Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

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Abstract In order to avoid most of the problems associated with classical Shannon's sampling theory, nowadays signals are assumed to belong to some shift-invariant subspace. In this work we consider a general shift-invariant space V_Φ^2 of $L^2(\mathbb{R}^d)$ with a set Φ of r stable generators. Besides, in many common situations the available data of a signal are samples of some filtered versions of the signal itself taken at a sub-lattice of $\mathbb{Z}^d$. This leads to the problem of generalized sampling in shift-invariant spaces. Assuming that the ℓ^2-norm of the generalized samples of any $f \in V_\Phi^2$ is stable with respect to the $L^2(\mathbb{R}^d)$-norm of the signal f, we derive frame expansions in the shift-invariant subspace allowing the recovery of the signals in V_Φ^2 from the available data. The mathematical technique used here mimics the Fourier duality technique which works for classical Paley-Wiener spaces.

1 By way of introduction

The classical Whittaker-Shannon-Kotel'nikov sampling theorem (WSK sampling theorem) [23, 50] states that any function f band-limited to $[-1/2,1/2]$, that is, $f(t)=\int_{-1/2}^{1/2}\widehat{f}(w)e^{2\pi itw}dw$ for each $t\in\mathbb{R}$, may be reconstructed from the sequence of samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

$$f(t)=\sum_{n=-\infty}^{\infty} f(n)\,\frac{\sin \pi(t-n)}{\pi(t-n)}\,, \quad t\in\mathbb{R}\,.$$

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Thus, the Paley-Wiener space $PW_{1/2}$ of band-limited functions to $[-1/2,1/2]$ is generated by the integer shifts of the cardinal sine function, $\operatorname{sinc}(t) := \sin \pi t/\pi t$. A simple proof of this result is given by using the Fourier duality technique which uses that the Fourier transform

$$\begin{array}{rcl}\mathscr{F} : PW_{1/2} & \longrightarrow & L^2[-1/2,1/2] \\ f & \longmapsto & \widehat{f}\end{array}$$

is an unitary operator from the Paley-Wiener space $PW_{1/2}$ of band-limited functions to $[-1/2,1/2]$ onto $L^2[-1/2,1/2]$. Thus, the Fourier series $\widehat{f}=\sum_{n=-\infty}^{\infty} f(n)\,\mathrm{e}^{-2\pi i n w}$ of $\widehat{f}$ in $L^2[-1/2,1/2]$, by applying the inverse Fourier transform $\mathscr{F}^{-1}$, gives

$$f(t)=\sum_{n=-\infty}^{\infty} f(n)\mathscr{F}^{-1}\left[\mathrm{e}^{-2\pi i n w}\chi_{[-\pi,\pi]}(w)\right](t)=\sum_{n=-\infty}^{\infty} f(n)\,\frac{\sin \pi(t-n)}{\pi(t-n)} \text{ in } L^2(\mathbb{R}).$$

The pointwise convergence comes from the fact that $PW_{1/2}$ is a reproducing kernel Hilbert space (written shortly as RKHS) where convergence in norm implies pointwise convergence (which is, in this case, uniform on $\mathbb{R}$); this comes out from the inequality: $|f(t)| \le \|f\|$ for each $t \in \mathbb{R}$ and $f \in PW_{1/2}$ (for the RKHS's theory and applications, see, for instance, Ref. [36]).

The WSK theorem has its d-dimensional counterpart. Any function f band-limited to the d-dimensional cube $[-1/2,1/2]^d$, i.e., $f(t)=\int_{[-1/2,1/2]^d} \widehat{f}(x)\mathrm{e}^{2\pi i x^\top t}dx$ for each $t \in \mathbb{R}^d$ (here we are using the notation $x^\top t := x_1t_1+\cdots+x_dt_d$ identifying elements in $\mathbb{R}^d$ with column vectors), may be reconstructed from the sequence of samples $\{f(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ as

$$f(t)=\sum_{\alpha\in\mathbb{Z}^d} f(\alpha)\frac{\sin \pi(t_1-\alpha_1)}{\pi(t_1-\alpha_1)}\cdots\frac{\sin \pi(t_d-\alpha_d)}{\pi(t_d-\alpha_d)}\,, \quad t=(t_1,\dots,t_d)\in\mathbb{R}^d\,.$$

Although Shannon's sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in Refs. [42, 43]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite duration signal; the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has a very slow decay at infinity which makes computation in the signal domain very inefficient. Besides, in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval. Moreover, many applied problems impose different a priori constraints on the type of signals. For this reason, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces; signals are assumed to belong to some shift-invariant space of the form: $V_\varphi^2 := \overline{\operatorname{span}}_{L^2}\{\varphi(t-\alpha) \,:\, \alpha\in\mathbb{Z}^d\}$ where the function φ in $L^2(\mathbb{R}^d)$ is called the generator of V_φ^2. See, for instance, Refs. [2, 3, 4, 6, 7, 10, 43, 45, 47, 48, 49, 51] and the references therein.

In this new context, the analogous of the WSK sampling theorem in a shift-invariant space V_φ^2 was first time proved by Walter in [45]:

1.1 Walter's sampling theorem in shift-invariant spaces

Let $\varphi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space V_φ^2 which means that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ^2. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis $\{x_n\}_{n=1}^\infty$ has a unique biorthogonal (dual) Riesz basis $\{y_n\}_{n=1}^\infty$, i.e., $\langle x_n, y_m\rangle_{\mathscr{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^{\infty} \langle x, y_n\rangle_{\mathscr{H}}\, x_n = \sum_{n=1}^{\infty} \langle x, x_n\rangle_{\mathscr{H}}\, y_n\,,$$

hold for every $x \in \mathscr{H}$ (see [11] for more details and proofs). Recall that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for V_φ^2 (see, for instance, [11, p. 143]) if and only if there exist two positive constants $0 < A \le B$ such that

$$A \le \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2 \le B\,, \quad \text{a.e. } w \in [0,1]\,.$$

Thus we have that $V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot - n) \,:\, \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R})\,.$

We assume that the functions in the shift-invariant space V_φ^2 are continuous on $\mathbb{R}$. This is equivalent to say that the generator φ is continuous on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ (see [40]). Thus, any $f \in V_\varphi^2$ is defined on $\mathbb{R}$ as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n \varphi(t-n)$ for each $t \in \mathbb{R}$.

On the other hand, the space V_φ^2 is the image of $L^2[0,1]$ by means of the isomorphism

$$\begin{aligned} \mathscr{T}_\varphi :\quad & L^2[0,1] \longrightarrow V_\varphi^2 \\ & \{\mathrm{e}^{-2\pi i n x}\}_{n\in\mathbb{Z}} \longmapsto \{\varphi(t-n)\}_{n\in\mathbb{Z}}\,, \end{aligned}$$

which maps the orthonormal basis $\{\mathrm{e}^{-2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2[0,1]$ onto the Riesz basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ for V_φ^2. For any $F \in L^2[0,1]$ we have

$$\mathscr{T}_\varphi F(t) = \sum_{n\in\mathbb{Z}} \langle F, \mathrm{e}^{-2\pi i n x}\rangle \varphi(t-n) = \langle F, \sum_{n\in\mathbb{Z}} \overline{\varphi(t-n)} \mathrm{e}^{-2\pi i n x}\rangle = \langle F, K_t\rangle_{L^2[0,1]}\,, \quad t \in \mathbb{R}\,,$$

where, for each $t \in \mathbb{R}$, the function $K_t \in L^2[0,1]$ is given by

$$K_t(x) = \sum_{n\in\mathbb{Z}} \overline{\varphi(t-n)} \mathrm{e}^{-2\pi i n x} = \overline{\sum_{n\in\mathbb{Z}} \varphi(t+n) \mathrm{e}^{-2\pi i n x}} = \overline{Z\varphi(t,x)}\,.$$

Here, $Z\varphi(t,x) := \sum_{n\in\mathbb{Z}} \varphi(t+n) \mathrm{e}^{-2\pi i n x}$ denotes the Zak transform of the function φ. See [11, 22] for properties and uses of the Zak transform.

As a consequence, the samples in $\{f(a+m)\}_{m\in\mathbb{Z}}$ of $f \in V_\varphi^2$, where $a \in [0,1)$ is fixed, can be expressed as

$$f(a+m) = \langle F, K_{a+m}\rangle = \langle F, \mathrm{e}^{-2\pi i m x} K_a\rangle\,, \quad m \in \mathbb{Z} \text{ where } F = \mathscr{T}_\varphi^{-1} f\,.$$

As a consequence, the stable recovery of $f \in V_\varphi^2$ from the sequence of its samples $\{f(a+m)\}_{m\in\mathbb{Z}}$ reduces to the study of the sequence $\{\mathrm{e}^{-2\pi imx}K_a(x)\}_{m\in\mathbb{Z}}$ in $L^2[0,1]$. The following theorem is easy to prove, having in mind that the operator $m_F : L^2[0,1] \to L^2[0,1]$ defined as: $m_F(f) = Ff$ is well-defined if and only if $F \in L^\infty[0,1]$; in this case, it is bounded and its norm $\|m_F\| = \|F\|_\infty$.

Theorem 1. *The sequence of functions* $\{\mathrm{e}^{-2\pi imx}K_a(x)\}_{m\in\mathbb{Z}}$ *is a Riesz basis for* $L^2[0,1]$ *if and only if the inequalities* $0 < \|K_a\|_0 \le \|K_a\|_\infty < \infty$ *hold, where* $\|K_a\|_0 := \operatorname{ess\,inf}_{x\in[0,1]} |K_a(x)|$ *and* $\|K_a\|_\infty := \operatorname{ess\,sup}_{x\in[0,1]} |K_a(x)|$. *Moreover, its biorthogonal Riesz basis is* $\{\mathrm{e}^{-2\pi imx}/\overline{K_a(x)}\}_{m\in\mathbb{Z}}$.

In particular, the sequence $\{\mathrm{e}^{-2\pi imx}K_a(x)\}_{m\in\mathbb{Z}}$ is an orthonormal basis in $L^2[0,1]$ if and only if $|K_a(x)| = 1$ a.e. in $[0,1]$.

Let a be a real number in $[0,1)$ such that $0 < \|K_a\|_0 \le \|K_a\|_\infty < \infty$; next we prove Walter's sampling theorem for V_φ^2 in [45]. Given $f \in V_\varphi^2$, we expand the function $F = \mathscr{T}_\varphi^{-1} f \in L^2[0,1]$ with respect to the Riesz basis $\{\mathrm{e}^{-2\pi inx}/\overline{K_a(x)}\}_{n\in\mathbb{Z}}$. Thus we get

$$F = \sum_{n\in\mathbb{Z}} \langle F, K_{a+n}\rangle \frac{\mathrm{e}^{-2\pi inx}}{\overline{K_a(x)}} = \sum_{n\in\mathbb{Z}} f(a+n) \frac{\mathrm{e}^{-2\pi inx}}{\overline{K_a(x)}} \quad \text{in } L^2[0,1].$$

Applying the operator $\mathscr{T}_\varphi$ to the above expansion we obtain

$$f = \sum_{n\in\mathbb{Z}} f(a+n)\mathscr{T}_\varphi(\mathrm{e}^{-2\pi inx}/\overline{K_a(x)}) = \sum_{n\in\mathbb{Z}} f(a+n) S_a(\cdot - n) \quad \text{in } L^2(\mathbb{R}),$$

where we have used the shifting property $\mathscr{T}_\varphi(\mathrm{e}^{-2\pi inx}F)(t) = (\mathscr{T}_\varphi F)(t-n)$, $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, satisfied by the isomorphism $\mathscr{T}_\varphi$ for the particular function $S_a := \mathscr{T}_\varphi(1/\overline{K_a}) \in V_\varphi^2$. As in the Paley-Wiener case, the shift-invariant space V_φ^2 is a reproducing kernel Hilbert space. Indeed, for each $t \in \mathbb{R}$, the evaluation functional at t is bounded:

$$|f(t)| \le \|F\|\|K_t\| \le \|\mathscr{T}_\varphi^{-1}\|\|K_t\|\|f\| = \|\mathscr{T}_\varphi^{-1}\| \Big(\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 \Big)^{1/2} \|f\|, \quad f \in V_\varphi^2.$$

Therefore, the L^2-convergence implies pointwise convergence which here is uniform on $\mathbb{R}$. The convergence is also absolute due to the unconditional convergence of a Riesz expansion. Thus, for each $f \in V_\varphi^2$ we get the sampling formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(a+n) S_a(t-n), \quad t \in \mathbb{R}.$$

This mathematical technique, which mimics the Fourier duality technique for Paley-Wiener spaces [23], has been successfully used in deriving sampling formulas in other sampling settings [14, 16, 17, 19, 21, 24, 30, 31]. Here, it will be used for obtaining generalized sampling formulas in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators.

1.2 Statement of the general problem

Assume that our functions (signals) belong to some shift-invariant space of the form:

$$V_\Phi^2 := \overline{\text{span}}_{L^2(\mathbb{R}^d)}\big\{\varphi_k(t-\alpha) \,:\, k=1,2,\ldots,r \text{ and } \alpha\in\mathbb{Z}^d\big\}\,,$$

where the functions in $\Phi := \{\varphi_1,\ldots,\varphi_r\}$ in $L^2(\mathbb{R}^d)$ are called a set of generators for V_Φ^2. Assuming that the sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,\, k=1,2\ldots,r}$ is a Riesz basis for V_Φ^2, the shift-invariant space V_Φ^2 can be described as

$$V_\Phi^2 = \Big\{\sum_{n\in\mathbb{Z}}\sum_{k=1}^{r} d_k(\alpha)\ \varphi_k(t-\alpha) \,:\, d_k\in\ell^2(\mathbb{Z}^d), k=1,2\ldots,r\Big\}. \tag{1}$$

See Refs. [8, 9, 35] for the general theory of shift-invariant spaces and their applications. These spaces and the scaling functions $\Phi=\{\varphi_1,\ldots,\varphi_r\}$ appear in the multiwavelet setting. Multiwavelets lead to multiresolution analyses and fast algorithms just as scalar wavelets, but they have some advantages: they can have short support coupled with high smoothness and high approximation order, and they can be both symmetric and orthogonal (see, for instance, Ref. [28]). Classical sampling in multiwavelet subspaces has been studied in Refs. [37, 41].

On the other hand, in many common situations the available data are samples of some filtered versions $f*\mathsf{h}_j$ of the signal f itself, where the average function h_j reflects the characteristics of the adquisition device. This leads to generalized sampling (also called average sampling) in V_Φ^2 (see, among others, Refs. [2, 5, 14, 16, 17, 29, 33, 34, 38, 39, 41]).

Suppose that s convolution systems (linear time-invariant systems or filters in engineering jargon) $\mathscr{L}_j$, $j=1,2,\ldots,s$, are defined on the shift-invariant subspace V_Φ^2 of $L^2(\mathbb{R}^d)$. Assume also that the sequence of samples $\{(\mathscr{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ for f in V_Φ^2 is available, where the samples are taken at the sub-lattice $M\mathbb{Z}^d$ of $\mathbb{Z}^d$, where M denotes a matrix of integer entries with positive determinant. If we sample any function $f\in V_\Phi^2$ on $M\mathbb{Z}^d$, we are using the sampling rate $1/r(\det M)$ and, roughly speaking, we will need, for the recovery of $f\in V_\Phi^2$, the sequence of generalized samples $\{(\mathscr{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ coming from $s\ge r(\det M)$ convolution systems $\mathscr{L}_j$.

Assume that the sequences of generalized samples satisfy the following stability condition: There exist two positive constants $0<A\le B$ such that

$$A\|f\|^2 \le \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}|\mathscr{L}_j f(M\alpha)|^2 \le B\|f\|^2 \quad \text{for all } f\in V_\Phi^2\,.$$

In [5] is said that the set of systems $\{\mathscr{L}_1,\mathscr{L}_2,\ldots,\mathscr{L}_s\}$ is an M-stable filtering sampler for V_Φ^2. The aim of this work is to obtain sampling formulas in V_Φ^2 having the form

$$f(t)=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)S_j(t-M\alpha)\,,\quad t\in\mathbb{R}^d\,, \tag{2}$$

such that the sequence of reconstruction functions $\{S_j(\cdot-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a frame for the shift-invariant space V_Φ^2. This will be done in the light of the frame theory for separable Hilbert spaces, by using a similar mathematical technique as in the above section.

Recall that a sequence $\{x_n\}$ is a frame for a separable Hilbert space $\mathscr{H}$ if there exist two constants $A,B>0$ (frame bounds) such that

$$A\|x\|^2\le\sum_n|\langle x,x_n\rangle|^2\le B\|x\|^2\ \text{ for all } x\in\mathscr{H}\,.$$

Given a frame $\{x_n\}$ for $\mathscr{H}$ the representation property of any vector $x\in\mathscr{H}$ as a series $x=\sum_n c_n x_n$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for overcomplete frames) is sacrificed. Suitable frame coefficients c_n, depending linearly and continuously on x, are obtained by using the dual frames $\{y_n\}$ of $\{x_n\}$, i.e., the sequence $\{y_n\}$ is another frame for $\mathscr{H}$ such that, for each $x\in\mathscr{H}$, the expansions $x=\sum_n\langle x,y_n\rangle x_n=\sum_n\langle x,x_n\rangle y_n$ hold. For more details on the frame theory see the superb monograph [11] and the references therein.

2 Preliminaries on $L^2(\mathbb{R}^d)$ shift-invariant subspaces

Let $\Phi:=\{\varphi_1,\varphi_2,\dots,\varphi_r\}$, where $\varphi_k\in L^2(\mathbb{R}^d)$ $k=1,2,\dots,r$, such that the sequence $\big\{\varphi_k(t-\alpha)\big\}_{\alpha\in\mathbb{Z}^d,\ k=1,2\dots,r}$ is a Riesz basis for the shift-invariant space

$$V_\Phi^2:=\Big\{\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}d_k(\alpha)\ \varphi_k(t-\alpha)\,:\,d_k\in\ell^2(\mathbb{Z}^d),k=1,2\dots,r\Big\}\subset L^2(\mathbb{R}^d)\,.$$

There exists a necessary and sufficient condition involving the Gramian matrix-function

$$G_\Phi(w):=\sum_{\alpha\in\mathbb{Z}^d}\widehat{\Phi}(w+\alpha)\overline{\widehat{\Phi}(w+\alpha)}^{\top}\,,\ \text{where}\ \widehat{\Phi}:=(\widehat{\varphi}_1,\widehat{\varphi}_2,\dots,\widehat{\varphi}_r)^{\top}\,,$$

which assures that the sequence $\{\varphi_k(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d,\ k=1,2\dots,r}$ is a Riesz basis for V_Φ^2; namely (see, for instance, [5]): There exist two positive constants c and C such that

$$c\,\mathbb{I}_r\le G_\Phi(w)\le C\,\mathbb{I}_r\quad\text{ a.e. } w\in[0,1)^d\,. \tag{3}$$

We assume throughout the paper that the functions in the shift-invariant space V_Φ^2 are continuous on $\mathbb{R}^d$. As in the case of one generator, this is equivalent to the generators Φ being continuous on $\mathbb{R}^d$ with $\sum_{\alpha\in\mathbb{Z}^d}|\Phi(t-\alpha)|^2$ uniformly bounded on $\mathbb{R}^d$. Thus, any $f\in V_\Phi^2$ is defined on $\mathbb{R}^d$ as the pointwise sum

$$f(t)=\sum_{k=1}^{r}\sum_{\alpha\in\mathbb{Z}^d} d_k(\alpha)\ \varphi_k(t-\alpha)\,,\quad t\in\mathbb{R}^d\,. \tag{4}$$

Besides, the space V_Φ^2 is a RKHS since the evaluation functionals, $E_t f := f(t)$ are bounded on V_Φ^2. Indeed, for each fixed $t\in\mathbb{R}^d$ we have

$$\begin{aligned}|f(t)|^2 &= \Big|\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r} d_k(\alpha)\ \varphi_k(t-\alpha)\Big|^2 \le \Big(\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}|d_k(\alpha)|^2\Big)\Big(\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}|\varphi_k(t-\alpha)|^2\Big)\\ &= \Big(\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}|d_k(\alpha)|^2\Big)\Big(\sum_{\alpha\in\mathbb{Z}^d}|\Phi(t-\alpha)|^2\Big) \le \frac{\|f\|^2}{c}\sum_{\alpha\in\mathbb{Z}^d}|\Phi(t-\alpha)|^2,\quad f\in V_\Phi^2\,,\end{aligned}$$

where we have used Cauchy-Schwarz's inequality in (4), and the inequality satisfied for any lower Riesz bound c of the Riesz basis $\{\varphi_k(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d,\ k=1,2\dots,r}$ for V_Φ^2, that is, $c\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}|d_k(\alpha)|^2\le\|f\|^2$.

Thus, the convergence in V_Φ^2 in the $L^2(\mathbb{R}^d)$-sense implies pointwise convergence which is uniform on $\mathbb{R}^d$.

The product space

$$L_r^2[0,1)^d := \big\{\mathbf{F}=(F_1,F_2,\dots,F_r)^\top : F_k\in L^2[0,1)^d\,,\ k=1,2,\dots,r\big\}$$

with its usual inner product

$$\langle\mathbf{F},\mathbf{H}\rangle_{L_r^2[0,1)^d} := \sum_{k=1}^{r}\langle F_k,H_k\rangle_{L^2[0,1)^d} = \int_{[0,1)^d}\mathbf{H}^*(w)\mathbf{F}(w)dw$$

becomes a Hilbert space. Similarly, we introduce the product Banach space $L_r^\infty[0,1)^d$.

The system $\big\{\mathrm{e}^{-2\pi i\alpha^\top w}\mathbf{e}_k\big\}_{\alpha\in\mathbb{Z}^d,\ k=1,2,\dots,r}$, where $\mathbf{e}_k$ denotes the vector of $\mathbb{R}^r$ with all the components null except the k-th component which is equal to one, is an orthonormal basis for $L_r^2[0,1)^d$.

The shift-invariant space V_Φ^2 is the image of $L_r^2[0,1)^d$ by means of the isomorphism

$$\begin{aligned}\mathscr{T}_\Phi : L_r^2[0,1)^d &\longrightarrow V_\Phi^2\\ \{\mathrm{e}^{-2\pi i\alpha^\top w}\mathbf{e}_k\}_{\alpha\in\mathbb{Z}^d,\ k=1,2,\dots,r} &\longmapsto \{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,\ k=1,2,\dots,r}\,,\end{aligned}$$

which maps the orthonormal basis $\big\{\mathrm{e}^{-2\pi i\alpha^\top w}\mathbf{e}_k\big\}_{\alpha\in\mathbb{Z}^d,\ k=1,2,\dots,r}$ for $L_r^2[0,1)^d$ onto the Riesz basis $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,\ k=1,2,\dots,r}$ for V_Φ^2. For each $\mathbf{F}=(F_1,\dots,F_r)^\top\in L_r^2[0,1)^d$ we have

$$\mathscr{T}_\Phi\mathbf{F}(t) := \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\big\langle F_k,\mathrm{e}^{-2\pi i\alpha^\top\cdot}\big\rangle_{L^2[0,1)^d}\varphi_k(t-\alpha)\,,\quad t\in\mathbb{R}^d\,. \tag{5}$$

The isomorphism $\mathscr{T}_\Phi$ can also be expressed by

$$f(t) = \mathscr{T}_\Phi \mathbf{F}(t) = \langle \mathbf{F}, \mathbf{K}_t \rangle_{L^2_r[0,1)^d}, \quad t \in \mathbb{R}^d,$$

where the kernel transform $\mathbb{R}^d \ni t \mapsto \mathbf{K}_t \in L^2_r[0,1)^d$ is defined as $\mathbf{K}_t(x) := \overline{\mathbf{Z}\Phi}(t,x)$, and $\mathbf{Z}\Phi$ denotes the Zak transform of Φ, i.e.,

$$(\mathbf{Z}\Phi)(t,w) := \sum_{\alpha\in\mathbb{Z}^d} \Phi(t+\alpha) \mathrm{e}^{-2\pi i \alpha^\top w}.$$

Note that $(\mathbf{Z}\Phi) = (Z\varphi_1, \dots, Z\varphi_r)^\top$ where Z denotes the usual Zak transform.

The following shifting property of $\mathscr{T}_\Phi$ will be used later: For $\mathbf{F} \in L^2_r[0,1)^d$ and $\alpha \in \mathbb{Z}^d$ we have

$$\mathscr{T}_\Phi \big[\mathbf{F}(\cdot) \mathrm{e}^{-2\pi i \alpha^\top \cdot}\big](t) = \mathscr{T}_\Phi \mathbf{F}(t-\alpha), \quad t \in \mathbb{R}^d. \tag{6}$$

2.1 The convolution systems $\mathscr{L}_j$ on V^2_Φ

We consider s convolution systems $\mathscr{L}_j f = f * \mathsf{h}_j$, $j = 1, 2, \dots, s$, defined for $f \in V^2_\Phi$ where each impulse response h_j belongs to one of the following three types:

(a) The impulse response h_j is a linear combination of partial derivatives of shifted delta functionals, i.e.,

$$\big(\mathscr{L}_j f\big)(t) := \sum_{|\beta|\le N_j} c_{j,\beta} D^\beta f(t + d_{j,\beta}), \quad t \in \mathbb{R}^d.$$

If there is a system of this type, we also assume that $\sum_{\alpha\in\mathbb{Z}^d} |D^\beta \varphi(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$ for $|\beta| \le N_j$.

(b) The impulse response h_j of $\mathscr{L}_j$ belongs to $L^2(\mathbb{R}^d)$. Thus, for any $f \in V^2_\varphi$ we have

$$\big(\mathscr{L}_j f\big)(t) := [f * \mathsf{h}_j](t) = \int_{\mathbb{R}^d} f(x) \mathsf{h}_j(t-x) dx, \quad t \in \mathbb{R}^d.$$

(c) The function $\widehat{\mathsf{h}}_j \in L^\infty(\mathbb{R}^d)$ whenever $H_{\varphi_k}(x) := \sum_{\alpha\in\mathbb{Z}^d} |\widehat{\varphi}_k(x+\alpha)| \in L^2[0,1)^d$ for all $k = 1, 2, \dots, r$.

Lemma 1. *Let $\mathscr{L}$ be a convolution system of the type (b) or (c). Then for each fixed $t \in \mathbb{R}^d$ the sequence $\{\big(\mathscr{L}\varphi_k\big)(t+\alpha)\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^2(\mathbb{Z}^d)$ for each $k = 1, \dots, r$.*

Proof. First assume that $\mathsf{h} \in L^2(\mathbb{R}^d)$; then we have

$$\sum_{\alpha\in\mathbb{Z}^d} |\mathscr{L}\varphi_k(t+\alpha)|^2 = \Big\| \sum_{\alpha\in\mathbb{Z}^d} \mathscr{L}\varphi_k(t+\alpha) \mathrm{e}^{-2\pi i \alpha^\top x} \Big\|^2_{L^2[0,1)^d} = \big\| Z\mathscr{L}\varphi_k(t,x) \big\|^2_{L^2[0,1)^d}$$
$$= \Big\| \sum_{\alpha\in\mathbb{Z}^d} \big(\widehat{\mathscr{L}\varphi_k}\big)(x+\alpha) \mathrm{e}^{2\pi i (x+\alpha)^\top t} \Big\|^2_{L^2[0,1)^d},$$

where, in the last equality, we have used a version of the Poisson summation formula [20, Lemma 2.1]. Notice that $\widehat{\varphi}_k, \widehat{\mathsf{h}} \in L^2(\mathbb{R}^d)$ implies, by Cauchy-Schwarz's inequality, that $\widehat{\varphi}_k \widehat{\mathsf{h}} = \widehat{\mathscr{L}\varphi_k} \in L^1(\mathbb{R}^d)$. Now,

$$\begin{aligned}
&\Big\| \sum_{\alpha\in\mathbb{Z}^d} \big(\widehat{\mathscr{L}\varphi_k}\big)(x+\alpha)\mathrm{e}^{2\pi i(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d} \\
=&\Big\| \sum_{\alpha\in\mathbb{Z}^d} \widehat{\varphi}_k(x+\alpha)\widehat{\mathsf{h}}(x+\alpha)\mathrm{e}^{2\pi i(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d} \\
\leq&\Big\|\Big(\sum_{\alpha\in\mathbb{Z}^d}|\widehat{\varphi}_k(x+\alpha)|^2\Big)^{1/2}\Big(\sum_{\alpha\in\mathbb{Z}^d}|\widehat{\mathsf{h}}(x+\alpha)|^2\Big)^{1/2}\Big\|^2_{L^2[0,1)^d} \leq C^{1/2}\|\mathsf{h}\|^2_{L^2[0,1)^d}\,,
\end{aligned}$$

where we have used (3) and the fact that $\|\mathsf{h}\|^2_{L^2(\mathbb{R}^d)} = \|\sum_{\alpha\in\mathbb{Z}^d}|\widehat{\mathsf{h}}(x+\alpha)|^2\|_{L^1[0,1)^d}$. Finally, assume that $H_{\varphi_k}\in L^2[0,1)^d$; since $\widehat{\varphi}_k\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ we obtain that $\widehat{\mathscr{L}\varphi_k} = \widehat{\varphi}_k\widehat{\mathsf{h}} \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Since $\sum_{\alpha\in\mathbb{Z}^d}|\mathscr{L}\varphi_k(x+\alpha)| \leq \|\widehat{\mathsf{h}}\|_{L^\infty(\mathbb{R}^d)}H_{\varphi_k}(x)$, using again [20, Lemma 2.1] we get

$$\begin{aligned}
\sum_{\alpha\in\mathbb{Z}^d}|\mathscr{L}\varphi_k(x+\alpha)|^2 &= \Big\|\sum_{\alpha\in\mathbb{Z}^d}\big(\widehat{\mathscr{L}\varphi_k}\big)(x+\alpha)\mathrm{e}^{2\pi i(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d} \\
&\leq \|\widehat{\mathsf{h}}\|^2_{L^\infty(\mathbb{R}^d)}\|H_{\varphi_k}\|^2_{L^2[0,1)^d}\,.
\end{aligned}$$

□

Lemma 2. *Let $\mathscr{L}$ be a convolution system of the type (a), (b) or (c). Then, for each $f\in V^2_\Phi$ we have*

$$\big(\mathscr{L}f\big)(t) = \langle \mathbf{F}, \big(\overline{\mathbf{Z}\mathscr{L}\Phi}\big)(t,\cdot)\rangle_{L^2_r[0,1)^d}\,, \quad \textit{where} \quad \mathbf{F} = \mathscr{T}_\Phi^{-1}f\,.$$

Proof. Assume that $\mathscr{L}$ is a convolution system of type (a). Under our hypothesis on $\mathscr{L}$, for $m=0,1,2\dots,N$ we have that

$$f^{(m)}(t) = \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^r\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\rangle\varphi_k^{(m)}(t-\alpha)\,.$$

Having in mind we have assumed that $\sum_{\alpha\in\mathbb{Z}^d}|\Phi^{(m)}(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$, we obtain that

$$\begin{aligned}
(\mathscr{L}f)(t) &= \sum_{m=0}^N c_m f^{(m)}(t+d_m) = \sum_{m=0}^N c_m\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^r\big\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\big\rangle\varphi_k^{(m)}(t+d_m-\alpha) \\
&= \sum_{k=1}^r\Big\langle F_k, \sum_{m=0}^N\overline{c_m}\sum_{\alpha\in\mathbb{Z}^d}\overline{\varphi_k}^{(m)}(t+d_m-\alpha)\mathrm{e}^{-2\pi i\alpha^\top\cdot}\Big\rangle_{L^2[0,1)^d} \\
&= \sum_{k=1}^r\Big\langle F_k, \sum_{\alpha\in\mathbb{Z}^d}\overline{\mathscr{L}\varphi_k}(t-\alpha)\mathrm{e}^{-2\pi i\alpha^\top\cdot}\Big\rangle = \sum_{k=1}^r\big\langle F_k, (\overline{Z\mathscr{L}\varphi_k})(t,\cdot)\big\rangle_{L^2[0,1)^d}\,.
\end{aligned}$$

Assume now that $\mathscr{L}$ is a convolution system of the type (b) or (c). For each $t \in \mathbb{R}^d$, considering the function $\psi(x) := \overline{\mathsf{h}(-x)}$, we have

$$
\begin{aligned}
(\mathscr{L}f)(t) = \langle f, \psi(\cdot - t)\rangle_{L^2(\mathbb{R}^d)} &= \Big\langle \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r} \langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\rangle \varphi_k(\cdot-\alpha), \psi(\cdot - t)\Big\rangle_{L^2(\mathbb{R}^d)} \\
&= \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r} \langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\rangle_{L^2[0,1)^d} \Big\langle \varphi_k, \psi(\cdot - t+\alpha)\Big\rangle_{L^2(\mathbb{R}^d)} \\
&= \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r} \langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\rangle_{L^2[0,1)^d} \mathscr{L}\varphi_k(t-\alpha)\,.
\end{aligned}
$$

Since the sequence $\{(\mathscr{L}\varphi_k)(t+\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, Parseval's equality gives

$$
(\mathscr{L}f)(t) = \sum_{k=1}^{r} \langle F_k, \sum_{\alpha\in\mathbb{Z}^d} \overline{\mathscr{L}\varphi_k}(t-\alpha)\,\mathrm{e}^{-2\pi i\alpha^\top\cdot}\rangle_{L^2[0,1)^d} = \langle \mathbf{F}, (\overline{\mathbf{Z}\mathscr{L}\Phi})(t,\cdot)\rangle_{L^2_r(0,1)}\,,
$$

which ends the proof. □

2.2 Sampling at a lattice of $\mathbb{Z}^d$: An expression for the samples

Given a nonsingular matrix M with integer entries, we consider the lattice in $\mathbb{Z}^d$ generated by M, i.e.,

$$\Lambda_M := \{M\alpha : \alpha \in \mathbb{Z}^d\} \subset \mathbb{Z}^d\,.$$

Without loss of generality we can assume that $\det M > 0$; otherwise we can consider $M' = ME$ where E is some $d\times d$ integer matrix satisfying $\det E = -1$. Trivially, $\Lambda_M = \Lambda'_M$. We denote by $M^\top$ and $M^{-\top}$ the transpose matrices of M and M^{-1} respectively. The following useful generalized orthogonal relationship holds (see [44]):

$$
\sum_{p\in\mathscr{N}(M^\top)} \mathrm{e}^{-2\pi i\alpha^\top M^{-T}p} = \begin{cases} \det M, & \alpha\in\Lambda_M \\ 0 & \alpha\in\mathbb{Z}^d\setminus\Lambda_M \end{cases} \tag{7}
$$

where

$$\mathscr{N}(M^\top) := \mathbb{Z}^d \cap \{M^\top x : x\in[0,1)^d\} \tag{8}$$

The set $\mathscr{N}(M^\top)$ has $\det M$ elements (see [44] or [46]). One of these elements is zero, say $i_1 = 0$; we denote the rest of elements by $i_2,\dots,i_{\det M}$ ordered in any form; from now on, $\mathscr{N}(M^\top) = \{i_1 = 0, i_2,\dots,i_{\det M}\} \subset \mathbb{Z}^d$.

Note that the sets, defined as $Q_l := M^{-\top}i_l + M^{-\top}[0,1)^d$, $l = 1,2,\dots,\det M$, satisfy (see [46, p. 110]):

$$Q_l \cap Q_{l'} = \emptyset \ \text{ if } \ l\neq l' \quad \text{and} \quad \mathrm{Vol}\Big(\bigcup_{l=1}^{\det M} Q_l\Big) = 1\,.$$

Thus, $\int_{[0,1)^d} F(x)dx = \sum_{l=1}^{\det M} \int_{Q_l} F(x)dx$, for any function F integrable in $[0,1)^d$ and $\mathbb{Z}^d$-periodic.

Now assume that we sample the filtered versions $\mathscr{L}_j f$ of $f \in V_\Phi^2$, $j = 1,2,\dots,s$, at a lattice Λ_M. Having in mind Lemma 2, for $j = 1,2,\dots,s$ and $\alpha \in \mathbb{Z}^d$ we obtain that

$$\big(\mathscr{L}_j f\big)(M\alpha) = \langle \mathbf{F}, \overline{\mathbf{Z}\mathscr{L}_j \Phi}(M\alpha,\cdot)\rangle = \langle \mathbf{F}, \overline{\mathbf{Z}\mathscr{L}_j \Phi}(0,\cdot)\mathrm{e}^{-2\pi i \alpha^\top M^\top \cdot}\rangle_{L_r^2[0,1)^d}\,, \qquad (9)$$

where $\mathbf{F} = \mathscr{T}_\Phi^{-1} f \in L_r^2[0,1)^d$. Denote

$$\mathbf{g}_j(x) := \mathbf{Z}\mathscr{L}_j \Phi(0,x)\,, \quad j = 1,2,\dots,s\,, \qquad (10)$$

in other words, $\mathbf{g}_j^\top(x) := \big(g_{j,1}(x), g_{j,2}(x), \dots, g_{j,r}(x)\big)$, where $g_{j,k}(x) = Z\mathscr{L}_j \varphi_k(0,x)$ for $1 \le j \le s$ and $1 \le k \le r$.

As a consequence of expression (9) for generalized samples, a challenge problem is to study the completeness, Bessel, frame, or Riesz basis properties of any sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\big\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ in $L_r^2[0,1)^d$. To this end we introduce the $s \times r(\det M)$ matrix of functions

$$\mathbb{G}(x) := \begin{bmatrix} \mathbf{g}_1^\top(x) & \mathbf{g}_1^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_1^\top(x + M^{-\top} i_{\det M}) \\ \mathbf{g}_2^\top(x) & \mathbf{g}_2^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_2^\top(x + M^{-\top} i_{\det M}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_s^\top(x) & \mathbf{g}_s^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_s^\top(x + M^{-\top} i_{\det M}) \end{bmatrix}, \qquad (11)$$

and its related constants

$$A_{\mathbb{G}} := \operatorname*{ess\,inf}_{x \in [0,1)^d} \lambda_{\min}[\mathbb{G}^*(x)\mathbb{G}(x)], \quad B_{\mathbb{G}} := \operatorname*{ess\,sup}_{x \in [0,1)^d} \lambda_{\max}[\mathbb{G}^*(x)\mathbb{G}(x)]\,,$$

where $\mathbb{G}^*(x)$ denotes the transpose conjugate of the matrix $\mathbb{G}(x)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbb{G}^*(x)\mathbb{G}(x)$. Observe that $0 \le A_{\mathbb{G}} \le B_{\mathbb{G}} \le \infty$. Note that in the definition of the matrix $\mathbb{G}(x)$ we are considering the $\mathbb{Z}^d$-periodic extension of the involved functions $\mathbf{g}_j$, $j = 1,2,\dots,s$. Regardless the functions $\mathbf{g}_j$ in $L_r^2[0,1)^d$, $j = 1,2,\dots,s$, are given by (10), the following result holds:

Lemma 3. *Let $\mathbf{g}_j$ be in $L_r^2[0,1)^d$ for $j = 1,2,\dots,s$ and let $\mathbb{G}(x)$ be its associated matrix as in* (11). *Then,*

(a) The sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\big\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ is a complete system for $L_r^2[0,1)^d$ if and only if the rank of the matrix $\mathbb{G}(x)$ is $r(\det M)$ a.e. in $[0,1)^d$.

(b) The sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\big\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Bessel sequence for $L_r^2[0,1)^d$ if and only if $\mathbf{g}_j \in L_r^\infty[0,1)^d$ (or equivalently $B_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $B_{\mathbb{G}}/(\det M)$.

(c) The sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ *is a frame for* $L^2_r[0,1)^d$ *if and only if* $0<A_\mathbb{G}\le B_\mathbb{G}<\infty$. *In this case, the optimal frame bounds are* $A_\mathbb{G}/(\det M)$ *and* $B_\mathbb{G}/(\det M)$.

(d) The sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ *is a Riesz basis for* $L^2_r[0,1)^d$ *if and only if it is a frame and* $s=r(\det M)$.

Proof. For any $\mathbf{F}\in L^2_r[0,1)^d$ we have

$$
\begin{aligned}
\langle \mathbf{F}(x),\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d} &= \int_{[0,1)^d}\sum_{k=1}^{r}F_k(x)g_{j,k}(x)\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,dx\\
&=\sum_{k=1}^{r}\sum_{l=1}^{\det M}\int_{Q_l}F_k(x)g_{j,k}(x)\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,dx\\
&=\sum_{k=1}^{r}\int_{M^{-\top}[0,1)^d}\sum_{l=1}^{\det M}F_k(x+M^{-\top}i_l)g_{j,k}(x+M^{-\top}i_l)\,\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,dx \qquad (12)\\
&=\int_{M^{-\top}[0,1)^d}\sum_{k=1}^{r}\sum_{l=1}^{\det M}F_k(x+M^{-\top}i_l)g_{j,k}(x+M^{-\top}i_l)\,\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,dx\\
&=\int_{M^{-\top}[0,1)^d}\sum_{l=1}^{\det M}\mathbf{g}_j^\top(x+M^{-\top}i_l)\mathbf{F}(x+M^{-\top}i_l)\,\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,dx\,,
\end{aligned}
$$

where we have considered the $\mathbb{Z}^d$-periodic extension of $\mathbf{F}$. Then,

$$
\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\langle \mathbf{F}(x),\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\Big|^2 =
\frac{1}{\det M}\sum_{j=1}^{s}\Big\|\sum_{l=1}^{\det M}\mathbf{g}_j^\top(x+M^{-\top}i_l)\mathbf{F}(x+M^{-\top}i_l)\Big\|^2_{L^2_r(M^{-\top}[0,1)^d)}.
$$

Denoting $\mathbb{F}(x):=\big[\mathbf{F}^\top(x),\mathbf{F}^\top(x+M^{-\top}i_2),\cdots,\mathbf{F}^\top(x+M^{-\top}i_{\det M})\big]^\top$, the equality above reads

$$
\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\langle \mathbf{F}(x),\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\Big|^2=\frac{1}{\det M}\big\|\mathbb{G}(x)\mathbb{F}(x)\big\|^2_{L^2_s(M^{-\top}[0,1)^d)}. \qquad (13)
$$

On the other hand, using that the function $\mathbf{g}_j$ is $\mathbb{Z}^d$-periodic, we obtain that the set $\big\{\mathbf{g}_j(x+M^{-\top}i_l+M^{-\top}i_1),\mathbf{g}_j(x+M^{-\top}i_l+M^{-\top}i_2),\dots,\mathbf{g}_j(x+M^{-\top}i_l+M^{-\top}i_{\det M})\big\}$ has the same elements as $\big\{\mathbf{g}_j(x+M^{-\top}i_1),\mathbf{g}_j(x+M^{-\top}i_2),\dots,\mathbf{g}_j(x+M^{-\top}i_{\det M})\big\}$. Thus the matrix $\mathbb{G}(x+M^{-\top}i_l)$ has the same columns of $\mathbb{G}(x)$, possibly in a different order. Hence, $\operatorname{rank}\mathbb{G}(x)=r(\det M)$ a.e. in $[0,1)^d$ if and only if $\operatorname{rank}\mathbb{G}(x)=r(\det M)$ a.e. in $M^{-\top}[0,1)^d$. Moreover,

$$
A_\mathbb{G}=\operatorname*{ess\,inf}_{x\in M^{-\top}[0,1)^d}\lambda_{\min}[\mathbb{G}^*(x)\mathbb{G}(x)],\quad B_\mathbb{G}=\operatorname*{ess\,sup}_{x\in M^{-\top}[0,1)^d}\lambda_{\max}[\mathbb{G}^*(x)\mathbb{G}(x)]\,. \qquad (14)
$$

To prove (a), assume that there exists a set $\Omega \subseteq M^{-\top}[0,1)^d$ with positive measure such that $\operatorname{rank}\mathbb{G}(x) < r(\det M)$ for ech $x \in \Omega$. Then, there exists a measurable function $v(x)$, $x \in \Omega$, such that $\mathbb{G}(x)v(x) = 0$ and $\|v(x)\|_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)} = 1$ in Ω. This function can be constructed as in [27, Lemma 2.4]. Define $\mathbf{F} \in L^2_r[0,1)^d$ such that $\mathbb{F}(x) = v(x)$ if $x \in \Omega$, and $\mathbb{F}(x) = 0$ if $x \in M^{-\top}[0,1)^d \setminus \Omega$. Hence, from (13) we obtain that the system is not complete. Conversely, if the system is not complete, by using (13) we obtain a $\mathbb{F}(x)$ different from 0 in a set with positive measure such that $\mathbb{G}(x)\mathbb{F}(x) = 0$. Thus $\operatorname{rank}\mathbb{G}(x) < r(\det M)$ on a set with positive measure.
To prove (b) notice that

$$\begin{aligned}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\Big|^2 &= \frac{1}{\det M}\big\|\mathbb{G}(x)\mathbb{F}(x)\big\|^2_{L^2_s(M^{-\top}[0,1)^d)} \\ = \frac{1}{\det M}\int_{M^{-\top}[0,1)^d}\mathbb{F}^*(x)\mathbb{G}^*(x)\mathbb{G}(x)\mathbb{F}(x)dx\,. \end{aligned}\tag{15}$$

If $B_{\mathbb{G}} < \infty$ then, for each $\mathbb{F}$, we have

$$\begin{aligned}\frac{1}{\det M}\int_{M^{-\top}[0,1)^d}\mathbb{F}^*(x)\mathbb{G}^*(x)\mathbb{G}(x)\mathbb{F}(x)dx &\le \frac{B_{\mathbb{G}}}{\det M}\|\mathbb{F}\|^2_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)} \\ &= \frac{B_{\mathbb{G}}}{\det M}\|\mathbf{F}\|^2_{L^2_r[0,1)^d}\,,\end{aligned}\tag{16}$$

from which the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Bessel sequence and its optimal Bessel bound is less than or equal to $B_{\mathbb{G}}/(\det M)$.
Let $K < B_{\mathbf{G}}$; there exists a set $\Omega_K \subset M^{-\top}[0,1)^d$ with positive measure such that $\lambda_{\max_{x\in\Omega_K}}[\mathbb{G}^*(x)\mathbb{G}(x)] \ge K$. Let $\mathbf{F} \in L^2_r[0,1)^d$ such that its associated vector function $\mathbb{F}$ is 0 if $x \in M^{-\top}[0,1)^d \setminus \Omega_K$ and $\mathbb{F}$ is an eigenvector of norm 1 associated with the largest eigenvalue of $\mathbb{G}^*(x)\mathbb{G}(x)$ if $x \in \Omega_K$. Using (15), we obtain

$$\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\Big|^2 \ge \frac{K}{\det M}\|\mathbf{F}\|^2_{L^2_r[0,1)^d}\,.$$

Therefore if $B_{\mathbf{G}} = \infty$ the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is not a Bessel sequence, and the optimal Bessel bound is $B_{\mathbb{G}}/(\det M)$.
To prove (c) assume first that $0 < A_{\mathbb{G}} \le B_{\mathbb{G}} < \infty$. By using part (b), the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Bessel sequence in $L^2_r[0,1)^d$. Moreover, using (15) and the Rayleigh-Ritz theorem (see [25, p. 176]), for each $\mathbf{F} \in L^2_r[0,1)^d$ we obtain

$$\begin{aligned}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\Big|^2 &\ge \frac{A_{\mathbb{G}}}{\det M}\|\mathbb{F}\|^2_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)} \\ &= \frac{A_{\mathbb{G}}}{\det M}\|\mathbf{F}\|^2_{L^2_r[0,1)^d}\end{aligned}\tag{17}$$

Hence the sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a frame with optimal lower bound larger that or equal to $A_{\mathbb{G}}/(\det M)$.

Conversely if $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a frame for $L^2_r[0,1)^d$ we know by part (b) that $B_{\mathbb{G}}<\infty$. In order to prove that $A_{\mathbb{G}}>0$, consider any constant $K>A_{\mathbb{G}}$. Then there exists a set $\Omega_K\subset M^{-\top}[0,1)^d$ with positive measure such that $\lambda_{\min_{x\in\Omega_K}}[\mathbb{G}^*(x)\mathbb{G}(x)]\le K$. Let $\mathbf{F}\in L^2_r[0,1)^d$ such that its associated $\mathbb{F}(x)$ is 0 if $x\in M^{-\top}[0,1)^d\setminus\Omega_K$ and $\mathbb{F}(x)$ is an eigenvector of norm 1 associated with the smallest eigenvalue of $\mathbb{G}^*(x)\mathbb{G}(x)$ if $x\in\Omega_K$. Since $\mathbb{F}$ is bounded, we have that $\mathbb{G}(x)\mathbb{F}(x)\in L^2_s(M^{-\top}[0,1)^d)$. From (15) we get

$$\begin{aligned}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\langle\mathbf{F}(x),\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\Big|^2 &\le \frac{K}{\det M}\|\mathbb{F}\|^2_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)}\\ &=\frac{K}{\det M}\|\mathbf{F}\|^2_{L^2_r[0,1)^d}\,.\end{aligned}\tag{18}$$

Denoting by A the optimal lower frame bound of $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$, we have obtained that $K/(\det M)\ge A$ for each $K>A_{\mathbb{G}}$; thus $A_{\mathbb{G}}/(\det M)\ge A$ and consequently, $A_{\mathbb{G}}>0$. Moreover, under the hypotheses of part (c) we deduce that $A_{\mathbb{G}}/(\det M)$ and $B_{\mathbb{G}}/(\det M)$ are the optimal frame bounds.

The proof of (d) is based in the following result ([11, Theorem 6.1.1]): A frame is a Riesz basis if and only if it has a biorthogonal sequence. Assume that the sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Riesz basis for $L^2_r[0,1)^d$ being the sequence $\{\mathbf{h}_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ its biorthogonal sequence. Using (12) we get

$$\begin{aligned}&\int_{M^{-\top}[0,1)^d}\sum_{l=1}^{\det M}\mathbf{g}_j^\top(x+M^{-\top}i_l)\mathbf{h}_{j',0}(x+M^{-\top}i_l)\,\mathrm{e}^{2\pi i\alpha^\top M^\top x}dx\\ &\qquad=\langle\mathbf{h}_{j',0}(\cdot),\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\rangle=\delta_{j,j'}\delta_{\alpha,0}\,.\end{aligned}$$

Therefore,

$$\sum_{l=1}^{\det M}\mathbf{g}_j^\top(x+M^{-\top}i_l)\mathbf{h}_{j',0}(x+M^{-\top}i_l)\,\mathrm{e}^{2\pi i\alpha^\top M^\top x}=(\det M)\delta_{j,j'}\quad\text{a.e. in } M^{-\top}[0,1)^d\,.$$

Thus the matrix $\mathbb{G}(x)$ has a right inverse a.e. in $M^{-\top}[0,1)^d$ and, in particular, $s\le r(\det M)$. On the other hand, $A_{\mathbb{G}}>0$ implies that $\det[\mathbb{G}^*(x)\mathbb{G}(x)]>0$, a.e. in $M^{-\top}[0,1)^d$, and there exists the matrix $[\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$ a.e. in $M^{-\top}[0,1)^d$. This matrix is a left inverse of the matrix $\mathbb{G}(x)$ which implies $s\ge r(\det M)$. Thus, we obtain that $r(\det M)=s$.

Conversely, assume that $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a frame for $L^2_r[0,1)^d$ and $r(\det M)=s$. In this case $\mathbb{G}(x)$ is a square matrix and $\det[\mathbb{G}(x)^*(x)\mathbf{G}(x)(x)]>0$ a.e. in $M^{-\top}[0,1)^d$ implies that $\det\mathbb{G}(x)\neq 0$ a.e. in $M^{-\top}[0,1)^d$. Having in mind the structure of $\mathbb{G}(x)$ its inverse must be the $r(\det M)\times s$ matrix

$$\mathbb{G}^{-1}(x)=\begin{bmatrix} \mathbf{c}_1(x) & \dots & \mathbf{c}_s(x) \\ \mathbf{c}_1(x+M^{-\top}i_2) & \dots & \mathbf{c}_s(x+M^{-\top}i_2) \\ \vdots & & \vdots \\ \mathbf{c}_1(x+M^{-\top}i_{\det M}) & \dots & \mathbf{c}_s(x+M^{-\top}i_{\det M}) \end{bmatrix},$$

where, for each $j=1,2,\dots,s$, the function $\mathbf{c}_j\in L^2_r[0,1)^d$.
It is easy to verify that the sequence $\big\{(\det M)\mathbf{c}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a biorthogonal sequence of $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ and therefore, it is a Riesz basis for $L^2_r[0,1)^d$. □

3 Generalized regular sampling in V^2_Φ

In this section we prove that expression (9) allows us to obtain $\mathbf{F}=\mathscr{T}_\Phi^{-1}f$ from the generalized samples $\{\mathscr{L}_jf(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$; as a consequence, applying the isomorphism $\mathscr{T}_\Phi$ we recover the function f in V^2_Φ.

Assume that the functions $\mathbf{g}_j$ given in (10) belong to $\in L^\infty_r[0,1)^d$ for $j=1,2,\dots,s$; thus, $\mathbf{g}_j^\top(x)\mathbf{F}(x)\in L^2[0,1)^d$. Having in mind (7) and the expression (9) for the generalized samples, we have that

$$\begin{aligned}
&(\det M)\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_jf\big)(M\alpha)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\\
&=\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_jf\big)(\alpha)\mathrm{e}^{-2\pi i\alpha^\top x}\sum_{p\in\mathscr{N}(M^\top)}\mathrm{e}^{-2\pi i\alpha^\top M^{-\top}p}\\
&=\sum_{p\in\mathscr{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_jf\big)(\alpha)\mathrm{e}^{-2\pi i\alpha^\top(x+M^{-\top}p)}\\
&=\sum_{p\in\mathscr{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\langle\mathbf{F},\overline{\mathbf{g}_j(\cdot)}\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\rangle_{L^2_r[0,1)^d}\,\mathrm{e}^{-2\pi i\alpha^\top(x+M^{-\top}p)}\\
&=\sum_{p\in\mathscr{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\Big(\int_{[0,1)^d}\sum_{k=1}^{r}F_k(y)g_{j,k}(y)\mathrm{e}^{-2\pi i\alpha^\top M^\top y}dy\Big)\mathrm{e}^{-2\pi i\alpha^\top(x+M^{-\top}p)}\\
&=\sum_{p\in\mathscr{N}(M^\top)}\sum_{k=1}^{r}F_k(x+M^{-\top}p)g_{j,k}(x+M^{-\top}p)\\
&=\sum_{p\in\mathscr{N}(M^\top)}\mathbf{g}_j^\top(x+M^{-\top}p)\,\mathbf{F}(x+M^{-\top}p)\,.
\end{aligned}$$

Defining $\mathbb{F}(x):=\big[\mathbf{F}^\top(x),\mathbf{F}^\top(x+M^{-\top}i_2),\dots,\mathbf{F}^\top(x+M^{-\top}i_{\det M})\big]^\top$, the above equality allows us to writte, in matrix form, that $\mathbb{G}(x)\,\mathbb{F}(x)$ equals to

$$(\det M)\Big[\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_1 f\big)(M\alpha)\mathrm{e}^{-2\pi i\alpha^\top M^\top x},\dots,\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_s f\big)(M\alpha)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\Big]^\top.$$

In order to recover the function $\mathbf{F}=\mathscr{T}_\Phi^{-1}f$, assume the existence of an $r\times s$ matrix $\mathbf{a}(x):=[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)]$, with entries in $L^\infty[0,1)^d$, such that

$$\big[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)\big]\ \mathbb{G}(x)=[\mathbb{I}_r,\mathbb{O}_{(\det M-1)r\times r}]\quad \text{a.e. in } [0,1)^d.$$

If we left multiply $\mathbb{G}(x)\mathbb{F}(x)$ by $\mathbf{a}(x)$, we get

$$\mathbf{F}(x)=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(M\alpha)\,\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\quad \text{in}\ L^2_r[0,1)^d. \tag{19}$$

Finally, the isomorphism $\mathscr{T}_\Phi$ gives

$$f(t)=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(M\alpha)(\mathscr{T}_\Phi\mathbf{a}_j)(t-M\alpha),\quad t\in\mathbb{R}^d,$$

where we have used the shifting property (6) and that the space V^2_Φ is a RKHS. Much more can be said about the above sampling result. In fact, the following theorem holds:

Theorem 2. *Assume that the functions $\mathbf{g}_j$ given in* (10) *belong to $L^\infty_r[0,1)^d$ for each $j=1,2,\dots,s$. Let $\mathbb{G}(x)$ be the associated matrix defined in $[0,1)^d$ as in* (11)*. The following statements are equivalents:*

(a) $A_\mathbb{G}>0$.
(b) There exists an $r\times s$ matrix $\mathbf{a}(x):=\big[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)\big]$ with columns $\mathbf{a}_j\in L^\infty_r[0,1)^d$ satisfying

$$[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)]\mathbb{G}(x)=[\mathbb{I}_r,\mathbb{O}_{(\det M-1)r\times r}]\quad \textit{a.e. in } [0,1)^d. \tag{20}$$

(c) There exists a frame for V^2_Φ having the form $\{S_{j,\mathbf{a}}(\cdot-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ such that for any $f\in V^2_\Phi$

$$f=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)\,S_{j,\mathbf{a}}(\cdot-M\alpha)\quad \textit{in } L^2(\mathbb{R}^d). \tag{21}$$

(d) There exists a frame $\{S_{j,\alpha}(\cdot)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ for V^2_Φ such that for any $f\in V^2_\Phi$

$$f=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)\,S_{j,\alpha}\quad \textit{in } L^2(\mathbb{R}^d). \tag{22}$$

Proof. First we prove that (a) implies (b). As the determinant of the semipositive definite matrix $\mathbb{G}^*(x)\mathbb{G}(x)$ is equal to the product of its eigenvalues, condition (a) implies that $\operatorname{ess\,inf}_{x\in\mathbb{R}^d}\det[\mathbb{G}^*(x)\mathbb{G}(x)]>0$. Hence, there exists the left

pseudo-inverse matrix $\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$, a.e. in $[0,1)^d$, and it satisfies $\mathbb{G}^\dagger(x)\mathbb{G}(x) = \mathbb{I}_{r(\det M)}$. The first r rows of $\mathbb{G}^\dagger(x)$ form an $r \times s$ matrix $[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)]$ which satisfies (20). Moreover, the functions $\mathbf{a}_j(x)$, $j = 1,2,\dots,s$, are essentially bounded since the condition $\operatorname{ess\,inf}_{x\in[0,1)^d} \det[\mathbb{G}^*(x)\mathbb{G}(x)] > 0$ holds.
Next, we prove that (b) implies (c). For $j = 1,2,\dots,s$, let $\mathbf{a}_j(x)$ be a function in $L^\infty_r[0,1)^d$, and satisfying $[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)]\mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}]$. In (19) we have proved that, for each $\mathbf{F} = \mathscr{T}_\Phi^{-1}(f) \in L^2_r[0,1)^d$, we have the expansion

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(M\alpha)\,\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x} \quad \text{in } L^2_r[0,1)^d\,,$$

from which

$$f = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(M\alpha)\,S_{j,\mathbf{a}}(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d)\,,$$

where $S_{j,\mathbf{a}} := \mathscr{T}_\Phi \mathbf{a}_j$ for $j = 1,2,\dots,s$. Since we have assumed that $\mathbf{g}_j \in L^\infty_r[0,1)^d$ for each $j = 1,2,\dots,s$, the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Bessel sequence in $L^2_r[0,1)^d$ by using part (b) in Lemma 3. The same argument proves that the sequence $\big\{(\det M)\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is also a Bessel sequence in $L^2_r[0,1)^d$. These two Bessel sequences satisfy for each $\mathbf{F} \in L^2_r[0,1)^d$

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\langle \mathbf{F}, \overline{\mathbf{g}_j}\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\rangle \mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x} \quad \text{in } L^2_r[0,1)^d\,.$$

Hence, they are a pair of dual frames for $L^2_r[0,1)^d$ (see [11, Lemma 5.6.2]). Since $\mathscr{T}_\Phi$ is an isomorphism, the sequence $\big\{S_{j,\mathbf{a}}(t - M\alpha)\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a frame for V^2_Φ; hence (b) implies (c). Statement (c) implies (d) trivially.
Assume condition (d), applying the isomorphism $\mathscr{T}_\Phi^{-1}$ to the expansion (22) we get

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\langle \mathbf{F}, \overline{\mathbf{g}_j}\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\rangle \mathscr{T}_\Phi^{-1}(S_{j,\alpha})(x) \quad \text{in } L^2_r[0,1)^d\,, \tag{23}$$

where $\big\{\mathscr{T}_\Phi^{-1}S_{j,\alpha}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a frame for $L^2_r[0,1)^d$. By using Lemma 3, the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Bessel sequence; expansion (23) implies that is also a frame (see [11, Lemma 5.6.2]). Hence, by using again Lemma 3, condition (a) holds. □

In the case that the functions $\mathbf{g}_j$, $j = 1,2,\dots,s$, are continuous on $\mathbb{R}^d$ (for instance, if the sequences of generalized samples $\big\{\mathscr{L}_j\varphi_k(\alpha)\big\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^1(\mathbb{Z}^d)$ for $1 \le j \le s$ and $1 \le k \le r$), the following corollary holds:

Corollary 1. *Assume that the functions* $\mathbf{g}_j$, $j = 1,2,\dots,s$, *in* (10) *are continuous on* $\mathbb{R}^d$. *Then, the following assertions are equivalents:*

(a) rank $\mathbb{G}(x) = r(\det M)$ *for all* $x \in \mathbb{R}^d$.
(b) There exists a frame $\{S_{j,\mathbf{a}}(\cdot - rn)\}_{n\in\mathbb{Z},\ j=1,2,\dots,s}$ *for* V_Φ^2 *satisfying the sampling formula* (21).

Proof. Whenever the functions $\mathbf{g}_j$, $j = 1,2,\dots,s$, are continuous on $\mathbb{R}^d$, condition $A_\mathbb{G} > 0$ is equivalent to that $\det\left[\mathbb{G}^*(x)\mathbb{G}(x)\right] \neq 0$ for all $x \in \mathbb{R}^d$. Indeed, if $\det \mathbb{G}^*(x)\mathbb{G}(x) > 0$ then the r first rows of the matrix $\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$, give an $r\times s$ matrix $\mathbf{a}(x) = [\mathbf{a}_1(x),\mathbf{a}_2(x),\dots,\mathbf{a}_s(x)]$ satisfying statement (b) in Theorem 2, and therefore $A_\mathbb{G} > 0$.

The reciprocal follows from the fact that $\det\left[\mathbb{G}^*(x)\mathbb{G}(x)\right] \geq A_\mathbb{G}^{r(\det M)}$ for all $x \in \mathbb{R}^d$. Since $\det\left[\mathbb{G}^*(x)\mathbb{G}(x)\right] \neq 0$ is equivalent to rank $\mathbb{G}(x) = r(\det M)$ for all $x \in \mathbb{R}^d$, the result is a consequence of Theorem 2. □

The reconstruction functions $S_{j,\mathbf{a}}$, $j = 1,2,\dots,s$, are determined from the Fourier coefficients of the components of $\mathbf{a}_j(x) := [a_{1,j}(x), a_{2,j}(x),\dots,a_{r,j}]^\top$, $j = 1,2,\dots,s$. More specifically, if $\widehat{a}_{k,j}(\alpha) := \int_{[0,1)^d} a_{k,j}(x)\mathrm{e}^{2\pi i\alpha^\top x}dx$ we get

$$S_{j,\mathbf{a}}(t) = \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r} \widehat{a}_{k,j}(\alpha)\varphi_k(t-\alpha)\,, \quad t \in \mathbb{R}^d\,. \tag{24}$$

The Fourier transform in (24) gives $\widehat{S}_{j,\mathbf{a}}(x) = \sum_{k=1}^r a_{k,j}(x)\widehat{\varphi}_k(x)$.

Assume that the $r\times s$ matrix $\mathbf{a}(x) = \left[\mathbf{a}_1(x),\mathbf{a}_2(x),\dots,\mathbf{a}_s(x)\right]$ satisfies (20). We consider the periodic extension of $a_{k,j}$, i.e., $a_{k,j}(x+\alpha) = a_{k,j}(x)$, $\alpha \in \mathbb{Z}^d$. For all $x \in [0,1)^d$, the $r(\det M)\times s$ matrix

$$\mathbb{A}^\top(x) := \begin{bmatrix} \mathbf{a}_1(x) & \mathbf{a}_2(x) & \cdots & \mathbf{a}_s(x) \\ \mathbf{a}_1(x+M^{-\top}i_2) & \mathbf{a}_2(x+M^{-\top}i_2) & \cdots & \mathbf{a}_s(x+M^{-\top}i_2) \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_1(x+M^{-\top}i_{\det M}) & \mathbf{a}_2(x+M^{-\top}i_{\det M}) & \cdots & \mathbf{a}_s(x+M^{-\top}i_{\det M}) \end{bmatrix} \tag{25}$$

is a left inverse matrix of $\mathbb{G}(x)$, i.e., $\mathbb{A}^\top(x)\mathbb{G}(x) = \mathbb{I}_{r(\det M)}$.

Provided that condition (20) is satisfied, it can be easily checked that all matrices $\mathbf{a}(x)$ with entries in $L^\infty[0,1)^d$, and satisfying (20) correspond to the first r rows of the matrices of the form

$$\mathbb{A}^\top(x) = \mathbb{G}^\dagger(x) + \mathbb{U}(x)\left[\mathbb{I}_s - \mathbb{G}(x)\mathbb{G}^\dagger(x)\right], \tag{26}$$

where $\mathbb{U}(x)$ is any $r(\det M)\times s$ matrix with entries in $L^\infty[0,1)^d$, and $\mathbb{G}^\dagger$ denotes the left pseudo-inverse $\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$.

Notice that if $s = r(\det M)$ there exists a unique matrix $\mathbf{a}(x)$, given by the first r rows of $\mathbb{G}^{-1}(x)$; if $s > r(\det M)$ there are many solutions according to (26).

Moreover, the sequence $\left\{(\det M)\mathbf{a}_j^\dagger(\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$, associated with the $r\times s$ matrix $[\mathbf{a}_1^\dagger(x),\mathbf{a}_2^\dagger(x),\dots,\mathbf{a}_s^\dagger(x)]$ obtained from the r first rows of $\mathbb{G}^\dagger(x)$, gives precisely the canonical dual frame of the frame $\left\{\overline{\mathbf{g}_j}(\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$. Indeed, the frame operator $\mathscr{S}$ associated to $\left\{\overline{\mathbf{g}_j}(\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is given by

$$\mathscr{S}\mathbf{F}(x)=\frac{1}{\det M}\left[\overline{\mathbf{g}_1}(x),\overline{\mathbf{g}_2}(x),\dots,\overline{\mathbf{g}_s}(x)\right]\mathbb{G}(x)\mathbb{F}(x)\,,\quad \mathbf{F}\in L^2_r[0,1)^d\,,$$

from which one gets

$$\mathscr{S}\left[(\det M)\mathbf{a}_j^\dagger(\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\right](x)=\overline{\mathbf{g}_j}(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\,,\quad j=1,2,\dots,s \text{ and } \alpha\in\mathbb{Z}^d\,.$$

Something more can be said in the case where $s=r(\det M)$:

Theorem 3. *Assume that the functions $\mathbf{g}_j$, $j=1,2,\dots,s$, given in* (10) *belong to $L^\infty_r[0,1)^d$ and $s=r(\det M)$. The following statements are equivalent:*

(a) $A_\mathbb{G}>0$.
(b) There exists a Riesz basis $\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ for V^2_Φ such that for any $f\in V^2_\Phi$, the expansion

$$f=(\det M)\sum_{\alpha\in\mathbb{Z}^d}\sum_{j=1}^{s}\big(\mathscr{L}_j f\big)(M\alpha)\ S_{j,\alpha}\,,\tag{27}$$

holds in $L^2(\mathbb{R}^d)$.

In case the equivalent conditions are satisfied, necessarily $S_{j,\alpha}(t)=S_{j,\mathbf{a}}(t-M\alpha)$, $t\in\mathbb{R}^d$, where $S_{j,\mathbf{a}}=\mathscr{T}_\Phi(\mathbf{a}_j)$, $j=1,2,\dots,s$, and the $r\times s$ matrix $\mathbf{a}:=[\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_s]$ is formed with the r first rows of the inverse matrix $\mathbb{G}^{-1}$. The sampling functions $S_{j,\mathbf{a}}$, $j=1,2,\dots,s$, satisfy the interpolation property $(\mathscr{L}_{j'}S_{j,\mathbf{a}})(M\alpha)=\delta_{j,j'}\delta_{\alpha,0}$, where $j,j'=1,2,\dots,s$ and $\alpha\in\mathbb{Z}^d$.

Proof. Assume that $A_\mathbb{G}>0$; since $\mathbb{G}(x)$ is a square matrix, this implies that $\operatorname{ess\,inf}_{x\in\mathbb{R}^d}|\det\mathbb{G}(x)|>0$. Therefore, the r first rows of $\mathbb{G}^{-1}(x)$ gives a solution of the equation $[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)]\mathbb{G}(x)=[\mathbb{I}_r,\mathbb{O}_{(\det M-1)r\times r}]$ with $\mathbf{a}_j\in L^\infty_r[0,1)^d$ for $j=1,2,\dots,s$. According to Theorem 2, the sequence

$$\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}:=\{S_{j,\mathbf{a}}(t-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}\,,$$

where $S_{j,\mathbf{a}}=\mathscr{T}_\Phi(\mathbf{a}_j)$, satisfies the sampling formula (27). Moreover, the sequence

$$\{(\det M)\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}=\{\mathscr{T}_\Phi^{-1}S_{j,\mathbf{a}}(\cdot-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$$

is a frame for $L^2_r[0,1)^d$. Since $r(\det M)=s$, according to Lemma 3 it is a Riesz basis for $L^2_r[0,1)^d$. Hence, the sequence $\{S_{j,\mathbf{a}}(t-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Riesz basis for V^2_Φ and condition (b) is proved.

Conversely, assume now that $\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Riesz basis for V^2_Φ satisfying (27). From the uniqueness of the coefficients in a Riesz basis, we get that the interpolatory condition $(\mathscr{L}_{j'}S_{j,\alpha})(M\alpha')=\delta_{j,j'}\delta_{\alpha,\alpha'}$ holds for $j,j'=1,2,\dots,s$ and

$\alpha, \alpha' \in \mathbb{Z}^d$. Since $\mathscr{T}_\Phi^{-1}$ is an isomorphism, $\{\mathscr{T}_\Phi^{-1} S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Riesz basis for $L^2_r[0,1)^d$. Expanding the function $\overline{\mathbf{g}_{j'}(x)}\mathrm{e}^{-2\pi i\alpha'^\top M^\top x}$ with respect to the dual basis of $\{\mathscr{T}_\Phi^{-1} S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$, denoted by $\{G_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$, we obtain

$$\begin{aligned}\overline{\mathbf{g}_{j'}(x)}\mathrm{e}^{-2\pi i\alpha'^\top M^\top x} &= \sum_{\alpha\in\mathbb{Z}^d}\sum_{j=1}^{s}\langle\overline{\mathbf{g}_{j'}(\cdot)}\mathrm{e}^{-2\pi i\alpha'^\top M^\top \cdot}, \mathscr{T}_\Phi^{-1} S_{j,\alpha}\rangle_{L^2[0,1)^d} G_{j,\alpha}(x)\\ &= \sum_{\alpha\in\mathbb{Z}^d}\overline{\mathscr{L}_{j'}S_{j,\alpha}}(M\alpha')G_{j,\alpha}(x) = G_{j',\alpha'}(x)\,.\end{aligned}$$

Therefore, the sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is the dual basis of the Riesz basis $\{\mathscr{T}_\Phi^{-1} S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$. In particular it is a Riesz basis for $L^2_r[0,1)^d$, which implies, according to Lemma 3, that $A_\mathbf{G} > 0$; this proves (a). Moreover, the sequence $\{\mathscr{T}_\Phi^{-1} S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is necessarily the unique dual basis of the Riesz basis $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$. Therefore, this proves the uniqueness of the Riesz basis $\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ for V^2_Φ satisfying (27). □

3.1 Reconstruction functions with prescribed properties

The generalized sampling formula in the shift-invariant space V^2_Φ

$$f(t) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)S_{j,\mathbf{a}}(t-M\alpha)\,, \quad t\in\mathbb{R}^d\,, \tag{28}$$

can be read as a filter bank. Indeed, introducing the expression for the sampling functions $S_{j,\mathbf{a}}(t) = \sum_{\beta\in\mathbb{Z}^d}\sum_{k=1}^{r}\widehat{a}_{k,j}(\beta)\varphi_k(t-\beta)$, $t\in\mathbb{R}^d$, the change $\gamma = \beta + M\alpha$ in the summation's index gives

$$f(t) = (\det M)\sum_{k=1}^{r}\sum_{\gamma\in\mathbb{Z}^d}\left\{\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)\widehat{a}_{k,j}(\gamma-M\alpha)\right\}\varphi_k(t-\gamma)\,, \quad t\in\mathbb{R}^d\,.$$

Thus, the relevant data

$$d_k(\gamma) := \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)\widehat{a}_{k,j}(\gamma-M\alpha)\,, \quad \gamma\in\mathbb{Z}^d\,, \quad 1\le k\le r\,,$$

for the recovery of the signal $f\in V^2_\Phi$ is obtained by means of r filter banks whose impulse responses involve the Fourier coefficients of the entries of the $r\times s$ matrix $\mathbf{a} := [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s]$ in (20), and the input is given by the sampling data.

Notice that reconstruction functions $S_{j,\mathbf{a}}$ with compact support in the above sampling formula implies low computational complexities and avoids truncation errors. This occurs whenever the generators φ_k have compact support and the sum in (24) is finite. These sums are finite if and only if the entries of the $r \times s$ matrix $\mathbf{a}$ are trigonometric polynomials. In this case, all the filter banks involved in the reconstruction process are FIR (finite impulse response) filters.

Before to give a necessary and sufficient condition assuring compactly supported reconstruction functions $S_{j,\mathbf{a}}$ in formula (28), we introduce first some complex notation, more convenient for this study. We denote $\mathbf{z}^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_d^{\alpha_d}$ for $\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$, and the d-torus by $\mathbb{T}^d := \{\mathbf{z} \in \mathbb{C}^d : |z_1| = |z_2| = \dots = |z_d| = 1\}$. For $1 \le j \le s$ and $1 \le k \le r$ we define

$$\mathsf{g}_{j,k}(\mathbf{z}) := \sum_{\mu \in \mathbb{Z}^d} \mathscr{L}_j \varphi_k(\mu) \mathbf{z}^{-\mu}, \quad \mathsf{g}_j^\top(\mathbf{z}) := \big(\mathsf{g}_{j,1}(\mathbf{z}), \mathsf{g}_{j,2}(\mathbf{z}), \dots, \mathsf{g}_{j,r}(\mathbf{z})\big),$$

and the $s \times r(\det M)$ matrix

$$\mathsf{G}(\mathbf{z}) := \Big[\mathsf{g}_j^\top (z_1 \mathrm{e}^{2\pi i m_1^\top i_l}, \dots, z_d \mathrm{e}^{2\pi i m_d^\top i_l})\Big]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,r;\ l=1,2,\dots,\det M}} \tag{29}$$

where $m_1, \dots, m_d$ denote the columns of the matrix M^{-1}. Note that for the values $x = (x_1, \dots, x_d) \in [0,1)^d$ and $\mathbf{z} = (\mathrm{e}^{2\pi i x_1}, \dots, \mathrm{e}^{2\pi i x_d}) \in \mathbb{T}^d$ we have $\mathbb{G}(x) = \mathsf{G}(\mathbf{z})$.

Provided that the functions $\mathbf{g}_j$ are continuous on $\mathbb{R}^d$, Corollary 1 can be reformulated as follows: There exists an $r \times s$ matrix $\mathsf{a}(\mathbf{z}) = \big[\mathsf{a}_1(\mathbf{z}), \dots, \mathsf{a}_s(\mathbf{z})\big]$ with entries essentially bounded in the torus $\mathbb{T}^d$ and satisfying

$$\mathsf{a}(\mathbf{z})\mathsf{G}(\mathbf{z}) = [\mathbb{I}_r, \mathbb{O}_{(\det M - 1)r \times r}] \quad \text{for all } \mathbf{z} \in \mathbb{T}^d \tag{30}$$

if and only if

$$\operatorname{rank} \mathsf{G}(\mathbf{z}) = r(\det M) \quad \text{for all } \mathbf{z} \in \mathbb{T}^d. \tag{31}$$

Denoting the columns of the matrix $\mathsf{a}(\mathbf{z})$ as $\mathsf{a}_j^\top(\mathbf{z}) = \big(\mathsf{a}_{1,j}(\mathbf{z}), \dots, \mathsf{a}_{r,j}(\mathbf{z})\big)$, $j = 1, 2, \dots, s$, the corresponding reconstruction functions $S_{j,\mathsf{a}}$ in sampling formula (28) are

$$S_{j,\mathsf{a}}(t) = \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^r \widehat{\mathsf{a}}_{k,j}(\alpha) \varphi(t - \alpha), \quad t \in \mathbb{R}^d, \tag{32}$$

where $\widehat{\mathsf{a}}_{k,j}(\alpha)$, $\alpha \in \mathbb{Z}^d$, are the Laurent coefficients of the functions $\mathsf{a}_{k,j}(\mathbf{z})$, that is,

$$\mathsf{a}_{k,j}(\mathbf{z}) = \sum_{\alpha \in \mathbb{Z}^d} \widehat{\mathsf{a}}_{k,j}(\alpha) \mathbf{z}^{-\alpha}. \tag{33}$$

Note that, in order to obtain compactly supported reconstruction functions $S_{j,\mathsf{a}}$ in (28) we need an $r \times s$ matrix $\mathsf{a}(\mathbf{z})$ whose entries are Laurent polynomials, i.e., the sum in (33) is finite. The following result, which proof can be found in [16] under minor changes, holds:

Theorem 4. *Assume that the generators φ_k and the functions $\mathscr{L}_j \varphi_k$, $1 \le k \le r$ and $1 \le j \le s$, have compact support. Then, there exists an $r(\det M) \times s$ matrix $\mathsf{a}(\mathbf{z})$ whose entries are Laurent polynomials and satisfying* (30) *if and only if*

$$\textit{rank}\ \mathsf{G}(\mathbf{z}) = r(\det M) \ \textit{for all}\ \mathbf{z} \in (\mathbb{C} \setminus \{0\})^d .$$

The reconstruction functions $S_{j,\mathsf{a}}$, $j = 1,2,\ldots,s$, obtained from such matrix $\mathsf{a}(\mathbf{z})$ through Eq. (32) *have compact support.*

From one of these $r \times s$ matrices, say $\widetilde{\mathsf{a}}(\mathbf{z}) = [\widetilde{\mathsf{a}}_1(\mathbf{z}),\ldots,\widetilde{\mathsf{a}}_s(\mathbf{z})]$, we can get all of them. Indeed, it is easy to check that they are given by the r first rows of the $r(\det M) \times s$ matrices of the form

$$\mathsf{A}(\mathbf{z}) = \widetilde{\mathsf{A}}(\mathbf{z}) + \mathsf{U}(\mathbf{z})\left[\mathbb{I}_s - \mathsf{G}(\mathbf{z})\widetilde{\mathsf{A}}(\mathbf{z})\right], \tag{34}$$

where

$$\widetilde{\mathsf{A}}(\mathbf{z}) := \left[\widetilde{\mathsf{a}}_j(z_1 \mathrm{e}^{2\pi i m_1^\top i_l},\ldots,z_d \mathrm{e}^{2\pi i m_d^\top i_l})\right]_{\substack{k=1,2,\ldots,r;\ l=1,2,\ldots,\det M \\ j=1,2,\ldots,s}},$$

and $\mathsf{U}(\mathbf{z})$ is any $r(\det M) \times s$ matrix with Laurent polynomial entries. Remember that $m_1,\ldots,m_d$ denote the columns of the sampling matrix M, and $i_1,\ldots,i_{\det M}$ the elements of $\mathscr{N}(M^\top)$ defined in (8).

Next we study the existence of reconstruction functions $S_{j,\mathsf{a}}$, $j = 1,2,\ldots,s$, in (28) having exponential decay; it means that there exist constants $C > 0$ and $q \in (0,1)$ such that $|S_{j,\mathsf{a}}(t)| \leq Cq^{|t|}$ for each $t \in \mathbb{R}^d$. In so doing, we introduce the algebra $\mathscr{H}(\mathbb{T}^d)$ of all holomorphic functions in a neighborhood of the d-torus $\mathbb{T}^d$. Note that the elements in $\mathscr{H}(\mathbb{T}^d)$ are characterized as admitting a Laurent series where the sequence of coefficients decays exponentially fast [26].

The following theorem, which proof can be found in [16] under minor changes, holds:

Theorem 5. *Assume that the generators φ_k and the functions $\mathscr{L}_j\varphi_k$, $j = 1,2,\ldots,s$ and $k = 1,2,\ldots,r$, have exponential decay. Then, there exists an $r \times s$ matrix $\mathsf{a}(\mathbf{z}) = [\mathsf{a}_1(\mathbf{z}),\ldots,\mathsf{a}_s(\mathbf{z})]$ with entries in $\mathscr{H}(\mathbb{T}^d)$ and satisfying* (30) *if and only if rank $\mathsf{G}(\mathbf{z}) = r(\det M)$ for all $\mathbf{z} \in \mathbb{T}^d$.*

In this case, all of such matrices $\mathsf{a}(\mathbf{z})$ are given as the first r rows of a $r(\det M) \times s$ matrix $\mathsf{A}(\mathbf{z})$ of the form

$$\mathsf{A}(\mathbf{z}) = \mathsf{G}^\dagger(\mathbf{z}) + \mathsf{U}(\mathbf{z})\left[\mathbb{I}_s - \mathsf{G}(\mathbf{z})\mathsf{G}^\dagger(\mathbf{z})\right], \tag{35}$$

where $\mathsf{U}(\mathbf{z})$ denotes any $r(\det M) \times s$ matrix with entries in the algebra $\mathscr{H}(\mathbb{T}^d)$ and $\mathsf{G}^\dagger(\mathbf{z}) := \left[\mathsf{G}^*(\mathbf{z})\mathsf{G}(\mathbf{z})\right]^{-1}\mathsf{G}^*(\mathbf{z})$. The corresponding reconstruction functions $S_{j,\mathsf{a}}$, $j = 1,2,\ldots,s$, given by (32) *have exponential decay.*

3.2 Some illustrative examples

We include here some examples illustrating Theorem 4, a particular case of Theorem 2, by taking B-splines as generators; they certainly are important for practical purposes [42].

3.2.1 The case $d = 1$, $r = 1$, $M = 2$ and $s = 3$

Let $N_3(t) := \chi_{[0,1)} * \chi_{[0,1)} * \chi_{[0,1)}(t)$ be the quadratic B-spline, where $\chi_{[0,1)}$ denotes the characteristic function of the interval $[0,1)$, and let $\mathscr{L}_j,\ j = 1,2,3$, be the systems:

$$\mathscr{L}_1 f(t) = f(t); \quad \mathscr{L}_2 f(t) = f(t+\frac{2}{3}) \quad \text{and} \quad \mathscr{L}_3 f(t) = f(t+\frac{4}{3}).$$

Since the functions $\mathscr{L}_j N_3,\ j = 1,2,3$, have compact support, then the entries of the 3×2 matrix $\mathsf{G}(z)$ in (29) are Laurent polynomials and we can try to search a vector $\mathbf{a}(z) := [\mathbf{a}_1(z), \mathbf{a}_2(z), \mathbf{a}_3(z)]$ satisfying (30) with Laurent polynomials entries also. This implies reconstruction functions $S_{j,\mathbf{a}},\ j = 1,2,3$, with compact support. Proceeding as in [14] we obtain that any function $f \in V^2_{N_3}$ can be recovered through the sampling formula:

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{3} \mathscr{L}_j f(2n)\, S_{j,\mathbf{a}}(t-2n), \quad t \in \mathbb{R},$$

where the reconstruction functions, according to (32), are given by

$$\begin{aligned}
S_{1,\mathbf{a}}(t) &= \frac{1}{16}\big[N_3(t+3) - 3N_3(t+2) - 3N_3(t+1) + N_3(t)\big],\\
S_{2,\mathbf{a}}(t) &= \frac{1}{16}\big[27N_3(t+1) - 9N_3(t)\big],\\
S_{3,\mathbf{a}}(t) &= \frac{1}{16}\big[-9N_3(t+1) + 27N_3(t)\big], \quad t \in \mathbb{R}.
\end{aligned}$$

3.2.2 The case $d = 1$, $r = 2$, $M = 1$ and $s = 3$

Consider the Hermite cubic splines defined as

$$\varphi_1(t) = \begin{cases} (t+1)^2(1-2t), & t \in [-1,0] \\ (1-t)^2(1+2t), & t \in [0,1] \\ 0, & |t| > 1 \end{cases} \quad \text{and} \quad \varphi_2(t) = \begin{cases} (t+1)^2 t, & t \in [-1,0] \\ (1-t)^2 t, & t \in [0,1] \\ 0, & |t| > 1 \end{cases}.$$

They are stable generators for the space $V^2_{\varphi_1,\varphi_2}$ (see Ref. [12]). Consider the sampling period $M = 1$ and the systems $\mathscr{L}_j,\ j = 1,2,3$, defined by

$$\mathscr{L}_1 f(t) := \int_t^{t+1/3} f(u)du, \quad \mathscr{L}_2 f(t) := \mathscr{L}_1 f\big(t+\frac{1}{3}\big), \quad \mathscr{L}_3 f(t) := \mathscr{L}_1 f\big(t+\frac{2}{3}\big).$$

Since the functions $\mathscr{L}_j \varphi_k,\ j = 1,2,3$ and $k = 1,2$, have compact support, then the entries of the 3×2 matrix $\mathsf{G}(z)$ in (29) are Laurent polynomials and we can try to search an 2×3 matrix $\mathbf{a}(z) := [\mathbf{a}_1(z), \mathbf{a}_2(z), \mathbf{a}_3(z)]$ satisfying (30) with Laurent polynomials entries also. This leads to reconstruction functions $S_{j,\mathbf{a}}$, $j=1,2,3$, with compact support. Proceeding as in [17] we obtain in $V^2_{\varphi_1,\varphi_2}$ the following sampling formula:

$$f(t)=\sum_{n\in\mathbb{Z}}\sum_{j=1}^{3}\mathscr{L}_j f(n)\,S_{j,\mathbf{a}}(t-n)\,,\quad t\in\mathbb{R}\,,$$

where the sampling functions, according to (32), are

$$\begin{aligned}
S_{1,\mathbf{a}}(t) &:= \frac{85}{44}\varphi_1(t)+\frac{1}{11}\varphi_1(t-1)+\frac{85}{4}\varphi_2(t)-\varphi_2(t-1)\,,\\
S_{2,\mathbf{a}}(t) &:= \frac{-23}{44}\varphi_1(t)-\frac{23}{44}\varphi_1(t-1)-\frac{23}{4}\varphi_2(t)+\frac{23}{4}\varphi_2(t-1)\,,\\
S_{3,\mathbf{a}}(t) &:= \frac{1}{11}\varphi_1(t)+\frac{85}{44}\varphi_1(t-1)+\varphi_2(t)-\frac{85}{4}\varphi_2(t-1)\,,\quad t\in\mathbb{R}\,.
\end{aligned}$$

3.3 L^2-approximation properties

Consider an $r\times s$ matrix $\mathbf{a}(x):=\big[\mathbf{a}_1(x),\mathbf{a}_2(x),\dots,\mathbf{a}_s(x)\big]$ with entries $a_{k,j}\in L^\infty[0,1)^d$, $1\le k\le r$, $1\le j\le s$, and satisfying (20). Let $S_{j,\mathbf{a}}$ be the associated reconstruction functions, $j=1,2,\dots,s$, given in Theorem 2. The aim of this section is to show that if the set of generators Φ satisfies the Strang-Fix conditions of order ℓ, then the scaled version of the sampling operator

$$\Gamma_{\mathbf{a}}f(t):=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)S_{j,\mathbf{a}}(t-M\alpha)\,,\quad t\in\mathbb{R}^d\,,$$

gives L^2- approximation order ℓ for any smooth function f (in a Sobolev space). In do doing, we take advantage of the good approximation properties of the scaled space $\sigma_{1/h}V^2_{\Phi}$, where for $h>0$ we are using the notation: $\sigma_h f(t):=f(ht)$, $t\in\mathbb{R}^d$.

The set of generators $\Phi=\{\varphi_k\}_{k=1}^r$ is said to satisfy the Strang-Fix conditions of order ℓ if there exist r finitely supported sequences $b_k:\mathbb{Z}^d\to\mathbb{C}$ such that the function $\varphi(t)=\sum_{k=1}^r\sum_{\alpha\in\mathbb{Z}^d}b_k(\alpha)\varphi_k(t-\alpha)$ satisfies the Strang-Fix conditions of order ℓ, i.e.,

$$\widehat{\varphi}(0)\neq 0,\quad D^\beta\widehat{\varphi}(\alpha)=0,\quad |\beta|<\ell,\quad \alpha\in\mathbb{Z}^d\setminus\{0\}\,. \tag{36}$$

We denote by $W_2^\ell(\mathbb{R}^d):=\{f\,:\,\|D^\gamma f\|_2<\infty,\,|\gamma|\le\ell\}$ the usual Sobolev space, and by $|f|_{\ell,2}:=\sum_{|\beta|=\ell}\|D^\beta f\|_2$ the corresponding seminorm of a function $f\in W_2^\ell(\mathbb{R}^d)$. When $2\ell>d$ we identify $f\in W_2^\ell(\mathbb{R}^d)$ with its continuous choice (see [1]).

It is well-known that if Φ satisfies the Strang-Fix conditions of order ℓ, and the generators φ_k satisfy a suitable decay condition, the space V^2_Φ provides L^2-approximation order ℓ for any function f regular enough. For instance, Lei et al. proved in [32, Theorem 5.2] the following result: If a set $\Phi=\{\varphi_k\}_{k=1}^r$ of stable generators satisfies the Strang-Fix conditions of order ℓ, and the decay condition

$\varphi_k(t) = O\big([1+|t|]^{-d-\ell-\varepsilon}\big)$ for each $k = 1,2,\dots,r$ and some $\varepsilon > 0$, then, for any $f \in W_2^\ell(\mathbb{R}^d)$, there exists a function $f_h \in \sigma_{1/h}V_\Phi^2$ such that

$$\|f - f_h\|_2 \le C\,|f|_{\ell,2}\,h^\ell\,, \tag{37}$$

where the constant C does not depend on h and f.

In this section we assume that all the systems $\mathscr{L}_j$, $j = 1,2,\dots,s$, are of type (a), i.e., $\mathscr{L}_j f = f * \mathsf{h}_j$, belonging the impulse response h_j to the Hilbert space $\mathscr{L}^2(\mathbb{R}^d)$. Recall that a Lebesgue measurable function $\mathsf{h} : \mathbb{R}^d \longrightarrow \mathbb{C}$ belongs to the Hilbert space $\mathscr{L}^2(\mathbb{R}^d)$ if

$$|\mathsf{h}|_2 := \bigg(\int_{[0,1)^d}\Big(\sum_{\alpha\in\mathbb{Z}^d}|\mathsf{h}(t-\alpha)|\Big)^2 dt\bigg)^{1/2} < \infty.$$

Notice that $\mathscr{L}^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$ and $\mathsf{h} \in \mathscr{L}^2(\mathbb{R}^d)$, the following inequality holds: $\big\|\{\mathsf{h} * f(\alpha)\}_{\alpha\in\mathbb{Z}^d}\big\|_2 \le |\mathsf{h}|_2\,\|f\|_2$ (see [26, Theorem 3.1]); thus the sequence of generalized samples $\{(\mathscr{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ belongs to $\ell^2(\mathbb{Z}^d)$ for any $f \in L^2(\mathbb{R}^d)$.

First we note that the operator $\Gamma_{\mathbf{a}} : \big(L^2(\mathbb{R}^d), \|\cdot\|_2\big) \longrightarrow \big(V_\Phi^2, \|\cdot\|_2\big)$ given by

$$(\Gamma_{\mathbf{a}} f)(t) := (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)S_{j,\mathbf{a}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

is a well-defined bounded operator onto V_Φ^2. Besides, $\Gamma_{\mathbf{d}} f = f$ for all $f \in V_\Phi^2$.

Under appropriate hypotheses we prove that the scaled operator $\Gamma_{\mathbf{a}}^h := \sigma_{1/h}\Gamma_{\mathbf{a}}\sigma_h$ approximates, in the L^2-norm sense, any function f in the Sobolev space $W_2^\ell(\mathbb{R}^d)$ as $h \to 0^+$. Specifically we have:

Theorem 6. *Assume $2\ell > d$ and that all the systems $\mathscr{L}_j$ satisfy $\mathscr{L}_j f = f * \mathsf{h}_j$ with $\mathsf{h}_j \in \mathscr{L}^2(\mathbb{R}^d)$, $j = 1,\dots,s$. Then,*

$$\|f - \Gamma_{\mathbf{a}}^h f\|_2 \le (1 + \|\Gamma_{\mathbf{a}}\|)\inf_{g\in\sigma_{1/h}V_\Phi^2}\|f - g\|_2, \quad f \in W_2^\ell(\mathbb{R}^d),$$

where $\|\Gamma_{\mathbf{a}}\|$ denotes the norm of the sampling operator $\Gamma_{\mathbf{a}}$. If the set of generators $\Phi = \{\varphi_k\}_{k=1}^r$ satisfies the Strang-Fix conditions of order ℓ and, for each $k = 1,2,\dots,r$, the decay condition $\varphi_k(t) = O\big([1+|t|]^{-d-\ell-\varepsilon}\big)$ for some $\varepsilon > 0$, then

$$\|f - \Gamma_{\mathbf{a}}^h f\|_p \le C\,|f|_{\ell,2}\,h^\ell\,, \quad \text{for all } f \in W_2^\ell(\mathbb{R}^d),$$

where the constant C does not depend on h and f.

Proof. Using that $\Gamma_{\mathbf{a}}^h g = g$ for each $g \in \sigma_{1/h}V_\Phi^2$ then, for each $f \in L^2(\mathbb{R}^d)$ and $g \in \sigma_{1/h}V_\Phi^2$, Lebesgue's Lemma [13, p. 30] gives

$$\|f - \Gamma_{\mathbf{a}}^h f\|_2 \le \|f - g\|_2 + \|\Gamma_{\mathbf{a}}^h g - \Gamma_{\mathbf{a}}^h f\|_2 \le (1 + \|\Gamma_{\mathbf{a}}\|)\inf_{g\in\sigma_{1/h}V_\Phi^2}\|f - g\|_2\,,$$

where we have used that $\|\Gamma_{\mathbf{a}}^h\| = \|\Gamma_{\mathbf{a}}\|$ for $h > 0$. Now, for each $f \in W_2^\ell(\mathbb{R}^d)$ and $h > 0$, there exists a function $f_h \in \sigma_{1/h} V_\Phi^2$ such that (37) holds, from which we obtain the desired result. □

More results on approximation by means of generalized sampling formulas can be found in Refs. [15, 18].

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