Finite sampling in multiple generated U-invariant subspaces

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Abstract

The relevance in sampling theory of U-invariant subspaces of a Hilbert space $\mathcal{H}$, where U denotes a unitary operator on $\mathcal{H}$, is nowadays a recognized fact. Indeed, shift-invariant subspaces of $L^2(\mathbb{R})$ become a particular example; periodic extensions of finite signals provide also a remarkable example. As a consequence, the availability of an abstract U-sampling theory becomes a useful tool to handle these problems. In this paper we derive a sampling theory for finite dimensional multiple generated U-invariant subspaces of a Hilbert space $\mathcal{H}$. As the involved samples are identified as frame coefficients in a suitable euclidean space, the relevant mathematical technique is that of finite frame theory. Since finite frames are nothing but spanning sets of vectors, the used technique naturally meets matrix analysis.

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1 Statement of the problem

In this paper a sampling theory for multiple generated finite U-invariant subspaces of a Hilbert space $\mathcal{H}$ is developed; as usual, U denotes a unitary operator on $\mathcal{H}$. If we denote the set of generators as $\mathbf{a} := \{a_1, a_2, \dots, a_L\}$ and their respective orders as $N_1, N_2, \dots, N_L$, i.e., $U^{N_i} a_i = a_i$, $i = 1, 2, \dots, L$, the U-invariant subspace $\mathcal{A}_\mathbf{a}$ is defined by

$$\mathcal{A}_\mathbf{a} := \operatorname{span}\left\{a_i, Ua_i, U^2 a_i, \dots, U^{N_i-1} a_i\right\}_{i=1}^{L}.$$

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Linear independence of the vectors above gives an $N_1 + N_2 + \cdots + N_L$ dimensional subspace in $\mathcal{H}$ described as

$$\mathcal{A}_{\mathbf{a}} = \Big\{ \sum_{i=1}^{L} \sum_{k=0}^{N_i-1} \alpha_k^i \, U^k a_i \mid \alpha_k^i \in \mathbb{C} \Big\}.$$

These spaces are very natural in signal theory. As an academic example, consider the three dimensional subspace $\mathcal{A}$ of all 4-periodic signals depicted in Fig. 1; signals in $\mathcal{A}$ are obtained by a symmetrization process. If we consider the shift operator

$$U : \{x(m)\} \longmapsto \{x(m-2)\},$$

the U-invariant subspace generated by the two 4-periodic signals $a_1 = (1, 0, 0, 0)$ and $a_2 = (0, 1, 0, 1)$ coincides with $\mathcal{A}$. Indeed, they have orders 2 and 1 respectively, i.e., $U^2 a_1 = a_1$ and $U a_2 = a_2$; besides, the set $\{a_1, U a_1, a_2\}$ forms a basis for $\mathcal{A}$.

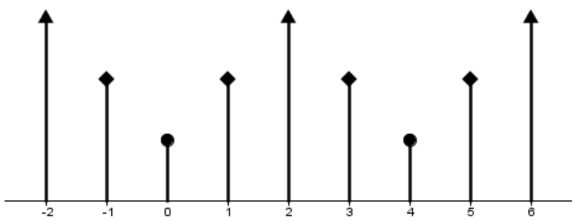

Figure 1: Symmetric extension of a finite signal $M = 4$

Formally, a sampling formula in $\mathcal{A}_{\mathbf{a}}$ is an expression as

$$x = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn) \, U^{rn} c_j \,, \quad x \in \mathcal{A}_{\mathbf{a}}\,, \tag{1}$$

where $c_j \in \mathcal{A}_{\mathbf{a}}$, $j = 1, 2, \ldots, s$, and $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\ldots,s \\ n=0,1,\ldots,\ell-1}}$ denotes the set of samples of $x \in \mathcal{A}_{\mathbf{a}}$. The meaning of all terms appearing in formula (1) is the following: the sequence of generalized samples $\{(\mathcal{L}_j x)(rm)\}_{\substack{j=1,2,\ldots,s \\ n=0,1,\ldots,\ell-1}}$ of $x \in \mathcal{A}_a$, where $r \in \mathbb{N}$ denotes the sampling period, is obtained from s elements $b_j \in \mathcal{H}$ as

$$(\mathcal{L}_j x)(rm) := \langle x, U^{rm} b_j \rangle_{\mathcal{H}}\,, \quad m \in \mathbb{Z}\,;\ j = 1, 2, \ldots, s\,. \tag{2}$$

For each $j = 1, 2, \ldots, s$, the number of samples is $\ell = N/r$, where N is the least common multiple of the orders $N_1, N_2, \ldots, N_L$, and the sampling period r divides N.

In the particular case of the shift operator $U : f(t) \mapsto f(t-1)$ in $L^2(\mathbb{R})$, the samples defined in (2) are nothing but

$$(\mathcal{L}_j f)(rm) = \langle f, U^{rm} b_j \rangle_{L^2(\mathbb{R})} = \big(f * h_j\big)(rm)\,,\ m \in \mathbb{Z}\,,\ \text{where } h_j(t) := \overline{b_j(-t)}\,.$$

The sequence $\{U^{rn}c_j\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ in formula (1) should be a frame for $\mathcal{A}_{\mathbf{a}}$. In finite dimension, frames are nothing but spanning sets of vectors, and they require control of certain condition numbers as well as over the spectrum of certain matrices. Thus, frame theory, firstly considered applied harmonic analysis, meets matrix analysis and numerical linear algebra. See, for instance, Ref. [4] for finite frames and their applications.

The frame concept for a general Hilbert space $\mathcal{H}$ was introduced by Duffin and Shaeffer in [7] while studying some problems in nonharmonic Fourier series; some years later it was revived by Daubechies, Grossman and Meyer in [6]. Nowadays, frames have become a tool in pure and applied mathematics, computer science, physics and engineering used to derive redundant, yet stable decompositions of a signal for analysis or transmission, while also promoting sparse expansions. As it is pointed out in the nice introduction of Chapter 1 in Ref. [4], traditionally, frames were used in signal and image processing, nonharmonic analysis, data compression, and sampling theory, but nowadays frame theory plays also a fundamental role in a wide variety of problems in both pure and applied mathematics, computer science, physics and engineering. The redundancy of frames, which gives flexibility and robustness, is the key to their significance for applications. For the frame theory, see, for instance, Ref. [5].

The use of frames in sampling theory is nowadays very fruitful: see, for instance, [1, 2, 3, 10, 18] and references therein. Recently, in [8, 9, 16] a generalization of the sampling theory for shift-invariant subspaces $V_\varphi^2 := \big\{\sum_{n\in\mathbb{Z}} \alpha_n\, \varphi(t-n)\ :\ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\big\}$ of $L^2(\mathbb{R})$ with generator φ has been obtained in the following sense: Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by $\mathcal{A}_a := \overline{\operatorname{span}}\big\{U^n a,\ n \in \mathbb{Z}\big\}$. In case that the (infinite) sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \Big\{\sum_{n\in\mathbb{Z}} \alpha_n\, U^n a :\ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\Big\}\,.$$

Under appropriate hypotheses it was proved in [9, 16], by using different techniques, the existence of frames in $\mathcal{A}_a$, having the form $\big\{U^{rm}c_j\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$, where $c_j \in \mathcal{A}_a$ for $j = 1, 2, \dots, s$, such that for each $x \in \mathcal{A}_a$ we get the sampling expansion

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, U^{rm} c_j \quad \text{in } \mathcal{H}\,. \tag{3}$$

The U-sampling problem was introduced, for the first time, in Refs. [14, 16]. It includes, in particular, sampling in shift-invariant subspaces V_φ^2 where $U : f(t) \mapsto f(t-1)$ is the shift operator in $L^2(\mathbb{R})$. Here, the sampling formula (3) for $f \in V_\varphi^2$ reads

$$f(t) = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \big(\mathcal{L}_j f\big)(rm)\, S_j(t-rm)\,, \quad t \in \mathbb{R}\,,$$

where the sequence of reconstruction functions $\{S_j(\cdot - rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for V_φ^2 (see, for instance, Ref. [10] and the references therein for the details).

Concerning U-sampling, in Refs. [8, 9, 16] it was implicitely assumed that the stationary sequence $\{U^n a\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ has infinite different elements. Later, in Ref. [11] it was first time handled the finite case with only one generator.

In the present paper we deal with the sampling theory associated to the case of L generators a_i with different orders N_i, $i=1,2,\dots,L$. The key point is to identify the samples in (1) as frame coefficients with respect to a suitable frame in $\mathbb{C}^{N_1+N_2+\cdots+N_L}$; this will be done in Section 2. Thus the problem reduces to find some particular dual frames in $\mathbb{C}^{N_1+N_2+\cdots+N_L}$ yielding, via an isomorphism $\mathcal{T}_{\mathbf{N},\mathbf{a}}$ between $\mathbb{C}^{N_1+N_2+\cdots+N_L}$ and $\mathcal{A}_{\mathbf{a}}$, the desired frames $\big\{U^{rn}c_j\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ for $\mathcal{A}_{\mathbf{a}}$ appearing in formula (1); this will be done in Section 3. As we will see, this is equivalent to searching for left-inverses of certain matrix associated to the sampling problem having a compatible U-structure. The obtained sampling theory has a nice application to the study of periodic extensions of finite signals; this is the aim of Section 4.

2 The mathematical setting

Let $\mathbf{a}:=\{a_1,a_2,\dots,a_L\}$ be a fixed subset of $\mathcal{H}$, and assume that there exist positive integers $N_1,N_2,\dots,N_L$ such that $U^{N_i}a_i=a_i$, $i=1,2,\dots,L$; suppose each N_i is the smallest index with this property; we denote N_i the order of the element a_i. Next, we consider the finite dimensional subspace of $\mathcal{H}$

$$\mathcal{A}_{\mathbf{a}}:=\operatorname{span}\big\{a_i,Ua_i,U^2a_i,\dots,U^{N_i-1}a_i\big\}_{i=1}^{L}.$$

In case of linear independence of the involved vectors, the $N_1+N_2+\cdots+N_L$ dimensional subspace $\mathcal{A}_{\mathbf{a}}$ of $\mathcal{H}$ can be described as

$$\mathcal{A}_{\mathbf{a}}=\Big\{\sum_{i=1}^{L}\sum_{k=0}^{N_i-1}\alpha_k^i\,U^k a_i \mid \alpha_k^i\in\mathbb{C}\Big\}.$$

Thus, we refer $\mathcal{A}_{\mathbf{a}}$ as a U-invariant subspace of $\mathcal{H}$ with L generators $a_1,a_2,\dots,a_L$ with respective orders $N_1,N_2,\dots,N_L$. The choice $N:=\text{l.c.m.}\,(N_1,N_2,\dots,N_L)$ yields the smallest positive integer N such that $U^Nx=x$ for all $x\in\mathcal{A}_{\mathbf{a}}$.

Linear independence

In order to characterize the linear independence of the set $\big\{a_i,Ua_i,U^2a_i,\dots,U^{N_i-1}a_i\big\}_{i=1}^{L}$ in $\mathcal{H}$ we consider the square matrix of order $N_1+N_2+\cdots+N_L$

$$\mathbf{C}_{\mathbf{a}}:=\begin{pmatrix}\mathbf{C}_{a_1,a_1} & \mathbf{C}_{a_2,a_1} & \dots & \mathbf{C}_{a_L,a_1}\\ \mathbf{C}_{a_1,a_2} & \mathbf{C}_{a_2,a_2} & \dots & \mathbf{C}_{a_L,a_2}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{C}_{a_1,a_L} & \mathbf{C}_{a_2,a_L} & \dots & \mathbf{C}_{a_L,a_L}\end{pmatrix},$$

where for $1\le i,j\le L$ each block $\mathbf{C}_{a_i,a_j}$ denotes the $N_j\times N_i$ matrix

$$\mathbf{C}_{a_i,a_j}:=\Big(\langle U^l a_i,U^k a_j\rangle\Big)_{0\le k\le N_j-1;\ 0\le l\le N_i-1}.$$

Proposition 1. *The set of vectors* $\{a_i, Ua_i, U^2a_i, \dots, U^{N_i-1}a_i\}_{i=1}^{L}$ *is linear independent in* $\mathcal{H}$ *if and only if* $\det \mathbf{C_a} \neq 0$.

Proof. Assume that the set is linearly independent; if $\det \mathbf{C_a} = 0$ then there exists a nonzero vector $\boldsymbol{\lambda}$ in $\mathbb{C}^{N_1+N_2+\cdots+N_L}$

$$\boldsymbol{\lambda} = \big(\lambda_{1,0}, \lambda_{1,1}, \dots, \lambda_{1,N_1-1}, \lambda_{2,0}, \lambda_{2,1}, \dots, \lambda_{2,N_2-1} \dots \lambda_{L,0}, \lambda_{L,1}, \dots, \lambda_{L,N_L-1}\big)^\top$$

such that $\mathbf{C_a}\boldsymbol{\lambda} = 0$. Consider the vector $b = \sum_{i=1}^{L}\sum_{k=0}^{N_i-1} \lambda_{i,k}U^k a_i$; condition $\mathbf{C_a}\boldsymbol{\lambda} = 0$ implies that b is orthogonal to every vector $U^r a_s$ with $1 \le s \le L$ and $0 \le r \le N_s - 1$. Thus $b = 0$, which implies $\boldsymbol{\lambda} = 0$, a contradiction.

Conversely, if $\sum_{i=1}^{L}\sum_{k=0}^{N_i-1} \lambda_{i,k}U^k a_i = 0$ for some $\boldsymbol{\lambda} \neq 0$ then the inner products $\big\langle \sum_{i=1}^{L}\sum_{k=0}^{N_i-1} \lambda_{i,k}U^k a_i, U^r a_s\big\rangle$ vanish for every $1 \le s \le L$ and $0 \le r \le N_s - 1$; this implies that $\mathbf{C_a}\boldsymbol{\lambda} = 0$ and consequently $\det \mathbf{C_a} = 0$.

□

The isomorphism $\mathcal{T}_{\mathbf{N},\mathbf{a}}$

As it was announced in Section 1, our technique consists of the translation of frame expansions in $\mathbb{C}^{N_1+N_2+\cdots+N_L}$ to the desired sampling formulas (1) in $\mathcal{A}_\mathbf{a}$. This will be done by means of the following isomorphism

$$\begin{array}{rcl} \mathcal{T}_{\mathbf{N},\mathbf{a}} : \mathbb{C}^{N_1+N_2+\cdots+N_L} & \longrightarrow & \mathcal{A}_\mathbf{a} \\ \boldsymbol{\alpha} = \Big((\alpha_k^1)_{k=0}^{N_1-1}, (\alpha_k^2)_{k=0}^{N_2-1}, \dots, (\alpha_k^L)_{k=0}^{N_L-1}\Big)^\top & \longmapsto & x = \displaystyle\sum_{i=1}^{L}\sum_{k=0}^{N_i-1} \alpha_k^i\, U^k a_i\,, \end{array} \tag{4}$$

where $\mathbf{N} := (N_1, N_2, \dots, N_L)$ denotes the involved orders. Besides, this isomorphism $\mathcal{T}_{\mathbf{N},\mathbf{a}}$ has a crucial shifting property. To state it, consider any vector $\mathbf{T}_0 \in \mathbb{C}^{N_1+N_2+\cdots+N_L}$ denoted by

$$\mathbf{T}_0 = \big(\mathbf{T}_0^1, \mathbf{T}_0^2, \dots, \mathbf{T}_0^L\big)^\top, \tag{5}$$

where each block, denoted by $\mathbf{T}_0^i = \big(T^i(0), T^i(1), \dots, T^i(N_i-1)\big)$, $1 \le i \le L$, is a row vector of dimension N_i. Set $N = \text{l.c.m.}\,(N_1, N_2, \dots N_L)$ and $1 \le m \le N-1$. Starting from $\mathbf{T}_0$ we define a new vector $\mathbf{T}_{N-m}$ in $\mathbb{C}^{N_1+N_2+\cdots+N_L}$ given by

$$\mathbf{T}_{N-m} := \big(\mathbf{T}_{N-m}^1, \mathbf{T}_{N-m}^2, \dots, \mathbf{T}_{N-m}^L\big)^\top, \tag{6}$$

where each block $\mathbf{T}_{N-m}^i = \big(T^i(N-m), T^i(N-m+1), \dots, T^i(N-m+N_i-1)\big)$, is obtained by assuming an N_i-periodic character in each $T^i(\cdot)$, $i = 1, 2, \dots, L$. Thus:

Lemma 1. *For any* $1 \le m \le N-1$*, consider the vectors* $\mathbf{T}_0$ *and* $\mathbf{T}_{N-m}$ *in* $\mathbb{C}^{N_1+N_2+\cdots+N_L}$ *respectively given by* (5) *and* (6)*. Then the following shifting property holds*

$$\mathcal{T}_{\mathbf{N},\mathbf{a}}(\mathbf{T}_{N-m}) = U^m\big(\mathcal{T}_{\mathbf{N},\mathbf{a}}(\mathbf{T}_0)\big)\,.$$

Proof. We have that

$$\mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\mathbf{T}_{N-m}\big) = \sum_{i=1}^{L}\sum_{k=0}^{N_i-1} T^i(N-m+k)\, U^k a_i\,.$$

The change of index $p = N - m + k$ and the N_i-periodic character of the corresponding i part of $\mathbf{T}_0$, for any $i = 1, 2, \dots, L$, give

$$\begin{aligned}\sum_{k=0}^{N_i-1} T^i(N-m+k)\, U^k a_i &= \sum_{p=N-m}^{N-m+N_i-1} T^i(p)\, U^{p-N+m} a_i = \sum_{p=N-m}^{N-m+N_i-1} T^i(p)\, U^{p+m} a_i \\ &= \sum_{q=0}^{N_i-1} T^i(q)\, U^{q+m} a_i = U^m\Big(\sum_{q=0}^{N_i-1} T^i(q)\, U^q a_i\Big)\,.\end{aligned}$$

Collecting all the pieces we get the desired result $\mathcal{T}_{\mathbf{N},\mathbf{a}}(\mathbf{T}_{N-m}) = U^m\big(\mathcal{T}_{\mathbf{N},\mathbf{a}}(\mathbf{T}_0)\big)$. $\square$

A suitable expression for the generalized samples

From now on we consider a sampling period r which divides $N = \text{l.c.m.}\,(N_1, N_2, \dots N_L)$ and $\ell := N/r$. Fixed s elements $b_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, for each $x \in \mathcal{A}_\mathbf{a}$ we consider its $s\ell$ generalized samples $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ with sampling period r defined by

$$\mathcal{L}_j x(rn) := \langle x, U^{rn} b_j\rangle_\mathcal{H}\,, \quad n = 0, 1, \dots, \ell-1 \text{ and } j = 1, 2, \dots, s\,.$$

Note that the samples are N-periodic. We refer $\mathcal{L}_j$ as the U-system associated with $b_j \in \mathcal{H}$.

The goal in this paper is to recover any $x = \sum_{i=1}^{L}\sum_{k=0}^{N_i-1} \alpha_k^i\, U^k a_i \in \mathcal{A}_\mathbf{a}$ from its finite sequence of samples $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ by means of suitable frames in $\mathcal{A}_\mathbf{a}$.

First, we obtain a convenient expression for the sample $\mathcal{L}_j x(rn)$; namely

$$\begin{aligned}\mathcal{L}_j x(rn) &= \langle x, U^{rn} b_j\rangle_\mathcal{H} = \sum_{i=1}^{L}\sum_{k=0}^{N_i-1} \alpha_k^i\, \langle U^k a_i, U^{rn} b_j\rangle_\mathcal{H} \\ &= \sum_{i=1}^{L} \Big\langle \sum_{k=0}^{N_i-1} \alpha_k^i\, \mathbf{e}_\mathbf{k}^\mathbf{i}, \sum_{k=0}^{N_i-1} \overline{\langle U^k a_i, U^{rn} b_j\rangle_\mathcal{H}}\, \mathbf{e}_\mathbf{k}^\mathbf{i}\Big\rangle_{\mathbb{C}^{N_i}} = \sum_{i=1}^{L} \big\langle \boldsymbol{\alpha}^{\boldsymbol{i}}, \mathbf{G}_{j,n}^i\big\rangle_{\mathbb{C}^{N_i}}\,,\end{aligned}$$

where $\big\{\mathbf{e}_\mathbf{k}^\mathbf{i}\big\}_{k=0}^{N_i-1}$ denotes the canonical basis for $\mathbb{C}^{N_i}$, $\boldsymbol{\alpha}^{\boldsymbol{i}} := (\alpha_0^i, \alpha_1^i, \dots, \alpha_{N_i-1}^i)^\top \in \mathbb{C}^{N_i}$, and $\mathbf{G}_{j,n}^i := \sum_{k=0}^{N_i-1} \overline{\langle U^k a_i, U^{rn} b_j\rangle}_\mathcal{H}\, \mathbf{e}_\mathbf{k}^\mathbf{i} \in \mathbb{C}^{N_i}$.

Finally, if we consider the vectors in $\mathbb{C}^{N_1+N_2+\cdots+N_L}$

$$\boldsymbol{\alpha} := (\boldsymbol{\alpha}^\mathbf{1}, \boldsymbol{\alpha}^\mathbf{2}, \dots, \boldsymbol{\alpha}^{\boldsymbol{L}})^\top \quad \text{and} \quad \mathbf{G}_{j,n} := (\mathbf{G}_{j,n}^1, \mathbf{G}_{j,n}^2, \dots, \mathbf{G}_{j,n}^L)^\top$$

(identifying the vector $\mathbf{G}_{j,n}^i$ with its coordinates in $\mathbb{C}^{N_i}$) we get the expression for the samples

$$\mathcal{L}_j x(rn) = \big\langle \boldsymbol{\alpha}, \mathbf{G}_{j,n}\big\rangle_{\mathbb{C}^{N_1+N_2+\cdots+N_L}}\,, \tag{7}$$

where $j = 1, 2, \dots, s$ and $n = 0, 1, \dots, \ell-1$. As a consequence of the above expression, having in mind the equivalence between spanning sets and frames (see, for instance, Refs. [4, 5]), we obtain that

Proposition 2. *Any $\boldsymbol{\alpha} \in \mathbb{C}^{N_1+N_2+\cdots+N_L}$ can be recovered from the sequence of samples $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ if and only if the set of vectors $\big\{\mathbf{G}_{j,n}\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ forms a spanning set for $\mathbb{C}^{N_1+N_2+\cdots+N_L}$, that is, a frame for $\mathbb{C}^{N_1+N_2+\cdots+N_L}$.*

In particular, we derive the necessary condition on the number of U-systems s:

$$s\ell \geq N_1 + N_2 + \cdots + N_L\,.$$

Via the isomorphism $\mathcal{T}_{\mathbf{N},\mathbf{a}}$ we conclude in $\mathcal{A}_{\mathbf{a}}$

Corollary 2. *Any element $x \in \mathcal{A}_{\mathbf{a}}$ can be recovered from the sequence of its generalized samples $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ if and only if the set of vectors $\big\{\mathbf{G}_{j,n}\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ is a frame for $\mathbb{C}^{N_1+N_2+\cdots+N_L}$.*

The *cross-covariance* between the stationary sequences $\{U^n a_i\}$ and $\{U^n b_j\}$ is the N_i-periodic sequence defined in [13] as

$$R_{a_i,b_j}(m) := \langle U^m a_i, b_j\rangle_{\mathcal{H}}\,, \quad m \in \mathbb{Z}\,. \tag{8}$$

The choice of $N = \text{l.c.m.}\,(N_1, N_2, \dots, N_L)$ allows to write, for each $i = 1, 2, \dots, L$,

$$\begin{aligned}\mathbf{G}^i_{j,n} &= \sum_{k=0}^{N_i-1} \overline{\langle U^{k-rn} a_i, b_j\rangle_{\mathcal{H}}}\, \mathbf{e}^{\mathbf{i}}_{\mathbf{k}} = \sum_{k=0}^{N_i-1} \overline{\langle U^{N+k-rn} a_i, b_j\rangle_{\mathcal{H}}}\, \mathbf{e}^{\mathbf{i}}_{\mathbf{k}} \\ &= \sum_{k=0}^{N_i-1} \overline{R_{a_i,b_j}(N+k-rn)}\, \mathbf{e}^{\mathbf{i}}_{\mathbf{k}} \in \mathbb{C}^{N_i}\,.\end{aligned}$$

Therefore, in terms of coordinates respect to the canonical basis for $\mathbb{C}^{N_1+N_2+\cdots+N_L}$ we have

$$\mathbf{G}_{j,n} = \Big(\mathbf{G}^1_{j,n}, \mathbf{G}^2_{j,n}, \dots, \mathbf{G}^L_{j,n}\Big)^\top \in \mathbb{C}^{N_1+N_2+\cdots+N_L}$$

where, for each $i = 1, 2, \dots, L$, $\mathbf{G}^i_{j,n}$ denotes the row vector (of dimension N_i)

$$\mathbf{G}^i_{j,n} = \Big(\overline{R_{a_i,b_j}(N-rn)}, \overline{R_{a_i,b_j}(N-rn+1)}, \dots, \overline{R_{a_i,b_j}(N-rn+N_i-1)}\Big)\,.$$

Next, for each $x \in \mathcal{A}_{\mathbf{a}}$ we denote the vector of its samples by

$$\boldsymbol{\mathcal{L}}_{\text{sam}} x := \big(\boldsymbol{\mathcal{L}}_1 x, \boldsymbol{\mathcal{L}}_2 x, \dots, \boldsymbol{\mathcal{L}}_s x\big)^\top \in \mathbb{C}^{s\ell}\,,$$

where, for each $j = 1, 2, \dots, s$, the row vector $\boldsymbol{\mathcal{L}}_j x := \big(\mathcal{L}_j x(0), \mathcal{L}_j x(r), \dots, \mathcal{L}_j x(r(\ell-1))\big)$ has dimension ℓ.
Then if we consider the vector $\boldsymbol{\alpha} := (\boldsymbol{\alpha}^{\mathbf{1}}, \boldsymbol{\alpha}^{\mathbf{2}}, \dots, \boldsymbol{\alpha}^{\boldsymbol{L}})^\top$ in $\mathbb{C}^{N_1+N_2+\cdots+N_L}$ associated to $x = \displaystyle\sum_{i=1}^{L} \sum_{k=0}^{N_i-1} \alpha^i_k\, U^k a_i \in \mathcal{A}_{\mathbf{a}}$, expression (7) for the samples can be written in a matrix form as

$$\boldsymbol{\mathcal{L}}_{\text{sam}} x = \mathbf{R}_{\mathbf{a},\mathbf{b}}\, \boldsymbol{\alpha}\,, \tag{9}$$

where the $s\ell \times (N_1 + N_2 + \cdots + N_L)$ matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ has the following block structure

$$\mathbf{R}_{\mathbf{a},\mathbf{b}} = \begin{pmatrix} \mathbf{R}_{a_1,b_1} & \mathbf{R}_{a_2,b_1} & \dots & \mathbf{R}_{a_L,b_1} \\ \mathbf{R}_{a_1,b_2} & \mathbf{R}_{a_2,b_2} & \dots & \mathbf{R}_{a_L,b_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}_{a_1,b_s} & \mathbf{R}_{a_2,b_s} & \dots & \mathbf{R}_{a_L,b_s} \end{pmatrix} \tag{10}$$

and, for each $1 \le i \le L$ and $1 \le j \le s$, the block $\mathbf{R}_{a_i,b_j}$ is the $\ell \times N_i$ matrix formed by using the cross-covariance $R_{a_i,b_j}(m)$ (8) between a_i and b_j as

$$\mathbf{R}_{a_i,b_j} = \begin{pmatrix} R_{a_i,b_j}(0) & R_{a_i,b_j}(1) & \dots & R_{a_i,b_j}(N_i-1) \\ R_{a_i,b_j}(N-r) & R_{a_i,b_j}(N-r+1) & \dots & R_{a_i,b_j}(N-r+N_i-1) \\ \vdots & \vdots & \ddots & \vdots \\ R_{a_i,b_j}(N-r(\ell-1)) & R_{a_i,b_j}(N-r(\ell-1)+1) & \dots & R_{a_i,b_j}(N-r(\ell-1)+N_i-1) \end{pmatrix}$$

Notice that, in sampling terms, the matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is formed from samples of the generators a_i, $i = 1, 2, \dots, L$, following the U-systems $\mathcal{L}_j$, $j = 1, 2, \dots, s$.
Corollary 2 can be now restated in terms of the matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ as

Corollary 3. *Any element $x \in \mathcal{A}_{\mathbf{a}}$ can be recovered from the sequence of its generalized samples $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ if and only if the matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ has rank $N_1+N_2+\cdots+N_L$.*

Whenever $\operatorname{rank} \mathbf{R}_{\mathbf{a},\mathbf{b}} = N_1 + N_2 + \cdots + N_L$, its Moore-Penrose pseudo-inverse is the $(N_1 + N_2 + \cdots + N_L) \times s\ell$ matrix $\mathbf{R}^\dagger_{\mathbf{a},\mathbf{b}} = \big[\mathbf{R}^*_{\mathbf{a},\mathbf{b}} \mathbf{R}_{\mathbf{a},\mathbf{b}}\big]^{-1} \mathbf{R}^*_{\mathbf{a},\mathbf{b}}$. Besides, any left-inverse $\mathbf{H}$ of the matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$, i.e., $\mathbf{H}\, \mathbf{R}_{\mathbf{a},\mathbf{b}} = \mathbf{I}_{N_1+N_2+\cdots+N_L}$, can be expressed as (see [15]):

$$\mathbf{H} = \mathbf{R}^\dagger_{\mathbf{a},\mathbf{b}} + \mathbf{U}\big[\mathbf{I}_{s\ell} - \mathbf{R}_{\mathbf{a},\mathbf{b}} \mathbf{R}^\dagger_{\mathbf{a},\mathbf{b}}\big]\,, \tag{11}$$

where $\mathbf{U}$ denotes any arbitrary $N \times s\ell$ matrix.

Let $\mathbf{H}$ be a left-inverse of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ and denote its columns by

$$\mathbf{H} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{H}_{1,0} & \dots & \mathbf{H}_{1,\ell-1} & \mathbf{H}_{2,0} & \dots & \mathbf{H}_{2,\ell-1} & \dots & \mathbf{H}_{s,0} & \dots & \mathbf{H}_{s,\ell-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{12}$$

A left multiplication by $\mathbf{H}$ in (9) gives

$$\boldsymbol{\alpha} = \mathbf{H}\, \boldsymbol{\mathcal{L}}_{\text{sam}} x = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \mathbf{H}_{j,n}\,. \tag{13}$$

Thus, the columns $\big\{\mathbf{H}_{j,n}\big\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ of any left-inverse $\mathbf{H}$ of the matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ give the family of all dual frames of the frame $\big\{\mathbf{G}_{j,n}\big\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ in $\mathbb{C}^{N_1+N_2+\cdots+N_L}$.

Applying the isomorphism $\mathcal{T}_{\mathbf{N},\mathbf{a}}$ in formula (13), for any $x = \sum_{i=1}^{L} \sum_{k=0}^{N_i-1} \alpha^i_k\, U^k a_i \in \mathcal{A}_{\mathbf{a}}$ we finally obtain the sampling formula:

$$x = \mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\boldsymbol{\alpha}\big) = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\mathbf{H}_{j,n}\big)\,. \tag{14}$$

The sampling functions $\mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\mathbf{H}_{j,n}\big)$ in the above formula do not have, in principle, any special U-structure as those sampling functions in (1) since the columns in $\mathbf{H}$ do not have any particular structure. However, in next section we find all the left-inverses matrices of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ compatible with the U-structure yielding sampling formulas like (1).

3 The sampling result

Assume at least for now that there exists a left-inverse $\mathbf{H}$ with the same structure that the matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ by columns. In other words, for $j = 1, 2, \dots, s$ and $n = 1, 2, \dots, \ell - 1$ its column $\mathbf{H}_{j,n}$ is obtained from the column $\mathbf{H}_{j,0}$ as follows. Denote the column $\mathbf{H}_{j,0}$ by

$$\mathbf{H}_{j,0} = \big(\mathbf{S}^1_{j,0}\,, \mathbf{S}^2_{j,0}\,, \dots, \mathbf{S}^L_{j,0}\,\big)^\top \in \mathbb{C}^{N_1+N_2+\cdots+N_L}$$

where, for each $i = 1, 2, \dots, L$, $\mathbf{S}^i_{j,0}$ denotes the row vector (of dimension N_i)

$$\mathbf{S}^i_{j,0} = \big(S^i_j(0), S^i_j(1), \dots, S^i_j(N_i - 1)\big)\,.$$

Then

$$\mathbf{H}_{j,n} = \big(\mathbf{S}^1_{j,n}\,, \mathbf{S}^2_{j,n}\,, \dots, \mathbf{S}^L_{j,n}\,\big)^\top \in \mathbb{C}^{N_1+N_2+\cdots+N_L}$$

where, for each $i = 1, 2, \dots, L$, $\mathbf{S}^i_{j,n}$ denotes the row vector (of dimension N_i)

$$\mathbf{S}^i_{j,n} = \big(S^i_j(N - rn), S^i_j(N - rn + 1), \dots, S^i_j(N - rn + N_i - 1)\big)\,.$$

As in Lemma 1, we are also extending each $S^i_j(\cdot)$ taking into account N_i-periodicity.

Therefore, the shifting property in Lemma 1 gives

$$\mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\mathbf{H}_{j,n}\big) = U^{rn}\big(\mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\mathbf{H}_{j,0}\big)\big) \quad \text{for } n = 0, 1, \dots, \ell - 1\,,$$

and, consequently, formula (14) reads as

$$\begin{aligned} x = \mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\boldsymbol{\alpha}\big) &= \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\mathbf{H}_{j,n}\big) \\ &= \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, U^{rn}\big(\mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\mathbf{H}_{j,0}\big)\big) = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, U^{rn} c_j\,, \end{aligned}$$

where $c_j = \mathcal{T}_{\mathbf{N},\mathbf{a}}\big(\mathbf{H}_{j,0}\big) \in \mathcal{A}_{\mathbf{a}}$, $j = 1, 2, \dots, s$.

Left-inverses of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ with its same structure

We now proceed to construct left-inverses of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ having the required structure in their columns. Indeed, let $\mathbf{H}$ be any left-inverse of the matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$. This matrix $\mathbf{H}$ has order $(N_1 + N_2 + \cdots + N_L) \times s\ell$ and it can be written in blocks as

$$\mathbf{H} = \begin{pmatrix} \mathbf{S}^1_1 & \mathbf{S}^1_2 & \dots & \mathbf{S}^1_s \\ \mathbf{S}^2_1 & \mathbf{S}^2_2 & \dots & \mathbf{S}^2_s \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}^L_1 & \mathbf{S}^L_2 & \dots & \mathbf{S}^L_s \end{pmatrix}\,,$$

where each block $\mathbf{S}_j^i$ denotes an $N_i \times \ell$ matrix, $i = 1, 2, \ldots, L$ and $j = 1, 2, \ldots, s$. Following [11, Section 3] the idea consists in defining a new left-inverse $\widetilde{\mathbf{H}}$ of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ to be the modification to $\mathbf{H}$ by replacing every block $\mathbf{S}_j^i$ such that $N_i > r$ by a suitable $N_i \times \ell$ block $\widetilde{\mathbf{S}}_j^i$. Namely, the required steps are:

1. If $N_i > r$, denote the first r rows of $\mathbf{S}_j^i$ as

$$\mathbf{S}_j^{i,r} := \begin{pmatrix} S_j^i(0) & S_j^i(N-r) & S_j^i(N-2r) & \ldots & S_j^i(N-r(\ell-1)) \\ S_j^i(1) & S_j^i(N-r+1) & S_j^i(N-2r+1) & \ldots & S_j^i(N-r(\ell-1)+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_j^i(r-1) & S_j^i(N-1) & S_j^i(N-2r+r-1) & \ldots & S_j^i(N-r(\ell-1)+r-1) \end{pmatrix}$$

2. Next we form a $N \times \ell$ matrix (note that $N = r\ell$)

$$\mathbf{S}_j^{i,N} := \begin{pmatrix} S_j^i(0) & S_j^i(N-r) & S_j^i(N-2r) & \ldots & S_j^i(r) \\ S_j^i(1) & S_j^i(N-r+1) & S_j^i(N-2r+1) & \ldots & S_j^i(r+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_j^i(r-1) & S_j^i(N-1) & S_j^i(N-r-1) & \ldots & S_j^i(2r-1) \\ S_j^i(r) & S_j^i(0) & S_j^i(N-r) & \ldots & S_j^i(2r) \\ S_j^i(r+1) & S_j^i(1) & S_j^i(N-r+1) & \ldots & S_j^i(2r+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_j^i(N-1) & S_j^i(N-r-1) & S_j^i(N-2r-1) & \ldots & S_j^i(r-1) \end{pmatrix}$$

following the rules in [11]:
• The first column of the matrix $\mathbf{S}_j^{i,N}$ is a concatenation of the columns 1, ℓ, $\ell-1$, …, and 2 of $\mathbf{S}_j^{i,r}$;
• The second column of $\mathbf{S}_j^{i,N}$ is a concatenation of the columns 2, 1, ℓ, …, and 3 of $\mathbf{S}_j^{i,r}$;
• The third column of $\mathbf{S}_j^{i,N}$ is a concatenation of the columns 3, 2, 1, …, and 4 of $\mathbf{S}_j^{i,r}$;
Repeating the process, finally,
• The column ℓ of $\mathbf{S}_j^{i,N}$ is a concatenation of the columns ℓ, $\ell-1$, $\ell-2$, …, and 1 of $\mathbf{S}_j^{i,r}$.

3. Take as the block $\widetilde{\mathbf{S}}_j^i$ the N_i first rows of the matrix $\mathbf{S}_j^{i,N}$.

4. If $N_i \leq r$, take $\widetilde{\mathbf{S}}_j^i := \mathbf{S}_j^i$.

5. Finally, consider the new $(N_1 + N_2 + \cdots + N_L) \times s\ell$ matrix $\widetilde{\mathbf{H}}$ written in blocks as

$$\widetilde{\mathbf{H}} = \begin{pmatrix} \widetilde{\mathbf{S}}_1^1 & \widetilde{\mathbf{S}}_2^1 & \ldots & \widetilde{\mathbf{S}}_s^1 \\ \widetilde{\mathbf{S}}_1^2 & \widetilde{\mathbf{S}}_2^2 & \ldots & \widetilde{\mathbf{S}}_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\mathbf{S}}_1^L & \widetilde{\mathbf{S}}_2^L & \ldots & \widetilde{\mathbf{S}}_s^L \end{pmatrix} \tag{15}$$

Thus, with the above procedure we have obtained a left-inverse matrix $\widetilde{\mathbf{H}}$ for $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ having the desired structure:

Lemma 4. *Let* $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ *and* $\widetilde{\mathbf{H}}$ *be the matrices defined in* (10) *and* (15) *respectively. Then we have that*

$$\widetilde{\mathbf{H}}\,\mathbf{R}_{\mathbf{a},\mathbf{b}} = \mathbf{I}_{N_1+N_2+\cdots+N_L}\,.$$

Proof. We represent by $\mathbf{H}^p$, $p=1,2,\ldots,L$, the extracted $N_p\times s\ell$ matrix from $\mathbf{H}$ taking the blocks $\mathbf{S}_1^p,\mathbf{S}_2^p,\ldots,\mathbf{S}_s^p$. Similarly, let $\mathbf{R}_{a_q,\mathbf{b}}$, $q=1,2,\ldots,L$, the extracted $s\ell\times N_q$ matrix from $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ taking the blocks $\mathbf{R}_{a_q,b_1},\mathbf{R}_{a_q,b_2},\ldots,\mathbf{R}_{a_q,b_s}$.
From

$$\mathbf{H}\,\mathbf{R}_{\mathbf{a},\mathbf{b}} = \mathbf{I}_{N_1+N_2+\cdots+N_L}$$

we derive that

$$\begin{cases}\mathbf{H}^q\,\mathbf{R}_{a_q,\mathbf{b}} = \mathbf{I}_{N_q}\\ \mathbf{H}^p\,\mathbf{R}_{a_q,\mathbf{b}} = \mathbf{O}_{N_p\times N_q} \quad \text{if} \quad p\neq q\end{cases}$$

where $p,q=1,2,\ldots,L$, and $\mathbf{I}_{N_p}$ and $\mathbf{O}_{N_p\times N_q}$ denote, respectively, the identity matrix of order N_p and the zero matrix of order $N_p\times N_q$.
Giving to the entries of each block $\mathbf{S}_j^{p,N}$ an N-periodic character and having in mind the N_q-periodic, and consequently N-periodic, character of the entries in matrix $\mathbf{R}_{a_q,\mathbf{b}}$, the product $\alpha_{m,k}^{p,q}$ of row $m+1$ of blocks $\widetilde{\mathbf{S}}_1^p,\widetilde{\mathbf{S}}_2^p,\ldots,\widetilde{\mathbf{S}}_s^p$, $p=1,2,\ldots,L$ and $m=0,1,\ldots,N_p-1$, with column $k+1$ of matrix $\mathbf{R}_{a_q,\mathbf{b}}$, $q=1,2,\ldots,L$ and $k=0,1,\ldots,N_q-1$, can be written as

$$\alpha_{m,k}^{p,q} = \sum_{j=1}^{s}\sum_{i=0}^{\ell-1} S_j^p(N-ir+m)R_{a_q,b_j}(N-ir+k)\,.$$

We now prove that $\alpha_{m,k}^{p,q}$ can be seen as the product of an appropriate row of $\mathbf{H}^p$ with the corresponding column of $\mathbf{R}_{a_q,\mathbf{b}}$. This is clear if $N_p\leq r$ or $m\leq r$ since then $\alpha_{m,k}^{p,q}$ is nothing but the product of row $m+1$ of blocks $\mathbf{S}_1^p,\mathbf{S}_2^p,\ldots,\mathbf{S}_s^p$, i.e, row $m+1$ of $\mathbf{H}^p$, with column $k+1$ of matrix $\mathbf{R}_{a_q,\mathbf{b}}$. In any case, note that for each $m\in\{0,1,\ldots,N_p-1\}$ there exist a unique $m'\in\{0,1,\ldots,r-1\}$ and a unique i' such that

$$N-i'r+m = N+m'\,.$$

In accord with k and i' take the unique $k'\in\{0,1,\ldots,N_q-1\}$ such that

$$N-i'r+k = k'+zN_q\,,$$

with $z\in\mathbb{Z}$. Thus, $\alpha_{m,k}^{p,q}=\alpha_{m',k'}^{p\ ,q}$ is just the product of row $m'+1$ of $\mathbf{H}^p$ with column $k'+1$ of $\mathbf{R}_{a_q,\mathbf{b}}$. Since $\mathbf{H}^p\,\mathbf{R}_{a_q,\mathbf{b}}=\mathbf{O}_{N_p\times N_q}$ for $p\neq q$, we have that $\alpha_{m,k}^{p,q}=0$ if $p\neq q$. Whenever $p=q$ we see that $m'=k'$ if and only if $m=k$. In fact, if $m'=k'$ it follows that $m-i'r=N-i'r+k-zN_q$ and consequently, as N is a multiple of N_q, we get that $m=N+k-zN_q=k+z''N_q$ with $z''\in\mathbb{Z}$. The equality $m=k$ follows from the fact that both m and k belong to $\{0,1,\ldots,N_q-1\}$. Finally from $\mathbf{H}^q\,\mathbf{R}_{a_q,\mathbf{b}}=\mathbf{I}_{N_q}$ we obtain that $\alpha_{m,k}^{q,q}=\delta_{m,k}$.

$\square$

Having in mind (11), we have to compute the pseudo-inverse $\mathbf{R}_{\mathbf{a},\mathbf{b}}^{\dagger}$. In practice it is done by using the singular value decomposition of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$; the singular values are the square root of the eigenvalues of the invertible and positive definite square matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}^{*}\,\mathbf{R}_{\mathbf{a},\mathbf{b}}$ of order $N_1+N_2+\cdots+N_L$ (see, for instance, [5, 12]). Note that the singular value decomposition of the matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is the most reliable method to reveal its rank in practice.

Collecting the pieces we have obtained until now allows to conclude our main result:

Theorem 5. *Given the $s\ell \times (N_1+N_2+\cdots+N_L)$ matrix of cross-covariances $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ defined in* (10)*, the following statements are equivalent:*

(a) rank $\mathbf{R}_{\mathbf{a},\mathbf{b}} = N_1+N_2+\cdots+N_L$

(b) There exists a structured left-inverse $\widetilde{\mathbf{H}}$ of matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ as in (15)*.*

(c) There exist $c_j \in \mathcal{A}_{\mathbf{a}}$, $j=1,2,\dots,s$ such that the sequence $\{U^{rn}c_j\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ is a frame for $\mathcal{A}_{\mathbf{a}}$, and for any $x \in \mathcal{A}_{\mathbf{a}}$ the expansion

$$x=\sum_{j=1}^{s}\sum_{n=0}^{\ell-1}\mathcal{L}_j x(rn)\,U^{rn}c_j \tag{16}$$

holds.

(d) There exists a frame $\{C_{j,n}\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ for $\mathcal{A}_{\mathbf{a}}$ such that, for each $x \in \mathcal{A}_{\mathbf{a}}$ the expansion

$$x=\sum_{j=1}^{s}\sum_{n=0}^{\ell-1}\mathcal{L}_j x(rn)\,C_{j,n}$$

holds.

Proof. That condition (a) implies condition (b) and condition (b) implies condition (c) have been proved above. Obviously, condition (c) implies condition (d): take $C_{j,n} = U^{rn}c_j$, $j=1,2,\dots,s$ and $n=0,1,\dots,\ell-1$. Finally, that condition (d) implies condition (a) is a consequence of Corollary 3. □

In the eventual case where $s\ell = N_1+N_2+\cdots+N_L$ we obtain:

Corollary 6. *Assume that $s\ell = N_1+N_2+\cdots+N_L$ and consider the cross-covariances square matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ of order $N_1+N_2+\cdots+N_L$ defined in* (10)*. The following statements are equivalent:*

(i) The square matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is invertible.

(ii) There exist s unique elements $c_j \in \mathcal{A}_{\mathbf{a}}$, $j=1,2,\dots,s$, such that the sequence $\{U^{rn}c_j\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ is a basis for $\mathcal{A}_a$, and the expansion of any $x \in \mathcal{A}_{\mathbf{a}}$ with respect to this basis is

$$x=\sum_{j=1}^{s}\sum_{n=0}^{\ell-1}\mathcal{L}_j x(rn)\,U^{rn}c_j\,.$$

In case the equivalent conditions are satisfied, the interpolation property $\mathcal{L}_j c_{j'}(rn) = \delta_{j,j'}\,\delta_{n,0}$, *whenever* $n = 0, 1, \ldots, \ell - 1$ *and* $j, j' = 1, 2, \ldots, s$, *holds.*

Proof. Notice that the inverse matrix $\mathbf{R}^{-1}_{\mathbf{a},\mathbf{b}}$ has necessarily the structure of the matrix $\widetilde{\mathbf{H}}$ in (15). The uniqueness of the expansion with respect to a basis gives the stated interpolation property. ◻

A filter-bank interpretation

Assume that rank of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ equals $N_1 + N_2 + \cdots + N_L$, and let $\widetilde{\mathbf{H}}$ be a structured left-inverse of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ as in (15) with columns $\mathbf{H}_{j,n}$, $j = 1, 2, \ldots, s$ and $n = 0, 1, \ldots, \ell - 1$, denoted as in expression (12). In the corresponding sampling formula (16) we have $c_j = \mathcal{T}_{N,a}\big(\mathbf{H}_{j,0}\big)$, $j = 1, 2, \ldots, s$; denote the component of $\mathbf{H}_{j,0}$ as

$$\mathbf{H}_{j,0} = \big(\beta^1_j(0), \beta^1_j(1), \ldots, \beta^1_j(N_1 - 1), \ldots, \beta^L_j(0), \beta^L_j(1), \ldots, \beta^L_j(N_L - 1)\big)^\top .$$

Substituting in (16), for $x \in \mathcal{A}_{\mathbf{a}}$ we get

$$\begin{aligned} x &= \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, U^{rn} \Big(\sum_{i=1}^{L} \sum_{k=0}^{N_i-1} \beta^i_j(k)\, U^k a_i \Big) \\ &= \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \Big(\sum_{i=1}^{L} \sum_{k=0}^{N_i-1} \beta^i_j(m)\, U^{rn+k} a_i \Big). \end{aligned}$$

The change of index $m := rn + k$ and assuming an $(N_1, N_2, \ldots, N_L)$-circulant character of each $\mathbf{H}_{j,0}$ gives

$$\begin{aligned} x &= \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \Big(\sum_{i=1}^{L} \sum_{m=rk}^{rk+N_i-1} \beta^i_j(m - rn)\, U^m a_i \Big) \\ &= \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \Big(\sum_{i=1}^{L} \sum_{m=0}^{N_i-1} \beta^i_j(m - rn)\, U^m a_i \Big) \\ &= \sum_{i=1}^{L} \sum_{m=0}^{N_i-1} \Big\{ \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \beta^i_j(m - rn) \Big\} U^m a_i . \end{aligned}$$

In other words, for $x = \sum_{i=1}^{L} \sum_{m=0}^{N_i-1} \alpha^i_m\, U^m a_i$, the coefficients α^i_m, $m = 0, 1, \ldots, N_i - 1$ are, for each $i = 1, 2, \ldots, L$, the output of a filter-bank

$$\alpha^i_m = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \beta^i_j(m - rn)\,, \quad m = 0, 1, \ldots, N_i - 1\,,$$

involving the data $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\ldots,s \\ n=0,1,\ldots,\ell-1}}$ and the columns $\mathbf{H}_{j,0}$, $j = 1, 2, \ldots, s$, of the matrix $\widetilde{\mathbf{H}}$ denoted as in expression (12).

4 An application to periodic extensions of finite signals

This final section is devoted to show how the usual periodic extensions of finite signals fit as an important example of the developed theory in this work. We are following the notations given in Ref. [17].

The wraparound extension

In this first case we merely extend a signal of length $M \in \mathbb{N}$ to an M-periodic signal (see Fig. 2). Here we consider the space $\ell^2_M(\mathbb{Z})$ of M-periodic signals with inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\ell^2_M} = \sum_{m=0}^{M-1} x(m)\, \overline{y(m)}$.

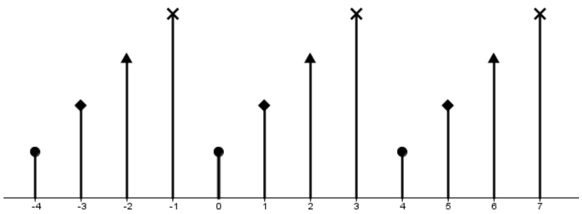

Figure 2: Wraparound extension $M = 4$

If we take, for instance, the M periodic signal $\mathbf{a} := (1, 0, \dots, 0)$ and the shift operator

$$U : \mathbf{x} = \big\{x(m)\big\} \longmapsto U\mathbf{x} := \big\{x(m-1)\big\}$$

in $\ell^2_M(\mathbb{Z})$, we trivially obtain that $U^M \mathbf{a} = \mathbf{a}$ and $\mathcal{A}_\mathbf{a} = \ell^2_M(\mathbb{Z})$.

Given $\mathbf{b}_j \in \ell^2_M$, $j = 1, 2, \dots, s$, each sample from $\mathbf{x} \in \ell^2_M$ in $\{\mathcal{L}_j \mathbf{x}(rn)\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ is obtained from the M-periodic convolution

$$\mathcal{L}_j \mathbf{x}(rn) = \langle \mathbf{x}, U^{rn} \mathbf{b}_j \rangle_{\ell^2_N} = \sum_{m=0}^{M-1} x(m)\, \overline{\mathbf{b}_j(m - rn)} = \big(\mathbf{x} * \mathbf{h}_j\big)(rn)\,,$$

where $\mathbf{h}_j(m) = \overline{\mathbf{b}_j(-m)}$, $m = 0, 1, \dots, M-1$.

As the cross-covariance $R_{a,b_j}(m) = \langle U^m \mathbf{a}, \mathbf{b}_j \rangle_{\ell^2_N} = \overline{b_j(m)}$, each $\ell \times M$ block $\mathbf{R}_{\mathbf{a},\mathbf{b_j}}$, $j = 1, 2, \dots, s$, of the $s\ell \times M$ matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ in (10) takes the form

$$\mathbf{R}_{\mathbf{a},\mathbf{b_j}} = \begin{pmatrix} \overline{b_j(0)} & \overline{b_j(1)} & \dots & \overline{b_j(M-1)} \\ \overline{b_j(M-r)} & \overline{b_j(M-r+1)} & \dots & \overline{b_j(2M-r-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b_j(M - r(\ell-1))} & \overline{b_j(M - r(\ell-1)+1)} & \dots & \overline{b_j(2M-1-r(\ell-1))} \end{pmatrix}$$

where the sampling period r divides M and $\ell = M/r$. In case the rank of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is M, which implies $s \geq r$ systems, from Theorem 5 we obtain in $\ell^2_M(\mathbb{Z})$ the sampling formula

$$\begin{aligned} \mathbf{x}(m) &= \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j \mathbf{x}(rn)\, \mathbf{c}_j(m - rn) \\ &= \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \big(\mathbf{x} * \mathbf{h}_j\big)(rn)\, \mathbf{c}_j(m - rn)\,, \qquad m = 0, 1, \dots, M-1\,. \end{aligned} \tag{17}$$

The sampling sequences in $\ell^2_M(\mathbb{Z})$ are $\mathbf{c}_j = \mathcal{T}_{M,a}(\mathbf{H}_{j,0})$, $j = 1, 2, \dots, s$, where $\mathbf{H}_{j,0}$ are the corresponding columns (denoted as in expression (12)) of a structured left-inverse $\widetilde{\mathbf{H}}$ of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ constructed as (15). Note that $\mathbf{c}_j$ is nothing but the M-periodic sequence derived from $\mathbf{H}_{j,0} \in \mathbb{C}^M$.

Via Theorem 5, the sampling result (17) has a straightforward interpretation in terms of *Perfect Reconstruction Filter Banks*. In fact, it gives a full characterization of the perfect reconstruction filter banks in $\ell^2_M(\mathbb{Z})$ in terms of matrix analysis:

Given, for each $\mathbf{x} \in \ell^2_M(\mathbb{Z})$, $s \geq r$ subsampled sequences $\big\{(\mathbf{x} * \mathbf{h}_j)(rn)\big\}_{n=0,1,\dots,\ell-1}$ with $j = 1, 2, \dots, s$, then the condition rank $\mathbf{R}_{\mathbf{a},\mathbf{b}} = M$ on the impulse responses $\{\mathbf{h}_j\}_{j=1}^s$ is necessary and sufficient for the perfect reconstruction stated in (17)*. Furthermore, if the rank condition is fulfilled, all the reconstruction filters $\{\mathbf{c}_j\}_{j=1}^s$ can be derived by means of the structured left-inverses $\widetilde{\mathbf{H}}$ of the matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$.*

The whole-point symmetry

In this case we extend a signal of length $M \in \mathbb{N}$, without repeating the extreme values, to form a $(2M-2)$-periodic signal (see Fig. 3). Here we consider the space $\ell^2_{2M-2}(\mathbb{Z})$ with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\ell^2_{2M-2}} = \sum_{m=0}^{2M-3} x(m)\, \overline{y(m)}$.

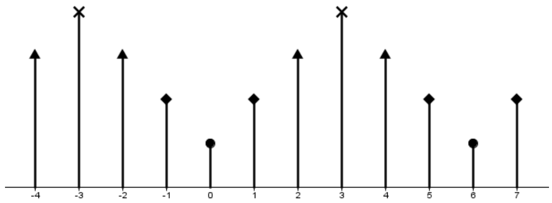

Figure 3: Whole-point symmetric extension $M = 4$

Now consider the subspace in $\ell^2_{2M-2}(\mathbb{Z})$ spanned by the following M signals:

$$\mathbf{a}_1 := (1, 0, 0, \dots, 0, 0)\,;\; \mathbf{a}_2 := (0, 1, 0, \dots, 0, 1)\,;\; \mathbf{a}_3 := (0, 0, 1, \dots, 1, 0)\,; \dots;$$
$$\mathbf{a}_i := (0, \dots, \underset{(i-1)}{1}, \dots, \underset{(2M-(i+1))}{1}, \dots, 0)\,; \dots\,;\; \mathbf{a}_M := (0, \dots, 0, \underset{(M-1)}{1}, 0, \dots, 0)\,,$$

i.e., for $i = 1, 2, \dots, M$ and $m = 0, 1, \dots, 2M-3$ we have

$$\mathbf{a}_i(m) = \begin{cases} 1 & \text{if } m = i-1 \text{ or } m = 2M-(i+1), \\ 0 & \text{otherwise.} \end{cases}$$

Since the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_M\}$ are linearly independent, they span an M-dimensional subspace $\mathcal{M}$ in $\ell^2_{2M-2}(\mathbb{Z})$. Now, we consider the shift operator U_{M-1} given by

$$U_{M-1} : \mathbf{x} = \big\{x(m)\big\} \longmapsto U_{M-1}\,\mathbf{x} := \big\{x(m-M+1)\big\}\,.$$

Under the above circumstances the following result holds:

Lemma 7. *For each $i = 1, 2, \dots, M$ we have:*

(i) $U_{M-1}\,\mathbf{a}_i = \mathbf{a}_{M+1-i}$

(ii) $U_{M-1}^2\,\mathbf{a}_i = \mathbf{a}_i$

(iii) *For* M *odd,* $U_{M-1}\big(\mathbf{a}_{\frac{M+1}{2}}\big) = \mathbf{a}_{\frac{M+1}{2}}$

Proof. It suffices to prove statement (i). To this end, note that the vector $U_{M-1}\,\mathbf{a}_i$ has the 1's in positions $i+M-2$ and $M-i$ which corresponds to the vector $\mathbf{a}_{M+1-i}$. $\square$

As a consequence, we conclude that:

- Whenever M is even, taking $\mathbf{a} := \big\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{M/2}\big\}$ we obtain that $\mathcal{M} = \mathcal{A}_\mathbf{a}$ where all the generators have order 2 with respect to U_{M-1}.
- Whenever M is odd, taking $\mathbf{a} := \big\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{\frac{M+1}{2}}\big\}$ we obtain that $\mathcal{M} = \mathcal{A}_\mathbf{a}$ where the generators have order 2 except for $\mathbf{a}_{\frac{M+1}{2}}$ which has order 1 with respect to U_{M-1}.

Concerning sampling results we distinguish two cases:
(1) M even. In this case we have $L = M/2$ generators of order 2 and $N = 2$.
For $r = 1$ and $\ell = 2$, the $2s \times M$ matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ in (10) has the blocks:

$$\mathbf{R}_{a_i,b_j} = \begin{pmatrix} R_{a_i,b_j}(0) & R_{a_i,b_j}(1) \\ R_{a_i,b_j}(1) & R_{a_i,b_j}(0) \end{pmatrix}, \qquad 1 \le i \le M/2 \text{ and } 1 \le j \le s\,.$$

If rank $\mathbf{R}_{\mathbf{a},\mathbf{b}} = M$, which implies $s \ge M/2$ systems, the corresponding sampling formula (16) for $\mathbf{x} \in \mathcal{A}_\mathbf{a}$ reads

$$\begin{aligned} \mathbf{x}(m) &= \sum_{j=1}^{s} \big[\mathcal{L}_j\mathbf{x}(0)\,\mathbf{c}_j(m) + \mathcal{L}_j\mathbf{x}(1)\,U_{M-1}\mathbf{c}_j(m)\big] \\ &= \sum_{j=1}^{s} \big[\mathcal{L}_j\mathbf{x}(0)\,\mathbf{c}_j(m) + \mathcal{L}_j\mathbf{x}(1)\,\mathbf{c}_j(m-M+1)\big], \quad 0 \le m \le 2M-3\,. \end{aligned} \tag{18}$$

For $r = 2$ and $\ell = 1$, the $s \times M$ matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ in (10) has the blocks:

$$\mathbf{R}_{a_i,b_j} = \begin{pmatrix} R_{a_i,b_j}(0) & R_{a_i,b_j}(1) \end{pmatrix}, \quad 1 \le i \le M/2 \text{ and } 1 \le j \le s\,.$$

If rank $\mathbf{R}_{\mathbf{a},\mathbf{b}} = M$, which implies $s \ge M$ systems, the corresponding sampling formula (16) for $\mathbf{x} \in \mathcal{A}_\mathbf{a}$ reads

$$\mathbf{x}(m) = \sum_{j=1}^{s} \mathcal{L}_j\mathbf{x}(0)\,\mathbf{c}_j(m), \quad 0 \le m \le 2M-3\,. \tag{19}$$

(2) M odd. In this case we have $L = \frac{M+1}{2}$ generators, $\frac{M-1}{2}$ of order 2 and one generator of order 1; again $N = 2$.
For $r = 1$ and $\ell = 2$; the $2s \times M$ matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ in (10) has the blocks:

$$\mathbf{R}_{a_i,b_j} = \begin{pmatrix} R_{a_i,b_j}(0) & R_{a_i,b_j}(1) \\ R_{a_i,b_j}(1) & R_{a_i,b_j}(0) \end{pmatrix}, \quad 1 \le i \le (M-1)/2 \text{ and } 1 \le j \le s\,,$$

and

$$\mathbf{R}_{a_{\frac{M+1}{2}},b_j} = \begin{pmatrix} R_{a_{\frac{M+1}{2}},b_j}(0) \\ R_{a_{\frac{M+1}{2}},b_j}(0) \end{pmatrix}, \quad 1 \le j \le s\,.$$

If rank $\mathbf{R}_{\mathbf{a},\mathbf{b}} = M$, which implies $s \ge M/2$ systems, we obtain a sampling formula in $\mathcal{A}_\mathbf{a}$ as (18).
For $r = 2$ and $\ell = 1$, the $s \times M$ matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ in (10) has the blocks:

$$\mathbf{R}_{a_i,b_j} = \big(R_{a_i,b_j}(0) \quad R_{a_i,b_j}(1)\big)\,,\ 1 \le i \le (M-1)/2\,, \quad \text{and} \quad \mathbf{R}_{a_{\frac{M+1}{2}},b_j} = \Big(R_{a_{\frac{M+1}{2}},b_j}(0)\Big)\,,$$

for $1 \le j \le s$. If rank $\mathbf{R}_{\mathbf{a},\mathbf{b}} = M$, which implies $s \ge M$ systems, we obtain a sampling formula in $\mathcal{A}_\mathbf{a}$ as (19).

The half-point symmetry

Here we extend a signal of length $M \in \mathbb{N}$, repeating the extreme values, to form a $2M$-periodic signal (see Fig.4). Here we consider the space $\ell^2_{2M}(\mathbb{Z})$ with the inner product $\langle \mathbf{x}, \mathbf{y}\rangle_{\ell^2_{2M}} = \sum_{m=0}^{2M-1} x(m)\,\overline{y(m)}$.

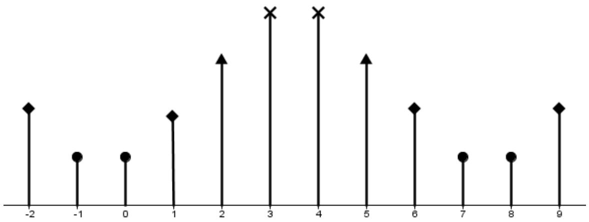

Figure 4: Half-point symmetric extension $M = 4$

Now consider the subspace in $\ell^2_{2M}(\mathbb{Z})$ spanned by the following M signals:

$$\mathbf{a}_1 := (1,0,0,\dots,0,1)\,;\ \mathbf{a}_2 := (0,1,0,\dots,1,0)\,;\dots;\ \mathbf{a}_M := (0,0,\dots,1,1,\dots,0,0)\,,$$

i.e., for $i = 1,2,\dots,M$ and $m = 0,1,\dots,2M-3$ we have

$$\mathbf{a}_i(m) = \begin{cases} 1 & \text{if } m = i-1 \text{ or } m = 2M-i, \\ 0 & \text{otherwise.} \end{cases}$$

Since the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_M\}$ are linearly independent, they span an M-dimensional subspace $\mathcal{M}$ in $\ell^2_{2M}(\mathbb{Z})$. Now, we consider the shift operator U_M given by

$$U_M : \mathbf{x} = \big\{x(m)\big\} \longmapsto U_M\,\mathbf{x} := \big\{x(m-M)\big\}\,.$$

Under the above circumstances the following result holds:

Lemma 8. *For each $i = 1,2,\dots,M$ we have:*

(i) $U_M\,\mathbf{a}_i = \mathbf{a}_{M+1-i}$

(ii) $U_M^2\,\mathbf{a}_i = \mathbf{a}_i$

(iii) For M odd, $U_M\big(\mathbf{a}_{\frac{M+1}{2}}\big) = \mathbf{a}_{\frac{M+1}{2}}$

Proof. It suffices to prove statement (i). To this end, note that the vector $U_M\,\mathbf{a}_i$ has the 1's in positions $i+M-1$ and $M-i$ which corresponds to the vector $\mathbf{a}_{M+1-i}$. $\square$

As a consequence, we conclude that:

- Whenever M is even, taking $\mathbf{a} := \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{M/2}\}$ we obtain that $\mathcal{M} = \mathcal{A}_\mathbf{a}$ where all the generators have order 2 with respect to U_M.
- Whenever M is odd, taking $\mathbf{a} := \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{\frac{M+1}{2}}\}$ we obtain that $\mathcal{M} = \mathcal{A}_\mathbf{a}$ where the generators have order 2 except for $\mathbf{a}_{\frac{M+1}{2}}$ which has order 1 with respect to U_M.

For the present case, analogous sampling results to those in the whole-point symmetry case can be derived in a similar way; we omit the details.

There is another extension of a finite signal which we are not considering here: the zero-padding extension, i.e., extending the signal by zeros to get a signal in $\ell^2(\mathbb{Z})$. Thus, we are interested in sampling formulas in $\ell^2(\mathbb{Z})$ involving only signals of finite support, i.e., with a finite number of nonzero terms. The corresponding sampling formulas can be easily characterized by using [9, Theorem 3.2].

A comment on the sampling period r

Once we have handled some examples, a comment on the sampling period r is in order. Along this paper we have assumed that the sampling period r divides N where $N = \text{l.c.m.}\,(N_1, N_2, \dots N_L)$.

Actually, the case where the sampling period does not divide N can be reduced to the case considered in the paper: we have taken N to be the least common multiple of $N_1, N_2, \dots N_L$ but everything works with any common multiple of $N_1, N_2, \dots N_L$. Thus, denoting as r' any sampling period, we could take $N' = \text{l.c.m.}\,(r', N_1, N_2, \dots N_L)$; the used techniques in Section 2 could be applied for N' and r' instead of N and r.

Moreover, this case reduces to the situation where r is the greatest common divisor of N and r' since $\ell = N/r = N'/r'$ and for each $x \in \mathcal{A}_\mathbf{a}$ and each $j = 1, 2, \dots, s$ the set of samples $\{\mathcal{L}_j x(r'n') \colon n' = 0, 1, \dots, \ell - 1\}$ with sampling period r' is nothing but the set of samples $\{\mathcal{L}_j x(rn) \colon n = 0, 1, \dots, \ell - 1\}$ with sampling period r. That is, the corresponding $s\ell \times (N_1 + N_2 + \dots + N_L)$ matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ in (10) for r' and N' is the same (up to a permutation of its rows) than those obtained for r and N.

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