\xi)]^\top$, $\widehat{\mathbf{c}}(\xi) := [\widehat{c}_1(\xi), \cdots, \widehat{c}_N(\xi)]^\top$ and $\widetilde{\mathbf{H}}_k(\xi) := \mathbf{G}_k(\xi)\mathbf{G}_k^*(\xi)$. On the other hand,

$$\|\mathbf{c}\|^2 = \frac{r}{2\pi}\|\widehat{\mathbf{c}}(r\xi)\|^2_{L^2_N(I)} = \frac{r}{2\pi}\sum_{k=1}^{m} \|\widehat{\mathbf{c}}(r\xi)\|^2_{L^2_N(B_k)}.$$

Hence, condition (3.21) is equivalent to

$$A\frac{r}{2\pi}\sum_{k=1}^{m}\|\widehat{\mathbf{c}}(r\xi)\|^2_{L^2_N(B_k)} \le \sum_{k\in\mathcal{P}}\langle\widetilde{\mathbf{H}}_k(\xi)\widehat{\mathbf{c}}(r\xi),\widehat{\mathbf{c}}(r\xi)\rangle_{L^2_N(B_k)} \le B\frac{r}{2\pi}\sum_{k=1}^{m}\|\widehat{\mathbf{c}}(r\xi)\|^2_{L^2(B_k)^N}\,, \tag{3.22}$$

which holds if and only if $\mathcal{P}=\{1,2,\cdots,m\}$ and $0<\widetilde{\alpha}_G\le\widetilde{\beta}_G<\infty$, where $\widetilde{\alpha}_G := \min_{1\le k\le m}\|\widetilde{\lambda}_{\min,k}\|_0$, $\widetilde{\beta}_G := \max_{1\le k\le m}\|\widetilde{\lambda}_{\max,k}\|_\infty$, and $\widetilde{\lambda}_{\min,k}$ (respectively $\widetilde{\lambda}_{\max,k}(\xi)$) is the smallest (respectively the largest) eigenvalue of the matrix $\widetilde{\mathbf{H}}_k(\xi)$.

Now assume that $\big\{\overline{g_j(\xi)}e^{-irn\xi} : 1\le j\le N, n\in\mathbb{Z}\big\}$ is a Riesz basis for $L^2[\sigma(V)]$. Since (3.11) holds, we deduce that $l(k)\le N$ for any $k\in\mathcal{P}$; but we also have (3.22) so that $N\le l(k)$ for any $1\le k\le m$. Hence, $l(k)=N$ for all $1\le k\le m$. Conversely, assume that (3.20) holds. Thus, $\big\{\overline{g_j(\xi)}e^{-irn\xi} : 1\le j\le N, n\in\mathbb{Z}\big\}$ is a complete set in $L^2[\sigma(V)]$ since it is a frame for $L^2[\sigma(V)]$. For each $1\le k\le m$, since $\alpha_G\,\mathbf{I}_N\le\mathbf{H}_k(\xi)\le\beta_G\,\mathbf{I}_N$, for any $\mathbf{F}_k\in L^2_N(B_k)$ we have

$$\alpha_G\|\mathbf{F}_k\|^2_{L^2_N(B_k)}\le\|\mathbf{G}_k(\xi)\mathbf{F}_k(\xi)\|^2_{L^2_N(B_k)}\le\beta_G\|\mathbf{F}_k\|^2_{L^2_N(B_k)}\,,$$

and there exists the inverse matrix $\mathbf{G}_k(\xi)^{-1}$ a.e. with entries essentially bounded. Then $\mathbf{G}_k(\xi)$ and $\mathbf{G}^*_k(\xi)$ are isomorphisms from $L^2_N(B_k)$ onto $L^2_N(B_k)$. Hence, for any $k=1,2,\cdots,m$ we have

$$\alpha_G\|\mathbf{F}_k\|^2_{L^2_N(B_k)}\le\|\mathbf{G}^*_k(\xi)\mathbf{F}_k(\xi)\|^2_{L^2_N(B_k)}=\langle\widetilde{\mathbf{H}}_k(\xi)\mathbf{F}_k(\xi),\mathbf{F}_k(\xi)\rangle_{L^2_N(B_k)}\le\beta_G\|\mathbf{F}_k\|^2_{L^2_N(B_k)}\,,$$

for any $\mathbf{F}_k\in L^2_N(B_k)$. Thus (3.22) or, equivalently, (3.21) holds, from which we deduce that the sequence $\big\{\overline{g_j(\xi)}e^{-irn\xi} : 1\le j\le N, n\in\mathbb{Z}\big\}$ is a Riesz basis for $L^2[\sigma(V)]$. □

For the particular case $N=1$, Theorem 3.7 reads:

Corollary 3.9. *Let $\mathcal{L}$ be an LTI system of type (i), (ii) or (iii). Then, there exists a Riesz basis $\{S_n(t) : n\in\mathbb{Z}\}$ for $V(\varphi)$ such that, for any $f\in V(\varphi)$, the sampling formula*

$$f(t)=\sum_{n\in\mathbb{Z}}(\mathcal{L}f)(a+rn)\,S_n(t),\quad t\in\mathbb{R}\,, \tag{3.23}$$

holds if and only if

$$0<\|Z_\psi(a,\xi)\|_{L^0[\sigma(V)]}\le\|Z_\psi(a,\xi)\|_{L^\infty[\sigma(V)]}<\infty \text{ and } l(k)=1 \text{ for all } 1\le k\le m. \tag{3.24}$$

Moreover, in this case:

- $S_n(t)=S(t-rn),\ n\in\mathbb{Z}$;
- $(\mathcal{L}S)(a+rn)=\delta_{n,0},\ n\in\mathbb{Z}$;
- $|\sigma(V)|=\frac{2\pi}{r}$.

Proof. Assume $l(k)=1$ for all $1\le k\le m$; for each $k=1,2,\cdots,m$, there is a unique integer n_k with $1\le n_k\le r$ such that $\widetilde{B}_k=B_k+(n_k-1)\frac{2\pi}{r}\subseteq\sigma(V)$. Thus, $\mathbf{G}_k(\xi)=D_k(g)(\xi)=\frac{1}{2\pi}Z_\psi(a,\xi+(n_k-1)\frac{2\pi}{r})$ and $\mathbf{H}_k(\xi)=\frac{1}{(2\pi)^2}|Z_\psi(a,\xi+(n_k-1)\frac{2\pi}{r})|^2$ for $\xi\in B_k$. Hence, $0<\alpha_G\le\beta_G<\infty$ if and only if $0<\|Z_\psi(a,\xi)\|_{L^0[\sigma(V)]}\le\|Z_\psi(a,\xi)\|_{L^\infty[\sigma(V)]}<\infty$ and, as a consequence, Corollary 3.9 follows from Theorem 3.7. □

Furthermore, if $r=1$ in Corollary 3.9, then φ must be a Riesz generator since $\sigma(V)=[0,2\pi]$ and $\widehat{S}(\xi)=\widehat{\varphi}(\xi)/[Z_\psi(a,\xi)]$.

Finally, it is worth to notice that in sampling formula (3.2) we may allow a rational sampling period $r = \frac{p}{q}$, where p and q are coprime positive integers, since

$$\big\{(\mathcal{L}_j f)(a_j + rn) : n \in \mathbb{Z}\big\} = \big\{(\mathcal{L}_j f)(a_j + r(k-1) + pn) : 1 \le k \le q \text{ and } n \in \mathbb{Z}\big\}\,.$$

4. An illustrative example

Let $\varphi(t) = \frac{1}{2}\mathrm{sinc}\frac{t}{2} = \frac{\sin \pi \frac{t}{2}}{\pi t}$ so that $\widehat{\varphi}(\xi) = \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi)$. On $[0, 2\pi]$ we have,

$$G_\varphi(\xi) = \begin{cases} 1 & \text{on } [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \\ 0 & \text{on } (\frac{\pi}{2}, \frac{3\pi}{2}) \end{cases}$$

so that φ is a continuous frame generator of $V(\varphi)$ and $\sigma(V) = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$. By the Poisson summation formula, we also have

$$\begin{aligned} C_\varphi(t) = \sum_{n\in\mathbb{Z}} |\varphi(t+n)|^2 &= \frac{1}{2\pi}\big\|Z_\varphi(t,\cdot)\big\|^2_{L^2[0,2\pi]} = \frac{1}{2\pi}\Big\|\sum_{n\in\mathbb{Z}} \widehat{\varphi}(\cdot + 2n\pi)e^{it(\cdot+2n\pi)}\Big\|^2_{L^2[0,2\pi]} \\ &\le \frac{1}{2\pi}\Big\|\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\cdot + 2n\pi)|\Big\|^2_{L^2[0,2\pi]} = \frac{1}{2}, \quad t \in \mathbb{R}. \end{aligned}$$

(a) First take $N = 2$, $\widehat{h}_j(\xi) = (i\xi)^{j-1}\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi)$ for $j = 1, 2$, $r = 4$ and $a_1 = a_2 = 0$. For any $f \in V(\varphi)$

$$\mathcal{L}_j[f](t) = f^{(j-1)}(t) \text{ for } j = 1, 2\,.$$

For $\psi_j(t) = \mathcal{L}_j[\varphi](t)$, the Poisson summation formula gives

$$Z_{\psi_j}(0,\xi) = \sum_{n\in\mathbb{Z}} \psi_j(n)e^{-in\xi} = \sum_{n\in\mathbb{Z}} \hat{\psi}_j(\xi + 2n\pi),\ j = 1, 2\,,$$

so that

$$Z_{\psi_1}(0,\xi) = \begin{cases} 1 & \text{on } [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \\ 0 & \text{on } (\frac{\pi}{2}, \frac{3\pi}{2}) \end{cases}$$

and

$$Z_{\psi_2}(0,\xi) = \begin{cases} i\xi & \text{on } [0, \frac{\pi}{2}] \\ 0 & \text{on } (\frac{\pi}{2}, \frac{3\pi}{2}) \\ i(\xi - 2\pi) & \text{on } [\frac{3\pi}{2}, 2\pi]\,. \end{cases}$$

Hence, $Z_{\psi_j}(0,\xi) \in L^\infty[0, 2\pi]$ for $j = 1, 2$.

On the other hand, since $\sigma(V) = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$ and $I = [0, \frac{\pi}{2}]$, $m = 1$ and $L(1) = \{1, 4\}$ so that $l(1) = 2$. Hence,

$$2\pi\mathbf{G}_1(\xi) = \begin{bmatrix} 1 & 1 \\ i\xi & i(\xi - \frac{\pi}{2}) \end{bmatrix},\ 0 \le \xi \le \frac{\pi}{2}$$

and consequently,

$$(2\pi)^2\mathbf{H}_1(\xi) = \begin{bmatrix} 1 + \xi^2 & 1 + \xi(\xi - \frac{\pi}{2}) \\ 1 + \xi(\xi - \frac{\pi}{2}) & 1 + (\xi - \frac{\pi}{2})^2 \end{bmatrix},\ 0 \le \xi \le \frac{\pi}{2}.$$

Hence, $\det \mathbf{H}_1(\xi) = |\det \mathbf{G}_1(\xi)|^2 = 1/(64\pi^2)$ and we deduce that $\alpha_G > 0$. Therefore, by using Theorem 3.7, there exists a Riesz basis $\{S_j(t - 4n) : j = 1, 2 \text{ and } n \in \mathbb{Z}\}$ for $V(\varphi)$ such that, for any $f \in V(\varphi)$

$$f(t) = \sum_{n\in\mathbb{Z}} \big\{f(4n)S_1(t - 4n) + f'(4n)S_2(t - 4n)\big\}, \quad t \in \mathbb{R}\,.$$

(b) We now take $N=3$, $\widehat{h}_j(\xi)=(i\xi)^{j-1}\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi)$ for $j=1,2,3$, $r=5$ and $a_1=a_2=a_3=0$. For $f\in V(\varphi)$,

$$\mathcal{L}_j[f](t)=f^{(j-1)}(t)\ \text{for}\ j=1,2,3\,,$$

and

$$Z_{\psi_3}(0,\xi)=\begin{cases}-\xi^2 & \text{on } [0,\frac{\pi}{2}]\\ 0 & \text{on } (\frac{\pi}{2},\frac{3\pi}{2})\\ -(\xi-2\pi)^2 & \text{on } [\frac{3\pi}{2},2\pi]\,.\end{cases}$$

so that $Z_{\psi_j}(0,\xi)\in L^\infty[0,2\pi]$ for $j=1,2,3$.

Since $\sigma(V)=[0,\frac{\pi}{2}]\cup[\frac{3\pi}{2},2\pi]$ and $I=[0,\frac{2\pi}{5}]$, $m=3$ and $\{t_j\}_{j=0}^3=\{0,\frac{\pi}{10},\frac{3\pi}{10},\frac{2\pi}{5}\}$, so that $L(1)=\{1,2,5\}$, $L(2)=\{1,5\}$, $L(3)=\{1,4,5\}$. We then have

$$2\pi\mathbf{G}_1(\xi)=\begin{bmatrix}1 & 1 & 1\\ i\xi & i(\xi+\frac{2\pi}{5}) & i(\xi-\frac{2\pi}{5})\\ -\xi^2 & -(\xi+\frac{2\pi}{5})^2 & -(\xi-\frac{2\pi}{5})^2\end{bmatrix},\ \xi\in B_1=(0,\frac{\pi}{10});$$

$$2\pi\mathbf{G}_2(\xi)=\begin{bmatrix}1 & 1\\ i\xi & i(\xi-\frac{2\pi}{5})\\ -\xi^2 & -(\xi-\frac{2\pi}{5})^2\end{bmatrix},\ \xi\in B_2=(\frac{\pi}{10},\frac{3\pi}{10});$$

$$2\pi\mathbf{G}_3(\xi)=\begin{bmatrix}1 & 1 & 1\\ i\xi & i(\xi-\frac{4\pi}{5}) & i(\xi-\frac{2\pi}{5})\\ -\xi^2 & -(\xi-\frac{4\pi}{5})^2 & -(\xi-\frac{2\pi}{5})^2\end{bmatrix},\ \xi\in B_3=(\frac{3\pi}{10},\frac{2\pi}{5}).$$

Thus, for $\mathbf{H}_j(\xi)=\mathbf{G}_j^*(\xi)\mathbf{G}_j(\xi)$, $j=1,2,3$, we have $\det\mathbf{H}_1(\xi)=\det\mathbf{H}_3(\xi)=(2/125)^2$ and

$$(2\pi)^4\det\mathbf{H}_2(\xi)=(x^2\xi-x\xi^2)^2+(x^2-\xi^2)^2+(x-\xi)^2\ge(x-\xi)^2=\frac{4\pi^2}{25},$$

where $x=\xi-\frac{2\pi}{5}$; hence $\alpha_G>0$. Therefore, by Theorem 3.3, there exists a frame $\{S_j(t-5n): j=1,2,3 \text{ and } n\in\mathbb{Z}\}$ for $V(\varphi)$ such that, for each $f\in V(\varphi)$ we have

$$f(t)=\sum_{n\in\mathbb{Z}}\left\{f(5n)S_1(t-5n)+f'(5n)S_2(t-5n)+f''(5n)S_3(t-5n)\right\},\quad t\in\mathbb{R}\,.$$

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