Sampling in Λ-shift-invariant subspaces of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$

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Abstract

The translation of an operator is defined by using conjugation with time-frequency shifts. Thus, one can define Λ-shift-invariant subspaces of Hilbert-Schmidt operators, finitely generated, with respect to a lattice Λ in $\mathbb{R}^{2d}$. These spaces can be seen as a generalization of classical shift-invariant subspaces of square integrable functions. Obtaining sampling results for these subspaces appears as a natural question that can be motivated by the problem of channel estimation in wireless communications. These sampling results are obtained in the light of the frame theory in a separable Hilbert space.

Keywords: Hilbert-Schmidt operators; Weyl transform; Kohn-Nirenberg transform; Translation of operators; Λ-shift-invariant subspaces; Sampling Hilbert-Schmidt operators.
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1 Introduction

In this paper we obtain sampling results in shift-invariant-like subspaces of the class $\mathcal{HS}(\mathbb{R}^d)$ of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$. To be more precise, these subspaces are obtained by translation in a lattice $\Lambda \subset \mathbb{R}^{2d}$ of a fixed set of *Hilbert-Schmidt* operators $S_1, S_2, \dots, S_N$. *The translation of an operator* S by $z \in \mathbb{R}^{2d}$ is defined by using conjugation with the *time-frequency shift* $\pi(z)$, where $z = (x, \omega)$ belongs to the *phase space* $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ (which in the sequel will be identified with $\mathbb{R}^{2d}$) by

$$\alpha_z(S) := \pi(z) S\, \pi(z)^*\,, \quad z \in \mathbb{R}^{2d}\,.$$

Recall that the time-frequency shift acts on $f \in L^2(\mathbb{R}^d)$ as $\pi(z)f(t) = \mathrm{e}^{2\pi i \omega \cdot t} f(t-x)$. The set of translations $\{\alpha_z\}_{z\in\mathbb{R}^{2d}}$ is a *unitary representation* of the group $\mathbb{R}^{2d}$ on the Hilbert space $\mathcal{HS}(\mathbb{R}^d)$.

If we take a *full rank lattice* Λ in $\mathbb{R}^{2d}$, i.e., $\Lambda = A\mathbb{Z}^d$ where A is a $2d \times 2d$ real invertible matrix, such that the sequence $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ is a *Riesz sequence* in $\mathcal{HS}(\mathbb{R}^d)$ we consider the subspace of $\mathcal{HS}(\mathbb{R}^d)$ given by

$$V_{\mathbf{S}}^2 = \Big\{ \sum_{n=1}^{N} \sum_{\lambda\in\Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \ : \ \{c_n(\lambda)\}_{\lambda\in\Lambda} \in \ell^2(\Lambda)\,,\, n = 1, 2, \dots, N \Big\}\,.$$

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From now on, the subspaces $V^2_{\mathbf{S}}$ obtained in this way will be called Λ-*shift-invariant subspaces* in $\mathcal{HS}(\mathbb{R}^d)$. These spaces are a generalization of the classical shift-invariant subspaces in $L^2(\mathbb{R}^d)$:

$$V^2_\Phi := \Big\{ \sum_{n=1}^N \sum_{\alpha\in\mathbb{Z}^d} c_n(\alpha)\, \varphi_n(t-\alpha) \ : \ \{c_n(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)\,,\, n=1,2,\dots,N \Big\}\,,$$

where $\Phi=\{\varphi_1,\varphi_2,\dots,\varphi_N\}$ denotes a set of generators of V^2_Φ. Sampling in the shift-invariant subspace V^2_Φ usually involves, for each $f\in V^2_\Phi$, pointwise samples $\{f(\alpha+\beta_m)\}_{\alpha\in\mathbb{Z}^d}$ and/or average samples $\{\langle f,\psi_m(\cdot-\alpha)\rangle\}_{\alpha\in\mathbb{Z}^d}$, where ψ_m is an *average function* in $L^2(\mathbb{R}^d)$, which not necessarily belong to V^2_φ. Any stable sampling in V^2_Φ will involve, necessarily, $M\geq N$ sequences of samples (see, for instance, [1, 13] and references therein).

A challenge problem here is to choose an appropriate set of samples that should be used for operators in $V^2_{\mathbf{S}}$. Inspired in Ref. [18] and motivated by the problem of channel estimation in wireless communications, in this paper we propose for any $T\in V^2_{\mathbf{S}}$ its *diagonal channel samples* at the lattice $\Lambda\subset\mathbb{R}^{2d}$ defined by

$$s_{T,m}(\lambda) := \big\langle \alpha_{-\lambda}(T)g_m, \widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}\,, \quad \lambda\in\Lambda\,,\ \ m=1,2,\dots,M\,, \tag{1}$$

where $g_m,\widetilde{g}_m$, $m=1,2,\dots,M$, are $2M$ fixed functions in $L^2(\mathbb{R}^d)$ (we will see that necessarily $M\geq N$). The name *diagonal channel samples* coined for these samples will become clear later on where a little explanation will be done for both, the choice of Hilbert-Schmidt operators (in $V^2_{\mathbf{S}}$) to be sampled, and the choice of the above samples for any $T\in V^2_{\mathbf{S}}$. As we will see in Section 3.3 the samples defined in (1) are nothing but the *lower symbol of the operator* T with respect $g_m,\widetilde{g}_m\in L^2(\mathbb{R}^d)$ and lattice Λ, i.e., $\big\langle T\pi(\lambda)g_m,\pi(\lambda)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}$, $\lambda\in\Lambda$, or the samples of the *Berezin transform* $\mathcal{B}^{g_m,\widetilde{g}_m}\,T(z) := \big\langle T\pi(z)g_m,\pi(z)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}$, $z\in\mathbb{R}^{2d}$, at the lattice Λ (see Ref. [21]). These samples are also a particular case of the *average samples* $\big\langle T,\alpha_\lambda(Q_m)\big\rangle_{\mathcal{HS}}$ where the *average operator* Q_m is the rank-one operator $\widetilde{g}_m\otimes g_m$; average sampling has been used previously in Refs. [6, 12].

The main aim here is the stable recovery of any $T\in V^2_{\mathbf{S}}$ from its samples (1) by means of a sampling formula in $V^2_{\mathbf{S}}$ having the form

$$T=\sum_{m=1}^M\sum_{\lambda\in\Lambda} s_{T,m}(\lambda)\,\alpha_\lambda(H_m) \quad \text{in } \mathcal{HS}\text{-norm}\,,$$

for each $T\in V^2_{\mathbf{S}}$. The operators H_m, $m=1,2,\dots,M$, above belong to $V^2_{\mathbf{S}}$ and satisfy that the sequence $\{\alpha_\lambda(H_m)\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ is a *frame* for the Hilbert space $V^2_{\mathbf{S}}$.

For sampling in classical shift-invariant spaces see, for instance, Refs. [1, 13, 14] and references therein. See also Ref. [8] for the case where other unitary representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ is used instead of the classical one given by translations. For the less known topic on sampling operators, see Refs. [6, 12, 18, 20, 22, 23].

The used techniques in this work are those of the frame theory in a separable Hilbert space. To be precise, the samples used along this paper will be expressed as a discrete convolution system in the product Hilbert space $\ell^2_N(\Lambda) := \ell^2(\Lambda)\times\cdots\times\ell^2(\Lambda)$ (N times), and then it will be used the close relationship between a discrete convolution system and a sequence of translates in $\ell^2_N(\Lambda)$ (see, for instance, Ref. [15]). The other involved tools are the Kohn-Nirenberg transform or the Weyl transform for Hilbert-Schmidt operators: both are unitary operators from $L^2(\mathbb{R}^{2d})$ onto $\mathcal{HS}(\mathbb{R}^d)$ which respect the translations in the sense that, if we denote any of them by $\mathcal{L}$, we have $\mathcal{L}(T_z f) = \alpha_z(\mathcal{L}f)$ for $f \in L^2(\mathbb{R}^{2d})$ and $z \in \mathbb{R}^{2d}$.

Now we briefly explain a practical motivation for considering the samples defined in Eq. (1) for the elements in $V^2_{\mathbf{S}}$. It is a well-known fact in mobile wireless channels that the relative location between transmitter and receiver is varying with time and consequently the input-output relation is modeled by a *time-varying system* $x \mapsto Hx$ that can be expressed as the integral operator

$$Hx(t) = \int_{\mathbb{R}^d} h_t(s)\, x(t-s)\, ds = \int_{\mathbb{R}^d} \sigma(t,\omega)\, \widehat{x}(\omega)\, \mathrm{e}^{2\pi i\, \omega\cdot t} d\omega\,,$$

where $\sigma(t,\omega) = \mathcal{F}(h_t)(\omega)$, i.e., the Fourier transform with respect to the last d variables in $h(t,s) := h_t(s)$. In this last formulation, operator H becomes a *pseudodifferential operator* with *Kohn-Nirenberg symbol* σ (see, for instance, Refs. [16, 25]).

As it was pointed out in Ref. [18], in *orthogonal frequency-division multiplexing* (OFDM) the digital information, i.e., a sequence of numbers $\{c_\lambda\}$, λ in the lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ $(a, b > 0)$, is used as the coefficients of the input signal $x(t) = \sum_{\mu\in\Lambda} c_\mu\, \pi(\lambda)g(t)$ of a time-varying system H producing the output $y(t) = Hx(t)$. Then, the sequence of numbers

$$d_\lambda = \big\langle y, \pi(\lambda)\widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)} = \sum_{\mu\in\Lambda} c_\mu \,\big\langle H\pi(\mu)g, \pi(\lambda)\widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)}\,, \quad \lambda \in \Lambda\,, \tag{2}$$

is considered. The main task of the engineer is to recover the original data $\{c_\lambda\}$ from the received data $\{d_\lambda\}$. The matrix $A = [a_{\lambda,\mu}]$, where $a_{\lambda,\mu} = \big\langle H\pi(\mu)g, \pi(\lambda)\widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)}$, which appears in Eq. (2), involving H and the time-frequency shifts of a pair of fixed functions $g, \widetilde{g} \in L^2(\mathbb{R}^d)$, is the so-called *channel matrix* associated with H and the functions $g, \widetilde{g}$ in $L^2(\mathbb{R}^d)$. As it will be proved in Section 3.3 (see Eq. (8) below), we have that

$$\big\langle H\pi(\lambda)g, \pi(\lambda)\widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)} = \big\langle \alpha_{-\lambda}(H)g, \widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)}\,, \quad \lambda \in \Lambda\,,$$

i.e., the samples $\big\langle \alpha_{-\lambda}(H)g, \widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)}$, $\lambda \in \Lambda$, coincide with the diagonal entries of the channel matrix associated with H and windows $g, \widetilde{g}$. This is the reason to consider the samples defined in Eq. (1) and to name them as the *diagonal channel samples* of the operator H with respect to the fixed functions $g, \widetilde{g} \in L^2(\mathbb{R}^d)$ and lattice Λ.

Besides, a simple class of operators H describing time-varying systems, and allowing to live in the Hilbert space setting, is given by the class of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$. A Hilbert-Schmidt operator H on $L^2(\mathbb{R}^d)$ is a compact operator on $L^2(\mathbb{R}^d)$ having the integral representation

$$Hx(t) = \int_{\mathbb{R}^d} \kappa(t,s)\, x(s) ds = \int_{\mathbb{R}^d} \kappa(t, t-s)\, x(t-s) ds\,,$$

with kernel $\kappa \in L^2(\mathbb{R}^{2d})$. Although only Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$ can be described as integral operators with kernel in $L^2(\mathbb{R}^{2d})$, every bounded operator on $L^2(\mathbb{R}^d)$ can be uniquely described, via the *Schwartz kernel theorem*, by a distributional kernel in $\mathcal{S}'(\mathbb{R}^{2d})$ (see, for instance, Ref. [16]).

The paper is organized as follows: Section 2 introduces, for the sake of completeness, some preliminaries needed in the sequel; they comprise Hilbert-Schmidt operators and their Kohn-Nirenberg and Weyl transforms, the concept of translation of an operator, and *symplectic Fourier series*. For the theory of bases and frames in a Hilbert space we cite Ref. [3]. Section 3 contains the main sampling results for the multiple generated subspace $V^2_{\mathbf{S}}$ of $\mathcal{HS}(\mathbb{R}^d)$. They rely on the expression of the involved samples as the output of a bounded discrete convolution system $\ell^2_{_N}(\Lambda) \to \ell^2_{_M}(\Lambda)$, and its relationship with a frame of translates for $\ell^2_{_N}(\Lambda)$.

2 Some preliminaries

Next we briefly introduce some mathematical tools used throughout the work. For the needed theory of bases and frames in a Hilbert space we merely make reference to [3]; it mainly comprises Riesz sequences, dual Riesz bases and frames and its duals in a separable Hilbert space. The results for discrete convolution systems and their relationship with frames of translates in $\ell^2_{_N}(\Lambda)$ can be found, for instance, in Ref. [15].

The Kohn-Nirenberg and Weyl transforms in the class of Hilbert-Schmidt operators

The class of Hilbert-Schmidt operators in a Hilbert space, $L^2(\mathbb{R}^d)$ in our case, can be introduced by using the *Schmidt decomposition* (singular value decomposition) of a compact operator on $L^2(\mathbb{R}^d)$ (see, for instance, Ref. [4]). Namely, for a compact operator S on $L^2(\mathbb{R}^d)$ there exist two orthonormal sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ and a bounded sequence of positive numbers $\{s_n(S)\}_{n\in\mathbb{N}}$ (*singular values* of S) such that

$$S = \sum_{n\in\mathbb{N}} s_n(S)\, x_n \otimes y_n\,,$$

with convergence of the series in the operator norm. Here, $x_n \otimes y_n$ denotes the rank-one operator defined by $\big(x_n \otimes y_n\big)(e) = \big\langle e, y_n\big\rangle_{L^2} x_n$ for $e \in L^2(\mathbb{R}^d)$. For $1 \le p < \infty$ we define the *Schatten-p class* $\mathcal{T}^p$ by

$$\mathcal{T}^p := \big\{ S \text{ compact on } L^2(\mathbb{R}^d) \;:\; \{s_n(S)\}_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}) \big\}\,.$$

The Schatten-p class $\mathcal{T}^p$ is a Banach space endowed with the norm $\|S\|^p_{\mathcal{T}^p} = \sum_{n\in\mathbb{N}} s^p_n(S)$.

In particular, for $p = 1$ we obtain the so-called *trace class operators* $\mathcal{T}^1$. The *trace* defined by $\operatorname{tr}(S) = \sum_{n\in\mathbb{N}} \langle Se_n, e_n\rangle_{L^2}$ is a well-defined bounded linear functional on $\mathcal{T}^1$, and independent of the used orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$.

For $p = 2$ we obtain the class of *Hilbert-Schmidt operators* $\mathcal{HS}(\mathbb{R}^d) := \mathcal{T}^2$. The space $\mathcal{HS}(\mathbb{R}^d)$ endowed with the inner product $\big\langle S, T\big\rangle_{\mathcal{HS}} = \operatorname{tr}(ST^*)$ becomes a Hilbert space. For the norm of $S \in \mathcal{HS}(\mathbb{R}^d)$ we have

$$\|S\|^2_{\mathcal{HS}} = \operatorname{tr}(SS^*) = \sum_{n\in\mathbb{N}} \|S^*(e_n)\|^2_{L^2} = \sum_{n\in\mathbb{N}} \|S(e_n)\|^2_{L^2} = \sum_{n\in\mathbb{N}} s^2_n(S)\,.$$

A Hilbert-Schmidt operator $S \in \mathcal{HS}(\mathbb{R}^d)$ can be seen also as a compact operator on $L^2(\mathbb{R}^d)$ defined for each $f \in L^2(\mathbb{R}^d)$ by

$$Sf(t) = \int_{\mathbb{R}^d} \kappa_S(t,x) f(x) dx \quad \text{a.e. } t \in \mathbb{R}^d \,,$$

with kernel $\kappa_S \in L^2(\mathbb{R}^{2d})$. Besides, $\big\langle S, T \big\rangle_{\mathcal{HS}} = \big\langle \kappa_S, \kappa_T \big\rangle_{L^2(\mathbb{R}^{2d})}$ for $S, T \in \mathcal{HS}(\mathbb{R}^d)$.

Now, we briefly introduce the Kohn-Nirenberg and Weyl transforms in $L^2(\mathbb{R}^{2d})$, the setting where they will be used in this paper. More information and details about these transforms, also valid in more general settings, can be found in Refs. [7, 9, 16, 24, 26].

The *Kohn-Nirenberg transform* $L^2(\mathbb{R}^{2d}) \ni \sigma \longmapsto K_\sigma \in \mathcal{HS}(\mathbb{R}^d)$ is a unitary operator where $K_\sigma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the Hilbert-Schmidt operator defined in weak sense by

$$\big\langle K_\sigma \phi, \psi \big\rangle_{L^2(\mathbb{R}^d)} = \big\langle \sigma, R(\psi,\phi) \big\rangle_{L^2(\mathbb{R}^{2d})} \,, \quad \phi, \psi \in L^2(\mathbb{R}^d) \,; \tag{3}$$

here

$$R(\psi,\phi)(x,\omega) = \psi(x)\, \overline{\widehat{\phi}(\omega)}\, \mathrm{e}^{-2\pi i\, x\cdot\omega} \,, \quad (x,\omega) \in \mathbb{R}^{2d} \,,$$

is the *Rihaczek distribution* of the functions $\psi, \phi \in L^2(\mathbb{R}^d)$ (see [16, Theorem 14.6.1]).

Thus, for each operator $S \in \mathcal{HS}(\mathbb{R}^d)$ there exists a unique function $\sigma_S \in L^2(\mathbb{R}^{2d})$, called its *Kohn-Nirenberg symbol*, i.e. $S = K_{\sigma_S}$, and such that

$$\langle S, T \rangle_{\mathcal{HS}} = \langle \sigma_S, \sigma_T \rangle_{L^2(\mathbb{R}^{2d})} \quad \text{for each } S, T \in \mathcal{HS}(\mathbb{R}^d) \,.$$

The Weyl transform $L^2(\mathbb{R}^{2d}) \ni f \longmapsto L_f \in \mathcal{HS}(\mathbb{R}^d)$ is also a unitary operator where $L_f : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the Hilbert-Schmidt operator defined in weak sense by

$$\big\langle L_f \phi, \psi \big\rangle_{L^2(\mathbb{R}^d)} = \big\langle f, W(\psi,\phi) \big\rangle_{L^2(\mathbb{R}^{2d})} \,, \quad \phi, \psi \in L^2(\mathbb{R}^d) \,; \tag{4}$$

here

$$W(\psi,\phi)(x,\omega) = \int_{\mathbb{R}^d} \psi\big(x + \frac{t}{2}\big)\, \overline{\phi\big(x - \frac{t}{2}\big)}\, \mathrm{e}^{-2\pi i\, \omega\cdot t} dt \,, \quad (x,\omega) \in \mathbb{R}^{2d} \,,$$

is the *cross-Wigner distribution* of the functions $\psi, \phi \in L^2(\mathbb{R}^d)$ (see Ref. [16, Theorem 14.6.1]).

Thus, for each operator $S \in \mathcal{HS}(\mathbb{R}^d)$ there exists a unique function $a_S \in L^2(\mathbb{R}^{2d})$, called its *Weyl symbol*, i.e. $S = L_{a_S}$, and such that

$$\langle S, T \rangle_{\mathcal{HS}} = \langle a_S, a_T \rangle_{L^2(\mathbb{R}^{2d})} \quad \text{for each } S, T \in \mathcal{HS}(\mathbb{R}^d) \,.$$

If a_S denotes the Weyl symbol of S, its Kohn-Nirenberg symbol σ_S is given by $U a_S$ where U is the unitary operator on $L^2(\mathbb{R}^{2d})$ such that $\widehat{Ua_S}(\xi, u) = \mathrm{e}^{\pi i u\cdot\xi}\, \widehat{a}_S(\xi, u)$, $(\xi, u) \in \mathbb{R}^{2d}$ (see the details in Ref. [16]).

The Kohn-Nirenberg (or Weyl) transform can be defined for σ (or f) in $\mathcal{S}'(\mathbb{R}^{2d})$, i.e., for tempered distributions by using the dualities $\big(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)\big)$ and $\big(\mathcal{S}(\mathbb{R}^{2d}), \mathcal{S}'(\mathbb{R}^{2d})\big)$ in Eq. (3) (or Eq. (4)); see, for instance, Refs. [16, 24].

Translation of operators

For $z=(x,\omega)\in\mathbb{R}^{2d}$, the *time-frequency shift* operator $\pi(z): L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is defined as

$$\pi(z)\varphi(t)=\mathrm{e}^{2\pi i\omega\cdot t}\varphi(t-x)\quad \text{for } \varphi\in L^2(\mathbb{R}^d)\,.$$

It is used to define the *short-time Fourier transform* (Gabor transform) $V_\psi\varphi$ of φ with window ψ, both in $L^2(\mathbb{R}^d)$, by

$$V_\psi\varphi(z)=\big\langle\varphi,\pi(z)\psi\big\rangle_{L^2(\mathbb{R}^d)}\,,\quad z\in\mathbb{R}^{2d}\,.$$

Its adjoint operator is $\pi(z)^*=\mathrm{e}^{-2\pi i x\cdot\omega}\,\pi(-z)$ for $z=(x,\omega)\in\mathbb{R}^{2d}$. By using conjugation with $\pi(z)$ one can define the translation by $z\in\mathbb{R}^{2d}$ of an operator $S\in\mathcal{HS}(\mathbb{R}^d)$. Namely,

$$\alpha_z(S):=\pi(z)\,S\,\pi(z)^*\,,\quad z\in\mathbb{R}^{2d}\,.$$

For instance, for $\varphi,\psi\in L^2(\mathbb{R}^d)$ we get $\alpha_z(\varphi\otimes\psi)=[\pi(z)\varphi]\otimes[\pi(z)\psi],\ z\in\mathbb{R}^{2d}$.

Since α_z defines a unitary operator on $\mathcal{HS}(\mathbb{R}^d)$, $\alpha_z\alpha_{z'}=\alpha_{z+z'}$ for $z,z'\in\mathbb{R}^{2d}$, and the map $z\mapsto\alpha_z(S)$ is continuous for each $S\in\mathcal{HS}(\mathbb{R}^d)$ we have that $\{\alpha_z\}_{z\in\mathbb{R}^{2d}}$ is a *unitary representation* of the group $\mathbb{R}^{2d}$ on the Hilbert space $\mathcal{HS}(\mathbb{R}^d)$. More properties and applications can be found, for instance, in Refs. [21, 24, 26].

Symplectic Fourier series

Let Λ be a *full rank lattice* in $\mathbb{R}^{2d}$, i.e., $\Lambda=A\mathbb{Z}^{2d}$ with $A\in GL(2d,\mathbb{R})$ and volume $|\Lambda|=\det A$. Its dual group $\widehat{\Lambda}$ is identified with $\mathbb{R}^{2d}/\Lambda^\circ$, where Λ° is the *annihilator group*

$$\Lambda^\circ=\big\{\lambda^\circ\in\mathbb{R}^{2d}\ :\ \mathrm{e}^{2\pi i\,\sigma(\lambda^\circ,\lambda)}=1\ \text{ for all }\lambda\in\Lambda\big\}\,,$$

where σ denotes here the *standard symplectic form* $\sigma(z,z')=\omega\cdot x'-\omega'\cdot x$ for $z=(x,\omega)$ and $z'=(x',\omega')$ in $\mathbb{R}^{2d}$. Notice that, since Λ is discrete its dual group $\widehat{\Lambda}$ is compact. The group Λ° is itself a lattice: the so-called *adjoint lattice* of Λ. The *symplectic characters* $\chi_z(z'):=\mathrm{e}^{2\pi i\,\sigma(z,z')}$ are the natural way of identifying the group $\mathbb{R}^{2d}$ with its dual group via the bijection $z\mapsto\chi_z$.

The Fourier transform of $c\in\ell^1(\Lambda)$ is the *symplectic Fourier series*

$$\mathcal{F}_s^\Lambda(c)(\dot z):=\sum_{\lambda\in\Lambda}c(\lambda)\,\mathrm{e}^{2\pi i\,\sigma(\lambda,z)}\,,\quad \dot z\in\mathbb{R}^{2d}/\Lambda^\circ\,,$$

where $\dot z$ denotes the image of z under the natural quotient map $\mathbb{R}^{2d}\to\mathbb{R}^{2d}/\Lambda^\circ$.

Since $\mathcal{F}_s^\Lambda$ is a Fourier transform it extends to a unitary mapping $\mathcal{F}_s^\Lambda:\ell^2(\Lambda)\to L^2(\widehat{\Lambda})$. It satisfies $\mathcal{F}_s^\Lambda(c*_\Lambda d)=\mathcal{F}_s^\Lambda(c)\,\mathcal{F}_s^\Lambda(d)$ for $c\in\ell^1(\Lambda)$ and $d\in\ell^2(\Lambda)$. Moreover, if $c,d\in\ell^2(\Lambda)$ with $c*_\Lambda d\in\ell^2(\Lambda)$, then $\mathcal{F}_s^\Lambda(c*_\Lambda d)=\mathcal{F}_s^\Lambda(c)\,\mathcal{F}_s^\Lambda(d)$. As usual, the convolution $*_\Lambda$ of two sequences c,d is defined by

$$\big(c*_\Lambda d\big)(\lambda)=\sum_{\mu\in\Lambda}c(\mu)\,d(\lambda-\mu),\quad \lambda\in\Lambda\,.$$

For more details, see, for instance, Refs. [5, 10, 11, 24].

3 Sampling in the case of multiple generators

For a fixed set $\mathbf{S} = \{S_1, S_2, \dots, S_N\} \subset \mathcal{HS}(\mathbb{R}^d)$, we are interested that the sequence of translates $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ forms a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$ where $\Lambda \subset \mathbb{R}^{2d}$ is a full rank lattice with dual group $\widehat{\Lambda}$.

3.1 Riesz sequences of translated operators in $\mathcal{HS}(\mathbb{R}^d)$

As it was said before, the Weyl transform $f \mapsto L_f$ is a unitary operator $L^2(\mathbb{R}^{2d}) \to \mathcal{HS}(\mathbb{R}^d)$ which respects translations in the sense that

$$L_{T_z f} = \alpha_z(L_f) \quad \text{for } f \in L^2(\mathbb{R}^{2d}) \text{ and } z \in \mathbb{R}^{2d}\,.$$

These two properties are very important throughtout this work. In particular, as it was pointed out in Refs. [6, 24], for fixed $S \in \mathcal{HS}(\mathbb{R}^d)$ with Weyl symbol $a_S \in L^2(\mathbb{R}^{2d})$ and lattice Λ in $\mathbb{R}^{2d}$, the sequence $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $\mathcal{HS}(\mathbb{R}^d)$, i.e., a Riesz basis for $V_S^2 := \overline{\text{span}}_{\mathcal{HS}}\big\{\alpha_\lambda(S)\big\}_{\lambda\in\Lambda}$, if and only if the sequence $\{T_\lambda(a_S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $L^2(\mathbb{R}^{2d})$, i.e., a Riesz basis for the shift-invariant subspace $V_{a_S}^2$ in $L^2(\mathbb{R}^{2d})$ generated by a_S.

A necessary and sufficient condition for $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ to be a Riesz sequence in $\mathcal{HS}(\mathbb{R}^d)$ is given in Ref. [24]. There, it is assumed that $S \in \mathcal{B}$, a Banach space of continuous operators with Weyl symbol a_S in the *Feichtinger's algebra* $\mathcal{S}_0(\mathbb{R}^{2d})$; in essence, $\mathcal{B}$ consists of trace class operators on $L^2(\mathbb{R}^d)$ with a norm-continuous inclusion $\iota : \mathcal{B} \hookrightarrow \mathcal{T}^1$ (see the details in Refs. [17, 24]).

Recall that the *Feichtinger's algebra* $\mathcal{S}_0(\mathbb{R}^d)$ is the space of all tempered distributions ψ in $\mathbb{R}^d$ such that

$$\|\psi\|_{\mathcal{S}_0} := \int_{\mathbb{R}^{2d}} |V_{\varphi_0}\psi(z)|dz < \infty\,,$$

where φ_0 denotes the L^2-normalized gaussian $\varphi_0(x) = 2^{d/4}\mathrm{e}^{-\pi x\cdot x}$ for $x \in \mathbb{R}^d$. With this norm, $\mathcal{S}_0(\mathbb{R}^d)$ is a Banach space of continuous functions and an algebra under multiplication and convolution; see the details in Refs. [16, 19, 24].

Theorem 1. *([24, Theorem 6.1]) Let Λ be a lattice and $S \in \mathcal{B}$. The sequence $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence in $\mathcal{HS}(\mathbb{R}^d)$ if and only if the function*

$$P_{\Lambda^\circ}\big(|\mathcal{F}_W(S)|^2\big)(\dot{z}) := \frac{1}{|\Lambda|}\sum_{\lambda^\circ\in\Lambda^\circ} |\mathcal{F}_W(S)(z+\lambda^\circ)|^2\,, \quad z \in \mathbb{R}^{2d}\,,$$

has no zeros in $\widehat{\Lambda}$.

It involves the *periodization operator* P_{Λ° in Λ° and the *Fourier-Wigner transform* $\mathcal{F}_W$ of an operator S. In this case, we have that $\mathcal{F}_W(S) = \mathcal{F}_s(a_S)$, where $\mathcal{F}_s$ denotes the *symplectic Fourier transform* of a_S defined by

$$\mathcal{F}_s(a_S)(z) := \int_{\mathbb{R}^{2d}} a_S(z')\,\mathrm{e}^{-2\pi i\,\sigma(z,z')}dz'\,, \quad z \in \mathbb{R}^{2d}\,,$$

where σ denotes here the standard symplectic form in $\mathbb{R}^{2d}$. The Fourier-Wigner transform of an operator S is defined as the function

$$\mathcal{F}_W(S)(z) := \mathrm{e}^{-\pi i\,x\cdot\omega}\,\mathrm{tr}[\pi(-z)S]\,, \quad z = (x,\omega) \in \mathbb{R}^{2d}\,.$$

See the details in Ref. [24]. A similar result to that in the above theorem for a rank-one operator $S = \psi \otimes \phi$, where $\psi, \phi \in L^2(\mathbb{R}^d)$, can be found in Refs. [2, 6].

In case $\{\alpha_\lambda(S)\}_{\lambda\in\Lambda}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$, the operator S is the *generator* of the Λ-*shift-invariant subspace* V_S^2 which can be described by

$$V_S^2 := \overline{\operatorname{span}}_{\mathcal{HS}}\big\{\alpha_\lambda(S)\big\}_{\lambda\in\Lambda} = \Big\{\sum_{\lambda\in\Lambda} c(\lambda)\,\alpha_\lambda(S) \;:\; \{c(\lambda)\}_{\lambda\in\Lambda} \in \ell^2(\Lambda)\Big\}.$$

Observe that operators in V_S^2 are nothing but *Gabor multipliers* in case $S = \varphi \otimes \psi$. Indeed, for $\eta \in L^2(\mathbb{R}^d)$ we have

$$\sum_{\lambda\in\Lambda} c(\lambda)\,\alpha_\lambda(S)(\eta) = \sum_{\lambda\in\Lambda} c(\lambda)\,\big(\pi(\lambda)\varphi \otimes \pi(\lambda)\psi\big)(\eta) = \sum_{\lambda\in\Lambda} c(\lambda)\,V_\psi\,\eta(\lambda)\pi(\lambda)\varphi\,,$$

that is, $\sum_{\lambda\in\Lambda} c(\lambda)\,\alpha_\lambda(S) = \mathcal{G}_{\mathbf{c}}^{\psi,\varphi}$, the Gabor multiplier with windows ψ, φ and mask $\mathbf{c}$ in $\ell^2(\Lambda)$ used in time-frequency analysis (see, for instance, Ref. [24]).

Analogously, a necessary and sufficient condition can be obtained for the multiply generated case. Indeed, let $\mathbf{S} = \{S_1, S_2, \dots, S_N\}$ be a fixed subset of $\mathcal{HS}(\mathbb{R}^d)$ and let Λ be a lattice in $\mathbb{R}^{2d}$. We are searching for a necessary and sufficient condition such that $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$, i.e., a Riesz basis for the closed subspace

$$V_{\mathbf{S}}^2 := \overline{\operatorname{span}}_{\mathcal{HS}}\big\{\alpha_\lambda(S_n)\big\}_{\lambda\in\Lambda;\, n=1,2,\dots,N} \subset \mathcal{HS}(\mathbb{R}^d)\,.$$

For the multiply generated case we have the following result:

Theorem 2. *Let Λ be a lattice and $S_n \in \mathcal{B}$, $n = 1, 2, \dots, N$. Then, $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$ if and only if there exist two constants $0 < m \le M$ such that*

$$m\,\mathbb{I}_N \le G_{\mathbf{S}}^W(z) \le M\,\mathbb{I}_N \quad \text{for any } z \in \mathbb{R}^{2d}\,,$$

where $G_{\mathbf{S}}^W(z)$ denotes the $N \times N$ matrix-valued function

$$G_{\mathbf{S}}^W(z) := \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(\mathbf{S})(z+\lambda^\circ)\,\overline{\mathcal{F}_W(\mathbf{S})(z+\lambda^\circ)}^\top, \quad z \in \mathbb{R}^{2d}\,,$$

and $\mathcal{F}_W(\mathbf{S}) = \big(\mathcal{F}_W(S_1), \mathcal{F}_W(S_2), \dots, \mathcal{F}_W(S_N)\big)^\top$.

Proof. As indicated above, it will be a Riesz sequence in $\mathcal{HS}(\mathbb{R}^d)$ if and only if the sequence $\{T_\lambda(a_{S_n})\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ is a Riesz sequence in $L^2(\mathbb{R}^{2d})$. To this end, we introduce the $N \times N$ matrix-valued function

$$G_{\mathbf{S}}^\sigma(z) := \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_s(a_{\mathbf{S}})(z+\lambda^\circ)\,\overline{\mathcal{F}_s(a_{\mathbf{S}})(z+\lambda^\circ)}^\top, \quad z \in \mathbb{R}^{2d}\,,$$

where $\mathcal{F}_s(a_{\mathbf{S}}) = \big(\mathcal{F}_s(a_{S_1}), \mathcal{F}_s(a_{S_2}), \dots, \mathcal{F}_s(a_{S_N})\big)^\top$. It is known (see, for instance, Ref. [1]) that the sequence $\{T_\lambda(a_{S_n})\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ is a Riesz sequence in $L^2(\mathbb{R}^{2d})$ if and only if there exist two constants $0 < m \le M$ such that $m\,\mathbb{I}_N \le G_{\mathbf{S}}^\sigma(z) \le M\,\mathbb{I}_N$, a.e. $z \in \mathbb{R}^{2d}$, where $\mathbb{I}_N$ denotes the $N \times N$ identity matrix. Assuming as before that $S_n \in \mathcal{B}$, $n = 1, 2, \dots, N$, the functions $\mathcal{F}_s(a_{S_n})$ are continuous and $\mathcal{F}_W(S_n) = \mathcal{F}_s(a_{S_n})$ for $n = 1, 2, \dots, N$. Hence, the above necessary and sufficient condition can be expressed in terms of the hermitian matrix $G_{\mathbf{S}}^W(z)$ as in the statement of the theorem. □

In this case, $\mathbf{S}=\{S_1,S_2,\dots,S_N\}$ is a *set of generators* for the Λ-*shift-invariant subspace* $V^2_{\mathbf{S}}:=\overline{\operatorname{span}}_{\mathcal{HS}}\big\{\alpha_\lambda(S_n)\big\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ which can be described by

$$V^2_{\mathbf{S}}=\Big\{\sum_{n=1}^{N}\sum_{\lambda\in\Lambda}c_n(\lambda)\,\alpha_\lambda(S_n)\ :\ \{c_n(\lambda)\}_{\lambda\in\Lambda}\in\ell^2(\Lambda)\,,\ n=1,2,\dots,N\Big\}.$$

3.2 The isomorphism $\mathcal{T}_{\mathbf{S}}$

Our sampling results rely on the following isomorphism $\mathcal{T}_{\mathbf{S}}$ which involves the spaces $\ell^2_{_N}(\Lambda)$, the shift-invariant subspace $V^2_{\sigma_{\mathbf{S}}}$ in $L^2(\mathbb{R}^{2d})$ generated by the Kohn-Nirenberg symbols $\sigma_{_{S_n}}$ of S_n, $n=1,2,\dots,N$, and the Λ-shift-invariant subspace $V^2_{\mathbf{S}}$. Namely,

$$\begin{array}{cccccc}\mathcal{T}_{\mathbf{S}}: & \ell^2_{_N}(\Lambda) & \longrightarrow & V^2_{\sigma_{\mathbf{S}}}\subset L^2(\mathbb{R}^{2d}) & \longrightarrow & V^2_{\mathbf{S}}\subset\mathcal{HS}(\mathbb{R}^d)\\ & (c_1,c_2,\dots,c_N)^\top & \longmapsto & \displaystyle\sum_{n=1}^{N}\sum_{\lambda\in\Lambda}c_n(\lambda)\,T_\lambda\sigma_{_{S_n}} & \longmapsto & \displaystyle\sum_{n=1}^{N}\sum_{\lambda\in\Lambda}c_n(\lambda)\,\alpha_\lambda(S_n)\,.\end{array}\tag{5}$$

The isomorphism $\mathcal{T}_{\mathbf{S}}$ is the composition of the isomorphism $\mathcal{T}_{\sigma_{\mathbf{S}}}:\ell^2_{_N}(\Lambda)\to V^2_{\sigma_{\mathbf{S}}}$ which maps the standard orthonormal basis $\{\boldsymbol{\delta}_\lambda\}_{\lambda\in\Lambda}$ for $\ell^2_{_N}(\Lambda)$ onto the Riesz basis $\{T_\lambda\sigma_{_{S_n}}\}_{\lambda\in\Lambda;\,n=1,2,\dots,N}$ for $V^2_{\sigma_{\mathbf{S}}}$, and the Kohn-Nirenberg transform transform between $V^2_{\sigma_{\mathbf{S}}}$ and $V^2_{\mathbf{S}}$.

Recall that the Kohn-Nirenberg transform $L^2(\mathbb{R}^{2d})\ni f\mapsto K_f\in\mathcal{HS}(\mathbb{R}^d)$ is a unitary operator which respects translations in the sense that $K_{T_zf}=\alpha_z(K_f)$ for $f\in L^2(\mathbb{R}^{2d})$ and $z\in\mathbb{R}^{2d}$. See, for instance, Ref. [7, 16].

3.3 An expression for the samples

For each $T=\sum_{n=1}^{N}\sum_{\mu\in\Lambda}c_n(\mu)\,\alpha_\mu(S_n)$ in $V^2_{\mathbf{S}}$ we define a set of *diagonal channel samples* as

$$\mathbf{s}_{_T}(\lambda):=\big(\langle\alpha_{-\lambda}(T)g_1,\widetilde{g}_1\rangle,\langle\alpha_{-\lambda}(T)g_2,\widetilde{g}_2\rangle,\dots,\langle\alpha_{-\lambda}(T)g_{_M},\widetilde{g}_{_M}\rangle\big)^\top,\quad\lambda\in\Lambda\,,\tag{6}$$

where $g_m,\widetilde{g}_m$, $m=1,2,\dots,M$, denote $2M$ fixed functions in $L^2(\mathbb{R}^d)$. For $m=1,2,\dots,M$ the above samples can be expressed by

$$\begin{aligned}s_{_{T,m}}(\lambda):=&\big\langle\alpha_{-\lambda}(T)g_m,\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}=\Big\langle\sum_{n=1}^{N}\sum_{\mu\in\Lambda}c_n(\mu)\alpha_{\mu-\lambda}(S_n)g_m,\widetilde{g}_m\Big\rangle_{L^2(\mathbb{R}^d)}\\ =&\sum_{n=1}^{N}\sum_{\mu\in\Lambda}c_n(\mu)\big\langle\alpha_{\mu-\lambda}(S_n)g_m,\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}=\sum_{n=1}^{N}\big(a_{m,n}*_\Lambda c_n\big)(\lambda)\,,\quad\lambda\in\Lambda\,,\end{aligned}\tag{7}$$

where $a_{m,n}(\mu):=\big\langle\alpha_{-\mu}(S_n)g_m,\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}$, $\mu\in\Lambda$. Observe that $a_{m,n}(\lambda)$, $\lambda\in\Lambda$, are precisely the samples $\mathbf{s}_{_{S_n}}(\lambda)$, $\lambda\in\Lambda$, of the generator S_n.

Lemma 3. *Concerning the samples defined in Eq.* (7) *we have:*

1. *For $m=1,2,\dots,M$ these samples can be written as*

$$\big\langle\alpha_{-\lambda}(T)g_m,\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}=\big\langle T\pi(\lambda)g_m,\pi(\lambda)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}=\big\langle T,\alpha_\lambda(\widetilde{g}_m\otimes g_m)\big\rangle_{_{\mathcal{HS}}},\ \lambda\in\Lambda\,.\tag{8}$$

2. *The sequences* $\{a_{m,n}(\lambda)\}_{\lambda\in\Lambda}$ *appearing in Eq.* (7) *belong to* $\ell^2(\Lambda)$ *for* $m=1,2,\dots,M$ *and* $n=1,2,\dots,N$.

Proof. For the first equality in (8) we have that

$$\begin{aligned} s_{T,m}(\lambda) &= \big\langle \alpha_{-\lambda}(T) g_m, \widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)} = \big\langle \pi(-\lambda) T \pi(-\lambda)^* g_m, \widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)} \\ &= \big\langle \sigma_T, R\big(\pi(-\lambda)^* \widetilde{g}_m, \pi(-\lambda)^* g_m\big) \big\rangle_{L^2(\mathbb{R}^{2d})}, \quad \lambda \in \Lambda . \end{aligned}$$

On the other hand, it is easy to check that for the Rihaczek distribution one gets

$$R\big(\pi(-\lambda)^*\widetilde{g}_m, \pi(-\lambda)^* g_m\big)(z) = R\big(\pi(\lambda)\widetilde{g}_m, \pi(\lambda) g_m\big)(z), \quad z \in \mathbb{R}^{2d}.$$

Hence, for each $\lambda\in\Lambda$ we obtain

$$\big\langle \alpha_{-\lambda}(T) g_m, \widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)} = \big\langle \sigma_T, R\big(\pi(\lambda)\widetilde{g}_m, \pi(\lambda) g_m\big) \big\rangle_{L^2(\mathbb{R}^{2d})} = \big\langle T\pi(\lambda) g_m, \pi(\lambda)\widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)} .$$

For the second equality we get

$$\begin{aligned} \big\langle T, \alpha_\lambda(\widetilde{g}_m \otimes g_m) \big\rangle_{\mathcal{HS}} &= \big\langle T, \pi(\lambda)\widetilde{g}_m \otimes \pi(\lambda) g_m \big\rangle_{\mathcal{HS}} = \big\langle \sigma_T, \sigma_{\pi(\lambda)\widetilde{g}_m \otimes \pi(\lambda) g_m} \big\rangle_{L^2(\mathbb{R}^{2d})} \\ &= \big\langle \sigma_T, R(\pi(\lambda)\widetilde{g}_m, \pi(\lambda) g_m) \big\rangle_{L^2(\mathbb{R}^{2d})} = \big\langle T\pi(\lambda) g_m, \pi(\lambda)\widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)} . \end{aligned}$$

We have used that the Kohn-Nirenberg symbol of $\pi(\lambda)\widetilde{g}_m \otimes \pi(\lambda) g_m$ coincides with the Rihaczek distribution of the pair of functions $\pi(\lambda)\widetilde{g}_m$ and $\pi(\lambda) g_m$ in $L^2(\mathbb{R}^d)$.

In particular we have proved that

$$a_{m,n}(\lambda) = \big\langle \alpha_{-\lambda}(S_n) g_m, \widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)} = \big\langle S_n, \alpha_\lambda(\widetilde{g}_m \otimes g_m) \big\rangle_{\mathcal{HS}} = \big\langle \alpha_{-\lambda}(S_n), \widetilde{g}_m \otimes g_m \big\rangle_{\mathcal{HS}}, \quad \lambda \in \Lambda .$$

Since $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$, it is in particular a Bessel sequence in $\mathcal{HS}(\mathbb{R}^d)$. Hence, the sequences $\big\{\big\langle \alpha_{-\lambda}(S_n), \widetilde{g}_m \otimes g_m \big\rangle_{\mathcal{HS}}\big\}_{\lambda\in\Lambda}$ belongs to $\ell^2(\Lambda)$ for $m=1,2,\dots,M$ and $n=1,2,\dots,N$. □

Once we have that $a_{m,n}\in\ell^2(\Lambda)$ for each $m=1,2,\dots,M$ and $n=1,2,\dots,N$, and denoting $A=[a_{m,n}]$ the corresponding $M\times N$ matrix with entries in $\ell^2(\Lambda)$, the sampling process in (6) is described by means of the discrete convolution system

$$T = \sum_{n=1}^{N} \sum_{\lambda\in\Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \in V_{\mathbf{S}}^2 \longmapsto \mathbf{s}_T(\lambda) = \big(A *_\Lambda \mathbf{c}\big)(\lambda) = \sum_{\mu\in\Lambda} A(\lambda-\mu)\, \mathbf{c}(\mu), \quad \lambda \in \Lambda ,$$

where $\mathbf{c}=(c_1,c_2,\dots,c_N)^\top \in \ell^2_N(\Lambda) := \ell^2(\Lambda)\times\cdots\times\ell^2(\Lambda)$ (N times). Note that the m-th entry of $A *_\Lambda \mathbf{c}$ is $\sum_{n=1}^N (a_{m,n} *_\Lambda c_n)$.

First of all, the mapping $\mathcal{A}:\ell^2_N(\Lambda)\to\ell^2_M(\Lambda)$ which maps $\mathbf{c}\mapsto A *_\Lambda \mathbf{c}$ is a well-defined bounded operator if and only if the $M\times N$ matrix-valued function $\widehat{A}(\xi) := \big[\mathcal{F}^\Lambda_s(a_{m,n})(\xi)\big]$, a.e. $\xi\in\widehat{\Lambda}$, has entries in $L^\infty(\widehat{\Lambda})$. The needed results on discrete convolution systems $\mathcal{A}:\ell^2_N(\Lambda)\to\ell^2_M(\Lambda)$, and their relationship with frames of translates in $\ell^2_N(\Lambda)$ can be found in Ref. [15]. Notice that the m-th component of $A *_\Lambda \mathbf{c}$ is

$$[A * \mathbf{c}]_m(\lambda) = \sum_{n=1}^{N} (a_{m,n} *_\Lambda c_n)(\lambda) = \big\langle \mathbf{c}, T_\lambda\, \mathbf{a}_m^* \big\rangle_{\ell^2_N(\Lambda)}, \quad \lambda \in \Lambda ,$$

where $a^*_{m,n}$ denotes the involution $a^*_{m,n}(\lambda) := \overline{a_{m,n}(-\lambda)}$, $\lambda \in \Lambda$. As a consequence, the operator $\mathcal{A}$ is the analysis operator of the sequence $\{T_\lambda \mathbf{a}^*_m\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ in $\ell^2_{_N}(\Lambda)$. Since the sequence $\{T_\lambda\, \mathbf{a}^*_m\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ is a frame for $\ell^2_{_N}(\Lambda)$ if and only if its bounded analysis operator is injective with a closed range (see Ref. [3]), it will be a frame for $\ell^2_N(\Lambda)$ if and only if

$$0 < \alpha_A := \operatorname*{ess\,inf}_{\xi\in\widehat{\Lambda}} \lambda_{\min}[\,\widehat{A}(\xi)^*\widehat{A}(\xi)] \le \beta_A := \operatorname*{ess\,sup}_{\xi\in\widehat{\Lambda}} \lambda_{\max}[\widehat{A}(\xi)^*\widehat{A}(\xi)] < +\infty\,, \tag{9}$$

where $\lambda_{\min}$ (respectively, $\lambda_{\max}$) denotes the smallest (respectively, the largest) eigenvalue of the positive semidefinite matrix $\widehat{A}(\xi)^*\widehat{A}(\xi)$ (see Ref. [15]).

Concerning the duals of $\{T_\lambda\, \mathbf{a}^*_m\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ having its same structure, consider two matrices $\widehat{A} \in \mathcal{M}_{_{M\times N}}(L^\infty(\widehat{\Lambda}))$ and $\widehat{B} \in \mathcal{M}_{_{N\times M}}(L^\infty(\widehat{\Lambda}))$, and let $\mathbf{b}_m$ denote the m-th column of the matrix B associated to $\widehat{B}$. Then, the sequences $\{T_\lambda\, \mathbf{a}^*_m\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ and $\{T_\lambda\, \mathbf{b}_m\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ form a pair of dual frames for $\ell^2_{_N}(\Lambda)$ if and only if $\widehat{B}(\xi)\, \widehat{A}(\xi) = \mathbb{I}_{_N}$, a.e. $\xi \in \widehat{\Lambda}$; equivalently, if and only if $\mathcal{B}\,\mathcal{A} = \mathcal{I}_{\ell^2_N(\Lambda)}$, i.e., the convolution system $\mathcal{B}$ with matrix B is a left-inverse of the convolution system $\mathcal{A}$ with matrix A. Thus, we have the frame expansion

$$\mathbf{c} = \sum_{m=1}^{M}\sum_{\lambda\in\Lambda} \big\langle \mathbf{c}, T_\lambda \mathbf{a}^*_m\big\rangle_{\ell^2_{_N}(\Lambda)}\, T_\lambda \mathbf{b}_m \quad \text{for each } \mathbf{c} \in \ell^2_{_N}(\Lambda)\,.$$

Observe that a possible left-inverse $\widehat{B}(\xi)$ of the matrix $\widehat{A}(\xi)$ is given by its Moore-Penrose pseudo-inverse $\widehat{A}(\xi)^\dagger = \big[\widehat{A}(\xi)^*\widehat{A}(\xi)\big]^{-1}\widehat{A}(\xi)^*$, a.e. $\xi \in \widehat{\Lambda}$.

3.4 The sampling results

Next we prove the main sampling result in this paper:

Theorem 4. *Suppose that for each $T \in V^2_{\mathbf{S}}$ we consider the samples defined by* (6)*, and such that the matrix $A = [a_{m,n}]$, where $a_{m,n}(\lambda) = \big\langle \alpha_{-\lambda}(S_n)g_m, \widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}$, $\lambda \in \Lambda$, satisfies conditions in Eq.* (9)*. Then, there exist $M \ge N$ elements $H_m \in V^2_{\mathbf{S}}$, $m = 1, 2, \dots, M$, such that the sampling formula*

$$T = \sum_{m=1}^{M}\sum_{\lambda\in\Lambda} s_{_{T,m}}(\lambda)\, \alpha_\lambda(H_m) \quad \textit{in } \mathcal{HS}\textit{-norm} \tag{10}$$

holds for each $T \in V^2_{\mathbf{S}}$ where $\{\alpha_\lambda(H_m)\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ is a frame for $V^2_{\mathbf{S}}$. The convergence of the series is unconditional in Hilbert-Schmidt norm.

Moreover, the ℓ^2-norm of the samples $\|\mathbf{s}_{_T}\|_{\ell^2_{_M}}$ defines an equivalent norm to $\|T\|_{\mathcal{HS}}$ in $V^2_{\mathbf{S}}$, and for each $f \in L^2(\mathbb{R}^d)$ we have the pointwise expansion

$$Tf = \sum_{m=1}^{M}\sum_{\lambda\in\Lambda} s_{_{T,m}}(\lambda)\, \alpha_\lambda(H_m) f \quad \textit{in } L^2(\mathbb{R}^d)\,.$$

Proof. Under the hypotheses of the theorem the sequence $\big\{T_\lambda\, \mathbf{a}_m^*\big\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ is a frame for $\ell^2_{_N}(\Lambda)$, and we can consider a dual frame $\big\{T_\lambda\, \mathbf{b}_m\big\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ with the same structure. As a consequence, for each $T=\sum_{n=1}^N \sum_{\lambda\in\Lambda} c_n(\lambda)\, \alpha_\lambda(S_n)$ in $V^2_{\mathbf{S}}$ we have

$$\mathbf{c}=\sum_{m=1}^M \sum_{\lambda\in\Lambda} \big\langle \mathbf{c}, T_\lambda \mathbf{a}_m^*\big\rangle_{\ell^2_{_N}(\Lambda)}\, T_\lambda \mathbf{b}_m=\sum_{m=1}^M \sum_{\lambda\in\Lambda} s_{_{T,m}}(\lambda)\, T_\lambda \mathbf{b}_m \quad \text{in } \ell^2_{_N}(\Lambda)\,, \tag{11}$$

where $\mathbf{c}=(c_1,c_2,\dots,c_N)^\top\in \ell^2_{_N}(\Lambda)$. Notice that the fact that $\big\{T_\lambda\, \mathbf{a}_m^*\big\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ is a frame for $\ell^2_{_N}(\Lambda)$ and the isomorphism $\mathcal{T}_{\mathbf{S}}$ in Eq. (5) give the equivalence of the norms.

The isomorphism $\mathcal{T}_{\mathbf{S}}$ defined by Eq. (5) applied in Eq. (11) gives the sampling expansion (10), where $H_m=K_{h_m}\in V^2_{\mathbf{S}}$ with Kohn-Nirenberg symbol $h_m=\mathcal{T}_{\sigma_{\mathbf{S}}}(\mathbf{b}_m)\in V^2_{\sigma_{\mathbf{S}}}$, $m=1,2,\dots,M$. Furthermore, since $\{\alpha_\lambda(H_m)\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ is a frame for $V^2_{\mathbf{S}}$ the convergence of the series in the Hilbert-Schmidt norm is unconditional. Notice that $\mathcal{T}_{\sigma_S}(T_\lambda \mathbf{b}_m)=T_\lambda(\mathcal{T}_{\sigma_S}\mathbf{b}_m)=T_\lambda(h_m)$, where the same symbol T_λ denotes both the translation by λ in $\ell^2_{_N}(\Lambda)$ and in $L^2(\mathbb{R}^{2d})$ respectively. Notice that if $\mathbf{b}_m=\big(b_{1,m}(\lambda), b_{2,m}(\lambda),\dots,b_{N,m}(\lambda)\big)^\top$, then

$$H_m=\sum_{n=1}^N \sum_{\lambda\in\Lambda} b_{n,m}(\lambda)\, \alpha_\lambda(S_n)\,,\quad m=1,2,\dots,M\,.$$

Since convergence in $\mathcal{HS}$-norm implies convergence in operator norm we deduce the pointwise expansion for each $f\in L^2(\mathbb{R}^d)$. ☐

Observe that, due to conditions (9) in Theorem 4 we have necessarily $M\geq N$. Whenever $M>N$, there are infinite dual frames $\big\{T_\lambda\, \mathbf{b}_m\big\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ of $\big\{T_\lambda\, \mathbf{a}_m^*\big\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ given by the samples (7). They are obtained from the left-inverses $\widehat{B}(\xi)$ of $\widehat{A}(\xi)$ which are deduced, from the Moore-Penrose pseudo-inverse $\widehat{A}(\xi)^\dagger$, as the $N\times M$ matrices

$$\widehat{B}(\xi):=\widehat{A}(\xi)^\dagger + C(\xi)\big[\mathbb{I}_{_M}-\widehat{A}(\xi)\widehat{A}(\xi)^\dagger\big]\,,\quad \text{a.e. } \xi\in\widehat{\Lambda}\,,$$

where C denotes any $N\times M$ matrix with entries in $L^\infty(\widehat{\Lambda})$.

More can be said in case $M=N$:

Corollary 5. *In case $M=N$, assume that the conditions*

$$0<\operatorname*{ess\,inf}_{\xi\in\widehat{\Lambda}} \big|\det[\widehat{A}(\xi)]\big| \leq \operatorname*{ess\,sup}_{\xi\in\widehat{\Lambda}} \big|\det[\widehat{A}(\xi)]\big|<+\infty \tag{12}$$

hold. Then, there exist N unique elements H_n, $n=1,2,\dots,N$, in $V^2_{\mathbf{S}}$ such that the associated sequence $\big\{\alpha_\lambda(H_n)\big\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ is a Riesz basis for $V^2_{\mathbf{S}}$ and the sampling formula

$$T=\sum_{n=1}^N \sum_{\lambda\in\Lambda} s_{_{T,n}}(\lambda)\, \alpha_\lambda(H_n) \quad \textit{in } \mathcal{HS}\textit{-norm}$$

holds for each $T\in V^2_{\mathbf{S}}$. Moreover, the interpolation property $\big\langle \alpha_{-\lambda}(H_m)g_n, \widetilde{g}_n\big\rangle=\delta_{m,n}\delta_{\lambda,0}$, where $\lambda\in\Lambda$ and $m,n=1,2,\dots,N$, holds.

Proof. In this case, the square matrix $\widehat{A}(\xi)$ is invertible and the statement (12) in corollary is equivalent to condition $0 < \alpha_A \le \beta_A < +\infty$ in (9); besides, any Riesz basis has a unique dual basis. The uniqueness of the coefficients in a Riesz basis expansion gives the interpolation property. $\square$

In particular, for the case $N = M = 1$ we have:

Corollary 6. *Assume that the sequence* $\mathbf{a} = \{a(\lambda)\}_{\lambda\in\Lambda}$*, where* $a(\lambda) = \big\langle \alpha_{-\lambda}(S)g, \widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)}$*,* $\lambda \in \Lambda$*, for a fixed pair of functions* $g, \widetilde{g} \in L^2(\mathbb{R}^d)$*, satisfies the conditions*

$$0 < \operatorname*{ess\,inf}_{\xi\in\widehat{\Lambda}} |\mathcal{F}_s^\Lambda(\mathbf{a})(\xi)| \le \operatorname*{ess\,sup}_{\xi\in\widehat{\Lambda}} |\mathcal{F}_s^\Lambda(\mathbf{a})(\xi)| < \infty\,. \tag{13}$$

Then, there exists a unique $H \in V_S^2$ *such that the sequence* $\big\{\alpha_\lambda(H)\big\}_{\lambda\in\Lambda}$ *is a Riesz basis for* V_S^2 *and the sampling formula*

$$T = \sum_{\lambda\in\Lambda} \big\langle \alpha_{-\lambda}(T)g, \widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)}\, \alpha_\lambda(H) \quad \textit{in } \mathcal{HS}\textit{-norm}$$

holds for each $T \in V_S^2$*. Moreover, the interpolation property* $\big\langle \alpha_{-\lambda}(H)g, \widetilde{g}\,\big\rangle = \delta_{\lambda,0}$*,* $\lambda \in \Lambda$*, holds; in particular,* $\big\langle Hg, \widetilde{g}\,\big\rangle = 1$*.*

It is worth to remark that in the above sampling result is not necessary that the operators in V_S^2 have a bandlimited Kohn-Nirenberg symbol as in Ref. [18, Theorem 2].

The bandlimited case is obtained as a particular case. Let $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ be a lattice in $\mathbb{R}^{2d}$ with $a, b > 0$. Assume that the generator S of V_S^2 is a bandlimited operator to $Q := [\frac{-1}{2a}, \frac{1}{2a}]^d \times [\frac{-1}{2b}, \frac{1}{2b}]^d$, i.e., it belongs to $OPW^2(Q) := \{T \in \mathcal{HS}(\mathbb{R}^d) \,:\, \operatorname{supp}\widehat{\sigma}_T \subseteq Q\}$. Then any $T \in V_S^2$ also belongs to $OPW^2(Q)$. In case conditions (13) are satisfied, any $T \in V_S^2$ can be recovered from its diagonal channel samples as

$$T = \sum_{\lambda\in\Lambda} \big\langle T\pi(\lambda)g, \pi(\lambda)\widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)}\, \alpha_\lambda(H) \quad \text{in } \mathcal{HS}\text{-norm}\,,$$

where $H = \sum_{\lambda\in\Lambda} b(\lambda)\,\alpha_\lambda(S)$ in V_S^2 is obtained from the sequence $\mathbf{b} = \{b(\lambda)\}_{\lambda\in\Lambda}$ in $\ell^2(\Lambda)$ such that $\mathcal{F}_s^\Lambda(\mathbf{b})(\xi)\,\mathcal{F}_s^\Lambda(\mathbf{a})(\xi) = 1$, a.e. $\xi \in \widehat{\Lambda}$.

In Ref. [18] the reconstruction of pseudodifferential operators with a bandlimited Kohn-Nirenberg symbol is considered. In particular, Theorem 2 of the same reference proves that, under some appropriate assumptions, for any $T \in OPW^2(Q)$ we have

$$\sigma_T = \frac{1}{(ab)^d} \sum_{\lambda\in\Lambda} \big\langle T\pi(\lambda)g, \pi(\lambda)g\,\big\rangle_{L^2(\mathbb{R}^d)}\, T_\lambda\big(\mathrm{sinc}_{a,b} * k\big) \quad \text{in } L^2(\mathbb{R}^{2d})\,,$$

where the function k, independent of T, belongs to $L^1(\mathbb{R}^{2d})$ and $\mathrm{sinc}_{\mathrm{a,b}}$ denotes the *sinc function* adapted to the lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, namely

$$\mathrm{sinc}_{a,b}(x) = \prod_{j=1}^{d} \frac{\sin \pi a x_j}{\pi a x_j} \prod_{j=d+1}^{2d} \frac{\sin \pi b x_j}{\pi b x_j}\,, \quad x \in \mathbb{R}^{2d}\,.$$

Using the Kohn-Nirenberg transfom, the above sampling formula for σ_T can be written as

$$T = \frac{1}{(ab)^d} \sum_{\lambda\in\Lambda} \big\langle T\pi(\lambda)g, \pi(\lambda)g \big\rangle_{L^2(\mathbb{R}^d)} \alpha_\lambda(k * K_{\text{sinc}_{\text{a,b}}}) \quad \text{in } \mathcal{HS}\text{-norm}\,,$$

where $k * K_{\text{sinc}_{a,b}}$ denotes the Hilbert-Schmidt operator obtained from the convolution of the function k and the operator $K_{\text{sinc}_{a,b}}$; we have also used the following result:

Lemma 7. *Let K_f be an operator in $\mathcal{HS}(\mathbb{R}^d)$ with Kohn-Nirenberg symbol $f \in L^2(\mathbb{R}^{2d})$, and let g a function in $L^1(\mathbb{R}^{2d})$. Then we have that $K_{g*f} = g * K_f$.*

Proof. Recall that the convolution $g * K_f$ is the operator in $\mathcal{HS}(\mathbb{R}^d)$ defined by the operator-valued integral (in weak sense)

$$g * K_f = \int_{\mathbb{R}^{2d}} g(z)\, \alpha_z(K_f) dz\,,$$

i.e.,

$$\Big\langle \Big(\int_{\mathbb{R}^{2d}} g(z)\, \alpha_z(K_f) dz \Big) \varphi, \psi \Big\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} g(z)\, \langle \alpha_z(K_f)\varphi, \psi \rangle dz\,, \quad \varphi, \psi \in L^2(\mathbb{R}^d)\,.$$

See the details in Refs. [21, 24, 26]. Since the map $\mathcal{K} : \mathcal{HS}(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ such that $\mathcal{K}(K_f) = f$ is a unitary operator and bounded operators conmute with convergent integrals [21, Proposition 2.4] we get

$$\mathcal{K}(g * K_f) = \int_{\mathbb{R}^{2d}} g(z)\, \mathcal{K}\big(\alpha_z(K_f)\big) dz = \int_{\mathbb{R}^{2d}} g(z)\, \mathcal{K}\big(K_{T_z f}\big) dz = \int_{\mathbb{R}^{2d}} g(z)\, f(\cdot - z) dz = g * f\,,$$

that is, $K_{g*f} = g * K_f$. □

In the same manner we can consider *average sampling* in $V^2_{\mathbf{S}}$. Namely, for any $T \in V^2_{\mathbf{S}}$, its *average samples* at Λ are defined by

$$\big\langle T, \alpha_\lambda(Q_m) \big\rangle_{\mathcal{HS}}\,, \quad \lambda \in \Lambda\,,\ \ m = 1, 2, \dots, M\,,$$

from M fixed operators $Q_1, Q_2, \dots, Q_M$ in $\mathcal{HS}(\mathbb{R}^d)$, not necessarily in $V^2_{\mathbf{S}}$. Observe that, having in mind Eq. (8) in Lemma 3, the diagonal channel samples defined in Eq. (6) are a particular case of average sampling where $Q_m = \widetilde{g}_m \otimes g_m$, $m = 1, 2, \dots, M$. The average samples of any $T = \sum_{n=1}^N \sum_{\mu\in\Lambda} c_n(\mu)\, \alpha_\mu(S_n)$ can be also expressed as a discrete convolution system in $\ell^2_N(\Lambda)$. Indeed, for $m = 1, 2, \dots, M$ we have

$$\begin{aligned}
\big\langle T, \alpha_\lambda(Q_m) \big\rangle_{\mathcal{HS}} &= \big\langle \sigma_T, T_\lambda \sigma_{Q_m} \big\rangle_{L^2(\mathbb{R}^{2d})} = \big\langle \sum_{n=1}^N \sum_{\mu\in\Lambda} c_n(\mu)\, T_\mu \sigma_{S_n}, T_\lambda \sigma_{Q_m} \big\rangle_{L^2(\mathbb{R}^{2d})} \\
&= \sum_{n=1}^N \sum_{\mu\in\Lambda} c_n(\mu) \langle T_\mu \sigma_{S_n}, T_\lambda \sigma_{Q_m} \rangle_{L^2(\mathbb{R}^{2d})} = \sum_{n=1}^N \sum_{\mu\in\Lambda} c_n(\mu) \langle \sigma_{S_n}, T_{\lambda-\mu} \sigma_{Q_m} \rangle_{L^2(\mathbb{R}^{2d})} \\
&= \sum_{n=1}^N \big(a_{m,n} *_\Lambda c_n \big)(\lambda)\,, \quad \lambda \in \Lambda\,,
\end{aligned}$$

where $a_{m,n}(\mu) := \langle \sigma_{S_n}, T_\mu \sigma_{Q_m} \rangle_{L^2(\mathbb{R}^{2d})} = \langle S_n, \alpha_\mu(Q_m) \rangle_{\mathcal{HS}}$, $\mu \in \Lambda$, and σ_{S_n}, σ_{Q_m} are the Kohn-Nirenberg symbols of S_n, Q_m respectively.

Observe that, for each $m = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$, the sequence $\{a_{m,n}(\lambda)\}_{\lambda \in \Lambda}$ belongs to $\ell^2(\Lambda)$ since, in particular, $\{T_\lambda \sigma_{S_n}\}_{\lambda \in \Lambda;\, n=1,2,\ldots,N}$ is a Bessel sequence in $L^2(\mathbb{R}^{2d})$.

Corollary 8. *Assume that the matrix $A = [a_{m,n}]$ with entries $a_{m,n}(\lambda) = \langle S_n, \alpha_\lambda(Q_m) \rangle_{\mathcal{HS}}$, $\lambda \in \Lambda$, satisfies conditions in (9). Then, there exist $M \geq N$ operators $H_m \in V^2_{\mathbf{S}}$, $m = 1, 2, \ldots, M$, such that the sampling formula*

$$T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} \langle T, \alpha_\lambda(Q_m) \rangle_{\mathcal{HS}}\, \alpha_\lambda(H_m) \quad \textit{in } \mathcal{HS}\textit{-norm}$$

holds for each $T \in V^2_{\mathbf{S}}$ where $\{\alpha_\lambda(H_m)\}_{\lambda \in \Lambda;\, m=1,2,\ldots,M}$ is a frame for $V^2_{\mathbf{S}}$. The convergence of the series is unconditional in Hilbert-Schmidt norm.

The above sampling formula was obtained in Ref. [12] by using the Weyl symbols of S_n and Q_m instead of their Kohn-Nirenberg symbols. Finally, it is worth to mention that each sampling result in this section admit a kind of converse result; see the details in Theorems 1-2 and Corollary 3 of Ref. [12].

An illustrative example

Assume that in $V^2_{\mathbf{S}}$ we have N stable generators of the form $S_n = \varphi_n \otimes \widetilde{\varphi}_n$ with $\varphi_n, \widetilde{\varphi}_n \in \mathcal{S}_0(\mathbb{R}^d)$, $n = 1, 2, \ldots, N$. In this regard, note that in order to apply Theorem 2 we have that $\mathcal{F}_W(\varphi_n \otimes \widetilde{\varphi}_n)(z) = \mathrm{e}^{\pi i x \cdot \omega}\, V_{\widetilde{\varphi}_n} \varphi_n(z)$, $z = (x, \omega) \in \mathbb{R}^{2d}$ (see Ref. [24]).

Next, for each $T \in V^2_{\mathbf{S}}$ we consider the diagonal channel samples $\langle T\pi(\lambda) g_m, \pi(\lambda)\widetilde{g}_m \rangle_{L^2(\mathbb{R}^d)}$, $\lambda \in \Lambda$ and $m = 1, 2, \ldots M$, with $g_m, \widetilde{g}_m \in \mathcal{S}_0(\mathbb{R}^d)$. In this case, for $m = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$, we get

$$\begin{aligned} a_{m,n}(\lambda) &= \langle \alpha_{-\lambda}(\varphi_n \otimes \widetilde{\varphi}_n) g_m, \widetilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \langle (\varphi_n \otimes \widetilde{\varphi}_n)\pi(\lambda) g_m, \pi(\lambda)\widetilde{g}_m \rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \langle \pi(\lambda) g_m, \widetilde{\varphi}_n \rangle \varphi_n, \pi(\lambda)\widetilde{g}_m \rangle_{L^2(\mathbb{R}^d)} = \overline{V_{g_m} \widetilde{\varphi}_n(\lambda)}\, V_{\widetilde{g}_m} \varphi_n(\lambda)\,, \quad \lambda \in \Lambda\,. \end{aligned}$$

From Proposition 4.1 in Ref. [24] we deduce that the sequences $\{a_{m,n}(\lambda)\}_{\lambda \in \Lambda}$ belong to $\ell^1(\Lambda)$ and, as a consequence, the entries in the transfer matrix $\widehat{A}$ are continuous functions on the compact $\widehat{\Lambda}$. In order to apply Theorem 4 conditions in Eq. (9) reduce to

$$\det[\widehat{A}(\xi)^* \widehat{A}(\xi)] \neq 0 \quad \text{for all } \xi \in \widehat{\Lambda}\,.$$

Under the above circumstances, any $T \in V^2_{\mathbf{S}}$, which is nothing but $T = \sum_{n=1}^{N} \mathcal{G}_{\mathbf{c}_n}^{\widetilde{\varphi}_n, \varphi_n}$ a finite sum of Gabor multipliers, can be recovered, in a stable way, from its diagonal channel samples $\langle T\pi(\lambda) g_m, \pi(\lambda)\widetilde{g}_m \rangle_{L^2(\mathbb{R}^d)}$, $\lambda \in \Lambda$ and $m = 1, 2, \ldots, M$.

3.5 Sampling in a sub-lattice of Λ

Let Λ' be a sub-lattice of Λ with finite index L, i.e., the quotient group Λ/Λ' has finite order L. We consider $\{\lambda_1, \lambda_2, \ldots, \lambda_L\}$ a set of representatives of the cosets of Λ'. That is, the lattice Λ be decomposed as

$$\Lambda = \bigcup_{l=1}^{L} (\lambda_l + \Lambda') \quad \text{with} \quad (\lambda_l + \Lambda') \cap (\lambda_{l'} + \Lambda') = \varnothing \text{ for } l \neq l' .$$

Thus, the space $V_{\mathbf{S}}^2$ can be written as

$$\begin{aligned} V_{\mathbf{S}}^2 &= \Big\{ \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \,:\, c_n \in \ell^2(\Lambda) \Big\} = \Big\{ \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{\mu \in \Lambda'} c_n(\lambda_l + \mu)\, \alpha_{\lambda_l + \mu}(S_n) \Big\} \\ &= \Big\{ \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{\mu \in \Lambda'} c_{nl}(\mu)\, \alpha_\mu(S_{nl}) \,:\, c_{nl} \in \ell^2(\Lambda') \Big\} , \end{aligned}$$

where $c_{nl}(\mu) := c_n(\lambda_l + \mu)$ and $S_{nl} := \alpha_{\lambda_l}(S_n)$, and the new index nl goes from 11 to NL. As a consequence, the subspace $V_{\mathbf{S}}^2$ can be rewritten as $V_{\widetilde{\mathbf{S}}}^2$ with NL generators $\widetilde{\mathbf{S}} = \{S_{nl}\}$ and coefficients c_{nl} in $\ell^2(\Lambda')$.

Let $T = \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{\nu \in \Lambda'} c_{nl}(\nu)\, \alpha_\nu(S_{nl})$ be in $V_{\mathbf{S}}^2$; its samples $\big\langle \alpha_{-\mu}(T) g_m, \widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)}$, $\mu \in \Lambda'$, can be expressed by

$$\begin{aligned} s_{T,m}(\mu) &:= \big\langle \alpha_{-\mu}(T) g_m, \widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)} = \Big\langle \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{\nu \in \Lambda'} c_{nl}(\mu)\, \alpha_{\nu-\mu}(S_{nl}) g_m, \widetilde{g}_m \Big\rangle_{L^2(\mathbb{R}^d)} \\ &= \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{\nu \in \Lambda'} c_{nl}(\nu) \big\langle \alpha_{\nu-\mu}(S_{nl}) g_m, \widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)} = \sum_{n=1}^{N} \sum_{l=1}^{L} \big(a_{m,nl} *_{\Lambda'} c_{nl} \big)(\mu) , \quad \mu \in \Lambda' , \end{aligned}$$

where $a_{m,nl}(\nu) := \big\langle \alpha_{-\nu}(S_{nl}) g_m, \widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)}$, $\nu \in \Lambda'$. Hence, Theorem 4 gives:

Corollary 9. *Let $A = [a_{m,nl}]$ be the $M \times NL$ matrix with entries*

$$a_{m,nl}(\nu) = \big\langle \alpha_{-\nu}(S_{nl}) g_m, \widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)} , \quad \nu \in \Lambda' ,$$

for $m = 1, 2, \dots, M$ and $nl = 11, 12, \dots, NL$. Assume that A satisfies conditions in (9) *with respect to the dual $\widehat{\Lambda}'$. Then, there exist $M \geq NL$ operators $H_m \in V_{\mathbf{S}}^2$, $m = 1, 2, \dots, M$, such that the sampling formula*

$$T = \sum_{m=1}^{M} \sum_{\mu \in \Lambda'} s_{T,m}(\mu)\, \alpha_\mu(H_m) \quad \text{in } \mathcal{HS}\text{-norm}$$

holds for each $T \in V_{\mathbf{S}}^2$ where $\{\alpha_\mu(H_m)\}_{\mu \in \Lambda';\, m=1,2,\dots,M}$ is a frame for $V_{\mathbf{S}}^2$. The convergence of the series is unconditional in Hilbert-Schmidt norm.

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