The zero-removing property in Hilbert spaces of entire functions arising in sampling theory*

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Abstract

In the topic of sampling in reproducing kernel Hilbert spaces, sampling in Paley-Wiener spaces is the paradigmatic example. A natural generalization of Paley-Wiener spaces is obtained by substituting the Fourier kernel with an analytic Hilbert-space-valued kernel K. Thus we obtain a reproducing kernel Hilbert space $\mathcal{H}_K$ of entire functions in which the Kramer property allows to prove a sampling theorem. A necessary and sufficient condition ensuring that this sampling formula can be written as a Lagrange-type interpolation series concerns the stability under removal of a finite number of zeros of the functions belonging to the space $\mathcal{H}_K$; this is the so-called zero-removing property. This work is devoted to the study of the zero-removing property in $\mathcal{H}_K$ spaces, regardless of the Kramer property, revealing its connections with other mathematical fields.

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1 Introduction

Sampling in reproducing kernel Hilbert spaces is nowadays an interesting mathematical topic (see, for instance, Refs. [8, 18, 20]). Besides, it has opened new research lines: sampling in unitarily translation invariant reproducing kernel Hilbert spaces or sampling in reproducing Banach spaces (see, for instance, Refs. [17, 19, 20]). The present work is intimately related with this subject, and an easy motivation can be found in the Lagrange-type interpolatory character of the Shannon sampling theorem which holds for Paley-Wiener spaces. Namely, the Paley-Wiener space PW_π of bandlimited functions to $[-\pi, \pi]$, i.e.,

$$PW_\pi := \left\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \mid \text{supp } \widehat{f} \subseteq [-\pi, \pi] \right\},$$

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where $\widehat{f}$ stands for the Fourier transform of f, coincides, via the classical Paley-Wiener theorem [27, p. 85], with the space of entire functions f such that $|f(z)| \leq A\,\mathrm{e}^{\pi|z|}$ on $\mathbb{C}$ for some positive constant A, and $f|_{\mathbb{R}} \in L^2(\mathbb{R})$. In PW_π the classical Shannon sampling theorem holds: Any $f \in PW_\pi$ can be expanded as

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}\,, \quad z \in \mathbb{C}\,. \tag{1}$$

The series converges absolutely and uniformly on horizontal strips of the complex plane. Moreover, the sampling expansion (1) can be written as the Lagrange-type interpolation series

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{P(z)}{(z-n)P'(n)}\,, \quad z \in \mathbb{C}\,,$$

where P stands for the entire function $P(z) = (\sin \pi z)/\pi$, which has only simple zeros at $\mathbb{Z}$.

Since any function $f \in PW_\pi$ can be written as

$$f(z) = \Big\langle \frac{\mathrm{e}^{iz\cdot}}{\sqrt{2\pi}}, F \Big\rangle_{L^2[-\pi,\pi]}\,, \quad z \in \mathbb{C}\,,$$

for some function $F \in L^2[-\pi, \pi]$, Shannon sampling theory admits a straighforward generalization by substituting the Fourier kernel

$$\mathbb{C} \ni z \longmapsto K(z) \in L^2[-\pi,\pi] \text{ such that } K(z)(w) := \mathrm{e}^{izw}/\sqrt{2\pi}\,, \quad w \in [-\pi,\pi]\,,$$

by another abstract kernel K valued in a Hilbert space $\mathcal{H}$. The analytic Kramer sampling theory accomplishes this generalization. Indeed, let $\mathcal{H}$ be a complex, separable Hilbert space with inner product $\langle \cdot\,, - \rangle_{\mathcal{H}}$ and suppose K is an $\mathcal{H}$-valued analytic function defined on $\mathbb{C}$. For each $x \in \mathcal{H}$, define the function $f_x(z) = \langle K(z), x\rangle_{\mathcal{H}}$ on $\mathbb{C}$, and let $\mathcal{H}_K$ denote the collection of all such functions f_x. Furthermore, each element in $\mathcal{H}_K$ is an entire function since K is analytic on $\mathbb{C}$. In this setting, an abstract version of the analytic Kramer theorem [15] is obtained by assuming the Kramer property, that is, the existence of two sequences, $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^\infty$ in $\mathbb{C} \setminus \{0\}$, and a Riesz basis for $\mathcal{H}$ $\{x_n\}_{n=1}^\infty$ such that $K(z_n) = a_n x_n$ for each $n \in \mathbb{N}$. Namely, for any $f_x \in \mathcal{H}_K$ we have

$$f_x(z) = \sum_{n=1}^{\infty} f_x(z_n)\, \frac{S_n(z)}{a_n}\,, \quad z \in \mathbb{C}\,, \tag{2}$$

where, for each $n \in \mathbb{N}$, $S_n(z) = \langle K(z), y_n\rangle$, $z \in \mathbb{C}$, and $\{y_n\}_{n=1}^\infty$ stands for the dual Riesz basis of $\{x_n\}_{n=1}^\infty$ (see Section 2 below for the details).

A challenging problem is to give a necessary and sufficient condition to ensure that the above sampling formula can be written as a Lagrange-type interpolation series, that is

$$f_x(z) = \sum_{n=1}^{\infty} f_x(z_n)\, \frac{P(z)}{(z-z_n)P(z_n)}\,, \quad z \in \mathbb{C}\,, \tag{3}$$

where P denotes an entire function having only simple zeros at all points of the sequence $\{z_n\}_{n=1}^\infty$. The necessary and sufficient condition ensuring when a Kramer sampling expansion

(2) can be written as a Lagrange-type interpolation series (3) was proved in [8] for orthogonal sampling formulas, and in [9] for non-orthogonal Riesz basis sampling formulas. Roughly speaking, the aforesaid necessary and sufficient condition concerns the stability of the functions belonging to the space $\mathcal{H}_K$ under removal of a finite number of their zeros; in other words,

$$f \in \mathcal{H}_K \text{ and } f(a) = 0 \text{ implies that } \frac{f(z)}{z-a} \in \mathcal{H}_K\,.$$

This is an ubiquitous algebraic property in the mathematical literature (see Section 2 below) and it will be called the zero-removing property (ZR property in short) throughout the paper. The main aim in this paper is to thoroughly study the ZR property in $\mathcal{H}_K$ spaces, regardless of the Kramer property, revealing its relationships with other mathematical fields. For instance, Paley-Wiener spaces are particular cases of de Branges spaces [4] where the ZR property holds, and de Branges spaces are particular cases of $\mathcal{H}_K$ spaces as well [10].

Next, we outline the organization of the paper, highlighting its significant contributions. In Section 2 we introduce the needed preliminaries on spaces $\mathcal{H}_K$: these spaces are reproducing kernel Hilbert spaces (RKHS in short) of entire functions; we briefly recall the sampling result in $\mathcal{H}_K$. In Section 3 we study some properties of $\mathcal{H}_K$ obtained from the Taylor coefficients of the kernel K at a fixed complex point. In particular, the relationship between $\mathcal{H}_K$ and the set $\mathcal{P}(\mathbb{C})$ of complex polynomials. In Section 4 we study the zero-removing property at a fixed point; this property can be reduced to a general moment problem. Thus, the zero-removing property at a fixed point depends on the continuity of a certain associated operator which looks like the classical shift operator. Moreover, we give a sufficient condition for the continuity of this operator. The section is closed by studying the local zero-removing property: If the zero-removing property holds for a fixed point, say 0, it also holds for any $a \in \mathbb{C}$ with $|a|$ small enough. This study is carried out by using the well-known Fredholm operator theory. Finally, in Section 5 we close the paper with an study of the differentiation operator in an $\mathcal{H}_K$ space.

2 Preliminaries on $\mathcal{H}_K$ spaces

Suppose we are given a separable complex Hilbert space and an abstract kernel K which is nothing but an $\mathcal{H}$-valued function on $\mathbb{C}$. For each $z \in \mathbb{C}$, set $f_x(z) := \langle K(z), x\rangle_{\mathcal{H}}$ for $z \in \mathbb{C}$, and denote by $\mathcal{H}_K$ the collection of all such functions f_x, $x \in \mathcal{H}$, and let $\mathcal{T}_K$ be the mapping

$$\mathcal{H} \ni x \xmapsto{\mathcal{T}_K} f_x \in \mathcal{H}_K \tag{4}$$

If we define the norm $\|f\|_{\mathcal{H}_K} := \inf\{\|x\|_{\mathcal{H}} \,:\, f = \mathcal{T}_K x\}$ in $\mathcal{H}_K$ (in the sequel we omit the subscript x in f_x), we obtain a reproducing kernel Hilbert space whose reproducing kernel is given by

$$k(z,w) = \langle K(z), K(w)\rangle_{\mathcal{H}}\,, \quad z,w \in \mathbb{C}\,.$$

(see [23] for the details). Notice that the mapping $\mathcal{T}_K$ is an antilinear mapping from $\mathcal{H}$ onto $\mathcal{H}_K$. It is injective if and only if the set $\{K(z)\}_{z\in\mathbb{C}}$ is complete in $\mathcal{H}$. In particular, if there exist sequences $\{z_n\}_{n=1}^\infty \subset \mathbb{C}$, $\{a_n\}_{n=1}^\infty \in \mathbb{C}\setminus\{0\}$ and a Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathcal{H}$ such that $K(z_n) = a_n\, x_n$ for any $n \in \mathbb{N}$, then the mapping $\mathcal{T}_K$ is an anti-linear isometry from $\mathcal{H}$ onto $\mathcal{H}_K$. Recall that a Riesz basis in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a boundedly invertible operator. Any Riesz basis $\{x_n\}_{n=1}^\infty$ has a unique biorthonormal (dual) Riesz basis $\{y_n\}_{n=1}^\infty$, i.e., $\langle x_n, y_m\rangle_{\mathcal{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^{\infty} \langle x, y_n\rangle_{\mathcal{H}}\, x_n = \sum_{n=1}^{\infty} \langle x, x_n\rangle_{\mathcal{H}}\, y_n$$

hold for every $x \in \mathcal{H}$ (see [6, 27] for more details and proofs).

The convergence in the norm $\|\cdot\|_{\mathcal{H}_K}$ implies pointwise convergence which is uniform on those subsets of $\mathbb{C}$ where the function $z \mapsto \|K(z)\|_{\mathcal{H}}$ is bounded; in particular, in compact subsets of $\mathbb{C}$ whenever K is a continuous kernel.

Like in the classical case the following result holds: The space $\mathcal{H}_K$ is a RKHS of entire functions if and only if the kernel K is analytic in $\mathbb{C}$ ([25, p. 266]). Another characterization of the analyticity of the functions in $\mathcal{H}_K$ is given in terms of Riesz bases. Suppose that a Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathcal{H}$ is given and let $\{y_n\}_{n=1}^\infty$ be its dual Riesz basis; expanding $K(z)$, where $z \in \mathbb{C}$ is fixed, with respect to the basis $\{x_n\}_{n=1}^\infty$ we obtain

$$K(z) = \sum_{n=1}^{\infty} \langle K(z), y_n\rangle_{\mathcal{H}}\, x_n = \sum_{n=1}^{\infty} S_n(z)\, x_n \quad \text{in } \mathcal{H}\,,$$

where the coefficients

$$S_n(z) := \langle K(z), y_n\rangle_{\mathcal{H}}, \quad z \in \mathbb{C}\,, \tag{5}$$

as functions in $z \in \mathbb{C}$, are in $\mathcal{H}_K$. The following result holds [10]: Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be a pair of dual Riesz bases for $\mathcal{H}$. Then, $\mathcal{H}_K$ is RKHS of entire functions if and only if all the functions S_n, $n \in \mathbb{N}$, are entire and the function $z \mapsto \|K(z)\|_{\mathcal{H}}$ is bounded on compact sets of $\mathbb{C}$.

2.1 Sampling and the zero-removing property in $\mathcal{H}_K$ spaces

Consider the data

$$\{z_n\}_{n=1}^\infty \in \mathbb{C} \quad \text{and} \quad \{a_n\}_{n=1}^\infty \in \mathbb{C}\setminus\{0\}. \tag{6}$$

Definition 1 *An analytic kernel $K : \mathbb{C} \longrightarrow \mathcal{H}$ is said to be an analytic Kramer kernel (with respect to the data (6)) if it satisfies $K(z_n) = a_n\, x_n$, $n \in \mathbb{N}$, for some Riesz basis $\{x_n\}_{n=1}^\infty$ of $\mathcal{H}$. A sequence $\{S_n\}_{n=1}^\infty$ of functions in $\mathcal{H}_K$ is said to have the interpolation property (with respect to the data (6)) if*

$$S_n(z_m) = a_n\, \delta_{n,m}\,. \tag{7}$$

An analytic kernel K is an analytic Kramer one if and only if the sequence of functions $\{S_n\}_{n=1}^\infty$ in $\mathcal{H}_K$ given by (5), where $\{y_n\}_{n=1}^\infty$ is the dual Riesz basis of $\{x_n\}_{n=1}^\infty$, has the interpolation property with respect to the same data (6).

Under the notation introduced so far an abstract version of the classical Kramer sampling theorem [15] holds: First notice that $\lim_{m\to\infty} |z_m| = +\infty$; otherwise we obtain that any entire function S_n is identically zero in $\mathbb{C}$. The anti-linear mapping $\mathcal{T}_K$ is a bijective isometry between $\mathcal{H}$ and $\mathcal{H}_K$. As a consequence, the functions $\big\{S_n = \mathcal{T}_K(y_n)\big\}_{n=1}^\infty$ form a Riesz basis for $\mathcal{H}_K$; the sequence $\{T_n := \mathcal{T}_K(x_n)\}_{n=1}^\infty$ is its dual Riesz basis. Expanding any $f \in \mathcal{H}_K$ with respect the basis $\big\{S_n\big\}_{n=1}^\infty$ we obtain

$$f(z) = \sum_{n=1}^{\infty} \big\langle f, T_n\big\rangle_{\mathcal{H}_K} S_n(z) \quad \text{in } \mathcal{H}_K\,.$$

Besides,

$$\langle f, T_n\rangle_{\mathcal{H}_K} = \overline{\langle x, x_n\rangle}_{\mathcal{H}} = \Big\langle \frac{K(z_n)}{a_n}, x \Big\rangle_{\mathcal{H}} = \frac{f(z_n)}{a_n}\,.$$

Since a Riesz basis is an unconditional basis, the sampling series will be pointwise unconditionally convergent and hence, absolutely convergent. The uniform convergence is a standard result in the setting of the RKHS theory since $z \mapsto \|K(z)\|_{\mathcal{H}}$ is bounded on compact subsets of $\mathbb{C}$. Thus we have proved an abstract version of the *classical Kramer sampling theorem* [15]:

Theorem 1 *Let $K : \mathbb{C} \longrightarrow \mathcal{H}$ be an analytic Kramer kernel, and assume that the interpolation property (7) holds for some sequences $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^\infty$ in $\mathbb{C} \setminus \{0\}$. Let $\mathcal{H}_K$ be the corresponding RKHS of entire functions. Then any $f \in \mathcal{H}_K$ can be recovered from the sequence of its samples $\{f(z_n)\}_{n=1}^\infty$ by means of the sampling series*

$$f(z) = \sum_{n=1}^{\infty} f(z_n)\,\frac{S_n(z)}{a_n}\,, \quad z \in \mathbb{C}\,. \tag{8}$$

This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Equivalently, the Kramer property in Definition 1 can be seen as a sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ such that the sequence of reproducing kernels $\{k(\cdot, z_n)\}_{n=1}^\infty$ is a Riesz basis for $\mathcal{H}_K$. An interesting problem is to characterize the sequences $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ having this property in some structured RKHS spaces of entire functions like Hardy or Bergman spaces (see, for instance, Refs. [2, 3, 24] and references therein).

Concerning the sampling formula (8) in $\mathcal{H}_K$, a challenging problem is to give a necessary and sufficient condition to ensure that it can be written as a Lagrange-type interpolation series (see, Eq. (9) below). As it was pointed out in the introduction, it concerns the stability of the functions belonging to the space $\mathcal{H}_K$ on removing a finite number of their zeros; it will be called the *zero-removing property*:

Definition 2 *A set $\mathcal{A}$ of entire functions has the zero-removing property (ZR property in short) if for any $g \in \mathcal{A}$ and any zero w of g the function $g(z)/(z - w)$ belongs to $\mathcal{A}$.*
A set $\mathcal{A}$ of entire functions has the zero-removing property at a point $a \in \mathbb{C}$ (ZR_a property in short) if for any $g \in \mathcal{A}$ with $g(a) = 0$ the function $g(z)/(z - a)$ belongs to $\mathcal{A}$.

In fact, the following result holds (see [8, 9] for the proof):

Theorem 2 *Let $\mathcal{H}_K$ be a RKHS of entire functions obtained from an analytic Kramer kernel K with respect to the data $\{z_n\}_{n=1}^\infty \subset \mathbb{C}$ and $\{a_n\}_{n=1}^\infty \in \mathbb{C} \setminus \{0\}$, i.e., $K(z_n) = a_n\, x_n$, $n \in \mathbb{N}$, for some Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathcal{H}$. Then, the sampling formula (8) for $\mathcal{H}_K$ can be written as a Lagrange-type interpolation series*

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{P(z)}{(z - z_n)P'(z_n)}\,, \quad z \in \mathbb{C}\,, \tag{9}$$

where P denotes an entire function having only simple zeros at $\{z_n\}_{n=1}^\infty$ if and only if the space $\mathcal{H}_K$ satisfies the ZR property.

The ZR property (also called the *division property*; see [11]) is ubiquitous in mathematics; for instance, the set $\mathcal{P}_N(\mathbb{C})$ of polynomials with complex coefficients of degree less or equal than N has the ZR property. Another more involved examples sharing this property are:

(a) The entire functions in the Pólya class have the ZR property [4, p. 15]. Recall that an entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half-plane, if $|E(x-iy)| \le |E(x+iy)|$ for $y>0$, and if $|E(x+iy)|$ is a nondecreasing function of $y>0$ for each fixed x.

(b) The Paley-Wiener space PW_π satisfies the ZR property; it follows immediately from its characterization as the space of entire functions f such that $|f(z)| \le A\,\mathrm{e}^{\pi|z|}$ on $\mathbb{C}$ for some positive constant A, and $f|_{\mathbb{R}} \in L^2(\mathbb{R})$, i.e., the classical Paley-Wiener theorem [27, p. 85]. For a direct proof, consider $f \in PW_\pi$ such that $f(a)=0$, i.e.,

$$f(z)=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \mathrm{e}^{iwz}\,\widehat{f}(w)\,dw\,,\ z\in\mathbb{C},\quad \text{such that}\quad \int_{-\pi}^{\pi}\mathrm{e}^{iwa}\,\widehat{f}(w)\,dw=0\,,$$

where $\widehat{f}$ stands for the Fourier transform of f. Consider the function $g(w)=\int_{-\pi}^{w}\mathrm{e}^{iax}\widehat{f}(x)\,dx$ which satisfies $g(-\pi)=g(\pi)=0$. Integrating by parts one obtains

$$\frac{f(z)}{z-a}=\frac{1}{\sqrt{2\pi}}\frac{1}{z-a}\int_{-\pi}^{\pi}\mathrm{e}^{i(z-a)w}\,\mathrm{e}^{iwa}\,\widehat{f}(w)\,dw=\frac{-1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\mathrm{e}^{iwz}\big(i\,\mathrm{e}^{-iaw}\,g(w)\big)\,dw\,.$$

In other words, since the function $-i\,\mathrm{e}^{-iaw}g(w)$ belongs to $L^2[-\pi,\pi]$, the function $f(z)/(z-a)$ belongs to PW_π.

(c) In general, a de Branges space $\mathcal{H}(E)$ with strict de Branges (structure) function E has the ZR property [4, p. 52]. Let E be an entire function verifying $|E(x-iy)|<|E(x+iy)|$ for all $y>0$. The de Branges space $\mathcal{H}(E)$ is the set of all entire functions f such that

$$\|f\|_E^2 := \int_{-\infty}^{\infty}\Big|\frac{f(t)}{E(t)}\Big|^2\,dt<\infty,$$

and such that both ratios f/E and f^*/E, where $f^*(z):=\overline{f(\overline{z})}$, $z\in\mathbb{C}$, are of bounded type and of nonpositive mean type in the upper half-plane. The structure function or de Branges function E has no zeros in the upper half plane. A de Branges function E is said to be strict if it has no zeros on the real axis. We require f/E and f^*/E to be of bounded type and nonpositive mean type in $\mathbb{C}^+$. A function is of bounded type if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$ and it is of nonpositive mean type if it grows no faster than $\mathrm{e}^{\varepsilon y}$ for each $\varepsilon>0$ as $y\to\infty$ on the positive imaginary axis $\{\mathrm{i}y\ :\ y>0\}$. Note that the Paley-Wiener space PW_π is a de Branges space with strict structure function $E_\pi(z)=\exp(-\mathrm{i}\pi z)$.

As a consequence of Theorem 2, any sampling formula like Eq. 8 in a de Branges space can be written as a Lagrange-type interpolation series.

(d) Whenever the space $\mathcal{H}_K$ is associated with a polynomial kernel $K(z):=\sum_{n=0}^{N}c_n z^n$, where $c_n\in\mathcal{H}$ and $c_N\neq 0$, it is easy to give a characterization for the ZR property in $\mathcal{H}_K$. Namely, the ZR property holds in $\mathcal{H}_K$ if and only if the set $\{c_0,c_1,\dots,c_N\}$ is linearly independent in $\mathcal{H}$ (see [9] for a proof). A more involved problem is to deal with a general entire $\mathcal{H}$-valued kernel $K(z)=\sum_{n=0}^{\infty}c_n z^n$, $z\in\mathbb{C}$; the aim of this paper is to obtain some results in this direction.

(e) In a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n\}_{n=0}^\infty$ consider the kernel

$$\begin{aligned} K_\gamma : \ \mathbb{C} &\longrightarrow \ \mathcal{H} \\ z &\longmapsto \ K_\gamma(z) := \sum_{n=0}^\infty \frac{e_n}{\gamma_n} z^n \,, \end{aligned}$$

where $\gamma := \{\gamma_n\}_{n=0}^\infty$ is a sequence of positive real numbers such that the sequence of quotients $\{\gamma_n/\gamma_{n+1}\}_{n\in\mathbb{N}_0}$ decreases to zero as n increases to infinity. The corresponding spaces $\mathcal{H}_{K_\gamma}$ constructed from this family of analytic kernels were introduced by Chan and Shapiro in [5]. Obviously, an entire function $f(z) = \sum_{n=0}^\infty \alpha_n z^n$ belongs to the space $\mathcal{H}_{K_\gamma}$ if and only if the sequence $\{\gamma_n \alpha_n\}_{n\in\mathbb{N}_0}$ belongs to $\ell^2(\mathbb{N}_0)$, where, as usual, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Therefore, it is straightforward to show that if $f \in \mathcal{H}_{K_\gamma}$, with $f(0) = 0$, then $f(z)/z$ belongs to $\mathcal{H}_{K_\gamma}$, i.e., the space $\mathcal{H}_{K_\gamma}$ satisfies the ZR_0 property. If the sequence $\{\gamma_{n+1}/\gamma_n\}_{n\in\mathbb{N}_0}$ is $O(1/n)$ as $n \to \infty$, then, for any $a \in \mathbb{C}$, the translation operator given by $T_a f(z) := f(z-a)$, $z \in \mathbb{C}$, is a well-defined bounded operator $T_a : \mathcal{H}_{K_\gamma} \longrightarrow \mathcal{H}_{K_\gamma}$ (see [5] for the details). As a consequence of this fact, the space $\mathcal{H}_{K_\gamma}$ satisfies the ZR property (for the details, see Eq. (20) below).

Next we include some examples of spaces $\mathcal{H}_K$ where the ZR property fails:

(f) Let $K : \mathbb{C} \to \mathcal{H}$ be an analytic kernel and assume that there exist two distinct points z_1 and $z_2 \in \mathbb{C}$ such that $K(z_1) = K(z_2)$. Then the space $\mathcal{H}_K$ does not hold the ZR property. Indeed, for $x \neq 0$ in $\mathcal{H}$, orthogonal to $K(z_1)$, consider the function $f(z) = \langle K(z), x\rangle$, $z \in \mathbb{C}$. Assume that r is the order of the zero z_1 of f. If the property ZR holds in $\mathcal{H}_K$, the function

$$g(z) = \frac{f(z)}{(z-z_1)^r}\,, \quad z \in \mathbb{C}$$

belongs to $\mathcal{H}_K$, and $g(z_1) \neq 0$. Let $y \in \mathcal{H}$ be such that $g(z) = \langle K(z), y\rangle$, $z \in \mathbb{C}$. Since $g(z_2) = 0$ we have that y is orthogonal to $K(z_2)$; but $g(z_1) \neq 0$ implies that y is not orthogonal to $K(z_1)$, that is, a contradiction.

(g) Finally, we exhibit a nontrivial example taken from [9] of a RKHS $\mathcal{H}_K$, built from the Sobolev Hilbert space $\mathcal{H} := H^1(-\pi,\pi)$, where the ZR property fails. Namely: consider the Sobolev Hilbert space $H^1(-\pi,\pi)$ with its usual inner product

$$\langle f, g\rangle_1 = \int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx + \int_{-\pi}^{\pi} f'(x)\,\overline{g'(x)}\,dx\,, \quad f, g \in H^1(-\pi,\pi)\,.$$

The sequence $\{e^{inx}\}_{n\in\mathbb{Z}} \cup \{\sinh x\}$ forms an orthogonal basis for $H^1(-\pi,\pi)$: It is straightforward to prove that the orthogonal complement of $\{e^{inx}\}_{n\in\mathbb{Z}}$ in $H^1(-\pi,\pi)$ is a one-dimensional space for which $\sinh x$ is a basis. For a fixed $a \in \mathbb{C} \setminus \mathbb{Z}$ we define a kernel

$$\begin{aligned} K_a : \ \mathbb{C} &\longrightarrow \ H^1(-\pi,\pi) \\ z &\longmapsto \ K_a(z)\,, \end{aligned}$$

by setting

$$[K_a(z)](x) = (z-a)\,e^{izx} + \sin \pi z \, \sinh x \quad \text{for } x \in (-\pi,\pi)\,.$$

Clearly, K_a defines an analytic Kramer kernel. Expanding $K_a(z) \in H^1(-\pi,\pi)$ in the former orthogonal basis we obtain

$$K_a(z) = [1 - i(z-a)]\, \sin \pi z \, \sinh x + (z-a) \sum_{n=-\infty}^{\infty} \frac{1+zn}{1+n^2}\, \mathrm{sinc}(z-n) e^{inx} \quad \text{in } H^1(-\pi,\pi)\,.$$

where sinc denotes the cardinal sine function $\operatorname{sinc}(z) = \sin \pi z/\pi z$, if $z \neq 0$, and $\operatorname{sinc}(0) = 1$. As a consequence, Theorem 1 gives the following sampling result in $\mathcal{H}_{K_a}$: Any function $f \in \mathcal{H}_{K_a}$ can be recovered from its samples $\{f(a)\} \cup \{f(n)\}_{n\in\mathbb{Z}}$ by means of the sampling formula

$$f(z) = [1 - i(z-a)]\,\frac{\sin \pi z}{\sin \pi a}\, f(a) + \sum_{n=-\infty}^{\infty} f(n)\,\frac{z-a}{n-a}\,\frac{1+zn}{1+n^2}\,\operatorname{sinc}(z-n)\,, \quad z \in \mathbb{C}\,.$$

The function $(z-a)\operatorname{sinc} z$ belongs to $\mathcal{H}_{K_a}$ since $(z-a)\operatorname{sinc} z = \langle K_a(z), 1/2\pi\rangle_1$ for each $z \in \mathbb{C}$. However, by using the above sampling formula for $\mathcal{H}_{K_a}$ it is straightforward to check that the function $\operatorname{sinc} z$ does not belong to $\mathcal{H}_{K_a}$. Analogously, one can prove that the zero-removing property also fails for any $n \in \mathbb{Z}$ by considering the function $f(z) = \sin \pi z$ which belongs to $\mathcal{H}_{K_a}$.

3 Some properties on $\mathcal{H}_K$ related to the kernel K

In this section we obtain some properties of the Hilbert space $\mathcal{H}_K$ derived from the sequence of Taylor coefficients of the entire kernel K at a point $a \in \mathbb{C}$. Indeed, for each $a \in \mathbb{C}$ we have the Taylor expansion

$$K(z) = \sum_{n=0}^{\infty} c_n(a)(z-a)^n\,, \quad z \in \mathbb{C}\,,$$

where the coefficient $c_n(a) \in \mathcal{H}$ for each $n \in \mathbb{N}_0$. By using Cauchy's integral formula for derivatives (see [25, p. 268] we have

$$c_n(a) = \frac{1}{n!}K^{(n)}(a) = \frac{1}{2\pi i}\int_{|z-a|=R} \frac{K(z)}{(z-a)^{n+1}}\,dz\,, \quad n = 0, 1, \dots\,,$$

from which

$$\|c_n(a)\|_{\mathcal{H}} \leq \frac{1}{R^{n+1}} \sup_{|z-a|=R} \|K(z)\|_{\mathcal{H}} = \frac{M_R(a)}{R^{n+1}}\,, \tag{10}$$

where $M_R(a) := \sup_{|z-a|=R} \|K(z)\|_{\mathcal{H}}$. Taking $R > 1$, the above inequality shows that the sequence $\{\|c_n(a)\|\}_{n\in\mathbb{N}_0}$ belongs to $\ell^1(\mathbb{N}_0) \subset \ell^2(\mathbb{N}_0)$.

Proposition 1 *Let $\{c_n(a)\}_{n\in\mathbb{N}_0}$ be the sequence of Taylor coefficients of K at any $a \in \mathbb{C}$.*

1. *The sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is a Bessel sequence for $\mathcal{H}$.*
2. *Assume that the mapping $\mathcal{T}_K$ in (4) is injective. Then the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is a complete sequence in $\mathcal{H}$.*

Proof. For any $x \in \mathcal{H}$ we have $|\langle c_n(a), x\rangle|^2 \leq \|c_n(a)\|^2_{\mathcal{H}}\|x\|^2_{\mathcal{H}}$ for each $n \in \mathbb{N}_0$. Thus, having in mind (10) we obtain

$$\sum_{n=0}^{\infty} |\langle c_n(a), x\rangle|^2 \leq \left(\sum_{n=0}^{\infty} \|c_n(a)\|^2_{\mathcal{H}}\right)\|x\|^2_{\mathcal{H}} \leq B\|x\|^2_{\mathcal{H}}\,,$$

where $B := \frac{M_R^2(a)}{R^2-1}$ and $R > 1$.

Assume now that $\langle c_n(a), x\rangle = 0$ for all $n \in \mathbb{N}_0$. For the function $f(z) := \langle K(z), x\rangle$, $z \in \mathbb{C}$, we have the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \langle c_n(a), x\rangle (z-a)^n = 0 \quad \text{for all } z \in \mathbb{C}\,.$$

Since the anti-linear mapping $\mathcal{T}_K$ is injective we deduce that $x = 0$. □

The Bessel property in Proposition 1 implies that the space $\mathcal{H}_K$ is a subspace of the Hardy space $H^2(\mathbb{D})$ with continuous inclusion (see [21]). It will be a closed subspace if and only if the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is a frame for $\mathcal{H}$ (see, for instance, [6, 22]). In this paper we often assume that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is also minimal (see Definition 3 below); as a consequence, $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is a Riesz basis where necessarily $0 < m \le \|c_n(0)\| \le M < \infty$ for all $n \in \mathbb{N}_0$ (see [6, p. 124]). This is not the case in our setting since the sequence of Taylor coefficients $c_n(0) \to 0$ in $\mathcal{H}$ as $n \to \infty$. In other words, the space $\mathcal{H}_K$ is not, in general, a closed subspace of the Hardy space $H^2(\mathbb{D})$.

As it was mentioned in Section 2, whenever K is a polynomial kernel with coefficients in $\mathcal{H}$, a necessary and sufficient condition for $\mathcal{H}_K$ satisfying the ZR property is the linear independence in $\mathcal{H}$ of the coefficients of K. In the general case, the linear independence of the Taylor coefficients $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of K at 0 is only a necessary condition for the ZR_0 property (clearly it is not a sufficient condition; see, for instance, example (g) in Section 2):

Proposition 2 *Assume that the space $\mathcal{H}_K$ satisfies the ZR_0 property and consider the Taylor expansion $K(z) = \sum_{n=0}^{\infty} c_n(0) z^n$ of K around 0. Then, the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is linearly independent in $\mathcal{H}$.*

Proof. Assume that there exists an index N such that the coefficient $c_N(0)$ depends linearly on $\{c_0(0), c_1(0), \dots, c_{N-1}(0)\}$, and consider a non-zero $x \in \{c_0(0), c_1(0), \dots, c_{N-1}(0)\}^{\perp}$. Then, the function $\langle K(z), x\rangle$ satisfies

$$\langle K(z), x\rangle = z^m \big(\langle c_m(0), x\rangle + \langle c_{m+1}(0), x\rangle z + \langle c_{m+2}(0), x\rangle z^2 + \cdots\big)$$

with $m \ge N+1$ and $\langle c_m(0), x\rangle \ne 0$. If $\mathcal{H}_K$ satisfies the ZR_0 property, then the entire function

$$g(z) = z^N (\langle c_m(0), x\rangle + \langle c_{m+1}(0), x\rangle z + \langle c_{m+2}(0), x\rangle z^2 + \cdots)$$

belongs to $\mathcal{H}_K$, that is, there exists $y \in \mathcal{H}$ such that

$$\langle c_0(0), y\rangle = \langle c_1(0), y\rangle = \cdots = \langle c_{N-1}(0), y\rangle = 0$$

and

$$\langle c_{m+k}(0), x\rangle = \langle c_{N+k}(0), y\rangle \quad \text{for all } k \ge 0\,.$$

Since $c_N(0)$ depends linearly on $\{c_0(0), c_1(0), \dots, c_{N-1}(0)\}$ and $\langle c_N(0), y\rangle \ne 0$ we get a contradiction. □

As a consequence of the above result, if the space $\mathcal{H}_K$ satisfies the ZR property then, for each $a \in \mathbb{C}$, the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent in $\mathcal{H}$. In other words, if there exists $a \in \mathbb{C}$ such that $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly dependent in $\mathcal{H}$, then the ZR property does not hold in $\mathcal{H}_K$.

A classical problem in a de Branges space $\mathcal{H}(E)$ is to determine when the set of polynomials $\mathcal{P}(\mathbb{C})$ is included in $\mathcal{H}(E)$ (see [1] and references therein). Next, we study the relationship between the set of polynomials $\mathcal{P}(\mathbb{C})$ and our spaces $\mathcal{H}_K$ via the Taylor coefficients $\{c_n(a)\}_{n\in\mathbb{N}_0}$ of the kernel K at a point $a \in \mathbb{C}$.

Definition 3 *A sequence $\{c_n\}_{n\in\mathbb{N}_0}$ is said to be minimal in $\mathcal{H}$ if $c_m \notin \overline{\text{span}}\{c_n\}_{n\neq m}$ for each $m \in \mathbb{N}_0$. A sequence $\{c_n\}_{n\in\mathbb{N}_0}$ is said to be supercomplete in $\mathcal{H}$ if the sequence $\{c_n\}_{n\geq m}$ is complete in $\mathcal{H}$ for each $m \in \mathbb{N}_0$.*

Obviously, each minimal sequence $\{c_n\}_{n=0}^{\infty}$ is linearly independent in $\mathcal{H}$. In this section we will assume that the mapping $\mathcal{T}_K$ in (4) is injective; consequently, the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at any $a \in \mathbb{C}$ is a complete sequence in $\mathcal{H}$ (see Proposition 1).

Proposition 3 *The set of polynomials $\mathcal{P}(\mathbb{C})$ is contained in $\mathcal{H}_K$ if and only if the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at 0 is minimal in $\mathcal{H}$. Moreover, the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is minimal in $\mathcal{H}$ if and only if the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is minimal in $\mathcal{H}$ for each $a \in \mathbb{C}$.*

Proof. For each $n \in \mathbb{N}_0$ the monomial z^n belongs to $\mathcal{H}_K$ if and only if there exists $x_n \in \mathcal{H}$ such that $\langle c_m(0), x_n\rangle = \delta_{m,n}$, where $\delta_{m,n}$ denotes the Kronecker delta. Equivalently, $\{z^n\}_{n=0}^{\infty} \subset \mathcal{H}_K$ if and only if there exists a biorthogonal sequence $\{x_n\}_{n=0}^{\infty} \subset \mathcal{H}$ of $\{c_n(0)\}_{n\in\mathbb{N}_0}$. This is known to be equivalent to the minimality of $\{c_n(0)\}_{n\in\mathbb{N}_0}$ (see [27]).

Now, suppose that for some $a \in \mathbb{C}$ the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ fails to be minimal. Then there exists $N \in \mathbb{N}_0$ such that

$$c_N(a) \in \overline{\text{span}}\{c_0(a), \dots, c_{N-1}(a), c_{N+1}(a), \dots\}$$

Having in mind the completeness of the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ in $\mathcal{H}$ we deduce that the sequence $\{c_0(a), \dots, c_{N-1}(a), c_{N+1}(a), \dots\}$ is complete in $\mathcal{H}$. Therefore, if $x \in \{c_m\}_{m\neq N}^{\perp}$ then $x = 0$ and, consequently, the polynomial $(z-a)^N$ does not belong to $\mathcal{H}_K$. □

Proposition 4 *The sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at 0 is supercomplete in $\mathcal{H}$ if and only if the space $\mathcal{H}_K$ does not contain any non-zero polynomial. Moreover, the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is supercomplete in $\mathcal{H}$ if and only if the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is supercomplete in $\mathcal{H}$ for each $a \in \mathbb{C}$.*

Proof. A non-zero polynomial $a_N z^N + a_{N-1}z^{N-1} + \cdots + a_0$ belongs to $\mathcal{H}_K$ if and only if there exists $x \in \mathcal{H}$, $x \neq 0$, such that

$$\langle c_0(0), x\rangle = a_0, \langle c_1(0), x\rangle = a_1, \dots, \langle c_N(0), x\rangle = a_N \quad \text{and} \quad \langle c_m(0), x\rangle = 0 \ \text{ for } m > N\,.$$

Hence, a non-zero polynomial is in $\mathcal{H}_K$ if and only if the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is not supercomplete in $\mathcal{H}$.

Now, suppose that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is supercomplete in $\mathcal{H}$ and that, for some $b \in \mathbb{C}$, the sequence $\{c_n(b)\}_{n\in\mathbb{N}_0}$ is not supercomplete in $\mathcal{H}$. Then, there exists $N \in \mathbb{N}_0$ such that sequence $\{c_{N+1}(b), c_{N+2}(b), \dots\}$ is not complete in $\mathcal{H}$. Therefore, there exists a non-zero $x \in \mathcal{H}$ such that $\langle c_m(b), x\rangle = 0$ for all $m > N$. As a consequence, the non-zero polynomial

$$\langle c_0(b), x\rangle + \langle c_1(b), x\rangle(z-b) + \cdots + \langle c_N(b), x\rangle(z-b)^N$$

belongs to $\mathcal{H}_K$, that is, a contradiction. $\square$

In the Paley-Wiener case, the Fourier kernel $K(z)(w) = \frac{1}{\sqrt{2\pi}} e^{izw}$, $w \in [-\pi, \pi]$, can be expanded, around $a \in \mathbb{C}$, as

$$K(z)(w) = \frac{1}{\sqrt{2\pi}} e^{izw} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{iaw} \frac{(iw)^n}{n!} (z-a)^n \,, \quad z \in \mathbb{C} \,.$$

Hence, for $n \in \mathbb{N}_0$, we get that $c_n(0)(w) = \frac{1}{\sqrt{2\pi}} \frac{(iw)^n}{n!}$, $w \in [-\pi, \pi]$.

As a by-product, since the Paley-Wiener space PW_π does not contain any non-zero polynomial, from Proposition 4 we deduce that the sequence of monomials $\{1, w, w^2, \dots\}$ is supercomplete (and hence, it is not minimal) in $L^2[-\pi, \pi]$.

Concerning the ZR property in $\mathcal{H}_K$ and the relationship between the set $\mathcal{P}(\mathbb{C})$ of polynomials and $\mathcal{H}_K$ we have the following result:

Proposition 5 *Suppose that the space $\mathcal{H}_K$ satisfies the ZR property. Then, only one of the following three cases hold:*

a. *For any $a \in \mathbb{C}$ the sequence $\{c_n(a)\}_{n \in \mathbb{N}_0}$ is minimal in $\mathcal{H}$. In this case the space $\mathcal{H}_K$ contains any polynomial.*

b. *For any $a \in \mathbb{C}$ the sequence $\{c_n(a)\}_{n \in \mathbb{N}_0}$ is supercomplete in $\mathcal{H}$. In this case the space $\mathcal{H}_K$ does not contain non-zero polynomials.*

c. *There exists $N \in \mathbb{N}_0$ such that the polynomials belonging to $\mathcal{H}_K$ are precisely the set of polynomials of degree less or equal than N. In this case, for each $a \in \mathbb{C}$ the sequence $\{c_n(a)\}_{n \geq N+1}$ is supercomplete in its closed span and $c_r(a) \notin \overline{\operatorname{span}}\{c_n(a)\}_{n \neq r}$ for $r = 0, 1, \dots, N$.*

Proof. We denote by ∂p the degree of a polynomial p. Assume that there exists a polynomial p belonging to the space $\mathcal{H}_K$. If the space $\mathcal{H}_K$ satisfies the ZR property then the set of polynomials whose degree is less or equal than ∂p is included in $\mathcal{H}_K$.

If the case a. does not hold, consider $N := \max_{r \in \mathbb{N}_0}\{q \in \mathcal{H}_K \,|\, q \text{ polynomial and } \partial q = r\}$ which is finite. Since the ZR property holds, the set of polynomials of degree less or equal than N is included in $\mathcal{H}_K$ $\square$

4 The zero-removing property at a fixed point

In this section we study conditions under which, for a fixed $a \in \mathbb{C}$, the ZR_a property holds in $\mathcal{H}_K$. Reducing the ZR_a property to a moment problem, a sufficient condition assuring that the ZR_a property holds involves the continuity of a shift-type operator.

4.1 A sufficient condition for the ZR_a property

Consider a function $f \in \mathcal{H}_K$, i.e., $f(z) = \langle K(z), x \rangle_{\mathcal{H}}$ on $\mathbb{C}$ for some $x \in \mathcal{H}$, such that $f(a) = 0$. Then $\langle c_0(a), x \rangle = 0$ and

$$\frac{f(z)}{z-a} = \sum_{n=0}^{\infty} \langle c_{n+1}(a), x \rangle (z-a)^n \,, \quad z \in \mathbb{C} \,.$$

As a consequence, the space $\mathcal{H}_K$ satisfies the property ZR_a if and only if for each $x \in \{c_0(a)\}^\perp$ there exists $y \in \mathcal{H}$ such that

$$\langle c_n(a), y\rangle = \langle c_{n+1}(a), x\rangle\,, \quad n \in \mathbb{N}_0\,.$$

For the sake of completeness we include the following result on general moment problems whose proof can be found in [27, p. 126]:

Theorem 3 *Let $\{f_1, f_2, f_3, \dots\}$ be a sequence of vectors belonging to a Hilbert space $\mathcal{H}$ and $\{d_1, d_2, d_3, \dots\}$ a sequence of scalars. In order that the equations*

$$\langle f, f_n\rangle = d_n\,, \quad n \in \mathbb{N}$$

shall admit at least one solution $f \in \mathcal{H}$ for which $\|f\| \le M$, it is necessary and sufficient that

$$\Big|\sum_n a_n \overline{d}_n\Big| \le M \Big\|\sum_n a_n f_n\Big\|$$

for every finite sequence of scalars $\{a_n\}$. If the sequence $\{f_1, f_2, f_3, \dots\}$ is complete in $\mathcal{H}$, then the solution is unique.

As a consequence of the above result we obtain:

Proposition 6 *The space $\mathcal{H}_K$ satisfies the ZR_a property if and only if for each $x \in \{c_0(a)\}^\perp$ the linear functional $\mu_{a,x}$ defined on $Y_a := \mathrm{span}\{c_n(a)\}_{n\in\mathbb{N}_0}$ as*

$$\mu_{a,x}\Big(\sum_n a_n c_n(a)\Big) = \sum_n a_n \langle c_{n+1}(a), x\rangle$$

for every finite sequence of scalars $\{a_n\}$, is bounded.

Assume that the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent. The linear functional $\mu_{a,x} : Y_a \to \mathbb{C}$ can be decomposed as $T_{a,x} \circ S_a$ where $T_{a,x} : Y_a \to \mathbb{C}$ is the linear operator given by

$$T_{a,x}\left(\sum_n a_n c_n(a)\right) = \sum_n a_n \langle c_n(a), x\rangle$$

and $S_a : Y_a \to Y_a$, is the linear operator given by

$$S_a\left(\sum a_n c_n(a)\right) = \sum a_n c_{n+1}(a) \tag{11}$$

for every finite sequence of scalars $\{a_n\}$. Observe that S_a is a well-defined linear operator since the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent. From now on we will assume that the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent (see Proposition 2 above). Note that operator S_a is nothing but a generalization of the classical shift operator defined by means of an orthonormal basis [21]; here it is defined by means of a linear independent Bessel sequence in $\mathcal{H}$.

The operator $T_{a,x}$ is obviously bounded since

$$T_{a,x}\Big(\sum_n a_n c_n(a)\Big) = \sum_n a_n \langle c_n(a), x\rangle = \Big\langle \sum_n a_n c_n(a), x\Big\rangle\,.$$

Thus, we have obtained the following result:

Theorem 4 *Assume that, for each $a \in \mathbb{C}$, the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent and the corresponding operator S_a given by* (11) *is bounded. Then the space $\mathcal{H}_K$ satisfies the ZR property.*

Notice that, for the Paley-Wiener space PW_π, the corresponding operator S_a is bounded for any $a \in \mathbb{C}$. Indeed, for $a = 0$, we have that $\frac{d}{dw} c_{n+1}(0)(w) = ic_n(0)(w)$, $w \in (-\pi, \pi)$, from which we deduce that

$$S_0 f(w) := i \int_0^w f(s)\, ds \qquad \text{for any } f \in L^2[-\pi, \pi]\,.$$

Since $\|S_0 f\|_2 \leq 2\pi \|f\|_2$ for any $f \in L^2[-\pi, \pi]$, S_0 is bounded on $\overline{\operatorname{span}}\{c_n(0)\}_{n\in\mathbb{N}_0} = L^2[-\pi, \pi]$. For a non-zero $a \in \mathbb{C}$ we have that $S_a f(w) = \mathrm{e}^{iaw} S_0 f(w)$ and, as a consequence, the operator S_a is bounded for each $a \in \mathbb{C}$.

The reciprocal of Theorem 4 remains true under the hypothesis that the function $1 \in \mathcal{H}_K$.

Theorem 5 *Assume that the mapping $\mathcal{T}_K$ in* (4) *is injective, the sequence $\{c_n(a)\}_{n=1}^\infty$ is linearly independent for any $a \in \mathbb{C}$, and $1 \in \mathcal{H}_K$. Then, the space $\mathcal{H}_K$ satisfies the ZR property if and only if the operator S_a is bounded for each $a \in \mathbb{C}$.*

Proof. From Theorem 4 it is enough to show that if $1 \in \mathcal{H}_K$ and the ZR_a property holds, then the operator S_a is bounded. Let $\sum_{n=0}^L a_n c_n(a)$ be a vector in Y_a. We have,

$$\Big|\mu_{a,x}\big(\sum_{n=0}^L a_n c_n(a)\big)\Big| = \Big|\big\langle \sum_{n=0}^L a_n c_{n+1}(a), x\big\rangle\Big| = \Big\|S_a\big(\sum_{n=0}^L a_n c_n(a)\big)\Big\| \|x\| \Big|\cos\big(\sum_{n=0}^L a_n c_{n+1}(a), x\big)\Big|$$

First we prove that the function $1 \in \mathcal{H}_K$ if and only if the condition $c_0(a) \notin \overline{\operatorname{span}}\{c_n(a)\}_{n=1}^\infty$ holds for each $a \in \mathbb{C}$. Indeed, $1 = \langle K(z), x\rangle$ for some $x \in \mathcal{H}$ (necessarily $x \neq 0$) implies that, for each $a \in \mathbb{C}$, $\langle c_n(a), x\rangle = 0$, $n \geq 1$. From the completeness of $\{c_n(a)\}_{n\in\mathbb{N}_0}$ (see Prop. 1) we deduce that $c_0(a) \notin \overline{\operatorname{span}}\{c_n(a)\}_{n=1}^\infty$. For the sufficient condition, let $b \in \mathbb{C}$ such that $c_0(b) \notin \overline{\operatorname{span}}\{c_n(b)\}_{n=1}^\infty$; there exists $x \neq 0$ in $\big(\{c_n(b)\}_{n=1}^\infty\big)^\perp$ and, as a consequence, the non-zero constant $\langle K(z), x\rangle$ belongs to $\mathcal{H}_K$. Note that the condition $c_0(b) \notin \overline{\operatorname{span}}\{c_n(b)\}_{n=1}^\infty$ for some $b \in \mathbb{C}$ is equivalent to the condition $c_0(a) \notin \overline{\operatorname{span}}\{c_n(a)\}_{n=1}^\infty$ for every $a \in \mathbb{C}$. Therefore, the hypothesis $1 \in \mathcal{H}_K$ implies the existence of a positive number α such that

$$|\cos(\widehat{v, x})| = \frac{|\langle v, x\rangle|}{\|v\|\,\|x\|} > \alpha > 0 \text{ for any nonzero } v \in \operatorname{span}\{c_n(a)\}_{n=1}^\infty \text{ and any } x \in c_0(a)^\perp \setminus \{0\}\,.$$

Hence,

$$\Big\|S_a\big(\sum_{n=0}^L a_n c_n(a)\big)\Big\| \leq \frac{1}{\alpha\|x\|} \Big|\mu_{a,x}\big(\sum_{n=0}^L a_n c_n(a)\big)\Big|\,. \tag{12}$$

Since the ZR_a property holds, the linear functional $\mu_{a,x}$ is bounded in Y_a. Thus, the inequality (12) implies the boundedness of the operator S_a on Y_a. □

In the context of de Branges spaces, Baranov [1], improving a previous work in [13, 14], solved the problem of finding the structure functions E of zero exponential type for which $1 \in \mathcal{H}(E)$. Since de Branges spaces are particular cases of $\mathcal{H}_K$ spaces [10], the condition $1 \in \mathcal{H}(E)$ could be replaced by the equivalent geometric condition $c_0(a) \notin \overline{\operatorname{span}}\{c_n(a)\}_{n=1}^\infty$ for some $a \in \mathbb{C}$.

4.2 A sufficient condition for the global ZR property in an $\mathcal{H}_K$ space

In this section we give a sufficient condition on the continuity of the operator, say S_0, under the assumption of the minimality of the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ in $\mathcal{H}$, i.e., any polynomial belongs to $\mathcal{H}_K$ (see Proposition 3). Following Ref. [12, p. 27], the minimality of the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ in $\mathcal{H}$ implies that the numbers δ_k given by

$$\delta_k := \inf_{\theta\in\mathbb{R}} \rho\big(\mathrm{e}^{i\theta}\frac{c_k(0)}{\|c_k(0)\|}, \overline{\mathrm{span}}\{c_n(0)\}_{n\neq k}\big)\,, \quad k \in \mathbb{N}_0\,, \tag{13}$$

are strictly positive for every $k \in \mathbb{N}_0$. Note that the number δ_k denotes the inclination in $\mathcal{H}$ of the straight line spanned by $c_k(0)$ to the closed subspace $\overline{\mathrm{span}}\{c_n(0)\}_{n\neq k}$, being ρ the distance with respect to the metric given by the norm in $\mathcal{H}$.
Besides (see [12, pp. 27-28]), for any $x = \sum_k a_k c_k(0)$ (finite or convergent sum) the inequalities

$$|a_k| \leq \frac{\|x\|}{\delta_k\|c_k(0)\|} \quad \text{hold for each } k \in \mathbb{N}_0\,. \tag{14}$$

Lemma 6 *Assume that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at 0 is complete and minimal in $\mathcal{H}$. The convergence of the series*

$$\sum_{n=0}^{\infty} \frac{1}{\delta_n}\frac{\|c_{n+1}(0)\|}{\|c_n(0)\|}\,, \tag{15}$$

where the numbers $\delta_n > 0$, $n \in \mathbb{N}_0$, are given in (13)*, implies that the operator S_0 is bounded.*

Proof. For any finite sum $x = \sum_n a_n c_n(0)$, using inequalities (14), we have

$$\|S_0\, x\| \leq \sum_n |a_n|\|c_{n+1}(0)\| \leq \left(\sum_n \frac{1}{\delta_n}\frac{\|c_{n+1}(0)\|}{\|c_n(0)\|}\right)\|x\| \leq M\|x\|\,,$$

where M denotes the sum of the series in (15). Therefore, the operator S_0 is bounded on $\mathrm{span}\{c_n(0)\}_{n\in\mathbb{N}_0}$; the completeness of $\{c_n(0)\}_{n\in\mathbb{N}_0}$ proves that S_0 is bounded on $\mathcal{H}$. □

In fact, the following result holds:

Theorem 7 *Assume that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at 0 is complete and minimal in $\mathcal{H}$. Suppose also that the series in* (15) *converges and the sequence of quotients $\{\|c_{n+1}(0)\|/\|c_n(0)\|\}_{n\in\mathbb{N}_0}$ is monotonically decreasing. Then, the ZR property in $\mathcal{H}_K$ holds.*

Proof. By Lemma 6 the ZR_0 property holds. Let a be a nonzero complex number and let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in \mathcal{H}_K$ be such that $f(a) = 0$. Then,

$$g(z) = \frac{f(z)}{z-a} = -\frac{1}{a}\sum_{n=0}^{\infty} c_n z^n \quad \text{where} \quad c_n = \frac{1}{a^n}\sum_{k=0}^{n} \alpha_k a^k\,.$$

Since $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k = 0$, we have that $\sum_{k=0}^{n} \alpha_k a^k = -\sum_{k=n+1}^{\infty} \alpha_k a^k$. Hence,

$$g(z) = \sum_{n=0}^{\infty}\left(\frac{1}{a^{n+1}}\sum_{k=n+1}^{\infty} \alpha_k a^k\right) z^n$$

The entire function g belongs to $\mathcal{H}_K$ if and only if the linear functional defined on $Y_0 = \operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0}$ as

$$\nu_f\Big(\sum_n b_n c_n(0)\Big) = \sum_n b_n \Big(\frac{1}{a^{n+1}} \sum_{k=n+1}^{\infty} \alpha_k a^k\Big)\,,$$

for every finite sequence of scalars $\{b_n\}$, is bounded. For any $y = \sum_n b_n c_n(0)$, using inequalities (14) we have

$$|\nu_f(y)| \le \|y\|\|x\| \sum_n \frac{\|c_{n+1}(0)\|}{\delta_n \|c_n(0)\|} \sum_{m=0}^{\infty} \frac{\|c_{m+n+1}(0)\|}{\|c_{n+1}(0)\|} |a|^m\,,$$

where $f = \mathcal{T}_K x$. Applying the ratio test it is straightforward to prove that the series $\sum_{m=0}^{\infty} \frac{\|c_{m+n+1}(0)\|}{\|c_{n+1}(0)\|} z^m$ defines an entire function G_n fon any $n \in \mathbb{N}_0$. Moreover, since the sequence $\{\|c_{l+1}(0)\|/\|c_l(0)\|\}_{l\in\mathbb{N}_0}$ is monotonically decreseasing, we have that $G_n(|a|) \ge G_{n+1}(|a|)$ for any $a \in \mathbb{C}$. As a consequence, for any $y \in Y_0 = \operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0}$ we obtain that

$$|\nu_f(y)| \le \Big(\|x\| G_0(|a|) \sum_{n=0}^{\infty} \frac{\|c_{n+1}(0)\|}{\delta_n \|c_n(0)\|}\Big) \|y\| = M_{f,a} \|y\|\,,$$

i.e., the boundedness of ν_f. □

4.3 On the local zero-removing property

In this section we will assume that S_0 is well-defined bounded operator on $\operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0}$. As a consequence, the ZR_0 property holds in $\mathcal{H}_K$. This means that for each function $f \in \mathcal{H}_K$ with $f(0) = 0$, the function $f(z)/z$ belongs to $\mathcal{H}_K$. Our goal here is to prove that the ZR_a property also holds for $a \in \mathbb{C}$ with $|a|$ small enough.

We will also assume that the operator $\mathcal{T}_K : \mathcal{H} \longrightarrow \mathcal{H}_K$ given in (4) is injective. Therefore, for any $a \in \mathbb{C}$ the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is complete in $\mathcal{H}$. The completeness of the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ implies that S_0 can be extended to $\mathcal{H}$ as a bounded operator. Let S_0^* be its adjoint bounded operator, i.e., for each x, y in $\mathcal{H}$ we have $\langle x, S_0 y\rangle = \langle S_0^* x, y\rangle$.

By using the bijective anti-linear isometry $\mathcal{T}_K$, we define two bounded operators on $\mathcal{H}_K$ as $\mathfrak{S}_0 = \mathcal{T}_K\, S_0\, \mathcal{T}_K^{-1}$ and $\mathfrak{S}_0^* = \mathcal{T}_K\, S_0^*\, \mathcal{T}_K^{-1}$.

For each $x \in \mathcal{H}$, having in mind that $K(z) = \sum_{n=0}^{\infty} c_n(0) z^n$, we have

$$\langle K(z), S_0^* x\rangle = \langle S_0 K(z), x\rangle = \sum_{n=0}^{\infty} \langle c_{n+1}(0), x\rangle z^n = \frac{f(z) - f(0)}{z}\,,$$

being $f(z) = \langle K(z), x\rangle = \sum_{n=0}^{\infty} \langle c_n(0), x\rangle z^n$. Since $\mathcal{T}_K\, S_0^* x = \mathfrak{S}_0^*\, \mathcal{T}_K x = \mathfrak{S}_0^*(f)$, we deduce that

$$\mathfrak{S}_0^* f(z) = \frac{f(z) - f(0)}{z}\,, \quad z \in \mathbb{C}\,.$$

In general, assuming that S_a is bounded, the ZR_a property holds and the bounded operator $\mathfrak{S}_a^* := \mathcal{T}_K\, S_a^*\, \mathcal{T}_K^{-1}$ from $\mathcal{H}_K \to \mathcal{H}_K$ satisfies that

$$\mathfrak{S}_a^* f(z) = \frac{f(z) - f(a)}{z - a}\,, \quad z \in \mathbb{C}\,. \tag{16}$$

Notice that in the de Branges spaces theory, a natural question is whether the space is closed under forming difference quotients as in (16), which means that the function 1 is an associated function (see, for instance, [4, 26]).

For each $a \in \mathbb{C}$ we denote by $\mathcal{H}_a$ the set

$$\mathcal{H}_a := \big\{ f \in \mathcal{H}_K \text{ such that } f(a) = 0 \big\} .$$

It is straighforward to prove that $\mathcal{H}_a$ is a closed subspace of $\mathcal{H}_K$ for each $a \in \mathbb{C}$. Indeed, Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_a$ be a sequence converging to g in the $\mathcal{H}_K$ norm. Since $\mathcal{H}_K$ is a RKHS, for each $z \in \mathbb{C}$ we have that $f_n(z) \to g(z)$; in particular $g(a) = \lim_{n\to\infty} f_n(a) = 0$.

The following lemma relates de ZR_a property with the subspace $\mathcal{H}_a$ via the restriction of the operator $\mathfrak{S}_0^*$ to the subspace $\mathcal{H}_0$:

Lemma 8 *Assume that the operator S_0 is bounded. Let $\widetilde{\mathfrak{S}_0^*}$ be the restriction of the operator $\mathfrak{S}_0^*$ to the closed subspace $\mathcal{H}_0$ of $\mathcal{H}_K$. Given $a \in \mathbb{C}$, the ZR_a property holds in $\mathcal{H}_K$ if and only if the range of the operator $I - a\widetilde{\mathfrak{S}_0^*}$ is $\mathcal{H}_a$.*

Proof. Assume that the operator S_0 is bounded and, therefore, the ZR_0 property holds. The range of the operator $I_0 - a\widetilde{\mathfrak{S}_0^*}$ is a subspace included in $\mathcal{H}_a$, where $I_0 := I_{|\mathcal{H}_0}$. If the ZR_a property holds in $\mathcal{H}_K$ then any entire function g in $\mathcal{H}_a$ can be written as $g(z) = zh(z) - ah(z)$ where $h \in \mathcal{H}_K$. The entire function $zh(z)$ belongs to $\mathcal{H}_0$ and $g = (I - a\widetilde{\mathfrak{S}_0^*})(zh)$, i.e., g belongs to the range of $I - a\widetilde{\mathfrak{S}_0^*}$.

Now, suppose that the range of $I - a\widetilde{\mathfrak{S}_0^*}$ is $\mathcal{H}_a$. For any $g \in \mathcal{H}_a$ there exists $f \in \mathcal{H}_0$ such that $g(z) = f(z) - af(z)/z = (z-a)f(z)/z$. Hence, since the ZR_0 property holds, the entire function $g(z)/(z-a) = f(z)/z$ belongs to $\mathcal{H}_K$. □

In the sequel we follow the Fredholm operator theory as it appears in [7].

Theorem 9 *Assume that the operator S_0 is bounded. Then, there exists $\delta > 0$ such that the ZR_a property holds in $\mathcal{H}_K$ for $|a| < \delta$.*

Proof. The identity operator I restricted to $\mathcal{H}_0$, i.e., I_0, is a Fredholm operator. Indeed, I_0 is bounded; its range is $R(I_0) = \mathcal{H}_0$, hence, closed; the kernel of I_0, $N(I_0) = \{0\}$ is finite dimensional and the codimension of the range is finite and equal to 1 (recall that $\mathcal{H}_0^\perp$ is the subspace of $\mathcal{H}_K$ generated by $f_0(z) = \langle K(z), c_n(0)\rangle$, $z \in \mathbb{C}$). The index of I_0 is $\dim N(I_0) - \operatorname{codim} R(I_0) = -1$.

For any $a \in \mathbb{C}$ the operator $I_0 - a\widetilde{\mathfrak{S}_0^*}$ is injective. Indeed, let $f \in \mathcal{H}_0$ such that $(I_0 - a\widetilde{\mathfrak{S}_0^*})f = 0$ or, equivalently, such that $\frac{z-a}{z} f(z) = 0$, for any $z \in \mathbb{C}$. This implies that f is the zero function since f is an entire function. Following see [7, p. 34], there exists $\delta > 0$ such that if $|a| < \delta$ the operator $I_0 - a\widetilde{\mathfrak{S}_0^*}$ is Fredholm and its index verifies $\operatorname{ind}(I_0 - a\widetilde{\mathfrak{S}_0^*}) = \operatorname{ind} I_0 = -1$. Since $I_0 - a\widetilde{\mathfrak{S}_0^*}$ is an injective Fredholm operator we have that $R(I_0 - a\widetilde{\mathfrak{S}_0^*}) = \mathcal{H}_a$. Hence, by Lemma 8 the ZR_a property holds in $\mathcal{H}_K$ □

An estimation of the constant δ is given in next proposition:

Proposition 7 *Assume that the operator S_0 is bounded. Then, the ZR_a holds for each $a \in \mathbb{C}$ such that $|a| < \|\widetilde{\mathfrak{S}_0^*}\|^{-1}$.*

Proof. The *numerical range* of the operator $\widetilde{\mathfrak{S}_0^*}$ is defined by:

$$\Theta(\widetilde{\mathfrak{S}_0^*}) = \left\{\langle \widetilde{\mathfrak{S}_0^*} f, f\rangle \mid f \in \mathcal{H}_0 \text{ and } \|f\| = 1\right\}.$$

Since $\widetilde{\mathfrak{S}_0^*}$ is bounded we have that $\Theta(\widetilde{\mathfrak{S}_0^*})$ is bounded in $\mathbb{C}$; indeed, $|\langle \widetilde{\mathfrak{S}_0^*} F, F\rangle| \le \|\widetilde{\mathfrak{S}_0^*}\|$.

It is known that if $|\lambda| > \|\widetilde{\mathfrak{S}_0^*}\|$ then $\lambda I_0 - \widetilde{\mathfrak{S}_0^*}$ is an injective semi-Fredholm operator, whose range, $R(\lambda I_0 - \widetilde{\mathfrak{S}_0^*})$, is closed and $\operatorname{codim} R(\lambda I_0 - \widetilde{\mathfrak{S}_0^*})$ is constant in the set $\{\mu \in \mathbb{C}$ such that $|\mu| > \|\widetilde{\mathfrak{S}_0^*}\|\}$ (see [7, p. 100]).

Let $\lambda = a^{-1}$, taking into account that $a^{-1} I_0 - \widetilde{\mathfrak{S}_0^*} : \mathcal{H}_0 \to \mathcal{H}_a$ we obtain that if $|a| < \|\widetilde{\mathfrak{S}_0^*}\|^{-1}$ then $R(a^{-1} I_0 - \widetilde{\mathfrak{S}_0^*})$ is a closed subspace in $\mathcal{H}_a$, therefore, $\operatorname{codim} R(a^{-1} I_0 - \widetilde{\mathfrak{S}_0^*}) = C$ with $C \ge 1$ for each $a \neq 0$ satisfying $|a| < \|\widetilde{\mathfrak{S}_0^*}\|^{-1}$. From Theorem 9 we have that if $b \neq 0$ is close to 0 then $\operatorname{codim} R(I_0 - b\widetilde{\mathfrak{S}_0^*}) = \operatorname{codim} R(b^{-1} I_0 - \widetilde{\mathfrak{S}_0^*}) = 1$. Hence, $C = 1$ and the ZR_a property holds in $\mathcal{H}_K$ whenever $|a| < \|\widetilde{\mathfrak{S}_0^*}\|^{-1}$. □

Corollary 10 *Assume that the mapping $\mathcal{T}_K$ in (4) is injective, the sequence $\{c_n(a)\}_{n=1}^{\infty}$ is linearly independent for any $a \in \mathbb{C}$ and $1 \in \mathcal{H}_K$. Then the set*

$$\left\{b \in \mathbb{C} \mid \text{property } ZR_b \text{ holds}\right\}$$

is an open set in $\mathbb{C}$

Proof. It is a straightforward consequence of Theorems 5 and 9. □

Remark As far as Theorem 9 is concerned, one can construct kernels K such that the ZR_a property at a fixed point $a \in \mathbb{C}$ implies that the zero-removing property holds in $\mathcal{H}_K$ for every $b \in \mathbb{C}$. It remains the open question whether this is true for every space $\mathcal{H}_K$.

5 The differentiation operator in $\mathcal{H}_K$

In general, the differentiation operator $\mathcal{D} : \mathcal{H}_K \longrightarrow \mathcal{H}_K$ given by $\mathcal{D}(f) = f'$, $f \in \mathcal{H}_K$, is not well-defined as the following example shows. In example (e) in Section 2 with $\gamma := \{\sqrt{n!}\}_{n\in\mathbb{N}_0}$, an entire function $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ belongs to $\mathcal{H}_{K_\gamma}$ if and only if the sequence $\{\sqrt{n!}\,\alpha_n\}_{n=0}^{\infty}$ belongs to $\ell^2(\mathbb{N}_0)$. In particular, for the sequence $\alpha_n = 1/(n\sqrt{n!})$, $n \in \mathbb{N}_0$, the corresponding function f belongs to $\mathcal{H}_{K_\gamma}$; however its derivative f' does not belong to $\mathcal{H}_{K_\gamma}$. A sufficient condition is given in the next result:

Theorem 11 *Suppose that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at 0 is complete and minimal in $\mathcal{H}$. Consider the numbers $\delta_n > 0$, $n \in \mathbb{N}_0$, given in (13). If the series*

$$\sum_{n=0}^{\infty} \frac{(n+1)}{\delta_n} \frac{\|c_{n+1}(0)\|}{\|c_n(0)\|} \tag{17}$$

converges, then the differentiation operator $\mathcal{D}$ is a well-defined bounded operator on $\mathcal{H}_K$.

Proof. Let f be in $\mathcal{H}_K$; there exists $x \in \mathcal{H}$ such that $f(z) = \langle K(z), x\rangle$, for any $z \in \mathbb{C}$, and $f(z) = \sum_{n=0}^{\infty} \langle c_n(0), x\rangle z^n$. Therefore,

$$f'(z) = \sum_{n=1}^{\infty} \langle c_n(0), x\rangle n z^{n-1}, \quad z \in \mathbb{C}.$$

The derivative f' of the entire function f belongs to $\mathcal{H}_K$ if and only if there exists $y \in \mathcal{H}$ such that

$$\langle c_n(0), y\rangle = (n+1)\langle c_{n+1}(0), x\rangle \quad \text{for any } n \in \mathbb{N}_0 \,. \tag{18}$$

Proceeding as in the proof of Theorem 4, the set of equations (18) has a solution $y \in \mathcal{H}$ if and only if the operator

$$\begin{array}{rccc} \mathfrak{D}: & \operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0} & \longrightarrow & \operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0} \\ & c_n(0) & \longmapsto & (n+1)c_{n+1}(0)\,, \end{array}$$

is bounded. Let $u = \sum_n a_n\, c_n(0)$ be a finite sum in $\mathcal{H}$ with $\|u\|_{\mathcal{H}} = 1$. By using inequalities in (14), we obtain

$$\|\mathfrak{D}u\| = \Big\| \sum_n a_n (n+1) c_{n+1}(0) \Big\| \le \sum_n (n+1)|a_n| \|c_{n+1}(0)\| \le \sum_n \frac{(n+1)}{\delta_n} \frac{\|c_{n+1}(0)\|}{\|c_n(0)\|}\,.$$

Hence, the convergence of the series in (17) implies the continuity of the operator $\mathfrak{D}$. Moreover, the boundedness of the operator $\mathfrak{D}$ implies the boundedness of the differentiation operator $\mathcal{D}$. Indeed, if $\mathfrak{D}$ is bounded on $\operatorname{span}\{c_n(0)\}_{n=0}^{\infty}$ then it can be extended by continuity to the whole space $\mathcal{H}$. In this case, the adjoint operator of $\mathfrak{D}$, $\mathfrak{D}^* : \mathcal{H} \to \mathcal{H}$ is bounded and it is straighforward to prove that $\mathcal{D} = \mathcal{T}_K \mathfrak{D}^* \mathcal{T}_K^{-1}$ where $\mathcal{T}_K : \mathcal{H} \to \mathcal{H}_K$ is the anti-linear isometry defined in (4). □

Moreover, whenever the differentiation operator $\mathcal{D}$ is a well-defined bounded operator on $\mathcal{H}_K$, the translation operator given by $T_a f(z) := f(z-a)$, $z \in \mathbb{C}$, is also a well-defined bounded operator $T_a : \mathcal{H}_K \longrightarrow \mathcal{H}_K$ for each $a \in \mathbb{C}$. Indeed, adapting a result from [5] we obtain:

Proposition 8 *Suppose that the differentiation operator $\mathcal{D}$ defined as $\mathcal{D}(f) = f'$ is a well-defined bounded operator $\mathcal{D} : \mathcal{H}_K \to \mathcal{H}_K$. Then, for each $a \in \mathbb{C}$, the translation operator $T_a : \mathcal{H}_K \to \mathcal{H}_K$ is a well-defined, bounded operator. Moreover, we have the following expansion for T_a converging in the operator norm*

$$T_a = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \mathcal{D}^n \,. \tag{19}$$

Proof. It is a well-known result that (19) holds in $\mathcal{E}$, the space of entire functions endowed with the topology of the uniform convergence on compact sets (see, for instance, [5]). Since the differentiation operator $\mathcal{D}$ is bounded on the Hilbert space $\mathcal{H}_K$, the series on the rigth side of (19) converges absolutely, and hence in the operator norm to a bounded operator on $\mathcal{H}_K$. As the convergence in $\mathcal{H}_K$ implies convergence in the space $\mathcal{E}$, this operator must be T_a. □

Note that, under the hypotheses of Theorem 11, in the corresponding $\mathcal{H}_K$ space the ZR property holds. Indeed, by using Lemma (6)) the ZR_0 property holds. For $a \in \mathbb{C} \setminus \{0\}$, let g be an entire function in $\mathcal{H}_K$ such that $g(a) = 0$. The entire function $f = T_{-a} g$ belongs to $\mathcal{H}_K$ and $f(0) = g(a) = 0$. Since the ZR_0 property holds we have

$$h(z) = \frac{f(z)}{z} = \frac{g(z+a)}{z} \in \mathcal{H}_K \,, \tag{20}$$

and hence $g(z)/(z-a) = (T_a h)(z) \in \mathcal{H}_K$.

Closing the paper, it is worth to mention that the convergence of the series in (17) imposes a condition on the rate of decay of the sequence $\{\|c_n(0)\|\}_{n\in\mathbb{N}_0}$ and therefore, on the growth of the functions in $\mathcal{H}_K$. Indeed, let F be the entire function defined by $F(z) = \sum_{n=0}^{\infty} \|c_n(0)\| z^n$. Then, for any $f \in \mathcal{H}_K$, we have

$$|f(z)| \leq \|f\| F(|z|) = \|f\| \sum_{n=0}^{\infty} \|c_n(0)\| |z|^n \quad \text{for all } z \in \mathbb{C}\,. \tag{21}$$

In order to illustrate the relationship between the decaying of the sequence $\{\|c_n(0)\|\}_{n\in\mathbb{N}_0}$, the growth of functions in $\mathcal{H}_K$ and the ZR property, suppose that

$$\lim_{n\to\infty} n^r \frac{\|c_n(0)\|}{\|c_{n-1}(0)\|} = \alpha \qquad \text{for some } r > 2\,. \tag{22}$$

Assume that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is uniformly minimal, i.e., there exists δ such that $\delta_n > \delta > 0$ for any $n \in \mathbb{N}_0$ (see [12, p. 27]). Notice that the existence of the limit in (22) implies the convergence of the series $\sum_{n=0}^{\infty}(n+1)\frac{\|c_{n+1}(0)\|}{\|c_n(0)\|}$ and, the boundedness of the differentiation operator on $\mathcal{H}_K$; as a consequence, the ZR property holds.

Let $\gamma > \alpha$; following [5], condition (22) implies the existence of a positive constant C, depending only on γ, such that

$$\|c_n(0)\| \leq C \left(\frac{\mathrm{e}\,\gamma^{1/r}}{n}\right)^{nr}, \qquad n \in \mathbb{N}_0\,. \tag{23}$$

Now according to [16, p. 7], the entire function g defined by

$$g(z) = \sum_{n=1}^{\infty} \left(\frac{\mathrm{e}Mr^{-1}}{n}\right)^{nr} z^n\,,$$

where $M = r\gamma^{1/r}$, has order r^{-1}. Having in mind (23) we obtain that $F(|z|) \leq Cg(|z|)$ for any $z \in \mathbb{C}$. Hence, inequality (21) implies that any function f in $\mathcal{H}_K$ has order less or equal than $r^{-1} < 1/2$.

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