# Generalized sampling: from shift-invariant to $U$-invariant spaces

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**Abstract**

The aim of this article is to derive a sampling theory in $U$-invariant subspaces of a separable Hilbert space $\mathcal{H}$ where $U$ denotes a unitary operator defined on $\mathcal{H}$. To this end, we use some special dual frames for $L^2(0,1)$, and the fact that any $U$-invariant subspace with stable generator is the image of $L^2(0,1)$ by means of a bounded invertible operator. The used mathematical technique mimics some previous sampling work for shift-invariant subspaces of $L^2(\mathbb{R})$. Thus, sampling frame expansions in $U$-invariant spaces are obtained. In order to generalize convolution systems and deal with the time-jitter error in this new setting we consider a continuous group of unitary operators which includes the operator $U$.

**Keywords**: Stationary sequences; $U$-invariant subspaces; Frames; Dual frames; Time-jitter error; Group of unitary operators; Pseudo-dual frames.
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## 1 By way of motivation

The aim in this paper is to derive a generalized sampling theory for $U$-invariant subspaces of a separable Hilbert space $\mathcal{H}$, where $U : \mathcal{H} \to \mathcal{H}$ denotes a unitary operator. The motivation for our work can be found in the generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$; there $\mathcal{H} := L^2(\mathbb{R})$ and $U$ is the shift operator $T : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$. In that setting, the functions (signals) belong to some

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(principal) shift-invariant subspace $V_\varphi^2 := \overline{\text{span}}_{L^2(\mathbb{R})}\big\{\varphi(u-n),\ n\in\mathbb{Z}\big\}$, where the generator function $\varphi$ belongs to $L^2(\mathbb{R})$ and the sequence $\{\varphi(u-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$. Thus, the shift-invariant space $V_\varphi^2$ can be described as

$$V_\varphi^2 = \Big\{\sum_{n\in\mathbb{Z}} \alpha_n\ \varphi(u-n)\ :\ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\Big\}$$

On the other hand, in many common situations the available data are samples of some filtered versions $f * \mathsf{h}_j$ of the signal $f$ itself, where the average function $\mathsf{h}_j$ reflects the characteristics of the adquisition device.
For $s$ convolution systems (linear time-invariant systems or filters in engineering jargon) $\mathcal{L}_j f := f * \mathsf{h}_j$, $j = 1, 2, \dots, s$, defined on $V_\varphi^2$, and assuming also that the sequence of samples

$$\{(\mathcal{L}_j f)(rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}\,,$$

where $r \in \mathbb{N}$, is available for any $f$ in $V_\varphi^2$, the generalized sampling problem mathematically consists of the stable recovery of any $f \in V_\varphi^2$ from the above sequence of samples. In other words, it deals with the construction of sampling formulas in $V_\varphi^2$ having the form

$$f(u) = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \big(\mathcal{L}_j f\big)(rm)\, S_j(u - rm)\,, \quad u \in \mathbb{R}\,,$$

where the sequence of reconstruction functions $\{S_j(\cdot - rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for the shift-invariant space $V_\varphi^2$.

Sampling in shift-invariant spaces of $L^2(\mathbb{R})$ (or $L^2(\mathbb{R}^d)$), with one or multiple generators, has been profusely treated in the mathematical literature. A few selected references are: [4, 5, 9, 10, 11, 12, 13, 15, 18, 23, 27, 28, 29, 30, 31].

In the present work we provide a generalization of the above problem in the following sense. Let $U$ be a unitary operator in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by $\mathcal{A}_a := \overline{\text{span}}\big\{U^n a,\ n \in \mathbb{Z}\big\}$. In case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \Big\{\sum_{n\in\mathbb{Z}} \alpha_n\, U^n a :\ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\Big\}\,.$$

In order to generalize convolution systems and mainly to obtain some perturbation results in this new setting, we assume that the operator $U$ is included in a continuous *group of unitary operators* $\{U^t\}_{t\in\mathbb{R}}$ in $\mathcal{H}$ as $U := U^1$. Recall that $\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying (see Ref. [2, vol. 2; p. 29]):

(1) $U^t\, U^{t'} = U^{t+t'}$,

(2) $U^0 = I_{\mathcal{H}}$,

(3) $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of $t$ for any $x, y \in \mathcal{H}$.

Note that $(U^t)^{-1} = U^{-t}$, and since $(U^t)^* = (U^t)^{-1}$, we have $(U^t)^* = U^{-t}$.

Thus, for $b \in \mathcal{H}$ we consider the linear operator $\mathcal{H} \ni x \longmapsto \mathcal{L}_b x \in C(\mathbb{R})$ such that $\big(\mathcal{L}_b x\big)(t) := \langle x, U^t b\rangle_{\mathcal{H}}$ for every $t \in \mathbb{R}$. These operators $\mathcal{L}_b$, which will be called $U$-systems, can be seen as a generalization of the convolution systems in $L^2(\mathbb{R})$. Indeed, for the shift operator $U : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$ we have

$$\langle f, U^t b\rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(u)\overline{b(u-t)}du = (f * h)(t)\,, \quad t \in \mathbb{R}\,,$$

where $h(u) := \overline{b(-u)}$.

Given $U$-systems $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, corresponding to $s$ elements $b_j \in \mathcal{H}$, i.e., $\mathcal{L}_j \equiv \mathcal{L}_{b_j}$ for each $j = 1, 2, \ldots, s$, the generalized regular sampling problem in $\mathcal{A}_a$ consists of the stable recovery of any $x \in \mathcal{A}_a$ from the sequence of the samples

$$\big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s} \quad \text{where} \quad r \in \mathbb{N}\,, r \geq 1\,.$$

This $U$-sampling problem has been treated, for the first time, in some recent papers [22, 24]. Sampling in shift-invariant subspaces or in modulation-invariant subspaces of $L^2(\mathbb{R})$ becomes a particular case of $U$-sampling associated with the translation operator $T : f(u) \mapsto f(u-1)$ or with the modulation operator $M : f(u) \mapsto \mathrm{e}^{2\pi \mathrm{i} u} f(u)$ in $L^2(\mathbb{R})$ respectively.

In this paper we propose a completely different approach which allows to analyze in depth the $U$-sampling problem. In Section 3 we prove the existence of frames in $\mathcal{A}_a$, having the form $\big\{U^{rm} c_j\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$, where $c_j \in \mathcal{A}_a$ for $j = 1, 2, \ldots, s$, such that for each $x \in \mathcal{A}_a$ the sampling expansion

$$x = \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, U^{rm} c_j \quad \text{in } \mathcal{H} \tag{1}$$

holds. To this end, as in the shift-invariant case (see, for instance, Refs. [13, 15]), we use that the above sampling formula is intimately related with some special dual frames in $L^2(0,1)$ (see Section 2 below) via the isomorphism $\mathcal{T}_{U,a} : L^2(0,1) \longrightarrow \mathcal{A}_a$ which maps the orthonormal basis $\{e^{2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$. In [24] regular sampling expansions like (1) are obtained by using a completely different technique; basically, they use the cross-covariance function $R_{a,b_j}(n) := \langle U^n a, b_j\rangle_{\mathcal{H}}$ between the sequences $\{U^n a\}_{n\in\mathbb{Z}}$ and $\{U^n b_j\}_{n\in\mathbb{Z}}$, $j = 1, 2, \ldots, s$.

Strictly speaking, we do not need the formalism of the continuous group of unitary operators to derive the sampling results in Section 3 since we only use the discrete group $\{U^n\}_{n\in\mathbb{Z}}$ completely determined by $U$. However, for the study, in Section 4, of the time-jitter error in sampling formulas as in (1), the continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ becomes essential. In this case we dispose of a perturbed sequence of samples $\{(\mathcal{L}_j x)(rm + \epsilon_{mj})\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$, with errors $\epsilon_{mj} \in \mathbb{R}$, for the recovery of $x \in \mathcal{A}_a$. We prove that, for small enough errors $\epsilon_{mj}$, the stable recovery of any $x \in \mathcal{A}_a$ is still possible. Finally, in Section 5 we deal with the case of multiple stable generators. We only sketch the procedure since it is essentially identical to the one-generator case.

## 2 On sampling in $U$-invariant subspaces

For a fixed $a \in \mathcal{H}$, assume that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$. Recall that a *Riesz basis* in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis $\{x_n\}_{n\in\mathbb{Z}}$ has a unique biorthogonal (dual) Riesz basis $\{y_n\}_{n\in\mathbb{Z}}$, i.e., $\langle x_n, y_m\rangle_{\mathcal{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n\in\mathbb{Z}} \langle x, y_n\rangle_{\mathcal{H}}\, x_n = \sum_{n\in\mathbb{Z}} \langle x, x_n\rangle_{\mathcal{H}}\, y_n\,,$$

hold for every $x \in \mathcal{H}$. We state the definition by considering the integers set $\mathbb{Z}$ as the index set since throughout the paper most of sequences are indexed in $\mathbb{Z}$. A *Riesz sequence* in $\mathcal{H}$ is a Riesz basis for its closed span (see, for instance, [8]). Thus, the $U$-invariant subspace $\mathcal{A}_a := \overline{\text{span}}\{U^n a,\ n \in \mathbb{Z}\}$ can be expressed as

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a :\ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}\,.$$

For simplicity and ease of notation we are considering the one-generator setting; as we have already said, the same sampling results for the general case can be obtained by analogy, and it will be drawn in Section 5. The sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is an *stationary sequence* since the inner product $\langle U^n a, U^m a\rangle_{\mathcal{H}}$ depends only on the difference $n - m \in \mathbb{Z}$. Moreover, the *auto-covariance* $R_a$ of the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\mu_a(\theta)\,, \qquad k \in \mathbb{Z}\,,$$

in terms of a positive Borel measure $\mu_a$ on $(-\pi, \pi)$ called the *spectral measure* of the sequence (see [19]). This is obtained from the integral representation of the unitary operator $U$ on $\mathcal{H}$ (see, for instance, [2, 33]). The spectral measure $\mu_a$ can be decomposed into an absolute continuous and a singular part as $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$. A necessary and sufficient condition in order for the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ to be a Riesz sequence for $\mathcal{H}$ is given in next theorem in terms of the decomposition of the spectral measure $\mu_a$:

**Theorem 1.** *Let $\{U^n a\}_{n\in\mathbb{Z}}$ be a sequence obtained from a unitary operator in a separable Hilbert space $\mathcal{H}$ with spectral measure $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$, and let $\mathcal{A}_a$ be the closed subspace spanned by $\{U^n a\}_{n\in\mathbb{Z}}$. Then the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu_a^s \equiv 0$ and*

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \phi_a(\theta) \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \phi_a(\theta) < \infty\,.$$

Theorem 1 is just the one-generator case ($L = 1$) of Theorem 11 proved below. It is worth to mention that an straightforward computation shows that the dual Riesz basis of $\{U^n a\}_{n\in\mathbb{Z}}$ in $\mathcal{A}_a$ is given by $\{U^n b\}_{n\in\mathbb{Z}}$ whith $b = \sum_{k\in\mathbb{Z}} b_k\, U^k a \in \mathcal{A}_a$, where the terms of the sequence $\{b_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ are the Fourier coefficients of the function $1/\phi_a(\theta) \in L^2(-\pi, \pi)$. Indeed, for $b = \sum_{k\in\mathbb{Z}} b_k\, U^k a$ in $\mathcal{A}_a$, the biorthogonality between the sequences $\{U^n a\}_{n\in\mathbb{Z}}$ and $\{U^n b\}_{n\in\mathbb{Z}}$ means

$$\begin{aligned}\delta_{m,0} = \langle U^m a, b\rangle_{\mathcal{H}} &= \langle U^m a, \sum_{k\in\mathbb{Z}} b_k\, U^k a\rangle_{\mathcal{H}} = \sum_{k\in\mathbb{Z}} \bar{b}_k \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i(m-k)\theta}\phi_a(\theta)d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(\sum_{k\in\mathbb{Z}} \bar{b}_k \mathrm{e}^{-ik\theta}\big)\phi_a(\theta)\mathrm{e}^{im\theta}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} B(\theta)\phi_a(\theta)\mathrm{e}^{-im\theta}\, d\theta\,,\end{aligned}$$

where $B(\theta) := \sum_{k\in\mathbb{Z}} b_k \, \mathrm{e}^{ik\theta}$; in other words, we have $B(\theta)\phi_a(\theta) \equiv 1$ in $L^2(-\pi,\pi)$. Moreover, it is easy to deduce that $\phi_b(\theta) = 1/\phi_a(\theta)$, $\theta \in (-\pi,\pi)$; that is, for $k \in \mathbb{Z}$ we obtain $\langle U^k b, b\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} \frac{d\theta}{\phi_a(\theta)}$.

Finally, for the shift operator $T : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$, Theorem 1 allows to recover the classical necessary and sufficient condition for the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$, where $\varphi \in L^2(\mathbb{R})$, to be a Riesz basis for the corresponding shift-invariant subspace $\mathcal{A}_\varphi$ in $L^2(\mathbb{R})$. Indeed, consider the Fourier transform as $\widehat{\varphi}(\theta) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(t)\, \mathrm{e}^{-it\theta}\, d\theta$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$; using the Parseval's equality one easily gets

$$\begin{aligned}\langle T^k\varphi, \varphi\rangle_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} \varphi(u-k)\,\overline{\varphi(u)}\, du = \int_{-\infty}^{\infty} \widehat{\varphi(u-k)}(\theta)\overline{\widehat{\varphi}(\theta)}\, d\theta = \int_{-\infty}^{\infty} |\widehat{\varphi}(\theta)|^2\, \mathrm{e}^{-ik\theta}\, d\theta \\ &= \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2\, \mathrm{e}^{-ik\theta}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-ik\theta}\, 2\pi \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(-\theta+2\pi n)|^2\, d\theta\,,\end{aligned}$$

that is, $\phi_\varphi(\theta) = 2\pi\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(-\theta+2\pi n)|^2$, $\theta \in (-\pi,\pi)$. Thus, Theorem 1 yields the classical condition (see, for instance, [8]):

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2 \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2 < \infty\,.$$

The following isomorphism between $L^2(0,1)$ and $\mathcal{A}_a$ will be crucial along this paper:

### The isomorphism $\mathcal{T}_{U,a}$

We define the isomorphism $\mathcal{T}_{U,a}$ which maps the orthonormal basis $\{\mathrm{e}^{2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$, that is,

$$\begin{array}{llcl}\mathcal{T}_{U,a}: & L^2(0,1) & \longrightarrow & \mathcal{A}_a \\ & F = \sum_{n\in\mathbb{Z}} \alpha_n\, \mathrm{e}^{2\pi\mathrm{i}nw} & \longmapsto & x = \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a\,.\end{array}$$

The following $U$-shift property holds: For any $F \in L^2(0,1)$ and $N \in \mathbb{Z}$, we have

$$\mathcal{T}_{U,a}\Big(F\, \mathrm{e}^{2\pi\mathrm{i}Nw}\Big) = U^N\big(\mathcal{T}_{U,a}F\big)\,. \tag{2}$$

### The $U$-systems

For any fixed $b \in \mathcal{H}$ we define the $U$-system $\mathcal{L}_b$ as the linear operator between $\mathcal{H}$ and the set $C(\mathbb{R})$ of the continuous functions on $\mathbb{R}$ given by

$$\mathcal{H} \ni x \longmapsto \mathcal{L}_b x \in C(\mathbb{R}) \;\text{ such that }\; \mathcal{L}_b x(t) := \langle x, U^t b\rangle_{\mathcal{H}}\,, \quad t \in \mathbb{R}\,.$$

For any $x \in \mathcal{A}_a$ and $t \in \mathbb{R}$, by using the Plancherel equality for the orthonormal basis $\{\mathrm{e}^{2\pi\mathrm{i}nw}\}_{n\in\mathbb{Z}}$ in $L^2(0,1)$, we have

$$\begin{aligned}\mathcal{L}_b x(t) &= \langle x, U^t b\rangle_{\mathcal{H}} = \Big\langle \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a, U^t b\Big\rangle_{\mathcal{H}} = \sum_{n\in\mathbb{Z}} \alpha_n\, \overline{\langle U^t b, U^n a\rangle}_{\mathcal{H}} \\ &= \Big\langle F, \sum_{n\in\mathbb{Z}} \langle U^t b, U^n a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi\mathrm{i}nw}\Big\rangle_{L^2(0,1)} = \big\langle F, K_t\big\rangle_{L^2(0,1)}\,,\end{aligned} \tag{3}$$

where $\mathcal{T}_{U,a}F = x$, and the function

$$K_t(w) := \sum_{n\in\mathbb{Z}} \langle U^t b, U^n a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} n w} = \sum_{n\in\mathbb{Z}} \overline{\mathcal{L}_b a(t-n)}\, \mathrm{e}^{2\pi \mathrm{i} n w}$$

belongs to $L^2(0,1)$ since the sequence $\big\{\langle U^t b, U^n a\rangle_{\mathcal{H}}\big\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ for each $t\in\mathbb{R}$.

### An expression for the generalized samples

Suppose that $s$ vectors $b_j\in\mathcal{H}$, $j=1,2,\dots,s$, are given and consider their associated $U$-systems $\mathcal{L}_j := \mathcal{L}_{b_j}$, $j=1,2,\dots,s$. Our aim is the stable recovery of any $x\in\mathcal{A}_a$ from the sequence of samples $\big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ where $r\geq 1$. To this end, first we obtain a suitable expression for the samples. For $x\in\mathcal{A}_a$ let $F\in L^2(0,1)$ such that $\mathcal{T}_{U,a}F = x$; by using (3), for $j=1,2,\dots,s$ and $m\in\mathbb{Z}$ we have

$$\begin{aligned}\mathcal{L}_j x(rm) &= \Big\langle F, \sum_{n\in\mathbb{Z}} \langle U^{rm} b_j, U^n a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} n w}\Big\rangle_{L^2(0,1)} = \Big\langle F, \sum_{k\in\mathbb{Z}} \langle U^{k} b_j, a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} (rm-k) w}\Big\rangle_{L^2(0,1)}\\ &= \Big\langle F, \big[\sum_{k\in\mathbb{Z}} \overline{\langle a, U^k b_j\rangle_{\mathcal{H}}}\, \mathrm{e}^{-2\pi \mathrm{i} k w}\big]\, \mathrm{e}^{2\pi \mathrm{i} rm w}\Big\rangle_{L^2(0,1)},\end{aligned}$$

where the change in the summation's index $k := rm-n$ has been done. Hence,

$$\mathcal{L}_j x(rm) = \big\langle F, \overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rm w}\big\rangle_{L^2(0,1)} \quad \text{for } m\in\mathbb{Z} \text{ and } j=1,2,\dots,s\,, \tag{4}$$

where the function

$$g_j(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k)\, \mathrm{e}^{2\pi \mathrm{i} k w} \tag{5}$$

belongs to $L^2(0,1)$ for each $j=1,2,\dots,s$.

As a consequence of (4), the stable recovery of any $x\in\mathcal{A}_a$ depends on whether the sequence $\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rm w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ forms a frame for $L^2(0,1)$. Recall that a sequence $\{x_n\}_{n\in\mathbb{Z}}$ is a *frame* for a separable Hilbert space $\mathcal{H}$ if there exist two constants $A,B>0$ (frame bounds) such that

$$A\|x\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle x, x_n\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x\in\mathcal{H}\,.$$

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ satisfying only the right hand inequality above is said to be a *Bessel sequence* for $\mathcal{H}$. Given a frame $\{x_n\}_{n\in\mathbb{Z}}$ for $\mathcal{H}$ the representation property of any vector $x\in\mathcal{H}$ as a series $x=\sum_{n\in\mathbb{Z}} c_n\, x_n$ is retained, but, unlike the case of Riesz bases (*exact frames*), the uniqueness of this representation (for *overcomplete frames*) is sacrificed. Suitable frame coefficients $c_n$ which depend continuously and linearly on $x$ are obtained by using the dual frames $\{y_n\}_{n\in\mathbb{Z}}$ of $\{x_n\}_{n\in\mathbb{Z}}$, i.e., $\{y_n\}_{n\in\mathbb{Z}}$ is another frame for $\mathcal{H}$ such that $x=\sum_{n\in\mathbb{Z}}\langle x, y_n\rangle x_n = \sum_{n\in\mathbb{Z}}\langle x, x_n\rangle y_n$ for each $x\in\mathcal{H}$. For more details on frame theory see Ref. [8].

A deep study of sequences having the form of $\{\overline{g_j(w)}\,e^{2\pi i r m w}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ was done in Refs. [13, 15]. Namely, consider the $s\times r$ matrix of functions in $L^2(0,1)$

$$\mathbb{G}(w) := \begin{bmatrix} g_1(w) & g_1(w+\frac{1}{r}) & \cdots & g_1(w+\frac{r-1}{r}) \\ g_2(w) & g_2(w+\frac{1}{r}) & \cdots & g_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ g_s(w) & g_s(w+\frac{1}{r}) & \cdots & g_s(w+\frac{r-1}{r}) \end{bmatrix} = \Big[g_j\Big(w+\frac{k-1}{r}\Big)\Big]_{\substack{j=1,2,\dots,s\\ k=1,2,\dots,r}} \tag{6}$$

and its related constants

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)], \quad \beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,,$$

where $\mathbb{G}^*(w)$ denotes the transpose conjugate of the matrix $\mathbb{G}(w)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbb{G}^*(w)\mathbb{G}(w)$. Observe that $0\le\alpha_{\mathbb{G}}\le\beta_{\mathbb{G}}\le\infty$. Notice that in the definition of the matrix $\mathbb{G}(w)$ we are considering 1-periodic extensions of the involved functions $g_j$, $j=1,2,\dots,s$.

A complete characterization of the sequence $\{\overline{g_j(w)}\,e^{2\pi i r m w}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is given in the next lemma (see [13, Lemma 3] or [15, Lemma 2] for the proof):

**Lemma 2.** *For the functions $g_j\in L^2(0,1)$, $j=1,2,\dots,s$, consider the associated matrix $\mathbb{G}(w)$ given in* (6)*. Then, the following results hold:*

*(a) The sequence $\{\overline{g_j(w)}\,e^{2\pi i r n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a complete system for $L^2(0,1)$ if and only if the rank of the matrix $\mathbb{G}(w)$ is $r$ a.e. in $(0,1/r)$.*

*(b) The sequence $\{\overline{g_j(w)}\,e^{2\pi i r n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Bessel sequence for $L^2(0,1)$ if and only if $g_j\in L^\infty(0,1)$ (or equivalently $\beta_{\mathbb{G}}<\infty$). In this case, the optimal Bessel bound is $\beta_{\mathbb{G}}/r$.*

*(c) The sequence $\{\overline{g_j(w)}\,e^{2\pi i r n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0<\alpha_{\mathbb{G}}\le\beta_{\mathbb{G}}<\infty$. In this case, the optimal frame bounds are $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$.*

*(d) The sequence $\{\overline{g_j(w)}\,e^{2\pi i r n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Riesz basis for $L^2(0,1)$ if and only if is a frame and $s=r$.*

A comment about Lemma 2 in terms of the average sampling terminology introduced by Aldroubi et al. in [6] is in order. According to [6] we say that

1. The set $\{\mathcal{L}_1,\mathcal{L}_2,\dots,\mathcal{L}_s\}$ is an *$r$-determining $U$-sampler* for $\mathcal{A}_a$ if the only vector $x\in\mathcal{A}_a$, satisfying $\mathcal{L}_j x(rm)=0$ for all $j=1,2,\dots,s$ and $m\in\mathbb{Z}$, is $x=0$.
2. The set $\{\mathcal{L}_1,\mathcal{L}_2,\dots,\mathcal{L}_s\}$ is an *$r$-stable $U$-sampler* for $\mathcal{A}_a$ if there exist positive constants $A$ and $B$ such that

$$A\|x\|^2\le\sum_{j=1}^{s}\sum_{m\in\mathbb{Z}}|\mathcal{L}_j x(rm)|^2\le B\|x\|^2 \quad \text{for all } x\in\mathcal{A}_a.$$

Hence, parts $(a)$ and $(c)$ of Lemma 2 can be read, by using (4), as follows:

i. The set $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an $r$-determining $U$-sampler for $\mathcal{A}_a$ if and only if $\operatorname{rank} \mathbb{G}(w) = r$ a.e. in $(0,1)$ (and hence, necessarily, $s \geq r$).

ii. The set $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an $r$-stable $U$-sampler for $\mathcal{A}_a$ if and only if $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$.

An $r$-determining $U$-sampler for $\mathcal{A}_a$ can distinguish between two distinct elements in $\mathcal{A}_a$, but the recovery, if any, is not necessarily stable. If the system $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an $r$-stable $U$-sampler for $\mathcal{A}_a$, then any $x \in \mathcal{A}_a$ can be recovered, in a stable way, from the sequence of generalized samples $\big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$, where necessarily $s \geq r$. Roughly speaking, the operator which maps

$$\mathcal{A}_a \ni x \longmapsto \big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s} \in \ell^2_s(\mathbb{Z}) := \underset{(s \text{ times})}{\ell^2(\mathbb{Z}) \times \cdots \times \ell^2(\mathbb{Z})}$$

has a bounded inverse.

Having in mind (4), from the sequence of samples $\big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ we recover $F \in L^2(0,1)$, and by means of the isomorphism $\mathcal{T}_{U,a}$, the vector $x = \mathcal{T}_{U,a} F \in \mathcal{A}_a$. This will be the main goal in the next section:

## 3 Generalized regular sampling in $\mathcal{A}_a$

Along with the characterization of the sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ as a frame in $L^2(0,1)$, in [13] a family of dual frames are also given: Choose functions $h_j$ in $L^\infty(0,1)$, $j = 1,2,\dots,s$, such that

$$\big[h_1(w), h_2(w), \dots, h_s(w)\big]\mathbb{G}(w) = [1, 0, \dots, 0] \quad \text{a.e. in } (0,1)\,. \tag{7}$$

It was proven in [13] that the sequence $\{r h_j(w)\, \mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a dual frame of the sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ in $L^2(0,1)$. In other words, taking into account (4), we have for any $F \in L^2(0,1)$ the expansion

$$F = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, r h_j(w)\, \mathrm{e}^{2\pi \mathrm{i} rmw} \quad \text{in } L^2(0,1)\,. \tag{8}$$

Concerning to the existence of the functions $h_j$, $j = 1,2,\dots,s$, consider the first row of the $r \times s$ Moore-Penrose pseudo-inverse $\mathbb{G}^\dagger(w)$ of $\mathbb{G}(w)$ given by

$$\mathbb{G}^\dagger(w) := \big[\mathbb{G}^*(w)\, \mathbb{G}(w)\big]^{-1} \mathbb{G}^*(w)\,.$$

Its entries are essentially bounded in $(0,1)$ since the functions $g_j$, $j = 1,2,\dots,s$, and $\det^{-1}\big[\mathbb{G}^*(w)\, \mathbb{G}(w)\big]$ are essentially bounded in $(0,1)$, and (7) trivially holds. All the possible solutions of (7) are given by the first row of the $r \times s$ matrices given by

$$\mathbb{H}_{\mathbb{U}}(w) := \mathbb{G}^\dagger(w) + \mathbb{U}(w)\big[\mathbb{I}_s - \mathbb{G}(w)\mathbb{G}^\dagger(w)\big]\,, \tag{9}$$

where $\mathbb{U}(w)$ denotes any $r \times s$ matrix with entries in $L^\infty(0,1)$, and $\mathbb{I}_s$ is the identity matrix of order $s$.

Applying the isomorphism $\mathcal{T}_{U,a}$ in (8), for $x=\mathcal{T}_{U,a}F\in\mathcal{A}_a$ we obtain the sampling expansion:

$$
\begin{aligned}
x &= \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\,\mathcal{T}_{U,a}\big[rh_j(\cdot)\,\mathrm{e}^{2\pi i rm\,\cdot}\big] = \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\,U^{rm}\big[\mathcal{T}_{U,a}(rh_j)\big] \\
&= \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\,U^{rm}c_{j,h} \quad \text{in } \mathcal{H}\,,
\end{aligned} \tag{10}
$$

where $c_{j,h}:=\mathcal{T}_{U,a}(rh_j)\in\mathcal{A}_a$, $j=1,2,\dots,s$, and we have used the $U$-shift property (2). Besides, the sequence $\{U^{rm}c_{j,h}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$. In fact, the following result holds:

**Theorem 3.** *Let $b_j\in\mathcal{H}$ and let $\mathcal{L}_j$ be its associated $U$-system for $j=1,2,\dots,s$. Assume that the function $g_j$, $j=1,2,\dots,s$, given in (5) belongs to $L^\infty(0,1)$; or equivalently, that $\beta_{\mathbb{G}}<\infty$ for the associated $s\times r$ matrix $\mathbb{G}(w)$. The following statements are equivalent:*

*(a)* $\alpha_{\mathbb{G}}>0$.

*(b) There exists a vector $\big[h_1(w),h_2(w),\dots,h_s(w)\big]$ with entries in $L^\infty(0,1)$ satisfying*

$$\big[h_1(w),h_2(w),\dots,h_s(w)\big]\mathbb{G}(w)=[1,0,\dots,0] \quad \textit{a.e. in } (0,1)\,.$$

*(c) There exist $c_j\in\mathcal{A}_a$, $j=1,2,\dots,s$, such that the sequence $\{U^{rk}c_j\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$, and for any $x\in\mathcal{A}_a$ the expansion*

$$x=\sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\,U^{rk}c_j \quad \textit{in } \mathcal{H}\,, \tag{11}$$

*holds.*

*(d) There exists a frame $\{C_{j,k}\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ for $\mathcal{A}_a$ such that, for each $x\in\mathcal{A}_a$ the expansion*

$$x=\sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\,C_{j,k} \quad \textit{in } \mathcal{H}\,,$$

*holds.*

*Proof.* We have already proved that $(a)$ implies $(b)$ and that $(b)$ implies $(c)$. Obviously, $(c)$ implies $(d)$. As a consequence, we only need to prove that $(d)$ implies $(a)$. Applying the isomorphism $\mathcal{T}_{U,a}^{-1}$ to the expansion in $(d)$, and taking into account (4) we obtain

$$
\begin{aligned}
F=\mathcal{T}_{U,a}^{-1}x &= \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\,\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big) \\
&= \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \big\langle F,\overline{g_j(w)}\,\mathrm{e}^{2\pi i rkw}\big\rangle_{L^2(0,1)}\,\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big) \quad \text{in } L^2(0,1)\,,
\end{aligned}
$$

where the sequence $\{\mathcal{T}_{U,a}^{-1}(C_{j,k})\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ is a frame for $L^2(0,1)$. The sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi i r m w}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ is a Bessel sequence in $L^2(0,1)$ since $\beta_{\mathbb{G}}<\infty$, and satisfying the above expansion in $L^2(0,1)$. According to [8, Lemma 5.6.2] the sequences $\{\mathcal{T}_{U,a}^{-1}(C_{j,k})\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ and $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi i r k w}\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ form a pair of dual frames in $L^2(0,1)$; in particular, by using Lemma 2 we obtain that $\alpha_{\mathbb{G}}>0$ which concludes the proof. □

In case the functions $g_j$, $j=1,2,\dots,s$ are continuous on $\mathbb{R}$, condition $(a)$ in Theorem 3 can be expressed in terms of the rank of the matrix $\mathbb{G}(w)$; notice that this occurs, for example, whenever the sequences $\{\mathcal{L}_j a(k)\}_{k\in\mathbb{Z}}$, $j=1,2,\dots,s$, belong to $\ell^1(\mathbb{Z})$.

**Corollary 4.** *Assume that the* 1*-periodic extension of the functions $g_j$, $j=1,2,\dots,s$, given in* (5) *are continuous on $\mathbb{R}$. Then, the following conditions are equivalent:*

*(i)* $\operatorname{rank}\mathbb{G}(w)=r$ *for all $w\in\mathbb{R}$.*

*(ii) There exist $c_j\in\mathcal{A}_a$, $j=1,2,\dots,s$, such that the sequence $\{U^{rk}c_j\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$, and the sampling formula* (11) *holds for each $x\in\mathcal{A}_a$.*

*Proof.* Whenever the functions $g_j$, $j=1,2,\dots,s$, are continuous on $\mathbb{R}$, the condition $\alpha_{\mathbb{G}}>0$ is equivalent to $\det\big[\mathbb{G}^*(w)\mathbb{G}(w)\big]\neq 0$ for all $w\in\mathbb{R}$. Indeed, if $\det\mathbb{G}^*(w)\mathbb{G}(w)>0$ then the first row of the matrix $\mathbb{G}^\dagger(w):=[\mathbb{G}^*(w)\mathbb{G}(w)]^{-1}\mathbb{G}^*(w)$, gives a vector $[h_1,h_2,\dots,h_s]$ satisfying the statement (b) in Theorem 3 and, as a consequence, $\alpha_{\mathbb{G}}>0$. The converse follows from the fact that $\det\big[\mathbb{G}^*(w)\mathbb{G}(w)\big]\geq\alpha_{\mathbb{G}}^r$ for all $w\in\mathbb{R}$. Since, $\det\big[\mathbb{G}^*(w)\mathbb{G}(w)\big]\neq 0$ is equivalent to rank $\mathbb{G}(w)=r$ for all $w\in\mathbb{R}$, the result is a consequence of Theorem 3 □

Whenever the sampling period $r$ equals the number of $U$-systems $s$ we are in the presence of Riesz bases, and there exists a unique sampling expansion in Theorem 3:

**Corollary 5.** *Let $b_j\in\mathcal{H}$ for $j=1,2,\dots,r$, i.e., $r=s$ in Theorem* 3*. Let $\mathcal{L}_j$ be its associated $U$-system for $j=1,2,\dots,r$. Assume that the function $g_j$, $j=1,2,\dots,r$, given in* (5) *belongs to $L^\infty(0,1)$; or equivalently, $\beta_{\mathbb{G}}<\infty$ for the associated $r\times r$ matrix $\mathbb{G}(w)$. The following statements are equivalent:*

*(a)* $\alpha_{\mathbb{G}}>0$.

*(b) There exists a Riesz basis $\{C_{j,k}\}_{k\in\mathbb{Z};\,j=1,2,\dots,r}$ such that for any $x\in\mathcal{A}_a$ the expansion*

$$x=\sum_{j=1}^{r}\sum_{k\in\mathbb{Z}}\mathcal{L}_j x(rk)\,C_{j,k}\quad \text{in } \mathcal{H} \tag{12}$$

*holds.*

*In case the equivalent conditions are satisfied, necessarily there exist $c_j\in\mathcal{A}_a$, $j=1,2,\dots,r$, such that $C_{j,k}=U^{rk}c_j$ for $k\in\mathbb{Z}$ and $j=1,2,\dots,r$. Moreover, the interpolation property $\mathcal{L}_{j'}c_j(rk)=\delta_{j,j'}\,\delta_{k,0}$, where $k\in\mathbb{Z}$ and $j,j'=1,2,\dots,r$, holds.*

*Proof.* Assume that $\alpha_{\mathbb{G}} > 0$; since $\mathbb{G}(w)$ is a square matrix, this implies that

$$\operatorname*{ess\,inf}_{w\in\mathbb{R}} |\det \mathbb{G}(w)| > 0\,.$$

Therefore, the first row of $\mathbb{G}^{-1}(w)$ gives the unique solution $[h_1(w), h_2(w), \ldots, h_r(w)]$ of (7) with $h_j \in L^\infty(0,1)$ for $j = 1, 2, \ldots, r$.
According to Theorem 3, the sequence $\{C_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r} := \{U^{rk}c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$, where $c_j = \mathcal{T}_{U,a}(rh_j)$, satisfies the sampling formula (12). Moreover, the sequence $\{rh_j(w)\,\mathrm{e}^{2\pi\mathrm{i}rkw}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r} = \{\mathcal{T}_{U,a}^{-1}\big(U^{rk}c_j\big)\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is a frame for $L^2(0,1)$. Since $r = s$, according to Lemma 2, it is a Riesz basis. Hence, $\{U^{rk}c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is a Riesz basis for $\mathcal{A}_a$ and $(b)$ is proved.

Conversely, assume now that $\{C_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is a Riesz basis for $\mathcal{A}_a$ satisfying (12). From the uniqueness of the coefficients in a Riesz basis, we get that the interpolatory condition $(\mathcal{L}_{j'}C_{j,k})(rk') = \delta_{j,j'}\delta_{k,k'}$ holds for $j, j' = 1, 2, \ldots, r$ and $k, k' \in \mathbb{Z}$. Since $\mathcal{T}_{U,a}^{-1}$ is an isomorphism, the sequence $\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is a Riesz basis for $L^2(0,1)$. Expanding the function $\overline{g_{j'}(w)}\,\mathrm{e}^{-2\pi\mathrm{i}rk'w}$ with respect to the dual basis of $\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$, denoted by $\{D_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$, and having in mind (4) we obtain

$$\begin{aligned}\overline{g_{j'}(w)}\,\mathrm{e}^{2\pi\mathrm{i}rk'w} &= \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} \big\langle \overline{g_{j'}(\cdot)}\,\mathrm{e}^{2\pi\mathrm{i}rk'\cdot}, \mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\big\rangle_{L^2(0,1)} D_{j,k}(w) \\ &= \sum_{k\in\mathbb{Z}} \overline{\mathcal{L}_{j'}\,C_{j,k}(rk')}\, D_{j,k}(w) = D_{j',k'}(w)\,.\end{aligned}$$

Therefore, the sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi\mathrm{i}rkw}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is the dual basis of the Riesz basis $\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$. In particular, it is a Riesz basis for $L^2(0,1)$, which implies, according to Lemma 2, that $\alpha_{\mathbb{G}} > 0$, i.e., condition $(a)$. Moreover, the sequence $\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is necessarily the unique dual basis of the Riesz basis $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi\mathrm{i}rkw}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$. Therefore, this proves the uniqueness of the Riesz basis $\{C_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ for $\mathcal{A}_a$ satisfying (12). □

## Some comments on the sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$

Concerning Theorem 3, more can be said about the sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$, where the vectors $b_j \in \mathcal{H}$ define the $U$-systems $\mathcal{L}_j \equiv \mathcal{L}_{b_j}$, $j = 1, 2, \ldots, s$. Having in mind (4) and the isomorphism $\mathcal{T}_{U,a}$, we obtain that

$$\frac{\alpha_{\mathbb{G}}}{r}\|\mathcal{T}_{U,a}\|^{-2}\|x\|^2 \le \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} |\langle x, U^{rk}b_j\rangle|^2 \le \frac{\beta_{\mathbb{G}}}{r}\|\mathcal{T}_{U,a}^{-1}\|^2\|x\|^2 \quad \text{for all } x \in \mathcal{A}_a\,. \tag{13}$$

- In case that $b_j \in \mathcal{A}_a$ for each $j = 1, 2, \ldots, s$, we derive that $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a frame for $\mathcal{A}_a$, and it is dual to the frame $\big\{U^{rk}c_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ in $\mathcal{A}_a$. Thus, the sampling expansion (11) is nothing but a frame expansion in $\mathcal{A}_a$.

- In case that some $b_j \notin \mathcal{A}_a$, the sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ is not contained in $\mathcal{A}_a$. However, inequalities (13) hold. Therefore, the sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$

is a pseudo-dual frame for the frame $\{U^{rk}c_j\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ in $\mathcal{A}_a$ (see [20, 21]). Denoting by $P_{\mathcal{A}_a}$ the orthogonal projection onto $\mathcal{A}_a$, we derive from (13) that the sequence $\{P_{\mathcal{A}_a}(U^{rk}b_j)\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ is a dual frame of $\{U^{rk}c_j\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ in $\mathcal{A}_a$.

- Whenever $r=s$, according to the above cases, the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$ is a Riesz basis or a pseudo-Riesz basis for $\mathcal{A}_a$.

### Sampling formulas with prescribed properties

The sampling formula (11) can be thought as a filter-bank. Indeed, assume that for $j=1,2,\dots,s$ we have

$$c_{j,h}=\mathcal{T}_{U,a}(rh_j)=r\sum_{n\in\mathbb{Z}}\widehat{h}_j(n)\,U^n a \quad \text{where} \quad \widehat{h}_j(n)=\int_0^1 h_j(w)\,e^{-2\pi i n w}dw\,,\ n\in\mathbb{Z}\,.$$

Substituting in (11), after the change of summation index $m:=rk+n$ we obtain

$$x=\sum_{m\in\mathbb{Z}}\Big\{\sum_{j=1}^{s}\sum_{k\in\mathbb{Z}}r\mathcal{L}_j x(rk)\,\widehat{h}_j(m-rk)\Big\}\,U^m a\,,$$

that is, the relevant data is the output of a filter-bank:

$$\alpha_m:=\sum_{j=1}^{s}\sum_{k\in\mathbb{Z}}r\mathcal{L}_j x(rk)\,\widehat{h}_j(m-rk)\,,\quad m\in\mathbb{Z}\,.$$

where the input is the given samples and the impulse responses depend on the sampling vectors $c_{j,h}$, $j=1,2,\dots,s$. In the oversampling setting, i.e., $s>r$, according to (9) there exist infinitely many sampling vectors $c_{j,h}$, $j=1,2,\dots,s$, for which the sampling formula (11) holds. A natural question is whether we can choose the sampling vectors $c_{j,h}$, $j=1,2,\dots,s$, with prescribed properties.

For instance, a challenging problem is to ask under what conditions we are in the presence of a FIR (finite impulse response) filter-bank; i.e, $c_{j,h}=r\sum_{\text{finite}}\widehat{h}_j(n)\,U^n a$, $j=1,2,\dots,s$, or equivalently, when the functions $h_j$, $j=1,\dots,s$, are $2\pi$-periodic trigonometric polynomials. Instead, we deal with Laurent polynomials by using the variable $z=e^{2\pi i w}$, that is, $\mathsf{g}_j(z):=\sum_{k\in\mathbb{Z}}\mathcal{L}_j a(k)\,z^k$, $j=1,2,\dots,s$. We introduce the $s\times r$ matrix

$$\mathsf{G}(z):=\begin{bmatrix}\mathsf{g}_1(z) & \mathsf{g}_1(zW) & \cdots & \mathsf{g}_1(zW^{r-1})\\ \mathsf{g}_2(z) & \mathsf{g}_2(zW) & \cdots & \mathsf{g}_2(zW^{r-1})\\ \vdots & \vdots & & \vdots\\ \mathsf{g}_s(z) & \mathsf{g}_s(zW) & \cdots & \mathsf{g}_s(zW^{r-1})\end{bmatrix}=\Big[\mathsf{g}_j\big(zW^k\big)\Big]_{\substack{j=1,2,\dots,s\\ k=0,1,\dots,r-1}}\,,$$

where $W:e^{2\pi i/r}$. In case the functions $\mathsf{g}_j(z)$, $j=1,2,\dots,s$, are Laurent polynomials, the matrix $\mathsf{G}(z)$ has Laurent polynomials entries. Besides, the relationship $\mathbb{G}(w)=\mathsf{G}(e^{2\pi i w})$, $w\in(0,1)$, holds.

So that, we are interested in finding Laurent polynomials $\mathsf{h}_j(z)$, $j = 1, 2 \ldots, s$, satisfying

$$\big[\mathsf{h}_1(z), \mathsf{h}_2(z), \ldots, \mathsf{h}_s(z)\big]\, \mathsf{G}(z) = [1, 0, \ldots, 0]\,.$$

Thus, the trigonometric polynomials $h_j(w) := \mathsf{h}_j(\mathrm{e}^{2\pi i w})$, $j = 1, 2, \ldots, s$, satisfy (7), and the corresponding reconstruction vectors $c_{j,h} = \mathcal{T}_{U,a}(r h_j)$, $j = 1, 2, \ldots, s$, can be expanded in $\mathcal{A}_a$ with just a finite number of terms. Namely,

$$c_{j,h} = r \sum_{\text{finite}} \widehat{h}_j(n)\, U^n a\,, \quad \text{where} \quad \mathsf{h}_j(z) = \sum_{\text{finite}} \widehat{h}_j(n)\, z^n\,,\ j = 1, 2, \ldots, s\,.$$

The following result holds:

**Theorem 6.** *Assume that the sequences* $\big\{\mathcal{L}_j a(k)\big\}_{k \in \mathbb{Z}}$, $j = 1, 2, \ldots, s$, *contain only a finite number of nonzero terms. Then, there exists a vector* $\mathsf{h}(z) := [\mathsf{h}_1(z), \mathsf{h}_2(z), \ldots, \mathsf{h}_s(z)]$ *whose entries are Laurent polynomials, and satisfying* $\mathsf{h}(z)\, \mathsf{G}(z) = [1, 0, \ldots, 0]$ *if and only if*

$$\operatorname{rank} \mathsf{G}(z) = r \quad \textit{for all} \ \ z \in \mathbb{C} \setminus \{0\}\,.$$

*Proof.* This result is a consequence of the next lemma which proof can be found in [34, Theorems 5.1 and 5.6]:

**Lemma 7.** *Let* $\mathsf{G}(z)$ *be an* $s \times r$ *matrix whose entries are Laurent polynomials. Then, there exists an* $r \times s$ *matrix* $\mathsf{H}(z)$ *whose entries are also Laurent polynomials satisfying* $\mathsf{H}(z)\mathsf{G}(z) = \mathbb{I}_r$ *if and only if* $\operatorname{rank} \mathsf{G}(z) = r$ *for all* $z \in \mathbb{C} \setminus \{0\}$.

□

Analogously we can consider the case where the coefficients of the reconstruction vectors $c_{j,h} = r \sum_{n \in \mathbb{Z}} \widehat{h}_j(n)\, U^n a$, $j = 1, 2, \ldots, s$, have exponential decay, i.e., there exist $C > 0$ and $q \in (0, 1)$ such that $|\widehat{h}_j(n)| \le C q^{|n|}$, $n \in \mathbb{Z}$, $j = 1, 2, \ldots, s$. Assuming that the sequences $\big\{\mathcal{L}_j a(k)\big\}_{k \in \mathbb{Z}}$, $j = 1, 2, \ldots, s$, have exponential decay then, we can find reconstruction vectors $c_{j,h}$ such that the sequences $\{\widehat{h}_j(n)\}_{n \in \mathbb{Z}}$, $j = 1, 2, \ldots, s$, have exponential decay if and only if $\operatorname{rank} \mathsf{G}(z) = r$ for all $z \in \mathbb{C}$ such that $|z| = 1$. For the details, see [16] and references therein.

## 4 Time-jitter error: irregular sampling in $\mathcal{A}_a$

A close look to Section 3 shows that all the regular sampling results have been proved without the formalism of a continuous group of unitary operators $\{U^t\}_{t \in \mathbb{R}}$ in $\mathcal{H}$: we have only used the integer powers $\{U^n\}_{n \in \mathbb{Z}}$ which are completely determined from the unitary operator $U$. However, if we are concerned with the jitter-error in a sampling formula as (11), the group of unitary operators becomes essential. Here, we dispose of a perturbed sequence of samples $\{(\mathcal{L}_j x)(rm + \epsilon_{mj})\}_{m \in \mathbb{Z};\, j = 1, 2, \ldots, s}$, with errors $\epsilon_{mj} \in \mathbb{R}$, for the recovery of $x \in \mathcal{A}_a$. By using (4) and (3) we obtain:

$$\mathcal{L}_j x(rm) = \big\langle F, \overline{g_j(w)}\, \mathrm{e}^{2\pi i r m w} \big\rangle_{L^2(0,1)} \text{ and } \mathcal{L}_j x(rm + \epsilon_{mj}) = \big\langle F, \overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi i r m w} \big\rangle_{L^2(0,1)}\,,$$

where the functions

$$g_j(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k)\, \mathrm{e}^{2\pi i k w} \quad \text{and} \quad g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k+\epsilon_{mj})\, \mathrm{e}^{2\pi i k w}\,,$$

belong to $L^2(0,1)$. Let $\mathbb{G}(w)$ be the $s\times r$ matrix given in (6), associated with the functions $g_j$, $j=1,2,\dots,s$. In the case that $0<\alpha_{\mathbb{G}}\le\beta_{\mathbb{G}}<\infty$, the sequence $\big\{\overline{g_j(w)}\,\mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ with optimal frame bounds $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$. Thus, as in [14], we can see the sequence $\big\{\overline{g_{m,j}(w)}\,\mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ in $L^2(0,1)$ as a perturbation of the frame $\big\{\overline{g_j(w)}\,\mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ in $L^2(0,1)$. The following result on frame perturbation, which proof can be found in [8, p. 354] will be used later:

**Lemma 8.** *Let $\{x_n\}_{n=1}^\infty$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds $A$, $B$, and let $\{y_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R<A$ such that*

$$\sum_{n=1}^\infty \big|\langle x_n-y_n, x\rangle\big|^2 \le R\|x\|^2 \quad \textit{for each } x\in\mathcal{H}\,,$$

*then the sequence $\{y_n\}_{n=1}^\infty$ is also a frame for $\mathcal{H}$ with bounds $A\big(1-\sqrt{R/A}\big)^2$ and $B\big(1+\sqrt{R/B}\big)^2$. If the sequence $\{x_n\}_{n=1}^\infty$ is a Riesz basis, then the sequence $\{y_n\}_{n=1}^\infty$ is also a Riesz basis.*

## The time-jitter error sampling expansion

Given an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$, assume that the operator

$$\begin{array}{rccl} D_{\boldsymbol{\epsilon}}: & \ell^2(\mathbb{Z}) & \longrightarrow & \ell^2_s(\mathbb{Z}) \\ & c=\{c_l\}_{l\in\mathbb{Z}} & \longmapsto & D_{\boldsymbol{\epsilon}}\, c := \big(D_{\boldsymbol{\epsilon},1}\, c,\dots,D_{\boldsymbol{\epsilon},s}\, c\big)\,, \end{array}$$

is well-defined, where, for $j=1,2,\dots,s$,

$$D_{\boldsymbol{\epsilon},j}\, c := \Big\{\sum_{k\in\mathbb{Z}} \big[\mathcal{L}_j a(rm-k+\epsilon_{mj}) - \mathcal{L}_j a(rm-k)\big] c_k\Big\}_{m\in\mathbb{Z}}\,. \tag{14}$$

The operator norm (it could be infinity) is defined as usual

$$\|D_{\boldsymbol{\epsilon}}\| := \sup_{c\in\ell^2(\mathbb{Z})\setminus\{0\}} \frac{\|D_{\boldsymbol{\epsilon}}\, c\,\|_{\ell^2_s(\mathbb{Z})}}{\|c\,\|_{\ell^2(\mathbb{Z})}}\,,$$

where $\|D_\epsilon\, c\,\|^2_{\ell^2_s(\mathbb{Z})} := \sum_{j=1}^s \|D_{\epsilon,j}\, c\,\|^2_{\ell^2(\mathbb{Z})}$ for each $c\in\ell^2(\mathbb{Z})$.

**Theorem 9.** *Assume that for the functions $g_j$, $j=1,2,\dots,s$, given in* (5) *we have $0<\alpha_{\mathbb{G}}\le\beta_{\mathbb{G}}<\infty$. Let $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ be an error sequence satisfying the inequality $\|D_{\boldsymbol{\epsilon}}\|^2<\alpha_{\mathbb{G}}/r$. Then, there exists a frame $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ for $\mathcal{A}_a$ such that, for any $x\in\mathcal{A}_a$, the sampling expansion*

$$x = \sum_{j=1}^s \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm+\epsilon_{mj})\, C^{\boldsymbol{\epsilon}}_{j,m} \quad \textit{in } \mathcal{H}\,, \tag{15}$$

*holds. Moreover, when $r=s$ the sequence $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ is a Riesz basis for $\mathcal{A}_a$, and the interpolation property $(\mathcal{L}_l\, C^{\boldsymbol{\epsilon}}_{j,n})(rm+\epsilon_{mj}) = \delta_{j,l}\,\delta_{n,m}$ holds.*

*Proof.* The sequence $\big\{\overline{g_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame (a Riesz basis if $r=s$) for $L^2(0,1)$ with optimal frame (Riesz) bounds $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$. For any $F(w)=\sum_{l\in\mathbb{Z}} a_l\,\mathrm{e}^{2\pi \mathrm{i} lw}$ in $L^2(0,1)$ we have

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big|\big\langle \overline{g_{m,j}(\cdot)}\,\mathrm{e}^{2\pi \mathrm{i} rm\cdot}-\overline{g_j(\cdot)}\,\mathrm{e}^{2\pi \mathrm{i} rm\cdot}, F(\cdot)\big\rangle_{L^2(0,1)}\big|^2 \\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big|\big\langle \sum_{k\in\mathbb{Z}}\big(\overline{\mathcal{L}_j a(k+\epsilon_{mj})}-\overline{\mathcal{L}_j a(k)}\big)\mathrm{e}^{2\pi \mathrm{i}(rm-k)\cdot}, F(\cdot)\big\rangle_{L^2(0,1)}\big|^2 \\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big|\big\langle \sum_{k\in\mathbb{Z}}\big(\overline{\mathcal{L}_j a(rm-k+\epsilon_{mj})}-\overline{\mathcal{L}_j a(rm-k)}\big)\mathrm{e}^{2\pi \mathrm{i} k\cdot}, F(\cdot)\big\rangle_{L^2(0,1)}\big|^2 \qquad (16)\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big|\sum_{k\in\mathbb{Z}}\big(\overline{\mathcal{L}_j a(rm-k+\epsilon_{mj})}-\overline{\mathcal{L}_j a(rm-k)}\big)\,\overline{a}_k\big|^2 \\
&=\sum_{j=1}^{s}\|D_{\epsilon,j}\{a_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})}\le \|D_{\boldsymbol{\epsilon}}\|^2\|\{a_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})}=\|D_{\boldsymbol{\epsilon}}\|^2\|F\|^2_{L^2(0,1)}\,.
\end{aligned}
$$

By using Lemma 8 we obtain that the sequence $\big\{\overline{g_{m,j}(w)}\,\mathrm{e}^{2\pi \mathrm{i} rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ (a Riesz basis if $r=s$). Let $\{h^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ be its canonical dual frame. Hence, for any $F\in L^2(0,1)$

$$
\begin{aligned}
F&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big\langle F(\cdot),\overline{g_{m,j}(\cdot)}\,\mathrm{e}^{2\pi \mathrm{i} rm\cdot}\big\rangle_{L^2(0,1)}\,h^{\boldsymbol{\epsilon}}_{j,m}\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\mathcal{L}_j x(rm+\epsilon_{mj})\,h^{\boldsymbol{\epsilon}}_{j,m}\quad \text{in } L^2(0,1)\,.
\end{aligned}
$$

Applying the isomorphism $\mathcal{T}_{U,a}$, one gets (15), where $C^{\boldsymbol{\epsilon}}_{j,m}:=\mathcal{T}_{U,a}\big(h^{\boldsymbol{\epsilon}}_{j,m}\big)$ for $m\in\mathbb{Z}$ and $j=1,2,\dots,s$. Since $\mathcal{T}_{U,a}$ is an isomorphism between $L^2(0,1)$ and $\mathcal{A}_a$, the sequence $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$ (a Riesz basis if $r=s$). The interpolatory property in the case $r=s$ follows from the uniqueness of the coefficients with respect to a Riesz basis. □

Sampling formula (15) is useless from a practical point of view: it is impossible to determine the involved frame $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$. As a consequence, in order to recover $x\in\mathcal{A}_a$ from the sequence of samples $\big\{(\mathcal{L}_j x)(rm+\epsilon_{mj})\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ we should implement a frame algorithm in $\ell^2(\mathbb{Z})$ (see Ref. [14]); another possibility is given in Ref. [1].

In order to prove the existence of sequences $\boldsymbol{\epsilon}:=\{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,\dots,s}$ such that $\|D_{\boldsymbol{\epsilon}}\|^2<\alpha_{\mathbb{G}}/r$ we need some results from the group of unitary operators theory:

### A brief excursion on groups of unitary operators

Let $\{U^t\}_{t\in\mathbb{R}}$ denote a continuous *group of unitary operators* in $\mathcal{H}$. Classical Stone's theorem [26] assures us the existence of a self-adjoint operator $T$ (maybe unbounded) such that $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$. This self-adjoint operator $T$, defined on the dense domain of $\mathcal{H}$

$$D_T := \left\{x \in \mathcal{H} \text{ such that } \int_{-\infty}^{\infty} w^2\, d\|E_w x\|^2 < \infty\right\},$$

admits the *spectral representation* $T = \int_{-\infty}^{\infty} w\, dE_w$ which means:

$$\langle Tx, y\rangle = \int_{-\infty}^{\infty} w\, d\langle E_w x, y\rangle \quad \text{for any } x \in D_T \text{ and } y \in \mathcal{H},$$

where $\{E_w\}_{w\in\mathbb{R}}$ is the corresponding *resolution of the identity*, i.e., a one-parameter family of projection operators $E_w$ in $\mathcal{H}$ such that

(i) $E_{-\infty} := \lim_{w\to-\infty} E_w = O_{\mathcal{H}}, \quad E_\infty := \lim_{w\to\infty} E_w = I_{\mathcal{H}}$,

(ii) $E_{w^-} = E_w$ for any $-\infty < w < \infty$,

(iii) $E_u\, E_v = E_w$ where $w = \min\{u, v\}$.

Recall that $\|E_w x\|^2$ and $\langle E_w x, y\rangle$, as functions of $w$, have bounded variation and define, respectively, a positive and a complex Borel measure on $\mathbb{R}$.

Furthermore, for any $x \in D_T$ we have that $\lim_{t\to 0} \dfrac{U^t x - x}{t} = \mathrm{i}\, Tx$ and the operator $\mathrm{i}\, T$ is said to be the *infinitesimal generator* of the group $\{U^t\}_{t\in\mathbb{R}}$. For each $x \in D_T$, $U^t x$ is a continuous differentiable function of $t$. Notice that, whenever the self-adjoint operator $T$ is bounded, $D_T = \mathcal{H}$ and $\mathrm{e}^{\mathrm{i}tT}$ can be defined as the usual exponential series; in any case, $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$ means that

$$\langle U^t x, y\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}wt} d\langle E_w x, y\rangle\,, \quad t \in \mathbb{R}\,,$$

where $x \in D_T$ and $y \in \mathcal{H}$.
Finally, a comment on the continuity of a group of unitary operators: The group is said to be *strongly continuous* if, for each $x \in \mathcal{H}$ and $t_0 \in \mathbb{R}$, $U^t x \to U^{t_0} x$ as $t \to t_0$. If $\mathcal{H}$ is a separable Hilbert space, strong continuity can be deduced from continuity and even from weak measurability, i.e., $\langle U^t x, y\rangle_{\mathcal{H}}$ is a Lebesgue measurable function of $t$ for any $x, y \in \mathcal{H}$. See, for instance, Refs. [2, 7, 32, 33].

### On the existence of sequences $\boldsymbol{\epsilon}$ such that $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$

Assuming that $b_j \in D_T$, $j = 1, 2, \ldots, s$, the functions $\mathcal{L}_j a(t)$, $j = 1, 2, \ldots, s$, are continuously differentiable on $\mathbb{R}$. If, for instance, we demand in addition that, for each $j = 1, 2, \ldots, s$, there exists $\eta_j > 0$ such that

$$(\mathcal{L}_j a)'(t) = O(|t|^{-(1+\eta_j)}) \quad \text{whenever } |t| \to \infty\,, \tag{17}$$

then we can find out a finite bound for the norm $\|D_{\boldsymbol{\epsilon}}\|^2$. Indeed, for $j = 1, 2, \ldots, s$ and $n, m \in \mathbb{Z}$ denote

$$d_{m,k}^{(j)} := \mathcal{L}_j a(rm - k + \epsilon_{m,j}) - \mathcal{L}_j a(rm - k)\,.$$

Taking into account (14), for any sequence $c=\{c_k\}_{k\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$ we have

$$\|D_\epsilon c\|^2_{\ell^2_s(\mathbb{Z})}=\sum_{j=1}^{s}\sum_{m\in\mathbb{Z}}\Big|\sum_{k\in\mathbb{Z}}d^{(j)}_{m,k}\ c_k\Big|^2\leq\sum_{j=1}^{s}\sum_{m\in\mathbb{Z}}\sum_{l,k\in\mathbb{Z}}|d^{(j)}_{m,l}c_l\overline{d}^{(j)}_{m,k}\overline{c}_k|$$
$$=\sum_{j=1}^{s}\sum_{l,k\in\mathbb{Z}}|c_l|\ |c_k|\ \sum_{m\in\mathbb{Z}}|d^{(j)}_{m,l}d^{(j)}_{m,k}|\leq\sum_{j=1}^{s}\sum_{l,k\in\mathbb{Z}}\frac{|c_l|^2+|c_k|^2}{2}\ \sum_{m\in\mathbb{Z}}|d^{(j)}_{m,l}d^{(j)}_{m,k}| \tag{18}$$
$$=\sum_{j=1}^{s}\sum_{l\in\mathbb{Z}}|c_l|^2\sum_{k,m\in\mathbb{Z}}|d^{(j)}_{m,l}d^{(j)}_{m,k}|\,.$$

Under the decay conditions (17), for $|\gamma|\leq 1/2$ we define the continuous functions,

$$M_{(\mathcal{L}_ja)'}(\gamma):=\sum_{k\in\mathbb{Z}}\max_{t\in[k-\gamma,k+\gamma]}|(\mathcal{L}_ja)'(t)|\,,$$

and

$$N_{(\mathcal{L}_ja)'}(\gamma):=\max_{k=0,1,\dots,r-1}\sum_{m\in\mathbb{Z}}\max_{t\in[rm+k-\gamma,rm+k+\gamma]}|(\mathcal{L}_ja)'(t)|\,.$$

Notice that $N_{(\mathcal{L}_ja)'}(\gamma)\leq M_{(\mathcal{L}_ja)'}(\gamma)$ and for $r=1$ the equality holds.

**Theorem 10.** *Given an error sequence* $\boldsymbol{\epsilon}:=\{\epsilon_{mj}\}_{m\in\mathbb{Z};\,j=1,\dots,s}$*, define the constant* $\gamma_j:=\sup_{m\in\mathbb{Z}}|\epsilon_{mj}|$ *for each* $j=1,2,\dots,s$*. Then, the inequality*

$$\|D_{\boldsymbol{\epsilon}}\|^2\leq\sum_{j=1}^{s}\gamma_j^2\ N_{(\mathcal{L}_ja)'}(\gamma_j)M_{(\mathcal{L}_ja)'}(\gamma_j)$$

*holds and, as a consequence, condition*

$$\sum_{j=1}^{s}\gamma_j^2\ N_{(\mathcal{L}_ja)'}(\gamma_j)M_{(\mathcal{L}_ja)'}(\gamma_j)<\frac{\alpha_{\mathbf{G}}}{r}$$

*ensures the hypothesis* $\|D_{\boldsymbol{\epsilon}}\|^2<\alpha_{\mathbb{G}}/r$ *in Theorem 9.*

*Proof.* For each $j=1,2,\dots,s$, the mean value theorem gives

$$\sup_{d\in[-\gamma_j,\gamma_j]}\sum_{n\in\mathbb{Z}}|\mathcal{L}_ja(n+d)-\mathcal{L}_ja(n)|\leq\gamma_j\ M_{(\mathcal{L}_ja)'}(\gamma_j)\,, \tag{19}$$

and

$$\sup_{\substack{k=0,1,\dots,r-1\\ \{d_n\}\subset[-\gamma_j,\gamma_j]}}\sum_{n\in\mathbb{Z}}|\mathcal{L}_ja(rn+k+d_n)-\mathcal{L}_ja(rn+k)|\leq\gamma_j\ N_{(\mathcal{L}_ja)'}(\gamma_j)\,. \tag{20}$$

Thus, using (19) and (20), inequality (18) becomes

$$\|D_\epsilon c\|^2_{\ell^2_s(\mathbb{Z})}\leq\sum_{j=1}^{s}\sum_{l\in\mathbb{Z}}|c_l|^2\sum_{k,m\in\mathbb{Z}}|d^{(j)}_{m,l}d^{(j)}_{m,k}|\leq\sum_{j=1}^{s}\sum_{l\in\mathbb{Z}}|c_l|^2\sum_{m\in\mathbb{Z}}|d^{(j)}_{m,l}|\ \gamma_j\,M_{(\mathcal{L}_ja)'}(\gamma_j)$$
$$\leq\sum_{j=1}^{s}\sum_{l\in\mathbb{Z}}|c_l|^2\ (\gamma_j)^2\ M_{(\mathcal{L}_ja)'}(\gamma_j)N_{(\mathcal{L}_ja)'}(\gamma_j) \tag{21}$$
$$=\|c\|^2_{\ell^2(\mathbb{Z})}\sum_{j=1}^{s}\gamma_j^2\ N_{(\mathcal{L}_ja)'}(\gamma_j)M_{(\mathcal{L}_ja)'}(\gamma_j)\,,$$

which concludes the proof. $\square$

## 5 The case of multiple generators

The case of $L$ generators can be analogously derived. Indeed, consider the $U$-invariant subspace generated by $\mathbf{a} := \{a_1, a_2, \dots, a_L\} \subset \mathcal{H}$, i.e.,

$$\mathcal{A}_\mathbf{a} := \overline{\operatorname{span}}\big\{U^n a_l,\ n \in \mathbb{Z};\, l = 1, 2.\dots, L\big\}\,.$$

Assuming that the sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz sequence in $\mathcal{H}$, the $U$-invariant subspace $\mathcal{A}_\mathbf{a}$ can be expressed as

$$\mathcal{A}_\mathbf{a} = \Big\{ \sum_{l=1}^{L} \sum_{n\in\mathbb{Z}} \alpha_n^l\, U^n a : \{\alpha_n^l\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z});\, l = 1, 2.\dots, L \Big\}\,.$$

The sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ can be thought as an $L$-dimensional stationary sequence. Its *covariance matrix* $\mathbf{R}_\mathbf{a}(k)$ is the $L \times L$ matrix

$$\mathbf{R}_\mathbf{a}(k) := \Big[\langle U^k a_m, a_n\rangle_\mathcal{H}\Big]_{m,n=1,2,\dots,L}\,, \qquad k \in \mathbb{Z}\,.$$

Its admits the spectral representation [19]:

$$\mathbf{R}_\mathbf{a}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\boldsymbol{\mu}_\mathbf{a}(\theta)\,, \quad k \in \mathbb{Z}\,.$$

The *spectral measure* $\boldsymbol{\mu}_\mathbf{a}$ is an $L \times L$ matrix; its entries are the spectral measures associated with the cross-correlation functions $R_{m,n}(k) := \langle U^k a_m, a_n\rangle_\mathcal{H}$. It can be decomposed into an absolute continuous part and its singular part. Thus we can write

$$d\boldsymbol{\mu}_\mathbf{a}(\theta) = \mathbf{\Phi}_\mathbf{a}(\theta) d\theta + d\boldsymbol{\mu}^s_\mathbf{a}(\theta)\,.$$

In case that the singular part $\boldsymbol{\mu}^s_\mathbf{a} \equiv 0$, the hermitian $L \times L$ matrix $\mathbf{\Phi}_\mathbf{a}(\theta)$ is called the *spectral density* of the sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$. The following theorem holds:

**Theorem 11.** *Let $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ be a sequence obtained from a unitary operator in a separable Hilbert space $\mathcal{H}$ with spectral measure $d\boldsymbol{\mu}_\mathbf{a}(\theta) = \mathbf{\Phi}_\mathbf{a}(\theta) d\theta + d\boldsymbol{\mu}^s_\mathbf{a}(\theta)$, and let $\mathcal{A}_\mathbf{a}$ be the closed subspace spanned by $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$. Then the sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_\mathbf{a}$ if and only if the singular part $\boldsymbol{\mu}^s_\mathbf{a} \equiv 0$ and*

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \lambda_{\min}\big[\mathbf{\Phi}_\mathbf{a}(\theta)\big] \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \lambda_{\max}\big[\mathbf{\Phi}_\mathbf{a}(\theta)\big] < \infty\,. \tag{22}$$

*Proof.* For a fixed $\ell^2_L$-sequence $c := \{c_n^l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ we have

$$\begin{aligned}
\Big\| \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}} c_k^l U^k a_l \Big\|^2 &= \sum_{i,j=1}^{L}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_m^i \bar{c}_m^j \langle U^m a_i, U^n a_j\rangle \\
&= \sum_{i,j=1}^{L}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_m^i \bar{c}_n^j \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}m\theta}\mathrm{e}^{-\mathrm{i}n\theta} d\mu_{a_i,a_j}(\theta) \qquad (23)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} (\mathbf{c}_m\, \mathrm{e}^{\mathrm{i}m\theta})^\top d\boldsymbol{\mu}_\mathbf{a}(\theta) \bar{\mathbf{c}}_n\, \mathrm{e}^{-\mathrm{i}n\theta}\,,
\end{aligned}$$

where $\mathbf{c}_k = (c_k^1, c_k^2, \dots, c_k^L)^\top$ for every $k \in \mathbb{Z}$.

First we show that if the measure $\boldsymbol{\mu}_{\mathbf{a}}$ is not absolutely continuous with respect to Lebesgue measure $\lambda$ then $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is not a Riesz basis for $\mathcal{A}_{\mathbf{a}}$. Indeed, if the spectral measure $\boldsymbol{\mu}_{\mathbf{a}}$ is not absolutely continuous with respect to Lebesgue measure then there exists $i \in \{1, 2, \dots, L\}$ such that the positive spectral measure $\mu_{a_i,a_i}$ is not absolutely continuous with respect to Lebesgue measure; this comes from the fact that, if any spectral measure in the diagonal $\mu_{a_j,a_j}$ is absolutely continuous with respect to Lebesgue measure, the same occurs for each measure $\mu_{a_j,a_k}$ with $k \neq j$ (see [7, p. 137]). Then, $\mu_{a_i,a_i}(B) > 0$ for a (Lebesgue) measurable set $B \subset (-\pi, \pi)$ of Lebesgue measure zero. Bearing in mind that every measurable set is included in a Borel set, actually an intersection of a countable collection of open sets, having the same Lebesgue measure (see [25, p. 63]), we take $B$ to be a Borel set. Moreover, since every finite Borel measure on $(-\pi, \pi)$ is inner regular (see [25, p. 340]) we may also assume that $B$ is a compact set. For any $\varepsilon > 0$ there exists a sequence of disjoint open intervals $I_j \subset (-\pi, \pi)$ such that

$$B \subset \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda(I_j) \leq \lambda(B) + \varepsilon = \varepsilon\,,$$

(see [25, pp. 58 and 42]). Since $B$ is compact we may take the sequence to be finite. Hence, for every $N \in \mathbb{N}$ there exist open disjoint intervals $I_1^N, I_2^N, \dots, I_{j_N}^N$ in $(-\pi, \pi)$ such that

$$B \subset \bigcup_{j=1}^{j_N} I_j^N \quad \text{and} \quad \sum_{j=1}^{j_N} \lambda(I_j^N) \leq \frac{1}{3^N}\,.$$

Besides, $\sum_{j=1}^{j_N} \mu_{a_i,a_i}(I_j^N) \geq \mu_{a_i,a_i}(B)$. Consider the function $g_N \colon (-\pi, \pi) \to \mathbb{R}$, where $g_N = 2^{N/2}\chi_{\bigcup_{j=1}^{j_N} I_j^N}$, that satisfies

$$\|g_N\|_2^2 = 2^N \sum_{j=1}^{j_N} \lambda(I_j^N) \leq \frac{2^N}{3^N} < 1\,.$$

We modify and extend each $g_N$ to obtain a $2\pi$-periodic function $f_N \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $f_N$ and its derivative are continuous on $\mathbb{R}$, $\|f_N\|_2^2 \leq 1$ and $f_N(\theta) = g_N(\theta)$ for every $\theta \in \bigcup_{j=1}^{j_N} I_j^N$. Let $\sum_k c_k^N \mathrm{e}^{\mathrm{i}k\theta}$ be the Fourier series of $f_N$. First, by using Parseval's identity we have

$$\|c_k^N\|_2^2 = \frac{1}{2\pi}\|f_N\|_2^2 \leq \frac{1}{2\pi} \quad \text{for every } N \in \mathbb{N}\,,$$

so that $\{c^N\}_{N=1}^{\infty}$ is a bounded sequence in $\ell^2(\mathbb{Z})$. Besides, the regularity of each $f_N$ ensures that each Fourier series converges uniformly to $f_N$. Therefore each series $\sum_k c_k^N \mathrm{e}^{\mathrm{i}k\theta}$ converges to $f_N$ in $L^2_{\mu_{a_i,a_i}(-\pi,\pi)}$ and consequently,

$$\begin{aligned}\Big\| \sum_k c_k^N \,\mathrm{e}^{\mathrm{i}k\theta} \Big\|^2_{L^2_{\mu_{a_i,a_i}}(-\pi,\pi)} &= \int_{-\pi}^{\pi} |f_N|^2 d\mu_{a_i,a_i} \geq \int_{-\pi}^{\pi} |g_N|^2 d\mu_{a_i,a_i} = 2^N \sum_{j=1}^{j_N} \mu_{a_i,a_i}(I_j^N) \\ &\geq 2^N \mu_{a_i,a_i}(B)\,.\end{aligned}$$

For every $c^N \in \ell^2(\mathbb{Z})$ we consider the $\ell^2_L$-sequence $\{c_n^{Nl}\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ given by $c_n^{Ni} = c_n^N$ and $c_n^{Nl} = 0$ if $l \neq i$. Substituting each $\{c_n^{Nl}\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ in (23) we have that

$$\Big\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^{Nl} U^k a_l \Big\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| \sum_{k\in\mathbb{Z}} c_k^N \,\mathrm{e}^{\mathrm{i}k\theta} \Big|^2 d\mu_{a_i,a_i}(\theta)$$

tends to infinity with $N$, so $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ cannot be a Bessel sequence, therefore, not a Riesz basis.

For the remainder of the proof we assume that the singular part $\boldsymbol{\mu}_{\mathbf{a}}^s \equiv 0$ and that $d\boldsymbol{\mu}_{\mathbf{a}}(\theta) = \mathbf{\Phi}_{\mathbf{a}}(\theta) d\theta$. Then (23) yields that

$$\Big\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^l U^k a_l \Big\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \big( \sum_{m\in\mathbb{Z}} \mathbf{c}_m \,\mathrm{e}^{\mathrm{i}m\theta} \big)^\top \mathbf{\Phi}_{\mathbf{a}}(\theta) \overline{\sum_{n\in\mathbb{Z}} \mathbf{c}_n \,\mathrm{e}^{\mathrm{i}n\theta}} d\theta \,. \tag{24}$$

We have to show that $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_{\mathbf{a}}$ if and only if (22) holds. Rayleigh-Ritz theorem (see [17, p. 176]) provides the inequalities

$$\lambda_{\min}\big[\mathbf{\Phi}_{\mathbf{a}}(\theta)\big] \big| \sum_{k\in\mathbb{Z}} \mathbf{c}_k \,\mathrm{e}^{\mathrm{i}k\theta} \big|^2 \leq \big( \sum_{m\in\mathbb{Z}} \mathbf{c}_m \,\mathrm{e}^{\mathrm{i}m\theta} \big)^\top \mathbf{\Phi}_{\mathbf{a}}(\theta) \overline{\sum_{n\in\mathbb{Z}} \mathbf{c}_n \,\mathrm{e}^{\mathrm{i}n\theta}} \leq \lambda_{\max}\big[\mathbf{\Phi}_{\mathbf{a}}(\theta)\big] \big| \sum_{k\in\mathbb{Z}} \mathbf{c}_k \,\mathrm{e}^{\mathrm{i}k\theta} \big|^2$$

and taking into account (24) we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda_{\min}\big[\mathbf{\Phi}_{\mathbf{a}}(\theta)\big] \big| \sum_{k\in\mathbb{Z}} \mathbf{c}_k \,\mathrm{e}^{\mathrm{i}k\theta} \big|^2 d\theta &\leq \Big\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^l U^k a_l \Big\|^2 \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda_{\max}\big[\mathbf{\Phi}_{\mathbf{a}}(\theta)\big] \big| \sum_{k\in\mathbb{Z}} \mathbf{c}_k \,\mathrm{e}^{\mathrm{i}k\theta} \big|^2 d\theta \,, \end{aligned}$$

so that

$$\begin{aligned} \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \lambda_{\min}\big[\mathbf{\Phi}_{\mathbf{a}}(\theta)\big] \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |c_k^l|^2 &\leq \Big\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^l U^k a_l \Big\|^2 \\ &\leq \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \lambda_{\max}\big[\mathbf{\Phi}_{\mathbf{a}}(\theta)\big] \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |c_k^l|^2 \,. \end{aligned}$$

Therefore (22) implies that $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_{\mathbf{a}}$.

Conversely, if $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_{\mathbf{a}}$ then there exist constants $0 < A \leq B < \infty$ such that

$$A \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |c_k^l|^2 \leq \Big\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^l U^k a_l \Big\|^2 \leq B \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |c_k^l|^2 \tag{25}$$

for every $\ell^2_L$-sequence $c := \{c_n^l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$. Let us prove that

$$A \leq \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \lambda_{\min}\big[\mathbf{\Phi}_{\mathbf{a}}(\theta)\big] \leq \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \lambda_{\max}\big[\mathbf{\Phi}_{\mathbf{a}}(\theta)\big] \leq B \,. \tag{26}$$

Proceeding by contradiction, if (26) would not hold, then

$$A \le \lambda_{\min}\big[\mathbf{\Phi_a}(\theta)\big] \le \lambda_{\max}\big[\mathbf{\Phi_a}(\theta)\big] \le B$$

does not hold on a subset of $(-\pi,\pi)$ with positive Lebesgue measure. In case the set $\Gamma_B := \{\theta \in (-\pi,\pi) \colon \lambda_{\max}\big[\mathbf{\Phi_a}(\theta)\big] > B\}$ has positive Lebesgue measure we introduce the Fourier expansion of the function $F \in L^2_L(-\pi,\pi)$ ($L^2_L(-\pi,\pi)$ denotes the usual product Hilbert space $L^2(-\pi,\pi)\times\cdots\times L^2(-\pi,\pi)$ ($L$ times)) in (24), where $F(\theta) = \mathbf{X}(\theta)\,\chi_{\Gamma_B}(\theta)$ and $\mathbf{X}(\theta)$ is an eigenvector of norm 1 associated with the biggest eigenvalue of $\mathbf{\Phi_a}(\theta)$. We get

$$\Big\|\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}} c_k^l U^k a_l\Big\|^2 = \frac{1}{2\pi}\int_{\Gamma_B}\lambda_{\max}\big[\mathbf{\Phi_a}(\theta)\big]d\theta > \frac{1}{2\pi}\int_{\Gamma_B} B d\theta$$

which contradicts the right inequality in (25) for such a Fourier expansion. Whenever Lebesgue measure of the set $\Gamma_B$ is zero then we proceed in a similar way with the set of positive Lebesgue measure $\Gamma_A := \{\theta\in(-\pi,\pi)\colon \lambda_{\min}\big[\mathbf{\Phi_a}(\theta)\big] < A\}$. □

The above proof is similar to that of Lemma 2 in [24], except we do not exclude the case in which the singular measure is atomless. Another characterization for being $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ a Riesz basis for $\mathcal{A}_\mathbf{a}$ can be found in [3].

**The resulting regular sampling formulas**

As in the one-generator case, the space $\mathcal{A}_\mathbf{a}$ is the image of the usual product Hilbert space $L^2_L(0,1)$ by means of the isomorphism $\mathcal{T}_{U,\mathbf{a}} : L^2_L(0,1) \longrightarrow \mathcal{A}_\mathbf{a}$, which maps the orthonormal basis $\{\mathrm{e}^{-2\pi \mathrm{i} nw}\mathbf{e}_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ for $L^2_L(0,1)$ (here, $\{\mathbf{e}_l\}_{l=1}^L$ denotes the canonical basis for $\mathbb{C}^L$) onto the Riesz basis $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ for $\mathcal{A}_\mathbf{a}$, i.e.,

$$\mathcal{T}_{U,\mathbf{a}}\mathbf{F} := \sum_{l=1}^{L}\sum_{n\in\mathbb{Z}} \big\langle F_l, \mathrm{e}^{2\pi \mathrm{i} n\cdot}\big\rangle_{L^2(0,1)} U^n a_l = \sum_{l=1}^{L}\sum_{n\in\mathbb{Z}} \alpha_n^l\, U^n a_l\,, \tag{27}$$

where $\mathbf{F} = (F_1, F_2, \dots, F_L)^\top \in L^2_L(0,1)$.
Here, for $\mathbf{F}\in L^2_L(0,1)$ and $N\in\mathbb{Z}$ the $U$-shift property reads:

$$\mathcal{T}_{U,\mathbf{a}}\big(\mathbf{F}\mathrm{e}^{2\pi \mathrm{i} Nw}\big) = U^N\big(\mathcal{T}_{U,\mathbf{a}}\mathbf{F}\big)\,. \tag{28}$$

Concerning the representation of an $U$-system $\mathcal{L}_b$, for $x\in\mathcal{A}_\mathbf{a}$ we have

$$\begin{aligned}\mathcal{L}_b x(t) = \langle x, U^t b\rangle_\mathcal{H} &= \sum_{l=1}^{L}\sum_{n\in\mathbb{Z}} \alpha_n^l\,\overline{\langle U^t b, U^n a_l\rangle}_\mathcal{H} \\ &= \sum_{l=1}^{L}\Big\langle F_l, \sum_{n\in\mathbb{Z}}\langle U^t b, U^n a_l\rangle_\mathcal{H}\,\mathrm{e}^{2\pi \mathrm{i} nw}\Big\rangle_{L^2(0,1)} = \big\langle \mathbf{F}, \mathbf{K}_t\big\rangle_{L^2_L(0,1)}\,,\end{aligned}$$

where $\mathcal{T}_{U,a}\mathbf{F} = x$, $\mathbf{F} = \big(F_1, F_2, \dots, F_L\big)^\top \in L^2_L(0,1)$, and the function

$$\mathbf{K}_t(w) := \Big(\sum_{n\in\mathbb{Z}}\overline{\mathcal{L}_b a_1(t-n)}\,\mathrm{e}^{2\pi \mathrm{i} nw}, \sum_{n\in\mathbb{Z}}\overline{\mathcal{L}_b a_2(t-n)}\,\mathrm{e}^{2\pi \mathrm{i} nw}, \dots, \sum_{n\in\mathbb{Z}}\overline{\mathcal{L}_b a_L(t-n)}\,\mathrm{e}^{2\pi \mathrm{i} nw}\Big)^\top$$

belongs to $L^2_L(0,1)$. In particular, given $s$ $U$-systems $\mathcal{L}_j := \mathcal{L}_{b_j}$ associated with $b_j \in \mathcal{H}$, $j=1,2,\dots,s$, we get the expression for the samples $\{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$:

$$\mathcal{L}_j x(rm) = \big\langle \mathbf{F}, \overline{\mathbf{g}_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw} \big\rangle_{L^2_L(0,1)} \quad \text{for } m\in\mathbb{Z} \text{ and } j=1,2,\dots,s\,, \tag{29}$$

where $\mathcal{T}_{U,a}\mathbf{F} = x$ and

$$\mathbf{g}_j(w) := \Big(\sum_{k\in\mathbb{Z}} \mathcal{L}_j a_1(k)\,\mathrm{e}^{2\pi \mathrm{i} kw}, \sum_{k\in\mathbb{Z}} \mathcal{L}_j a_2(k)\,\mathrm{e}^{2\pi \mathrm{i} kw}, \dots, \sum_{k\in\mathbb{Z}} \mathcal{L}_j a_L(k)\,\mathrm{e}^{2\pi \mathrm{i} kw}\Big)^\top \in L^2_L(0,1)\,.$$

As in the one-generator case we must study the sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rmw}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ in $L^2_L(0,1)$. Consider the $s\times rL$ matrix of functions in $L^2(0,1)$

$$\mathbb{G}(w) := \begin{bmatrix} \mathbf{g}_1^\top(w) & \mathbf{g}_1^\top(w+\frac{1}{r}) & \cdots & \mathbf{g}_1^\top(w+\frac{r-1}{r}) \\ \mathbf{g}_2^\top(w) & \mathbf{g}_2^\top(w+\frac{1}{r}) & \cdots & \mathbf{g}_2^\top(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ \mathbf{g}_s^\top(w) & \mathbf{g}_s^\top(w+\frac{1}{r}) & \cdots & \mathbf{g}_s^\top(w+\frac{r-1}{r}) \end{bmatrix} = \Big[\mathbf{g}_j^\top\Big(w+\frac{k-1}{r}\Big)\Big]_{\substack{j=1,2,\dots,s\\ k=1,2,\dots,r}} \tag{30}$$

and its related constants

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)], \quad \beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,.$$

In [15, Lemma 2] one can find the proof of the following lemma:

**Lemma 12.** *Let $\mathbf{g}_j$ be in $L^2_L(0,1)$ for $j=1,2,\dots,s$ and let $\mathbb{G}(w)$ be its associated matrix given in* (30)*. Then, the following results hold:*

*(a) The sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a complete system for $L^2_L(0,1)$ if and only if the rank of the matrix $\mathbb{G}(w)$ is $rL$ a.e. in $(0,1/r)$.*

*(b) The sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a Bessel sequence for $L^2_L(0,1)$ if and only if $\mathbf{g}_j \in L^\infty_L(0,1)$ (or equivalently $\beta_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $\beta_{\mathbb{G}}/r$.*

*(c) The sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2_L(0,1)$ if and only if $0<\alpha_{\mathbb{G}}\le\beta_{\mathbb{G}}<\infty$. In this case, the optimal frame bounds are $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$.*

*(d) The sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a Riesz basis for $L^2_L(0,1)$ if and only if is a frame and $s=rL$.*

In case that the sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2_L(0,1)$ (here, necessarily $s\ge rL$), a dual frame is given by $\{r\mathbf{h}_j(w)\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$, where the functions $\mathbf{h}_j$, $j=1,2,\dots,s$, form an $L\times s$ matrix $\mathbf{h}(w) := [\mathbf{h}_1(w), \mathbf{h}_2(w), \dots, \mathbf{h}_s(w)]$ with entries in $L^\infty(0,1)$, and satisfying

$$[\mathbf{h}_1(w), \mathbf{h}_2(w), \dots, \mathbf{h}_s(w)]\,\mathbb{G}(w) = [\mathbb{I}_L, \mathbb{O}_{L\times(r-1)L}] \quad \text{a.e. in } (0,1)$$

(see Ref. [15] for the details). That is, the matrix $\mathbf{h}(w)$ is formed with the first $L$ rows of a left-inverse of the matrix $\mathbb{G}(w)$ having essentially bounded entries in $(0,1)$. In other words, all the dual frames of $\big\{\overline{\mathbf{g}_j(\mathrm{e}^{2\pi i r n w})}\big\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ with the above property are obtained by taking the first $L$ rows of the $rL\times s$ matrices given by

$$\mathbb{H}_{\mathbb{U}}(w) := \mathbb{G}^{\dagger}(w) + \mathbb{U}(w)\big[\mathbb{I}_s - \mathbb{G}(w)\mathbb{G}^{\dagger}(w)\big]\,,$$

where $\mathbb{U}(w)$ denotes any $rL\times s$ matrix with entries in $L^\infty(0,1)$.
Thus, any $\mathbf{F}\in L^2_L(0,1)$ can be expanded as

$$\mathbf{F} = \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \big\langle \mathbf{F}, \overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi i r n w}\big\rangle_{L^2_L(0,1)}\, r\mathbf{h}_j(w)\,\mathrm{e}^{2\pi i r n w} \quad \text{in } L^2_L(0,1)\,.$$

Applying the isomorphism $\mathcal{T}_{U,a}$ and taken into account (29), for each $x=\mathcal{T}_{U,a}\mathbf{F}\in\mathcal{A}_{\mathbf{a}}$ we get the sampling expansion

$$x = \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \mathcal{L}_j x(rn)\, U^{rn}\big[\mathcal{T}_{U,a}(r\mathbf{h}_j)\big] = \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \mathcal{L}_j x(rn)\, U^{rn} c_{j,\mathbf{h}} \quad \text{in } \mathcal{H}\,,$$

where $c_{j,\mathbf{h}} = \mathcal{T}_{U,a}(r\mathbf{h}_j)\in\mathcal{A}_{\mathbf{a}}$, $j=1,2,\dots,s$, and the sequence $\{U^{rn}c_{j,\mathbf{h}}\}_{n\in\mathbb{Z};\,j=1,2,\dots,s}$ is a frame for $\mathcal{A}_{\mathbf{a}}$. Proceeding as in Section 3, it is straighforward to state and prove the corresponding results.

**The time-jitter error sampling formulas**

Under appropriate slight changes, the time-jitter error results in Section 4 still remain valid for the case of multiple generators. Namely, given an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$, assume that the operator

$$\begin{array}{rccl} D_{\boldsymbol{\epsilon}}: & \ell^2_L(\mathbb{Z}) & \longrightarrow & \ell^2_s(\mathbb{Z}) \\ & \mathbf{c} & \longmapsto & D_{\boldsymbol{\epsilon}}\,\mathbf{c} := \big(D_{\boldsymbol{\epsilon},1}\,\mathbf{c},\dots,D_{\boldsymbol{\epsilon},s}\,\mathbf{c}\big)\,, \end{array}$$

is well-defined, where $\mathbf{c} := \big(\{c^1_k\}_{k\in\mathbb{Z}}, \{c^2_k\}_{k\in\mathbb{Z}},\dots,\{c^L_k\}_{k\in\mathbb{Z}}\big)\in\ell^2_L(\mathbb{Z})$ and, for $j=1,2,\dots,s$,

$$D_{\boldsymbol{\epsilon},j}\,\mathbf{c} := \Big\{\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}\big[\mathcal{L}_j a_l(rm-k+\epsilon_{mj}) - \mathcal{L}_j a_l(rm-k)\big]c^l_k\Big\}_{m\in\mathbb{Z}}.$$

The operator norm (it could be infinity) is defined as usual

$$\|D_{\boldsymbol{\epsilon}}\| := \sup_{\mathbf{c}\in\ell^2_L(\mathbb{Z})\setminus\{0\}} \frac{\|D_{\boldsymbol{\epsilon}}\,\mathbf{c}\,\|_{\ell^2_s(\mathbb{Z})}}{\|\mathbf{c}\,\|_{\ell^2_L(\mathbb{Z})}}\,,$$

where $\|D_\epsilon\,\mathbf{c}\,\|^2_{\ell^2_s(\mathbb{Z})} := \sum_{j=1}^{s}\|D_{\epsilon,j}\,\mathbf{c}\,\|^2_{\ell^2(\mathbb{Z})}$ and $\|\mathbf{c}\|^2_{\ell^2_L(\mathbb{Z})} = \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}|c^l_k|^2$ for each $\mathbf{c}\in\ell^2_L(\mathbb{Z})$. Assume that the matrix $\mathbb{G}$ in (30) satisfies $0<\alpha_{\mathbb{G}}\le\beta_{\mathbb{G}}<\infty$, and let $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ be an error sequence satisfying the inequality $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$. Then, proceeding as in Section 4, there exists a frame $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ for $\mathcal{A}_{\mathbf{a}}$ such that, for any $x\in\mathcal{A}_{\mathbf{a}}$ a sampling formula as in (15) holds.

Now assume that $b_j \in D_T$, $j = 1, 2, \ldots, s$; thus the functions $\mathcal{L}_{b_j} a_l(t) \equiv \mathcal{L}_j a_l(t)$, $j = 1, 2, \ldots, s$ and $l = 1, 2, \ldots, L$, are continuously differentiable on $\mathbb{R}$. Again, as in Section 4, under the decay condition (17) for each $(\mathcal{L}_j a_l)'(t)$, $j = 1, 2, \ldots, s$ and $l = 1, 2, \ldots, L$, one can easily prove that there exists $\delta > 0$ such that $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{mj}| < \delta$ for each $j = 1, 2, \ldots, s$, implies that $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$ for the error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$.

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