Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

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Abstract In order to avoid most of the problems associated with classical Shannon’s sampling theory, nowadays signals are assumed to belong to some shift-invariant subspace. In this work we consider a general shift-invariant space $V^2_\Phi$ of $L^2(\mathbb{R}^d)$ with a set $\Phi$ of $r$ stable generators. Besides, in many common situations the available data of a signal are samples of some filtered versions of the signal itself taken at a sub-lattice of $\mathbb{Z}^d$. This leads to the problem of generalized sampling in shift-invariant spaces. Assuming that the $\ell^2$-norm of the generalized samples of any $f \in V^2_\Phi$ is stable with respect to the $L^2(\mathbb{R}^d)$-norm of the signal $f$, we derive frame expansions in the shift-invariant subspace allowing the recovery of the signals in $V^2_\Phi$ from the available data. The mathematical technique used here mimics the Fourier duality technique which works for classical Paley-Wiener spaces.

1 By way of introduction

The classical Whittaker-Shannon-Kotel’nikov sampling theorem (WSK sampling theorem) [23, 50] states that any function $f$ band-limited to $[-1/2,1/2]$, that is, $f(t) = \int_{-1/2}^{1/2} \hat{f}(w)e^{2\pi iw}dw$ for each $t \in \mathbb{R}$, may be reconstructed from the sequence of samples $\{f(n)\}_{n \in \mathbb{Z}}$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (t-n)}{\pi (t-n)}, \quad t \in \mathbb{R}.$$
Thus, the Paley-Wiener space $PW_{1/2}$ of band-limited functions to $[-1/2, 1/2]$ is generated by the integer shifts of the cardinal sine function, $\text{sinc}(t) := \sin \pi t / \pi t$. A simple proof of this result is given by using the Fourier duality technique which uses that the Fourier transform

$$\mathcal{F} : PW_{1/2} \rightarrow L^2[-1/2, 1/2]$$

$$f \mapsto \hat{f}$$

is an unitary operator from the Paley-Wiener space $PW_{1/2}$ of band-limited functions to $[-1/2, 1/2]$ onto $L^2[-1/2, 1/2]$. Thus, the Fourier series $\hat{f} = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n w}$ of $\hat{f}$ in $L^2[-1/2, 1/2]$, by applying the inverse Fourier transform $\mathcal{F}^{-1}$, gives

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \mathcal{F}^{-1} [e^{-2\pi i n w} \chi_{[-\pi, \pi]}(w)](t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (t-n)}{\pi (t-n)} \text{ in } L^2(\mathbb{R}).$$

The pointwise convergence comes from the fact that $PW_{1/2}$ is a reproducing kernel Hilbert space (written shortly as RKHS) where convergence in norm implies pointwise convergence (which is, in this case, uniform on $\mathbb{R}$); this comes out from the inequality: $|f(t)| \leq ||f||$ for each $t \in \mathbb{R}$ and $f \in PW_{1/2}$ (for the RKHS’s theory and applications, see, for instance, Ref. [36]).

The WSK theorem has its $d$-dimensional counterpart. Any function $f$ band-limited to the $d$-dimensional cube $[-1/2, 1/2]^d$, i.e., $f(t) = \int_{[-1/2,1/2]^d} \hat{f}(x) e^{2\pi i x^\top t} dx$ for each $t \in \mathbb{R}^d$ (here we are using the notation $x^\top t := x_1 t_1 + \cdots + x_d t_d$ identifying elements in $\mathbb{R}^d$ with column vectors), may be reconstructed from the sequence of samples $\{f(\alpha)\}_{\alpha \in \mathbb{Z}^d}$ as

$$f(t) = \sum_{\alpha \in \mathbb{Z}^d} f(\alpha) \frac{\sin \pi (t_1 - \alpha_1)}{\pi (t_1 - \alpha_1)} \cdots \frac{\sin \pi (t_d - \alpha_d)}{\pi (t_d - \alpha_d)}, \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d.$$  

Although Shannon’s sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in Refs. [42, 43]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite duration signal; the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has a very slow decay at infinity which makes computation in the signal domain very inefficient. Besides, in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a $d$-dimensional interval. Moreover, many applied problems impose different a priori constraints on the type of signals. For this reason, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces; signals are assumed to belong to some shift-invariant space of the form: $V_\phi^2 := \text{span}_{t \in \mathbb{Z}^d} \{ \varphi(t - \alpha) : \alpha \in \mathbb{Z}^d \}$ where the function $\varphi$ in $L^2(\mathbb{R}^d)$ is called the generator of $V_\phi^2$. See, for instance, Refs. [2, 3, 4, 6, 7, 10, 43, 45, 47, 48, 49, 51] and the references therein.

In this new context, the analogous of the WSK sampling theorem in a shift-invariant space $V_\phi^2$ was first time proved by Walter in [45]:
1.1 Walter’s sampling theorem in shift-invariant spaces

Let \( \varphi \in L^2(\mathbb{R}) \) be a stable generator for the shift-invariant space \( V_\varphi^2 \) which means that the sequence \( \{ \varphi(\cdot - n) \}_{n \in \mathbb{Z}} \) is a Riesz basis for \( V_\varphi^2 \). A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis \( \{ x_n \}_{n \geq 1} \) has a unique biorthogonal (dual) Riesz basis \( \{ y_n \}_{n \geq 1} \), i.e., \( \langle x_n, y_m \rangle_{\mathcal{H}} = \delta_{n,m} \), such that the expansions

\[
x = \sum_{n=1}^{\infty} \langle x, y_n \rangle_{\mathcal{H}} x_n = \sum_{n=1}^{\infty} \langle x, x_n \rangle_{\mathcal{H}} y_n,
\]

hold for every \( x \in \mathcal{H} \) (see [11] for more details and proofs). Recall that the sequence \( \{ \varphi(\cdot - n) \}_{n \in \mathbb{Z}} \) is a Riesz sequence, i.e., a Riesz basis for \( V_\varphi^2 \) (see, for instance, [11, p. 143]) if and only if there exist two positive constants \( 0 < A \leq B \) such that

\[
A \leq \sum_{k \in \mathbb{Z}} |\hat{\varphi}(w+k)|^2 \leq B, \quad \text{a.e. } w \in [0,1].
\]

Thus we have that \( V_\varphi^2 = \left\{ \sum_{n \in \mathbb{Z}} a_n \varphi(\cdot - n) : \{a_n\} \in \ell^2(\mathbb{Z}) \right\} \subset L^2(\mathbb{R}) \).

We assume that the functions in the shift-invariant space \( V_\varphi^2 \) are continuous on \( \mathbb{R} \). This is equivalent to say that the generator \( \varphi \) is continuous on \( \mathbb{R} \) and the function \( \sum_{n \in \mathbb{Z}} |\varphi(t-n)|^2 \) is uniformly bounded on \( \mathbb{R} \) (see [40]). Thus, any \( f \in V_\varphi^2 \) is defined on \( \mathbb{R} \) as the pointwise sum \( f(t) = \sum_{n \in \mathbb{Z}} a_n \varphi(t-n) \) for each \( t \in \mathbb{R} \).

On the other hand, the space \( V_\varphi^2 \) is the image of \( L^2[0,1] \) by means of the isomorphism

\[
T_\varphi : L^2[0,1] \rightarrow V_\varphi^2 \quad \{e^{-2\pi i n t}\}_{n \in \mathbb{Z}} \mapsto \{\varphi(t-n)\}_{n \in \mathbb{Z}},
\]

which maps the orthonormal basis \( \{e^{-2\pi i n t}\}_{n \in \mathbb{Z}} \) for \( L^2[0,1] \) onto the Riesz basis \( \{\varphi(t-n)\}_{n \in \mathbb{Z}} \) for \( V_\varphi^2 \). For any \( F \in L^2[0,1] \) we have

\[
T_\varphi F(t) = \sum_{n \in \mathbb{Z}} \langle F, e^{-2\pi i n t} \rangle \varphi(t-n) = \langle F, \sum_{n \in \mathbb{Z}} \varphi(t-n)e^{-2\pi i n t} \rangle = \langle F, K_t \rangle_{L^2[0,1]}, \quad t \in \mathbb{R},
\]

where, for each \( t \in \mathbb{R} \), the function \( K_t \in L^2[0,1] \) is given by

\[
K_t(x) = \sum_{n \in \mathbb{Z}} \varphi(t-n)e^{-2\pi i n x} = \sum_{n \in \mathbb{Z}} \varphi(t+n)e^{-2\pi i n x} = Z\varphi(t,x).
\]

Here, \( Z\varphi(t,x) := \sum_{n \in \mathbb{Z}} \varphi(t+n)e^{-2\pi i n x} \) denotes the Zak transform of the function \( \varphi \). See [11, 22] for properties and uses of the Zak transform.

As a consequence, the samples in \( \{f(a+m)\}_{m \in \mathbb{Z}} \) of \( f \in V_\varphi^2 \), where \( a \in [0,1) \) is fixed, can be expressed as

\[
f(a+m) = \langle F, K_{a+m} \rangle = \langle F, e^{-2\pi i mx} K_{a} \rangle, \quad m \in \mathbb{Z}, \text{ where } F = T_\varphi^{-1} f.
\]
As a consequence, the stable recovery of \( f \in V_\phi^2 \) from the sequence of its samples \( \{ f(a + m) \}_{m \in \mathbb{Z}} \) reduces to the study of the sequence \( \{ e^{-2\pi imx} K_a(x) \}_{m \in \mathbb{Z}} \) in \( L^2[0, 1] \). The following theorem is easy to prove, having in mind that the operator \( m_F : L^2[0, 1] \to L^2[0, 1] \) defined as: \( m_F(f) = Ff \) is well-defined if and only if \( F \in L^\infty[0, 1] \); in this case, it is bounded and its norm \( \| m_F \| = \| F \|_\infty \).

**Theorem 1.** The sequence of functions \( \{ e^{-2\pi imx} K_a(x) \}_{m \in \mathbb{Z}} \) is a Riesz basis for \( L^2[0, 1] \) if and only if the inequalities \( 0 < \| K_a \|_0 \leq \| K_a \|_\infty < \infty \) hold, where \( \| K_a \|_0 := \text{essinf}_{x \in [0, 1]} | K_a(x) | \) and \( \| K_a \|_\infty := \text{esssup}_{x \in [0, 1]} | K_a(x) | \). Moreover, its biorthogonal Riesz basis is \( \{ e^{-2\pi imx} / K_a(x) \}_{m \in \mathbb{Z}} \).

In particular, the sequence \( \{ e^{-2\pi imx} K_a(x) \}_{m \in \mathbb{Z}} \) is an orthonormal basis in \( L^2[0, 1] \) if and only if \( | K_a(x) | = 1 \) a.e. in \([0, 1] \).

Let \( a \) be a real number in \([0, 1] \) such that \( 0 < \| K_a \|_0 \leq \| K_a \|_\infty < \infty \); next we prove Walter’s sampling theorem for \( V_\phi^2 \) in [45]. Given \( f \in V_\phi^2 \), we expand the function \( F = \mathcal{T}_\phi^{-1} f \in L^2[0, 1] \) with respect to the Riesz basis \( \{ e^{-2\pi imx} / K_a(x) \}_{m \in \mathbb{Z}} \). Thus we get

\[
F = \sum_{n \in \mathbb{Z}} \langle F, K_{a+n} \rangle \frac{e^{-2\pi imx}}{K_a(x)} = \sum_{n \in \mathbb{Z}} f(a + n) \frac{e^{-2\pi imx}}{K_a(x)} \text{ in } L^2[0, 1].
\]

Applying the operator \( \mathcal{T}_\phi \) to the above expansion we obtain

\[
f = \sum_{n \in \mathbb{Z}} f(a + n) \mathcal{T}_\phi(e^{-2\pi imx} / K_a(x)) = \sum_{n \in \mathbb{Z}} f(a + n) S_a(-n) \text{ in } L^2(\mathbb{R}),
\]

where we have used the shifting property \( \mathcal{T}_\phi(e^{-2\pi imx} F)(t) = (\mathcal{T}_\phi F)(t - n) \), \( t \in \mathbb{R} \) and \( n \in \mathbb{Z} \), satisfied by the isomorphism \( \mathcal{T}_\phi \) for the particular function \( S_a := \mathcal{T}_\phi(1/K_a) \in V_\phi^2 \). As in the Paley-Wiener case, the shift-invariant space \( V_\phi^2 \) is a reproducing kernel Hilbert space. Indeed, for each \( t \in \mathbb{R} \), the evaluation functional at \( t \) is bounded:

\[
|f(t)| \leq \| F \|_\mathcal{K} \leq \| \mathcal{T}_\phi^{-1} \| \| \mathcal{K} \| \| f \| = \| \mathcal{T}_\phi^{-1} \| \left( \sum_{n \in \mathbb{Z}} |f(t - n)|^2 \right)^{1/2} \| f \|, \quad f \in V_\phi^2.
\]

Therefore, the \( L^2 \)-convergence implies pointwise convergence which here is uniform on \( \mathbb{R} \). The convergence is also absolute due to the unconditional convergence of a Riesz expansion. Thus, for each \( f \in V_\phi^2 \) we get the sampling formula

\[
f(t) = \sum_{n=-\infty}^{\infty} f(a + n) S_a(t - n), \quad t \in \mathbb{R}.
\]

This mathematical technique, which mimics the Fourier duality technique for Paley-Wiener spaces [23], has been successfully used in deriving sampling formulas in other sampling settings [14, 16, 17, 19, 21, 24, 30, 31]. Here, it will be used for obtaining generalized sampling formulas in \( L^2(\mathbb{R}^d) \) shift-invariant subspaces with multiple stable generators.
1.2 Statement of the general problem

Assume that our functions (signals) belong to some shift-invariant space of the form:

$$V^2_\Phi := \text{span}_{L^2(\mathbb{R}^d)} \{ \phi_k(t - \alpha) : k = 1, 2, \ldots, r \text{ and } \alpha \in \mathbb{Z}^d \},$$

where the functions in $$\Phi := \{ \phi_1, \ldots, \phi_r \}$$ in $$L^2(\mathbb{R}^d)$$ are called a set of generators for $$V^2_\Phi$$. Assuming that the sequence $$\{ \phi_k(t - \alpha) \}_{\alpha \in \mathbb{Z}^d, k = 1, 2, \ldots, r}$$ is a Riesz basis for $$V^2_\Phi$$, the shift-invariant space $$V^2_\Phi$$ can be described as

$$V^2_\Phi = \left\{ \sum_{n \in \mathbb{Z}^d} \sum_{k=1}^r d_k(\alpha) \phi_k(t - \alpha) : d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2, \ldots, r \right\}.$$  \hfill (1)

See Refs. [8, 9, 35] for the general theory of shift-invariant spaces and their applications. These spaces and the scaling functions $$\Phi = \{ \phi_1, \ldots, \phi_r \}$$ appear in the multiwavelet setting. Multiwavelets lead to multiresolution analyses and fast algorithms just as scalar wavelets, but they have some advantages: they can have short support coupled with high smoothness and high approximation order, and they can be both symmetric and orthogonal (see, for instance, Ref. [28]). Classical sampling in multiwavelet subspaces has been studied in Refs. [37, 41].

On the other hand, in many common situations the available data are samples of some filtered versions $$f \ast h_j$$ of the signal $$f$$ itself, where the average function $$h_j$$ reflects the characteristics of the acquisition device. This leads to generalized sampling (also called average sampling) in $$V^2_\Phi$$ (see, among others, Refs. [2, 5, 14, 16, 17, 29, 33, 34, 38, 39, 41]).

Suppose that $$s$$ convolution systems (linear time-invariant systems or filters in engineering jargon) $$\mathcal{L}_j$$, $$j = 1, 2, \ldots, s$$, are defined on the shift-invariant subspace $$V^2_\Phi$$ of $$L^2(\mathbb{R}^d)$$. Assume also that the sequence of samples $$\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha \in \mathbb{Z}^d, j = 1, 2, \ldots, s}$$ for $$f \in V^2_\Phi$$ is available, where the samples are taken at the sub-lattice $$M\mathbb{Z}^d$$ of $$\mathbb{Z}^d$$, where $$M$$ denotes a matrix of integer entries with positive determinant. If we sample any function $$f \in V^2_\Phi$$ on $$M\mathbb{Z}^d$$, we are using the sampling rate $$1/\det(M)$$ and, roughly speaking, we will need, for the recovery of $$f \in V^2_\Phi$$, the sequence of generalized samples $$\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha \in \mathbb{Z}^d, j = 1, 2, \ldots, s}$$ coming from $$s \geq r(\det(M))$$ convolution systems $$\mathcal{L}_j$$.

Assume that the sequences of generalized samples satisfy the following stability condition: There exist two positive constants $$0 < A \leq B$$ such that

$$A \|f\|^2 \leq \sum_{j=1}^s \sum_{\alpha \in \mathbb{Z}^d} |\mathcal{L}_j f(M\alpha)|^2 \leq B \|f\|^2$$ for all $$f \in V^2_\Phi$$.

In [5] is said that the set of systems $$\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s\}$$ is an $$M$$-stable filtering sampler for $$V^2_\Phi$$. The aim of this work is to obtain sampling formulas in $$V^2_\Phi$$ having the form
\[ f(t) = (\det(M)) \sum_{\alpha \in \mathbb{Z}^d} \sum_{j=1}^s (\mathcal{L}_j f)(M\alpha) S_j(t - M\alpha), \quad t \in \mathbb{R}^d, \tag{2} \]

such that the sequence of reconstruction functions \( \{ S_j(\cdot - M\alpha) \}_{\alpha \in \mathbb{Z}^d, j=1,2,...,s} \) is a frame for the shift-invariant space \( V^2_{\mathcal{D}} \). This will be done in the light of the frame theory for separable Hilbert spaces, by using a similar mathematical technique as in the above section.

Recall that a sequence \( \{ x_n \} \) is a frame for a separable Hilbert space \( \mathcal{H} \) if there exist two constants \( A, B > 0 \) (frame bounds) such that

\[ A\|x\|^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \]

Given a frame \( \{ x_n \} \) for \( \mathcal{H} \) the representation property of any vector \( x \in \mathcal{H} \) as a series \( x = \sum_n c_n x_n \) is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for overcomplete frames) is sacrificed. Suitable frame coefficients \( c_n \), depending linearly and continuously on \( x \), are obtained by using the dual frames \( \{ y_n \} \) of \( \{ x_n \} \), i.e., the sequence \( \{ y_n \} \) is another frame for \( \mathcal{H} \) such that, for each \( x \in \mathcal{H} \), the expansions \( x = \sum_n \langle x, y_n \rangle x_n = \sum_n \langle x, x_n \rangle y_n \) hold. For more details on the frame theory see the superb monograph [11] and the references therein.

2 Preliminaries on \( L^2(\mathbb{R}^d) \) shift-invariant subspaces

Let \( \Phi := \{ \varphi_1, \varphi_2, \ldots, \varphi_r \} \), where \( \varphi_k \in L^2(\mathbb{R}^d) \) \( k = 1, 2, \ldots, r \), such that the sequence \( \{ \varphi_k(t - \alpha) \}_{\alpha \in \mathbb{Z}^d, k=1,2,...,r} \) is a Riesz basis for the shift-invariant space \( V^2_{\mathcal{D}} := \left\{ \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^r d_k(\alpha) \varphi_k(t - \alpha) : d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2, \ldots, r \right\} \subset L^2(\mathbb{R}^d) \).

There exists a necessary and sufficient condition involving the Gramian matrix-function

\[ G_\Phi(w) := \sum_{\alpha \in \mathbb{Z}^d} \hat{\Phi}(w + \alpha) \overline{\hat{\Phi}(w + \alpha)}^T, \quad \text{where } \hat{\Phi} := (\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_r)^T, \]

which assures that the sequence \( \{ \varphi_k(\cdot - \alpha) \}_{\alpha \in \mathbb{Z}^d, k=1,2,...,r} \) is a Riesz basis for \( V^2_{\mathcal{D}} \), namely (see, for instance, [5]): There exist two positive constants \( c \) and \( C \) such that

\[ cI_r \leq G_\Phi(w) \leq CI_r \quad \text{a.e. } w \in [0,1)^d. \tag{3} \]

We assume throughout the paper that the functions in the shift-invariant space \( V^2_{\mathcal{D}} \) are continuous on \( \mathbb{R}^d \). As in the case of one generator, this is equivalent to the generators \( \Phi \) being continuous on \( \mathbb{R}^d \) with \( \sum_{\alpha \in \mathbb{Z}^d} |\varphi(t - \alpha)|^2 \) uniformly bounded on \( \mathbb{R}^d \). Thus, any \( f \in V^2_{\mathcal{D}} \) is defined on \( \mathbb{R}^d \) as the pointwise sum
Besides, the space $V_\Phi^2$ is a RKHS since the evaluation functionals, $E_i f := f(t)$ are bounded on $V_\Phi^2$. Indeed, for each fixed $t \in \mathbb{R}^d$ we have

$$|f(t)|^2 = \left| \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha) \, \varphi_k(t - \alpha) \right|^2 \leq \left( \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^{r} |d_k(\alpha)|^2 \right) \left( \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^{r} |\varphi_k(t - \alpha)|^2 \right)$$

$$= \left( \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^{r} |d_k(\alpha)|^2 \right) \left( \sum_{\alpha \in \mathbb{Z}^d} |\Phi(t - \alpha)|^2 \right) \leq \frac{||f||^2}{c} \sum_{\alpha \in \mathbb{Z}^d} |\Phi(t - \alpha)|^2, \quad f \in V_\Phi^2,$$

where we have used Cauchy-Schwarz’s inequality in (4), and the inequality satisfied for any lower Riesz bound $c$ of the Riesz basis $\{\varphi_k(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^d}$, $k = 1,2,\ldots$, for $V_\Phi^2$, that is, $c \sum_{\alpha \in \mathbb{Z}^d} |d_k(\alpha)|^2 \leq ||f||^2$.

Thus, the convergence in $V_\Phi^2$ in the $L^2(\mathbb{R}^d)$-sense implies pointwise convergence which is uniform on $\mathbb{R}^d$.

The product space

$$L^2_\mathcal{F}(0,1)^d := \{ \mathbf{F} = (F_1, F_2, \ldots, F_r)^\top : F_k \in L^2(0,1)^d, \ k = 1,2,\ldots,r \}$$

with its usual inner product

$$\langle \mathbf{F}, \mathbf{H} \rangle_{L^2_\mathcal{F}(0,1)^d} := \sum_{k=1}^{r} \langle F_k, H_k \rangle_{L^2(0,1)^d} = \int_{0,1}^{d} \mathbf{H}^\top(w) \mathbf{F}(w) dw$$

becomes a Hilbert space. Similarly, we introduce the product Banach space $L^\infty_\mathcal{F}(0,1)^d$.

The system $\{ e^{-2\pi i \alpha^\top w} \mathbf{e}_k \}_{\alpha \in \mathbb{Z}^d, \ k = 1,2,\ldots,r}$, where $\mathbf{e}_k$ denotes the vector of $\mathbb{R}^r$ with all the components null except the $k$-th component which is equal to one, is an orthonormal basis for $L^2_\mathcal{F}(0,1)^d$.

The shift-invariant space $V_\Phi^2$ is the image of $L^2_\mathcal{F}(0,1)^d$ by means of the isomorphism

$$\mathcal{T}_\Phi : L^2_\mathcal{F}(0,1)^d \rightarrow V_\Phi^2, \quad \{ e^{-2\pi i \alpha^\top w} \mathbf{e}_k \}_{\alpha \in \mathbb{Z}^d, \ k = 1,2,\ldots,r} \mapsto \{ \varphi_k(t - \alpha) \}_{\alpha \in \mathbb{Z}^d, \ k = 1,2,\ldots,r},$$

which maps the orthonormal basis $\{ e^{-2\pi i \alpha^\top w} \mathbf{e}_k \}_{\alpha \in \mathbb{Z}^d, \ k = 1,2,\ldots,r}$ for $L^2_\mathcal{F}(0,1)^d$ onto the Riesz basis $\{ \varphi_k(t - \alpha) \}_{\alpha \in \mathbb{Z}^d, \ k = 1,2,\ldots,r}$ for $V_\Phi^2$. For each $\mathbf{F} = (F_1, \ldots, F_r)^\top \in L^2_\mathcal{F}(0,1)^d$ we have

$$\mathcal{T}_\Phi \mathbf{F}(t) := \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^{r} \langle F_k, e^{-2\pi i \alpha^\top \cdot} \rangle_{L^2(0,1)^d} \varphi_k(t - \alpha), \quad t \in \mathbb{R}^d.$$

The isomorphism $\mathcal{T}_\Phi$ can also be expressed by

\begin{align*}
\mathcal{G}_\Phi : L^2_\mathcal{F}(0,1)^d &\rightarrow V_\Phi^2, \\
\mathbf{F} = (F_1, F_2, \ldots, F_r)^\top &\mapsto \{ \varphi_k(t - \alpha) \}_{\alpha \in \mathbb{Z}^d, \ k = 1,2,\ldots,r},
\end{align*}
Lemma 1. Let \( \mathcal{L} \) be a convolution system of the type (b) or (c). Then for each fixed \( t \in \mathbb{R}^d \) the sequence \( \{ (\mathcal{L} \varphi_k)(t + \alpha) \}_{\alpha \in \mathbb{Z}^d} \) belongs to \( L^2(\mathbb{Z}^d) \) for each \( k = 1, \ldots, r \).

Proof. First assume that \( h \in L^2(\mathbb{R}^d) \); then we have

\[
\sum_{\alpha \in \mathbb{Z}^d} |\mathcal{L} \varphi_k(t + \alpha)|^2 = \| \sum_{\alpha \in \mathbb{Z}^d} \mathcal{L} \varphi_k(t + \alpha)e^{-2\pi i \alpha^\top x} \|^2_{L^2[0,1]^d} = \| Z \mathcal{L} \varphi_k(t,x) \|^2_{L^2[0,1]^d} = \| Z \mathcal{L} \varphi_k(x + \alpha)e^{2\pi i (x + \alpha)^\top t} \|^2_{L^2[0,1]^d},
\]

where the kernel transform \( \mathbb{R}^d \ni t \to K_t \in L^2[0,1]^d \) is defined as \( K_t(x) := Z \Phi(t,x) \), and \( Z \Phi \) denotes the Zak transform of \( \Phi \), i.e.,

\[
(Z \Phi)(t,w) := \sum_{\alpha \in \mathbb{Z}^d} \Phi(t + \alpha)e^{-2\pi i \alpha^\top w}.
\]

Note that \( (Z \Phi) = (Z \varphi_1, \ldots, Z \varphi_r)^\top \) where \( Z \) denotes the usual Zak transform.

The following shifting property of \( \mathcal{F}_\Phi \) will be used later: For \( F \in L^2[0,1]^d \) and \( \alpha \in \mathbb{Z}^d \) we have

\[
\mathcal{F}_\Phi[F(\cdot)e^{-2\pi i \alpha^\top \cdot}] = \mathcal{F}_\Phi[F(t - \alpha)], \quad t \in \mathbb{R}^d.
\]
where, in the last equality, we have used a version of the Poisson summation formula [20, Lemma 2.1]. Notice that $\hat{\psi}_k, \hat{h} \in L^2(\mathbb{R}^d)$ implies, by Cauchy-Schwarz’s inequality, that $\hat{\psi}_k \hat{h} = \mathcal{L} \psi_k \in L^1(\mathbb{R}^d)$. Now,

$$
\begin{align*}
\| \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L} \psi_k)(x+\alpha)e^{2\pi i (x+\alpha)^\top t} \|_{L^2([0,1]^d)}^2 &= || \sum_{\alpha \in \mathbb{Z}^d} \hat{\psi}_k(x+\alpha)\hat{h}(x+\alpha)e^{2\pi i (x+\alpha)^\top t} \|_{L^2([0,1]^d)}^2 \\
& \leq \left( \sum_{\alpha \in \mathbb{Z}^d} |\hat{\psi}_k(x+\alpha)|^2 \right)^{1/2} \left( \sum_{\alpha \in \mathbb{Z}^d} |\hat{h}(x+\alpha)|^2 \right)^{1/2} \leq C \| \hat{h} \|_{L^2([0,1]^d)}^2,
\end{align*}
$$

where we have used (3) and the fact that $\|h\|_{L^2(\mathbb{R}^d)}^2 = \|\sum_{\alpha \in \mathbb{Z}^d} |h(x+\alpha)|^2 \|_{L^1([0,1]^d)}^2$.

Finally, assume that $H_{\psi_k} \in L^2([0,1]^d)$; since $\hat{\psi}_k \hat{h} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we obtain that $\mathcal{L} \psi_k = \hat{\psi}_k \hat{h} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Since $\sum_{\alpha \in \mathbb{Z}^d} |\mathcal{L} \psi_k(x+\alpha)|^2 = \|\hat{h}\|_{L^2(\mathbb{R}^d)} H_{\psi_k}(x)$, using again [20, Lemma 2.1] we get

$$
\sum_{\alpha \in \mathbb{Z}^d} |\mathcal{L} \psi_k(x+\alpha)|^2 = \| \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L} \psi_k)(x+\alpha)e^{2\pi i (x+\alpha)^\top t} \|_{L^2([0,1]^d)}^2
\leq \|\hat{h}\|_{L^2(\mathbb{R}^d)}^2 \|H_{\psi_k}\|_{L^2([0,1]^d)}^2.
$$

\[ \square \]

**Lemma 2.** Let $\mathcal{L}$ be a convolution system of the type (a), (b) or (c). Then, for each $f \in V_0^2$ we have

$$
(\mathcal{L} f)(t) = \langle F, (\mathcal{L} \Phi)(t,\cdot) \rangle_{L^2([0,1]^d)}, \quad \text{where} \quad F = \mathcal{F}^{-1} f.
$$

**Proof.** Assume that $\mathcal{L}$ is a convolution system of type (a). Under our hypothesis on $\mathcal{L}$, for $m = 0, 1, 2, \ldots, N$ we have that

$$
f^{(m)}(t) = \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^r \langle F_k, e^{-2\pi i \alpha^\top} \rangle \psi_k^{(m)}(t - \alpha).
$$

Having in mind we have assumed that $\sum_{\alpha \in \mathbb{Z}^d} |\Phi^{(m)}(t - \alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$, we obtain that

$$
(\mathcal{L} f)(t) = \sum_{m=0}^N c_m f^{(m)}(t + d_m) = \sum_{m=0}^N c_m \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^r \langle F_k, e^{-2\pi i \alpha^\top} \rangle \psi_k^{(m)}(t + d_m - \alpha)
\leq \sum_{k=1}^r \langle F_k, \sum_{m=0}^N \sum_{\alpha \in \mathbb{Z}^d} \psi_k^{(m)}(t + d_m - \alpha) e^{-2\pi i \alpha^\top} \rangle_{L^2([0,1]^d)}
\leq \sum_{k=1}^r \langle F_k, \sum_{\alpha \in \mathbb{Z}^d} \mathcal{L} \psi_k(t - \alpha) e^{-2\pi i \alpha^\top} \rangle = \sum_{k=1}^r \langle F_k, (\mathcal{L} \mathcal{L} \Phi)(t,\cdot) \rangle_{L^2([0,1]^d)}.
$$
Assume now that $\mathcal{L}$ is a convolution system of the type (b) or (c). For each $t \in \mathbb{R}^d$, considering the function $\psi(x) := \overline{h(-x)}$, we have

\[
(\mathcal{L}f)(t) = \langle f, \psi(-t) \rangle_{L^2(\mathbb{R}^d)} = \left\langle \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^r \langle F_k, e^{-2\pi i \alpha^\top} \rangle \phi_k(\cdot - \alpha), \psi(-t) \right\rangle_{L^2(\mathbb{R}^d)}
\]

\[
= \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^r \langle F_k, e^{-2\pi i \alpha^\top} \phi_k(\cdot - \alpha), \psi(-t + \alpha) \rangle_{L^2(\mathbb{R}^d)}
\]

Since the sequence $\{\langle \mathcal{L} \phi_k \rangle_{(t+\alpha)}\}_{\alpha \in \mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$, Parseval’s equality gives

\[
(\mathcal{L}f)(t) = \sum_{k=1}^r \langle F_k, \sum_{\alpha \in \mathbb{Z}^d} \mathcal{L} \phi_k(t-\alpha) e^{-2\pi i \alpha^\top} \rangle_{L^2(\mathbb{R}^d)} = \langle F, (\mathcal{L} \Phi)(t, \cdot) \rangle_{L^2(\mathbb{R}^d)}
\]

which ends the proof. \hfill \square

### 2.2 Sampling at a lattice of $\mathbb{Z}^d$: An expression for the samples

Given a nonsingular matrix $M$ with integer entries, we consider the lattice in $\mathbb{Z}^d$ generated by $M$, i.e.,

\[ \Lambda_M := \{M\alpha : \alpha \in \mathbb{Z}^d\} \subset \mathbb{Z}^d. \]

Without loss of generality we can assume that $\det M > 0$; otherwise we can consider $M' = ME$ where $E$ is some $d \times d$ integer matrix satisfying $\det E = -1$. Trivially, $\Lambda_M = \Lambda_{M'}$. We denote by $M^\top$ and $M^{-\top}$ the transpose matrices of $M$ and $M^{-1}$ respectively. The following useful generalized orthogonal relationship holds (see [44]):

\[
\sum_{p \in \mathcal{N}(M^\top)} e^{-2\pi i \alpha^\top M^{-\top}p} = \begin{cases} 
\det M, & \alpha \in \Lambda_M \\
0, & \alpha \in \mathbb{Z}^d \setminus \Lambda_M 
\end{cases}
\]

where

\[ \mathcal{N}(M^\top) := \mathbb{Z}^d \cap \{M^\top x : x \in [0,1)^d\} \]

The set $\mathcal{N}(M^\top)$ has $\det M$ elements (see [44] or [46]). One of these elements is zero, say $i_1 = 0$; we denote the rest of elements by $i_2, \ldots, i_{\det M}$ ordered in any form; from now on, $\mathcal{N}(M^\top) = \{i_1 = 0, i_2, \ldots, i_{\det M}\} \subset \mathbb{Z}^d$.

Note that the sets, defined as $Q_l := M^{-\top} i_1 + M^{-\top} [0,1)^d, l = 1, 2, \ldots, \det M$, satisfy (see [46, p. 110]):

\[ Q_l \cap Q_{l'} = \emptyset \text{ if } l \neq l' \quad \text{and} \quad \text{Vol} \left( \bigcup_{l=1}^{\det M} Q_l \right) = 1. \]
Thus, \( f_{(0,1)^d} F(x) dx = \sum_{i=1}^{\det M} \int_{Q_i} F(x) dx \), for any function \( F \) integrable in \([0,1)^d\) and \( \mathbb{Z}^d \)-periodic.

Now assume that we sample the filtered versions \( \mathcal{L}_j f \) of \( f \in V_0^2 \), \( j = 1, 2, \ldots , s \), at a lattice \( \Lambda_M \). Having in mind Lemma 2, for \( j = 1, 2, \ldots , s \) and \( \alpha \in \mathbb{Z}^d \) we obtain that

\[
(\mathcal{L}_j f)(M\alpha) = \langle F, \mathcal{L}_j \Phi(M\alpha, \cdot) \rangle = \langle F, \mathcal{L}_j \Phi(0, \cdot)e^{-2\pi i \alpha^\top M^\top} \rangle_{L_r^2(0,1)^d},
\]

(9)

where \( F = \mathcal{L}_\mathcal{F}^{-1} f \in L_r^2(0,1)^d \). Denote

\[
g_j(x) := Z \mathcal{L}_j \Phi(0,x), \quad j = 1, 2, \ldots , s,
\]

(10)
in other words, \( g_j^T(x) := (g_{j,1}(x), g_{j,2}(x), \ldots , g_{j,s}(x)) \), where \( g_{j,k}(x) = Z \mathcal{L}_j \Phi_k(0,x) \) for \( 1 \leq j \leq s \) and \( 1 \leq k \leq r \).

As a consequence of expression (9) for generalized samples, a challenge problem is to study the completeness, Bessel, frame, or Riesz basis properties of any sequence \( \{g_j(x)e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d, j = 1, 2, \ldots , s} \) in \( L_r^2(0,1)^d \). To this end we introduce the \( s \times r(\det M) \) matrix of functions

\[
G(x) := \begin{bmatrix}
g_1^T(x) & g_1^T(x+M^{-1}I_2) & \cdots & g_1^T(x+M^{-1}I_{\det M}) \\
g_2^T(x) & g_2^T(x+M^{-1}I_2) & \cdots & g_2^T(x+M^{-1}I_{\det M}) \\
\vdots & \vdots & \ddots & \vdots \\
g_s^T(x) & g_s^T(x+M^{-1}I_2) & \cdots & g_s^T(x+M^{-1}I_{\det M})
\end{bmatrix},
\]

(11)

and its related constants

\[
A_G := \essinf_{x \in (0,1)^d} \lambda_{\min}[G^*(x)G(x)], \quad B_G := \esssup_{x \in (0,1)^d} \lambda_{\max}[G^*(x)G(x)],
\]

where \( G^*(x) \) denotes the transpose conjugate of the matrix \( G(x) \), and \( \lambda_{\min} \) (respectively \( \lambda_{\max} \)) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix \( G^*(x)G(x) \). Observe that \( 0 \leq A_G \leq B_G \leq \infty \). Note that in the definition of the matrix \( G(x) \) we are considering the \( \mathbb{Z}^d \)-periodic extension of the involved functions \( g_j \), \( j = 1, 2, \ldots , s \). Regardless the functions \( g_j \) in \( L_r^2(0,1)^d \), \( j = 1, 2, \ldots , s \), are given by (10), the following result holds:

**Lemma 3.** Let \( g_j \) be in \( L_r^2(0,1)^d \) for \( j = 1, 2, \ldots , s \) and let \( G(x) \) be its associated matrix as in (11). Then,

(a) The sequence \( \{g_j(x)e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d, j = 1, 2, \ldots , s} \) is a complete system for \( L_r^2(0,1)^d \) if and only if the rank of the matrix \( G(x) \) is \( r(\det M) \) a.e. in \((0,1)^d\).

(b) The sequence \( \{g_j(x)e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d, j = 1, 2, \ldots , s} \) is a Bessel sequence for \( L_r^2(0,1)^d \) if and only if \( g_j \in L_r^\infty(0,1)^d \) (or equivalently \( B_G < \infty \)). In this case, the optimal Bessel bound is \( B_G/(\det M) \).
(c) The sequence \( \{ g_j(x)e^{-2\pi i\alpha^TM^T x} \}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots} \) is a frame for \( L^2_{\mathbb{Z}}[0,1]^d \) if and only if \( 0 < A_G \leq B_G < \infty \). In this case, the optimal frame bounds are \( A_G/(\det M) \) and \( B_G/(\det M) \).

(d) The sequence \( \{ g_j(x)e^{-2\pi i\alpha^TM^T x} \}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots} \) is a Riesz basis for \( L^2_{\mathbb{Z}}[0,1]^d \) if and only if it is a frame and \( s = r(\det M) \).

Proof. For any \( F \in L^2_{\mathbb{Z}}[0,1]^d \) we have

\[
(F(x), \overline{g_j(x)e^{-2\pi i\alpha^TM^T x}})_{L^2_{\mathbb{Z}}[0,1]^d} = \int_{(0,1)^d} \sum_{k=1}^{r} F_k(x)g_{j,k}(x)e^{2\pi i\alpha^TM^T x} \, dx
\]

\[
= \sum_{k=1}^{r} \det M \int_Q F_k(x)g_{j,k}(x)e^{2\pi i\alpha^TM^T x} \, dx
\]

\[
= \int_{M^{-\top}(0,1)^d} \sum_{k=1}^{r} \det M \int_{M^{-\top}(0,1)^d} F_k(x+M^{-\top}i_1)g_{j,k}(x+M^{-\top}i_1)e^{2\pi i\alpha^TM^T x} \, dx
\]

\[
= \int_{M^{-\top}(0,1)^d} \sum_{k=1}^{r} \det M \sum_{l=1}^{\det M} F_k(x+M^{-\top}i_1)g_{j,k}(x+M^{-\top}i_1)e^{2\pi i\alpha^TM^T x} \, dx
\]

\[
= \int_{M^{-\top}(0,1)^d} \sum_{l=1}^{\det M} g_j^\top(x+M^{-\top}i_1)F(x+M^{-\top}i_1)e^{2\pi i\alpha^TM^T x} \, dx,
\]

where we have considered the \( \mathbb{Z}^d \)-periodic extension of \( F \). Then,

\[
\sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left\| (F(x), \overline{g_j(x)e^{-2\pi i\alpha^TM^T x}})_{L^2_{\mathbb{Z}}[0,1]^d} \right\|^2 = \frac{1}{\det M} \sum_{j=1}^{\det M} \left\| \sum_{i=1}^{\det M} g_j^\top(x+M^{-\top}i_1)F(x+M^{-\top}i_1) \right\|^2_{L^2_{\mathbb{Z}}(M^{-\top}(0,1)^d)}.
\]

Denoting \( F(x) := [F^\top(x), F^\top(x+M^{-\top}i_2), \cdots, F^\top(x+M^{-\top}i_{\det M})]^\top \), the equality above reads

\[
\sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left\| (F(x), \overline{g_j(x)e^{-2\pi i\alpha^TM^T x}})_{L^2_{\mathbb{Z}}[0,1]^d} \right\|^2 = \frac{1}{\det M} \left\| G(x)F(x) \right\|^2_{L^2_{\mathbb{Z}}(M^{-\top}(0,1)^d)},
\]

On the other hand, using that the function \( g_j \) is \( \mathbb{Z}^d \)-periodic, we obtain that the set \{ \( g_j(x+M^{-\top}i_1+M^{-\top}i_1), g_j(x+M^{-\top}i_1+M^{-\top}i_2), \ldots, g_j(x+M^{-\top}i_1+M^{-\top}i_{\det M}) \) \} has the same elements as \{ \( g_j(x+M^{-\top}i_1), g_j(x+M^{-\top}i_2), \ldots, g_j(x+M^{-\top}i_{\det M}) \) \}. Thus the matrix \( G(x+M^{-\top}i_1) \) has the same columns of \( G(x) \), possibly in a different order. Hence, \( \text{rank } G(x) = r(\det M) \) a.e. in \( [0,1)^d \) if and only if \( \text{rank } G(x) = r(\det M) \) a.e. in \( M^{-\top}[0,1)^d \). Moreover,

\[
A_G = \text{ess inf}_{x \in M^{-\top}[0,1)^d} \frac{\lambda_{\text{min}}[G^*(x)G(x)]}{\lambda_{\text{max}}[G^*(x)G(x)]}, \quad B_G = \text{ess sup}_{x \in M^{-\top}[0,1)^d} \frac{\lambda_{\text{min}}[G^*(x)G(x)]}{\lambda_{\text{max}}[G^*(x)G(x)]}.
\]
To prove (a), assume that there exists a set \( \Omega \subseteq M^{-\top}[0,1]^d \) with positive measure such that \( \mathbb{G}(x) < r(\det M) \) for each \( x \in \Omega \). Then, there exists a measurable function \( v(x), x \in \Omega \), such that \( \mathbb{G}(x)v(x) = 0 \) and \( \|v(x)\|_2^2 \leq 1 \) in \( \Omega \). This function can be constructed as in \([27, \text{Lemma 2.4}]\). Define \( F \in L_2^2[0,1]^d \) such that \( F(x) = v(x) \) if \( x \in \Omega \), and \( F(x) = 0 \) if \( x \in M^{-\top}[0,1]^d \setminus \Omega \). Hence, from (13) we obtain that the system is not complete. Conversely, if the system is not complete, by using (13) we obtain a \( F(x) \) different from 0 in a set with positive measure such that \( \mathbb{G}(x)F(x) = 0 \). Thus \( \text{rank} \mathbb{G}(x) < r(\det M) \) on a set with positive measure.

To prove (b), notice that

\[
\sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left| \langle F(x), \mathbf{g}_j(x)e^{-2\pi i \alpha^\top M^{-\top} x} \rangle_{L_2^2[0,1]^d} \right|^2 = \frac{1}{\det M} \int_{M^{-\top}[0,1]^d} \mathbb{G}(x) F(x) \mathbb{G}(x) F(x) dx.
\]

If \( B_G < \infty \) then, for each \( F \), we have

\[
\frac{1}{\det M} \int_{M^{-\top}[0,1]^d} \mathbb{G}(x) F(x) \mathbb{G}(x) F(x) dx \leq \frac{B_G}{\det M} \|F\|_{L_2^2(\det M)(M^{-\top}[0,1]^d)}^2
\]

from which the sequence \( \left\{ \mathbf{g}_j(x)e^{-2\pi i \alpha^\top M^{-\top} x} \right\}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots} \) is a Bessel sequence and its optimal Bessel bound is less than or equal to \( B_G/(\det M) \).

Let \( K < B_G \), there exists a set \( \Omega_K \subseteq M^{-\top}[0,1]^d \) with positive measure such that \( \lambda_{\max} \mathbb{G}(x) \mathbb{G}(x) \geq K \). Let \( F \in L_2^2[0,1]^d \) such that its associated vector function \( F \) is 0 if \( x \in M^{-\top}[0,1]^d \setminus \Omega_K \) and \( F \) is an eigenvector of norm 1 associated with the largest eigenvalue of \( \mathbb{G}(x) \mathbb{G}(x) \) if \( x \in \Omega_K \).

Using (15), we obtain

\[
\sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left| \langle F(x), \mathbf{g}_j(x)e^{-2\pi i \alpha^\top M^{-\top} x} \rangle_{L_2^2[0,1]^d} \right|^2 \geq \frac{K}{\det M} \|F\|_{L_2^2[0,1]^d}^2.
\]

Therefore if \( B_G = \infty \) the sequence \( \left\{ \mathbf{g}_j(x)e^{-2\pi i \alpha^\top M^{-\top} x} \right\}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots} \) is not a Bessel sequence, and the optimal Bessel bound is \( B_G/(\det M) \).

To prove (c) assume first that \( 0 < A_G \leq B_G < \infty \). By using part (b), the sequence \( \left\{ \mathbf{g}_j(x)e^{-2\pi i \alpha^\top M^{-\top} x} \right\}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots} \) is a Bessel sequence in \( L_2^2[0,1]^d \). Moreover, using (15) and the Rayleigh-Ritz theorem (see \([25, \text{p. 176}]\)), for each \( F \in L_2^2[0,1]^d \) we obtain

\[
\sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left| \langle F(x), \mathbf{g}_j(x)e^{-2\pi i \alpha^\top M^{-\top} x} \rangle_{L_2^2[0,1]^d} \right|^2 \geq \frac{A_G}{\det M} \|F\|_{L_2^2(\det M)(M^{-\top}[0,1]^d)}^2
\]

for each \( F \).
Hence the sequence \( \{ \overline{g_j}(x)e^{-2\pi i\alpha^TM^t x} \}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots,s} \) is a frame with optimal lower bound larger that or equal to \( A_G/(\det M) \).

Conversely if \( \{ \overline{g_j}(x)e^{-2\pi i\alpha^TM^t x} \}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots,s} \) is a frame for \( L^2(T^d) \) we know by part (b) that \( B_G < \infty \). In order to prove that \( A_G > 0 \), consider any constant \( K > A_G \). Then there exists a set \( \Omega_K \subset M^+ \subset [0,1)^d \) with positive measure such that

\[
\lambda_{\min(K^\ast(x)G(x))} \leq K.
\]

Let \( F \in L^2_T(0,1]^d \) such that its associated \( F(x) \) is 0 if \( x \in M^+ \setminus [0,1)^d \), \( \Omega_K \) and \( F(x) \) is an eigenvector of norm 1 associated with the smallest eigenvalue of \( G^\ast(x)G(x) \) if \( x \in \Omega_K \). Since \( F \) is bounded, we have that \( G(x)F(x) \in L^2_T(M^0,0,1)^d \). From (15) we get

\[
\sum_{j=1}^s \sum_{\alpha \in \mathbb{Z}^d} \left| \langle F(x), \overline{g_j}(x)e^{-2\pi i\alpha^TM^t x} \rangle_{L^2_T(0,1)} \right|^2 \leq \frac{K}{\det M} \| F \|_{L^2_T(M^0,0,1)^d}^2.
\]

Denoting by \( K \) the optimal lower bound of \( \{ \overline{g_j}(x)e^{-2\pi i\alpha^TM^t x} \}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots,s} \), we have obtained that \( K/(\det M) \geq A \) for each \( K > A_G \); thus \( A_G/(\det M) \geq A \) and consequently, \( A_G > 0 \). Moreover, under the hypotheses of part (c) we deduce that \( A_G/(\det M) \) and \( B_G/(\det M) \) are the optimal frame bounds.

The proof of (d) is based on the following result ([11, Theorem 6.1.1]): A frame is a Riesz basis if and only if it has a biorthogonal sequence. Assume that the sequence \( \{ \overline{g_j}(x)e^{-2\pi i\alpha^TM^t x} \}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots,s} \) is a Riesz basis for \( L^2_T(0,1)^d \) being the sequence \( \{ h_j, \alpha \}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots} \) its biorthogonal sequence. Using (12) we get

\[
\int_{M^+} \sum_{i=1}^{\det M} \overline{g_j}(x+M^{-t} i)h_{j',0}(x+M^{-t} i) e^{2\pi i\alpha^TM^t x} dx
\]

\[
= \langle h_{j',0}(\cdot), \overline{g_j}(x)e^{-2\pi i\alpha^TM^t x} \rangle = \delta_{j,j'} \delta_{\alpha,0}.
\]

Therefore,

\[
\sum_{i=1}^{\det M} \overline{g_j}(x+M^{-t} i)h_{j',0}(x+M^{-t} i) e^{2\pi i\alpha^TM^t x} = (\det M) \delta_{j,j'} \text{ a.e. in } M^0,0,1)^d.
\]

Thus the matrix \( G(x) \) has a right inverse a.e. in \( M^0,0,1)^d \) and, in particular, \( s \leq r(\det M) \). On the other hand, \( A_G > 0 \) implies that \( \det(\overline{G^\ast(x)G(x)}) \neq 0 \), a.e. in \( M^0,0,1)^d \), and there exists the matrix \( G^\ast(x)G(x) \) a.e. in \( M^0,0,1)^d \). This matrix is a left inverse of the matrix \( G(x) \) which implies \( s \geq r(\det M) \). Thus, we obtain that \( r(\det M) = s \).

Conversely, assume that \( \{ \overline{g_j}(x)e^{-2\pi i\alpha^TM^t x} \}_{\alpha \in \mathbb{Z}^d, j=1,2,\ldots,s} \) is a frame for \( L^2_T(0,1)^d \) and \( r(\det M) = s \). In this case \( G(x) \) is a square matrix and \( \det(\overline{G^\ast(x)G(x)}) \neq 0 \) a.e. in \( M^0,0,1)^d \) implies that \( \det G(x) \neq 0 \) a.e. in \( M^0,0,1)^d \). Having in mind the structure of \( G(x) \) its inverse must be the \( r(\det M) \times s \) matrix.
Defining \( \mathcal{G}^{-1} \) allows us to write, in matrix form, that

\[
\mathcal{G}^{-1}(x) = \begin{bmatrix}
  c_1(x) & \cdots & c_s(x) \\
  c_1(x + M^{-\top} l_2) & \cdots & c_s(x + M^{-\top} l_2) \\
  \vdots & \vdots & \vdots \\
  c_1(x + M^{-\top} i_{\det M}) & \cdots & c_s(x + M^{-\top} i_{\det M})
\end{bmatrix},
\]

where, for each \( j = 1, 2, \ldots, s \), the function \( c_j \in L_2^r[0, 1]^d \).

It is easy to verify that the sequence \( \{(\det M)c_j(x)e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d, \ j = 1, 2, \ldots, s} \) is a biorthogonal sequence of \( \{g_j(x)e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d, \ j = 1, 2, \ldots, s} \) and therefore, it is a Riesz basis for \( L_2^r[0, 1]^d \).

\[ \square \]

### 3 Generalized regular sampling in \( V_\Phi^2 \)

In this section we prove that expression (9) allows us to obtain \( F = \mathcal{T}_\Phi^{-1} f \) from the generalized samples \( \{\mathcal{L}_j f(M\alpha)\}_{\alpha \in \mathbb{Z}^d, \ j = 1, 2, \ldots, s} \); as a consequence, applying the isomorphism \( \mathcal{T}_\Phi \), we recover the function \( f \) in \( V_\Phi^2 \).

Assume that the functions \( g_j \) given in (10) belong to \( \in L_2^r[0, 1]^d \) for \( j = 1, 2, \ldots, s \); thus, \( g_j(x)F(x) \in L^2_2[0, 1]^d \). Having in mind (7) and the expression (9) for the generalized samples, we have that

\[
(\det M) \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha)e^{-2\pi i \alpha^\top M^\top x} = \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(\alpha)e^{-2\pi i \alpha^\top x} \sum_{p \in \mathcal{S}(M^\top)} e^{-2\pi i \alpha^\top M^\top p}
\]

\[
= \sum_{p \in \mathcal{S}(M^\top)} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(\alpha)e^{-2\pi i \alpha^\top (x + M^\top p)}
\]

\[
= \sum_{p \in \mathcal{S}(M^\top)} \sum_{\alpha \in \mathbb{Z}^d} \langle F, \overline{g_j(x)} \rangle e^{-2\pi i \alpha^\top (x + M^\top p)}
\]

\[
= \sum_{p \in \mathcal{S}(M^\top)} \sum_{\alpha \in \mathbb{Z}^d} \left( \int_{[0, 1]^d} \sum_{k=1}^r F_k(y)g_{j,k}(y)e^{-2\pi i \alpha^\top M^\top s dy} e^{-2\pi i \alpha^\top (x + M^\top p)}
\]

\[
= \sum_{p \in \mathcal{S}(M^\top)} \sum_{\alpha \in \mathcal{S}(M^\top)} F_k(x + M^\top p)g_{j,k}(x + M^\top p)
\]

\[
= \sum_{p \in \mathcal{S}(M^\top)} \mathcal{G}^\top_j (x + M^\top p) F(x + M^\top p).
\]

Defining \( F(x) := [F^\top(x), F^\top(x + M^{-\top} l_2), \ldots, F^\top(x + M^{-\top} i_{\det M})]^\top \), the above equality allows us to write, in matrix form, that \( \mathcal{G}(x) F(x) \) equals to
where we have used the shifting property (6) and that the space $\mathcal{S}(\varphi)$ is a RKHS. Much more can be said about the above sampling result. In fact, the following theorem gives

$$f(t) = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathcal{D}} (\mathcal{L}_j f)(M\alpha)(\mathcal{T}_{\varphi}a_j)(t - M\alpha), \quad t \in \mathbb{R}^d,$$

where we have used the shifting property (6) and that the space $V_\varphi^2$ is a RKHS. Much more can be said about the above sampling result. In fact, the following theorem holds:

**Theorem 2.** Assume that the functions $g_j$ given in (10) belong to $L^\infty_r[0,1)^d$ for each $j = 1, 2, \ldots, s$. Let $G(x)$ be the associated matrix defined in $[0,1)^d$ as in (11). The following statements are equivalents:

(a) $A_\varphi > 0$.
(b) There exists an $r \times s$ matrix $a(x) := [a_1(x), \ldots, a_s(x)]$ with columns $a_j \in L^\infty_r[0,1)^d$ satisfying

$$[a_1(x), \ldots, a_s(x)]G(x) = [I_r, \Omega(\det M - 1)_{r \times r}] \quad \text{a.e. in } [0,1)^d.$$

(c) There exists a frame for $V_\varphi^2$ having the form $\{S_j a(\cdot - M\alpha)\}_{\alpha \in \mathcal{D}}$, $j = 1, 2, \ldots, s$ such that for any $f \in V_\varphi^2$

$$f = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathcal{D}} (\mathcal{L}_j f)(M\alpha)S_j a(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d).$$

(d) There exists a frame $\{S_j a(\cdot)\}_{\alpha \in \mathcal{D}}$, $j = 1, 2, \ldots, s$ for $V_\varphi^2$ such that for any $f \in V_\varphi^2$

$$f = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathcal{D}} (\mathcal{L}_j f)(M\alpha)S_j a \quad \text{in } L^2(\mathbb{R}^d).$$

**Proof.** First we prove that (a) implies (b). As the determinant of the semipositive definite matrix $G^*(x)G(x)$ is equal to the product of its eigenvalues, condition (a) implies that $\text{ess inf}_{x \in \mathbb{R}^d} \det[G^*(x)G(x)] > 0$. Hence, there exists the left
Assume condition (d), applying the isomorphism \( T \). Hence, they are a pair of dual frames for each \( L^2(\mathbb{R}^d) \), where the sequence \( \{ g_j(x) \} \) is an isomorphism, the sequence \( \{ F_j(x) \} \) is also a Bessel sequence, and 17 \( \{ \sum_{i=1}^{s} M_i \} \) is a frame for \( s \). Since we have assumed that \( g_j \in L^2(\mathbb{R}^d) \) for each \( j = 1, \ldots, s \), the sequence \( \{ \sum_{i=1}^{s} M_i \} \) is a Bessel sequence in \( L^2(\mathbb{R}^d) \) by using part (b) in Lemma 3. The same argument proves that the sequence \( \{ \sum_{i=1}^{s} M_i \} \) is also a Bessel sequence in \( L^2(\mathbb{R}^d) \). These two Bessel sequences satisfy for each \( F \in L^2(\mathbb{R}^d) \)

\[
F(x) = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M \alpha) a_j(x) e^{-2\pi i \alpha^\top M^\top x} \quad \text{in } L^2(\mathbb{R}^d),
\]

from which

\[
f = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M \alpha) S_{j,\alpha}(\cdot - M \alpha) \quad \text{in } L^2(\mathbb{R}^d),
\]

where \( S_{j,\alpha} := \mathcal{F}_\Phi a_j \) for \( j = 1, \ldots, s \). Since we have assumed that \( g_j \in L^2(\mathbb{R}^d) \) for each \( j = 1, \ldots, s \), the sequence \( \{ \sum_{i=1}^{s} M_i \} \) is a Bessel sequence in \( L^2(\mathbb{R}^d) \) by using part (b) in Lemma 3. The same argument proves that the sequence \( \{ \sum_{i=1}^{s} M_i \} \) is also a Bessel sequence in \( L^2(\mathbb{R}^d) \). These two Bessel sequences satisfy for each \( F \in L^2(\mathbb{R}^d) \)

\[
F(x) = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (F, g_j)e^{-2\pi i \alpha^\top M^\top x} a_j(x) e^{-2\pi i \alpha^\top M^\top x} \quad \text{in } L^2(\mathbb{R}^d),
\]

Hence, they are a pair of dual frames for \( L^2(\mathbb{R}^d) \) (see [11, Lemma 5.6.2]). Since \( \mathcal{F}_\Phi \) is an isomorphism, the sequence \( \{ S_{j,\alpha}(t - M \alpha) \} \) is a frame for \( V^2 \); hence (b) implies (c). Statement (c) implies (d) trivially.

Assume condition (d), applying the isomorphism \( \mathcal{F}_\Phi^{-1} \) to the expansion (22) we get

\[
F(x) = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (F, g_j)e^{-2\pi i \alpha^\top M^\top x} \mathcal{F}_\Phi^{-1}(S_{j,\alpha})(x) \quad \text{in } L^2(\mathbb{R}^d),
\]

where \( \{ \mathcal{F}_\Phi^{-1} S_{j,\alpha} \} \) is a frame for \( L^2(\mathbb{R}^d) \). By using Lemma 3, the sequence \( \{ g_j(x) e^{-2\pi i \alpha^\top M^\top x} \} \) is a Bessel sequence; expansion (23) implies that is also a frame (see [11, Lemma 5.6.2]). Hence, by using again Lemma 3, condition (a) holds.

In the case that the functions \( g_j, j = 1, \ldots, s \), are continuous on \( \mathbb{R}^d \) (for instance, if the sequences of generalized samples \( \{ \mathcal{L}_j \phi_k(\alpha) \} \) belongs to \( \ell^1(\mathbb{Z}^d) \) for \( 1 \leq j \leq s \) and \( 1 \leq k \leq r \), the following corollary holds:

**Corollary 1.** Assume that the functions \( g_j, j = 1, \ldots, s \), in (10) are continuous on \( \mathbb{R}^d \). Then, the following assertions are equivalents:
(a) \( \text{rank } G(x) = r(\text{det } M) \) for all \( x \in \mathbb{R}^d \).

(b) There exists a frame \( \{S_j \alpha(\cdot - mn)\}_{n \in \mathbb{Z}} \) for \( V^2_\Phi \) satisfying the sampling formula \( (21) \).

**Proof.** Whenever the functions \( g_j, \ j = 1, 2, \ldots, s, \) are continuous on \( \mathbb{R}^d \), condition \( A_G > 0 \) is equivalent to that \( \text{det } [\tilde{G}^*(x)G(x)] \neq 0 \) for all \( x \in \mathbb{R}^d \). Indeed, if \( \text{det } G^*(x)G(x) > 0 \) then the \( r \) first rows of the matrix \( G^\dagger(x) := [G^*(x)G(x)]^{-1}G^*(x) \), give an \( r \times s \) matrix \( a(x) = [a_1(x), a_2(x), \ldots, a_s(x)] \) satisfying statement (b) in Theorem 2, and therefore \( A_G > 0 \).

The reciprocal follows from the fact that \( \text{det } [G^*(x)G(x)] \geq A_G^{r(\text{det } M)} \) for all \( x \in \mathbb{R}^d \). Since \( \text{det } [G^*(x)G(x)] \neq 0 \) is equivalent to \( \text{rank } G(x) = r(\text{det } M) \) for all \( x \in \mathbb{R}^d \), the result is a consequence of Theorem 2.

The reconstruction functions \( S_j \alpha, \ j = 1, 2, \ldots, s \), are determined from the Fourier coefficients of the components of \( a_j(x) = [a_{1,j}(x), a_{2,j}(x), \ldots, a_{r,j}(x)]^\top, \ j = 1, 2, \ldots, s \). More specifically, if \( \tilde{a}_{k,j}(\alpha) := \int_{[0,1]^d} a_{k,j}(x)e^{2\pi i \alpha^\top x}dx \) we get

\[
S_j \alpha(t) = \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^r \tilde{a}_{k,j}(\alpha) \phi_k(t - \alpha), \quad t \in \mathbb{R}^d. \tag{24}
\]

The Fourier transform in (24) gives \( \tilde{S}_j \alpha(x) = \sum_{k=1}^r \tilde{a}_{k,j}(x)\tilde{\phi}_k(x) \).

Assume that the \( r \times s \) matrix \( a(x) = [a_1(x), a_2(x), \ldots, a_s(x)] \) satisfies (20). We consider the periodic extension of \( a_{k,j}(x) \), i.e., \( a_{k,j}(x + \alpha) = a_{k,j}(x), \alpha \in \mathbb{Z}^d \). For all \( x \in [0,1)^d \), the \( r(\text{det } M) \times s \) matrix

\[
A^\top(x) := \begin{bmatrix}
    a_1(x) & a_2(x) & \cdots & a_s(x) \\
a_1(x + M^{-1}i_2) & a_2(x + M^{-1}i_2) & \cdots & a_s(x + M^{-1}i_2) \\
    \vdots & \vdots & \ddots & \vdots \\
a_1(x + M^{-1}i_{\text{det } M}) & a_2(x + M^{-1}i_{\text{det } M}) & \cdots & a_s(x + M^{-1}i_{\text{det } M})
\end{bmatrix} \tag{25}
\]

is a left inverse matrix of \( G(x) \), i.e., \( A^\top(x)G(x) = I_{r(\text{det } M)}. \)

Provided that condition (20) is satisfied, it can be easily checked that all matrices \( a(x) \) with entries in \( L^\infty[0,1]^d \), and satisfying (20) correspond to the first \( r \) rows of the matrices of the form

\[
A^\top(x) = G^\dagger(x) + U(x)[I_r - G(x)G^\dagger(x)], \tag{26}
\]

where \( U(x) \) is any \( r(\text{det } M) \times s \) matrix with entries in \( L^\infty[0,1]^d \), and \( G^\dagger \) denotes the left pseudo-inverse \( G^\dagger(x) := [G^*(x)G(x)]^{-1}G^*(x) \).

Notice that if \( s = r(\text{det } M) \) there exists a unique matrix \( a(x) \), given by the first \( r \) rows of \( G^{-1}(x) \); if \( s > r(\text{det } M) \) there are many solutions according to (26).

Moreover, the sequence \( \{ \langle (\text{det } M)a^\dagger_j(\cdot)e^{-2\pi i \alpha^\top M^{-1} \cdot} \rangle_{\alpha \in \mathbb{Z}^d}, \ j = 1, 2, \ldots, s \} \) associated with the \( r \times s \) matrix \( [a_1^\dagger(x), a_2^\dagger(x), \ldots, a_s^\dagger(x)] \) obtained from the \( r \) first rows of \( G^\dagger(x) \), gives...
precisely the canonical dual frame of the frame \( \{ \mathbf{g}_j(\cdot) e^{-2\pi i \alpha \cdot M^T}, \alpha \in \mathbb{Z}^d \} \)
Indeed, the frame operator \( \mathcal{S} \) associated to \( \{ \mathbf{g}_j(\cdot) e^{-2\pi i \alpha \cdot M^T}, \alpha \in \mathbb{Z}^d \} \) is given by

\[
\mathcal{S} \mathbf{F}(x) = \frac{1}{\det M} \left[ \mathbf{g}_1(x), \mathbf{g}_2(x), \ldots, \mathbf{g}_s(x) \right] \mathbf{G}(x) \mathbf{F}(x), \quad \mathbf{F} \in L_2^0[0,1]^d,
\]
from which one gets

\[
\mathcal{S} \left[ (\det M) \mathbf{a}_j^\top(\cdot) e^{-2\pi i \alpha \cdot M^T} \right](x) = \mathbf{g}_j(x) e^{-2\pi i \alpha \cdot M^T x}, \quad j = 1, 2, \ldots, s \text{ and } \alpha \in \mathbb{Z}^d.
\]

Something more can be said in the case where \( s = r(\det M) \):

**Theorem 3.** Assume that the functions \( \mathbf{g}_j, j = 1, 2, \ldots, s \), given in (10) belong to \( L_2^0[0,1]^d \) and \( s = r(\det M) \). The following statements are equivalent:

(a) \( A_G > 0 \).

(b) There exists a Riesz basis \( \{ S_{j,\alpha} \}_{\alpha \in \mathbb{Z}^d}, j = 1, 2, \ldots, s \) for \( V_2^d \) such that for any \( f \in V_2^d \), the expansion

\[
f = (\det M) \sum_{\alpha \in \mathbb{Z}^d} \sum_{j=1}^s (\mathcal{Z}_j f)(M\alpha) S_{j,\alpha},
\]
holds in \( L_2^1(\mathbb{R}^d) \).

In case the equivalent conditions are satisfied, necessarily \( S_{j,\alpha}(t) = S_{j,\alpha}(t-M\alpha), t \in \mathbb{R}^d \),
where \( S_{j,\alpha} = \mathcal{P}_\Phi(\mathbf{a}_j) \), \( j = 1, 2, \ldots, s \),
and the \( r \times s \) matrix \( \mathbf{a} := [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s] \)
is formed with the first rows of the inverse matrix \( \mathbf{G}^{-1} \).

The sampling functions \( S_{j,\alpha}, j = 1, 2, \ldots, s \), satisfy the interpolation property \( (\mathcal{Z}_j S_{j,\alpha})(M\alpha) = \delta_{j,j'} \delta_{\alpha,0}, \) where \( j, j' = 1, 2, \ldots, s \) and \( \alpha \in \mathbb{Z}^d \).

**Proof.** Assume that \( A_G > 0 \); since \( \mathbf{G}(x) \) is a square matrix, this implies that
\( \text{ess inf}_{x \in \mathbb{R}^d} |\det \mathbf{G}(x)| > 0 \). Therefore, the first row of \( \mathbf{G}^{-1}(x) \) gives a solution of the equation \( [\mathbf{a}_1(x), \ldots, \mathbf{a}_s(x)] \mathbf{G}(x) = [1, 0, \ldots, 0] \) with \( \mathbf{a}_j \in L_2^0[0,1]^d \) for \( j = 1, 2, \ldots, s \). According to Theorem 2, the sequence

\[
\{ S_{j,\alpha} \}_{\alpha \in \mathbb{Z}^d}, j = 1, 2, \ldots, s := \{ S_{j,\alpha}(t-M\alpha) \}_{\alpha \in \mathbb{Z}^d}, j = 1, 2, \ldots, s,
\]
where \( S_{j,\alpha} = \mathcal{P}_\Phi(\mathbf{a}_j) \), satisfies the sampling formula (27). Moreover, the sequence

\[
\{ (\det M) \mathbf{a}_j(x) e^{-2\pi i \alpha \cdot M^T x} \}_{\alpha \in \mathbb{Z}^d}, j = 1, 2, \ldots, s = \{ \mathbf{G}^{-1} S_{j,\alpha}(-M\alpha) \}_{\alpha \in \mathbb{Z}^d}, j = 1, 2, \ldots, s
\]
is a frame for \( L_2^0[0,1]^d \). Since \( r(\det M) = s \), according to Lemma 3 it is a Riesz basis for \( L_2^0[0,1]^d \). Hence, the sequence \( \{ S_{j,\alpha}(t-M\alpha) \}_{\alpha \in \mathbb{Z}^d}, j = 1, 2, \ldots, s \) is a Riesz basis for \( V_2^d \) and condition (b) is proved.

Conversely, assume now that \( \{ S_{j,\alpha} \}_{\alpha \in \mathbb{Z}^d}, j = 1, 2, \ldots, s \) is a Riesz basis for \( V_2^d \) satisfying (27). From the uniqueness of the coefficients in a Riesz basis, we get that the interpolatory condition \( (\mathcal{Z}_j S_{j,\alpha})(M\alpha') = \delta_{j,j'} \delta_{\alpha,\alpha'} \) holds for \( j, j' = 1, 2, \ldots, s \) and
Therefore, the sequence \(\{\mathbf{g}_f(x)e^{-2\pi i\alpha' \cdot M^\top x}\}_{\alpha' \in \mathbb{Z}^d}\) with respect to the dual basis of \(\{\mathcal{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha \in \mathbb{Z}^d}, j=1,2,\ldots,s^r\) is a Riesz basis for \(L^2(0,1)^d\). Expanding the function \(\mathbf{g}_f(x)e^{-2\pi i\alpha' \cdot M^\top x}\) in the summation’s index gives

\[
\mathbf{g}_f(x)e^{-2\pi i\alpha' \cdot M^\top x} = \sum_{\alpha' \in \mathbb{Z}^d} \sum_{j=1}^r \mathbf{g}_f(x)e^{-2\pi i\alpha' \cdot M^\top x} (\mathcal{T}_\Phi^{-1}S_{j,\alpha})_{\alpha \in \mathbb{Z}^d} G_{j,\alpha}(x) = \sum_{\alpha' \in \mathbb{Z}^d} \mathcal{T}_\Phi S_{j,\alpha}(M\alpha') G_{j,\alpha}(x) = G_{j,\alpha'}(x).
\]

Therefore, the sequence \(\{\mathbf{g}_f(x)e^{-2\pi i\alpha' \cdot M^\top x}\}_{\alpha' \in \mathbb{Z}^d}, j=1,2,\ldots,s^r\) is the dual basis of the Riesz basis \(\{\mathcal{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha \in \mathbb{Z}^d}, j=1,2,\ldots,s^r\). In particular it is a Riesz basis for \(L^2(0,1)^d\), which implies, according to Lemma 3, that \(A_G > 0\); this proves (a). Moreover, the sequence \(\{\mathcal{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha \in \mathbb{Z}^d}, j=1,2,\ldots,s^r\) is necessarily the unique dual basis of the Riesz basis \(\{\mathbf{g}_f(x)e^{-2\pi i\alpha' \cdot M^\top x}\}_{\alpha' \in \mathbb{Z}^d}, j=1,2,\ldots,s^r\). Therefore, this proves the uniqueness of the Riesz basis \(\{S_{j,\alpha}\}_{\alpha \in \mathbb{Z}^d}, j=1,2,\ldots,s^r\) for \(V^2_\Phi\) satisfying (27). \(\Box\)

### 3.1 Reconstruction functions with prescribed properties

The generalized sampling formula in the shift-invariant space \(V^2_\Phi\)

\[
f(t) = (\det M) \sum_{j=1}^s \sum_{\alpha \in \mathbb{Z}^d} \langle \mathcal{L}_j f \rangle (M\alpha) S_{j,a}(t-M\alpha), \quad t \in \mathbb{R}^d,
\]

(28)

can be read as a filter bank. Indeed, introducing the expression for the sampling functions \(S_{j,a}(t) = \sum_{\beta \in \mathbb{Z}^d} \sum_{k=1}^r \hat{a}_{k,j}(\beta) \Phi_k(t-\beta), \quad t \in \mathbb{R}^d\), the change \(\gamma = \beta + M\alpha\) in the summation’s index gives

\[
f(t) = (\det M) \sum_{k=1}^r \sum_{\gamma \in \mathbb{Z}^d} \left\{ \sum_{j=1}^s \sum_{\alpha \in \mathbb{Z}^d} \langle \mathcal{L}_j f \rangle (M\alpha) \hat{a}_{k,j}(\gamma-M\alpha) \right\} \Phi_k(t-\gamma), \quad t \in \mathbb{R}^d.
\]

Thus, the relevant data

\[
d_k(\gamma) := \sum_{j=1}^s \sum_{\alpha \in \mathbb{Z}^d} \langle \mathcal{L}_j f \rangle (M\alpha) \hat{a}_{k,j}(\gamma-M\alpha), \quad \gamma \in \mathbb{Z}^d, \quad 1 \leq k \leq r,
\]

for the recovery of the signal \(f \in V^2_\Phi\) is obtained by means of \(r\) filter banks whose impulse responses involve the Fourier coefficients of the entries of the \(r \times s\) matrix \(a := [a_1, a_2, \ldots, a_s]\) in (20), and the input is given by the sampling data.

Notice that reconstruction functions \(S_{j,a}\) with compact support in the above sampling formula implies low computational complexities and avoids truncation errors. This occurs whenever the generators \(\Phi_k\) have compact support and the sum in (24) is
finite. These sums are finite if and only if the entries of the \( r \times s \) matrix \( a \) are trigonometric polynomials. In this case, all the filter banks involved in the reconstruction process are FIR (finite impulse response) filters.

Before to give a necessary and sufficient condition assuring compactly supported reconstruction functions \( S_{j,a} \) in formula (28), we introduce first some complex notation, more convenient for this study. We denote \( z^d := z_1 z_2 \ldots z_d \) for \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \), \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d \), and the \( d \)-torus by \( \mathbb{T}^d := \{ z \in \mathbb{C}^d : |z_1| = |z_2| = \ldots = |z_d| = 1 \} \). For \( 1 \leq j \leq s \) and \( 1 \leq k \leq r \) we define

\[
  g_{j,k}(z) := \sum_{\mu \in \mathbb{Z}^d} \mathcal{L}_j \phi_k(\mu) z^{-\mu}, \quad g_j^T(z) := (g_{j,1}(z), g_{j,2}(z), \ldots, g_{j,r}(z)),
\]

and the \( s \times r(\det M) \) matrix

\[
  G(z) := \begin{bmatrix} g_j^T(z_1 e^{2\pi i m_1}, \ldots, z_d e^{2\pi i m_d}) \end{bmatrix}_{k=1}^{d \to r, l=1,2,\ldots,\det M} (30)
\]

where \( m_1, \ldots, m_d \) denote the columns of the matrix \( M^{-1} \). Note that for the values \( x = (x_1, \ldots, x_d) \in [0,1)^d \) and \( z = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}) \in \mathbb{T}^d \) we have \( G(x) = G(z) \).

Provided that the functions \( g_j \) are continuous on \( \mathbb{R}^d \), Corollary 1 can be reformulated as follows: There exists an \( r \times s \) matrix \( a(z) = [a_1(z), \ldots, a_s(z)] \) with entries essentially bounded in the torus \( \mathbb{T}^d \) and satisfying

\[
  a(z) G(z) = [I_r, O_{(\det M-1)\times r}] \quad \text{for all } z \in \mathbb{T}^d (30)
\]

if and only if

\[
  \text{rank } G(z) = r(\det M) \quad \text{for all } z \in \mathbb{T}^d. (31)
\]

Denoting the columns of the matrix \( a(z) \) as \( a_j^T(z) = (a_{1,j}(z), \ldots, a_{s,j}(z)) \), \( j = 1, 2, \ldots, s \), the corresponding reconstruction functions \( S_{j,a} \) in sampling formula (28) are

\[
  S_{j,a}(t) = \sum_{\alpha \in \mathbb{Z}^d} \sum_{l=1}^{s} \mathcal{h}_{k,j}(\alpha) \varphi(t - \alpha), \quad t \in \mathbb{R}^d, (32)
\]

where \( \mathcal{h}_{k,j}(\alpha), \alpha \in \mathbb{Z}^d \), are the Laurent coefficients of the functions \( a_{k,j}(z) \), that is,

\[
  a_{k,j}(z) = \sum_{\alpha \in \mathbb{Z}^d} \mathcal{h}_{k,j}(\alpha) z^{-\alpha}. (33)
\]

Note that, in order to obtain compactly supported reconstruction functions \( S_{j,a} \) in (28) we need an \( r \times s \) matrix \( a(z) \) whose entries are Laurent polynomials, i.e., the sum in (33) is finite. The following result, which proof can be found in [16] under minor changes, holds:

**Theorem 4.** Assume that the generators \( \phi_k \) and the functions \( \mathcal{L}_j \phi_k \), \( 1 \leq k \leq r \) and \( 1 \leq j \leq s \), have compact support. Then, there exists an \( r(\det M) \times s \) matrix \( a(z) \) whose entries are Laurent polynomials and satisfying (30) if and only if
\[ \text{rank } G(z) = r(\det M) \text{ for all } z \in (\mathbb{C} \setminus \{0\})^d. \]

The reconstruction functions \( S_{j,a} \), \( j = 1, 2, \ldots, s \), obtained from such matrix \( a(z) \) through Eq. (32) have compact support.

From one of these \( r \times s \) matrices, say \( \tilde{a}(z) = [\tilde{a}_1(z), \ldots, \tilde{a}_s(z)] \), we can get all of them. Indeed, it is easy to check that they are given by the first \( r \) rows of the \( r(\det M) \times s \) matrices of the form

\[ A(z) = \tilde{A}(z) + U(z) \left[ I_s - G(z) \tilde{A}(z) \right], \tag{34} \]

where

\[ \tilde{A}(z) := \left[ \tilde{a}_j(z e^{2\pi i m_j^1 i_1}, \ldots, z e^{2\pi i m_j^r i_r}) \right]_{k=1,2,\ldots,r, i=1,2,\ldots,\det M}, \]

and \( U(z) \) is any \( r(\det M) \times s \) matrix with Laurent polynomial entries. Remember that \( m_1, \ldots, m_d \) denote the columns of the sampling matrix \( M \), and \( i_1, \ldots, i_{\det M} \) the elements of \( \mathcal{N}(M^\top) \) defined in (8).

Next we study the existence of reconstruction functions \( S_{j,a} \), \( j = 1, 2, \ldots, s \), in (28) having exponential decay; it means that there exist constants \( C > 0 \) and \( q \in (0, 1) \) such that \( |S_{j,a}(t)| \leq Cq^{|t|} \) for each \( t \in \mathbb{R}^d \). In so doing, we introduce the algebra \( \mathcal{H}(\mathbb{T}^d) \) of all holomorphic functions in a neighborhood of the \( d \)-torus \( \mathbb{T}^d \). Note that the elements in \( \mathcal{H}(\mathbb{T}^d) \) are characterized as admitting a Laurent series where the sequence of coefficients decays exponentially fast [26].

The following theorem, which proof can be found in [16] under minor changes, holds:

**Theorem 5.** Assume that the generators \( \varphi_k \) and the functions \( \mathcal{L}_j \varphi_k, j = 1, 2, \ldots, s \) and \( k = 1, 2, \ldots, r \), have exponential decay. Then, there exists an \( r \times s \) matrix \( a(z) = [a_1(z), \ldots, a_s(z)] \) with entries in \( \mathcal{H}(\mathbb{T}^d) \) and satisfying (30) if and only if \( \text{rank } G(z) = r(\det M) \) for all \( z \in \mathbb{T}^d \).

In this case, all of such matrices \( a(z) \) are given as the first \( r \) rows of a \( r(\det M) \times s \) matrix \( A(z) \) of the form

\[ A(z) = G^\top(z) + U(z) \left[ I_s - G(z) G^\top(z) \right], \tag{35} \]

where \( U(z) \) denotes any \( r(\det M) \times s \) matrix with entries in the algebra \( \mathcal{H}(\mathbb{T}^d) \) and \( G^\top(z) := [G^\top(z) G(z)]^{-1} G^\top(z) \). The corresponding reconstruction functions \( S_{j,a} \), \( j = 1, 2, \ldots, s \), given by (32) have exponential decay.

### 3.2 Some illustrative examples

We include here some examples illustrating Theorem 4, a particular case of Theorem 2, by taking B-splines as generators; they certainly are important for practical purposes [42].
3.2.1 The case $d = 1, r = 1, M = 2$ and $s = 3$

Let $N_3(t) := \chi_{[0,1)} * \chi_{[0,1)} * \chi_{[0,1)}(t)$ be the quadratic B-spline, where $\chi_{[0,1)}$ denotes the characteristic function of the interval $[0,1)$, and let $L_j, j = 1, 2, 3$, be the systems:

\[
L_1 f(t) = f(t); \quad L_2 f(t) = f(t + \frac{2}{3}) \quad \text{and} \quad L_3 f(t) = f(t + \frac{4}{3}).
\]

Since the functions $L_j N_3, j = 1, 2, 3$, have compact support, then the entries of the $3 \times 2$ matrix $G(z)$ in (29) are Laurent polynomials and we can try to search a vector $a(z) := [a_1(z), a_2(z), a_3(z)]$ satisfying (30) with Laurent polynomials entries also. This implies reconstruction functions $S_{j,a}, j = 1, 2, 3$, with compact support. Proceeding as in [14] we obtain that any function $f \in V_N^2$ can be recovered through the sampling formula:

\[
f(t) = \sum_{n \in \mathbb{Z}} \sum_{j=1}^3 L_j f(2n) S_{j,a}(t-2n), \quad t \in \mathbb{R},
\]

where the reconstruction functions, according to (32), are given by

\[
\begin{align*}
S_{1,a}(t) &= \frac{1}{16} \left[ N_3(t+3) - 3N_3(t+2) - 3N_3(t+1) + N_3(t) \right], \\
S_{2,a}(t) &= \frac{1}{16} \left[ 27N_3(t+1) - 9N_3(t) \right], \\
S_{3,a}(t) &= \frac{1}{16} \left[ -9N_3(t+1) + 27N_3(t) \right], \quad t \in \mathbb{R}.
\end{align*}
\]

3.2.2 The case $d = 1, r = 2, M = 1$ and $s = 3$

Consider the Hermite cubic splines defined as

\[
\begin{align*}
\phi_1(t) &= \begin{cases} (t+1)^2(1-2t), & t \in [-1,0] \\
(1-t)^2(1+2t), & t \in [0,1] \\
0, & |t| > 1 \end{cases}, \\
\phi_2(t) &= \begin{cases} (t+1)^2t, & t \in [-1,0] \\
(1-t)^2t, & t \in [0,1] \\
0, & |t| > 1 \end{cases}.
\end{align*}
\]

They are stable generators for the space $V_{\phi_1,\phi_2}^2$ (see Ref. [12]). Consider the sampling period $M = 1$ and the systems $L_j, j = 1, 2, 3$, defined by

\[
L_1 f(t) := \int_t^{t+1/3} f(u) du, \quad L_2 f(t) := L_1 f(t + \frac{1}{3}), \quad L_3 f(t) := L_1 f(t + \frac{2}{3}).
\]

Since the functions $L_j \phi_k, j = 1, 2, 3$ and $k = 1, 2$, have compact support, then the entries of the $3 \times 2$ matrix $G(z)$ in (29) are Laurent polynomials and we can try to search an $2 \times 3$ matrix $a(z) := [a_1(z), a_2(z), a_3(z)]$ satisfying (30) with Laurent polynomials entries also.
polynomials entries also. This leads to reconstruction functions $S_{j,a}$, $j = 1, 2, 3$, with compact support. Proceeding as in [17] we obtain in $V_{\phi_1,\phi_2}^2$ the following sampling formula:

$$f(t) = \sum_{n \in \mathbb{Z}} \sum_{j=1}^{3} \mathcal{L}_j f(n) S_{j,a}(t-n), \quad t \in \mathbb{R},$$

where the sampling functions, according to (32), are

$$S_{1,a}(t) := \frac{85}{44} \phi_1(t) + \frac{1}{11} \phi_1(t-1) + \frac{85}{4} \phi_2(t) - \phi_2(t-1),$$

$$S_{2,a}(t) := -\frac{23}{44} \phi_1(t) - \frac{23}{44} \phi_1(t-1) - \frac{23}{4} \phi_2(t) + \frac{23}{4} \phi_2(t-1),$$

$$S_{3,a}(t) := \frac{1}{11} \phi_1(t) + \frac{85}{44} \phi_1(t-1) + \phi_2(t) - \frac{85}{4} \phi_2(t-1), \quad t \in \mathbb{R}.$$

### 3.3 $L^2$-approximation properties

Consider an $r \times s$ matrix $a(x) := [a_1(x), a_2(x), \ldots, a_s(x)]$ with entries $a_{k,j} \in L^\infty(0,1)^d$, $1 \leq k \leq r$, $1 \leq j \leq s$, and satisfying (20). Let $S_{j,a}$ be the associated reconstruction functions, $j = 1, 2, \ldots, s$, given in Theorem 2. The aim of this section is to show that if the set of generators $\Phi$ satisfies the Strang-Fix conditions of order $\ell$, then the scaled version of the sampling operator

$$\Gamma_{a}f(t) := \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,a}(t-M\alpha), \quad t \in \mathbb{R}^d,$$

gives $L^2$-approximation order $\ell$ for any smooth function $f$ (in a Sobolev space). In doing so, we take advantage of the good approximation properties of the scaled space $\sigma_{1/h}V_{\Phi}^2$, where for $h > 0$ we are using the notation: $\sigma_{h}f(t) := f(ht)$, $t \in \mathbb{R}^d$.

The set of generators $\Phi = \{\phi_k\}_{k=1}^r$ is said to satisfy the Strang-Fix conditions of order $\ell$ if there exist $r$ finitely supported sequences $b_k : \mathbb{Z}^d \to \mathbb{C}$ such that the function $\phi(t) = \sum_{k=1}^{r} \sum_{\alpha \in \mathbb{Z}^d} b_k(\alpha) \phi_k(t-\alpha)$ satisfies the Strang-Fix conditions of order $\ell$, i.e.,

$$\phi(0) \neq 0, \quad D^\beta \phi(\alpha) = 0, \quad |\beta| < \ell, \quad \alpha \in \mathbb{Z}^d \setminus \{0\}. \quad (36)$$

We denote by $W_{2}^{\ell}([R^d]) := \{f : ||D^\ell f||_2 < \infty, |\gamma| \leq \ell\}$ the usual Sobolev space, and by $|f|_{\ell,2} := \sum_{|\beta| = \ell} ||D^\beta f||_2$ the corresponding seminorm of a function $f \in W_{2}^{\ell}([R^d])$. When $2\ell > d$ we identify $f \in W_{2}^{\ell}([R^d])$ with its continuous choice (see [1]).

It is well-known that if $\Phi$ satisfies the Strang-Fix conditions of order $\ell$, and the generators $\phi_k$ satisfy a suitable decay condition, the space $V_{\phi}^2$ provides $L^2$-approximation order $\ell$ for any function $f$ regular enough. For instance, Lei et al. proved in [32, Theorem 5.2] the following result: If a set $\Phi = \{\phi_k\}_{k=1}^r$ of stable generators satisfies the Strang-Fix conditions of order $\ell$, and the decay condition
\( \varphi_k(t) = O((1 + |t|)^{-d - \ell - \varepsilon}) \) for each \( k = 1, \ldots, r \) and some \( \varepsilon > 0 \), then, for any \( f \in W^2_2(\mathbb{R}^d) \), there exists a function \( f_h \in \sigma_{1/h}V^2_\Phi \) such that

\[
\|f - f_h\|_2 \leq C |f|_{\ell, 2} h^\ell,
\]

(37)

where the constant \( C \) does not depend on \( h \) and \( f \).

In this section we assume that all the systems \( \mathcal{L}_j \), \( j = 1, \ldots, s \), are of type (a), i.e., \( \mathcal{L}_j f = f * h_j \), belonging the impulse response \( h_j \) to the Hilbert space \( \mathcal{L}^2(\mathbb{R}^d) \).

Recall that a Lebesgue measurable function \( h : \mathbb{R}^d \to \mathbb{C} \) belongs to the Hilbert space \( \mathcal{L}^2(\mathbb{R}^d) \) if

\[
|h|_2 := \left( \int_{[0,1]^d} \left( \sum_{\alpha \in \mathbb{Z}^d} |h(t - \alpha)| \right)^2 dt \right)^{1/2} < \infty.
\]

Notice that \( \mathcal{L}^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). For \( f \in L^2(\mathbb{R}^d) \) and \( h \in \mathcal{L}^2(\mathbb{R}^d) \), the following inequality holds: \( \| \{ h * f(\alpha) \}_{\alpha \in \mathbb{Z}^d} \|_2 \leq |h|_2 \| f \|_2 \) (see [26, Theorem 3.1]).

Thus the sequence of generalized samples \( \{ (\mathcal{L}_j f)(M\alpha) \}_{\alpha \in \mathbb{Z}^d}, j = 1, \ldots, s \), belongs to \( \ell^2(\mathbb{Z}^d) \) for any \( f \in L^2(\mathbb{R}^d) \).

First we note that the operator \( \Gamma_a : (L^2(\mathbb{R}^d), \| \cdot \|_2) \to (V^2_\Phi, \| \cdot \|_2) \) given by

\[
(\Gamma_a f)(t) := (\det M) \sum_{j=1}^s \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j t + M\alpha, \quad t \in \mathbb{R}^d,
\]

is a well-defined bounded operator onto \( V^2_\Phi \). Besides, \( \Gamma_a f = f \) for all \( f \in V^2_\Phi \).

Under appropriate hypotheses we prove that the scaled operator \( \Gamma^h_a := \sigma_{1/h} \Gamma_a \sigma_h \) approximates, in the \( L^2 \)-norm sense, any function \( f \) in the Sobolev space \( W^2_2(\mathbb{R}^d) \) as \( h \to 0^+ \). Specifically we have:

**Theorem 6.** Assume \( 2\ell > d \) and that all the systems \( \mathcal{L}_j \) satisfy \( \mathcal{L}_j f = f * h_j \) with \( h_j \in \mathcal{L}^2(\mathbb{R}^d) \), \( j = 1, \ldots, s \). Then,

\[
\|f - \Gamma^h_a f\|_2 \leq (1 + \|\Gamma_a\|) \inf_{g \in \sigma_{1/h}V^2_\Phi} \|f - g\|_2, \quad f \in W^2_2(\mathbb{R}^d),
\]

where \( \|\Gamma_a\| \) denotes the norm of the sampling operator \( \Gamma_a \). If the set of generators \( \Phi = \{ \varphi_k \}_{k=1}^r \) satisfies the Strang-Fix conditions of order \( \ell \) and, for each \( k = 1, 2, \ldots, r \), the decay condition \( \varphi_k(t) = O((1 + |t|)^{-d - \ell - \varepsilon}) \) for some \( \varepsilon > 0 \), then

\[
\|f - \Gamma^h_a f\|_r \leq C |f|_{\ell, 2} h^\ell, \quad \text{for all } f \in W^2_2(\mathbb{R}^d),
\]

where the constant \( C \) does not depend on \( h \) and \( f \).

**Proof.** Using that \( \Gamma^h_a g = g \) for each \( g \in \sigma_{1/h}V^2_\Phi \) then, for each \( f \in L^2(\mathbb{R}^d) \) and \( g \in \sigma_{1/h}V^2_\Phi \), Lebesgue’s Lemma [13, p. 30] gives

\[
\|f - \Gamma^h_a f\|_2 \leq \|f - g\|_2 + \|\Gamma^h_a g - \Gamma^h_a f\|_2 \leq (1 + \|\Gamma_a\|) \inf_{g \in \sigma_{1/h}V^2_\Phi} \|f - g\|_2.
\]
where we have used that $\| \Gamma_h^a \| = \| \Gamma_a \|$ for $h > 0$. Now, for each $f \in W_2^\ell(\mathbb{R}^d)$ and $h > 0$, there exists a function $f_h \in \sigma_{1/h}V_2^a$ such that (37) holds, from which we obtain the desired result. \hfill \Box

More results on approximation by means of generalized sampling formulas can be found in Refs. \cite{15,18}.

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**References**


