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The Use of Frames in Sampling Theory

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For those who enjoy maths…

Preface

When introducing a new book on sampling theory, the author should first justify the need for yet another title in an already long list of books and survey papers published on the subject. In spite of the fact that a rich mathematical literature in sampling already exists, the main reason for this book is to build a new narrative from my own experience in sampling research. Since it is almost impossible to be exhaustive, it is necessary to make a choice of contents giving a justification for this selection. Namely, my choice is guided by the idea that entitles the book: *The Use of Frames in Sampling Theory*.

Roughly speaking, sampling theory deals with the reconstruction of functions f, belonging, in general, to a Hilbert space $\mathcal{H}$ of continuous functions, through their values (samples) $\{f(t_n)\}$ at an appropriate sequence of points $\{t_n\}$ by means of a sampling expansion $f(t) = \sum_n f(t_n)\, S_n(t)$ involving these values. Frequently, the functions to be recovered belong to a reproducing kernel Hilbert space where the point values can be expressed in terms of the reproducing kernels $\{k_{t_n}\}$ as the inner products $f(t_n) = \langle f, k_{t_n} \rangle$. This leads us to study the properties of the sequence $\{k_{t_n}\}$ in the Hilbert space $\mathcal{H}$.

As we will see through this book, another similar situation is very frequent: the available data is a sequence $\{(\mathcal{L}f)(t_n)\}$, where $\mathcal{L}$ represents the linear device involved in the sampling process. The above samples can be expressed as a sequence of inner products $\{\langle x, x_n \rangle\}$ in an auxiliary Hilbert space $\mathcal{K}$, where $x \in \mathcal{K}$ is an element univocally related to the function $f \in \mathcal{H}$ to be recovered by means of an operator $\mathcal{T} : \mathcal{K} \to \mathcal{H}$—an isomorphism in general–, and $\{x_n\}$ is a fixed sequence in $\mathcal{K}$. This auxiliary space $\mathcal{K}$ will be usually an ℓ^2 or L^2-space.

Thus, the task is to recover x (and consequently f) uniquely from the available data $\{\langle x, x_n \rangle\}$ in a stable way, which means that the norm $\|\{\langle x, x_n \rangle\} - \{\langle y, x_n \rangle\}\|_{\ell^2}$ is small if and only if $\|x - y\|_{\mathcal{K}}$ is small.

This leads us to the concept of *frame* in a separable Hilbert space: the sequence $\{x_n\}$ should be a frame for the auxiliary space $\mathcal{K}$. Given a frame $\{x_n\}$ for $\mathcal{K}$, any $x \in \mathcal{K}$ can be expanded as the sum $x = \sum_n \langle x, x_n \rangle\, y_n$ where the sequence $\{y_n\}$ is a dual frame of $\{x_n\}$. Finally, the relationship $f = \mathcal{T}x$ will lead to a sampling formula like $f = \sum_n (\mathcal{L}f)(t_n)\, S_n(t)$, where $S_n = \mathcal{T}y_n$.

Whenever the space $\mathcal{H}$ where the sampling works has a predetermined structure, an appropriate choice, if possible, of the dual frame $\{y_n\}$ will lead us to a sampling formula sharing the same structure as the functions in $\mathcal{H}$. This is the case for shift-invariant spaces.

The main aim of this book is to exhibit how the use of frames, including Riesz and orthonormal bases, in the above sampling schemes comprises a variety of examples: from the classical Shannon sampling theorem up to average sampling in certain subspaces of Hilbert-Schmidt operators. Thus, the book constructs a unified narrative which helps to understand how sampling theory works. Naturally, it does not intend to be exhaustive and, as a consequence, some important topics are not included.

Although the ideas in the book are partially borrowed from some previous research of the author and his collaborators, this does not mean a mere cut and paste from previous work. The author has tried to give a unified and comprehensive treatment on the subject by completing, when necessary, the presentation of previous research.

The book is divided into seven chapters whose contents we now briefly describe:

In Chap. 1, sampling theory is presented in a non-rigorous way including a short historical note on who is considered its father, namely, Claude E. Shannon, without forgetting other important authors, such as J. M. Whittaker, V. Kotel'nikov, K. Ogura or I. Someya. Surely, the vast number of its applications has helped the growth of sampling theory in other related fields, not least in mathematics itself. In the second part of the chapter, an almost self-contained introduction to frame theory is included. Frame theory is the main mathematical axis around which the other chapters are constructed. Of course, frames include, as particular cases, both orthonormal and Riesz bases which probably lead to the more known examples.

Chapter 2 includes a brief introduction to reproducing kernel Hilbert spaces (hereafter RKHSs) leading to the study of an important example of spaces in sampling theory: Paley-Wiener spaces $PW_{\pi\sigma}$. Thus, this chapter may be also useful as a mathematical introductory course in sampling theory, avoiding perhaps some specialized topics. The needed mathematical background just consists of a classical course in harmonic analysis and a basic knowledge of Hilbert spaces, functional analysis and complex analysis.

Chapter 3 is a non-standard collection of results concerning regular sampling in shift-invariant subspaces of $L^2(\mathbb{R}^d)$. The first part is devoted to the search of Riesz bases giving Shannon-type sampling formulas in shift-invariant subspaces of $L^2(\mathbb{R}^d)$. To this end, we follow the main technique in the book. More precisely, in the particular $L^2(\mathbb{R})$ case by searching Riesz bases expansions in an auxiliary space—here $L^2(0, 1)$—having the available (regular) samples as coefficients. Finally, the inverse Fourier transform—the operator $\mathcal{T}$ in this case—gives the desired sampling results.

The second part of Chap. 3 deals with a topic that does not usually appear in the sampling literature. Very often, in interpolation theory, input data are transformed by using differences of the given data: thus we are skipping irrelevant information when changes between neighbors are not very significant. The (regular) Shannon-type sampling formulas obtained are (infinite) interpolation formulas; in intervals where a function does not change too much, the available data can be reduced by using regular samples with a bigger sampling period along with their differences. New sampling formulas can be obtained in a straightforward way by using the same technique as in Sect. 3.1.

Finally, the third part is devoted to generalizing shift-invariant subspaces: the shift operator can be replaced by another unitary operator U, leading to the concept of U-invariant subspaces. Some results are proven in this new setting that directly apply to the shift-invariant case.

Chapter 4 concerns to the so-called Kramer sampling theorem. The Kramer sampling theorem has been the cornerstone for a significant mathematical literature which has flourished for the past few years on the topic of sampling theorems associated with differential or difference problems. In this chapter, we take a unified and abstract point of view, enlightened with a variety of examples belonging to different areas in mathematics: orthogonal polynomials and moment problems, difference and differential problems, entire functions, reproducing kernel Hilbert spaces, Lagrange and Hermite interpolation, semi-inner-product Banach spaces, or Hilbert spaces of distributions with reproducing distribution. In short, this chapter shows, by means of a unified approach, how sampling theory is transversal to other mathematical fields.

In Chap. 5, a generalized sampling theory is proposed. It is based on dealing with convolution systems on discrete abelian groups as a unifying strategy. On the one hand, working in locally compact abelian groups allows us to gather all the classical groups along with their products covering most of practical situations. On the other hand, we see that most of linear sampling devices can be treated as convolution systems defined on appropriate spaces. Thus a sampling theory is developed by following the master line that guides the proposal of this book. A comprehensive list of examples showcases the richness of the followed approach.

Chapter 6 is devoted to the study of sampling-related frames in finite-dimensional U-invariant subspaces of a separable Hilbert space. In applications, frames in finite-dimensional spaces are required and, as in finite dimension a frame is nothing but a spanning set of vectors, finite frames require control of condition numbers and over the spectrum of certain matrices. Thus, frame theory, firstly considered as a part of applied harmonic analysis, meets matrix analysis and numerical linear algebra. As an important example, the theory developed in this chapter encompasses the problem of periodic extensions of finite signals. The chapter includes also a section devoted to finite sampling associated with a non-abelian group defined as the knit product of two groups.

Finally, Chap. 7 is focused on the average sampling problem in shift-invariant-like subspaces of Hilbert-Schmidt operators in $L^2(\mathbb{R}^d)$. This problem is motivated by the recovery of a time-varying system from its diagonal channel samples, a signal processing problem appearing in orthogonal frequency-division multiplexing. Hilbert-Schmidt operators are the simplest examples of time-varying systems whose models are, in general, pseudo-differential operators. Having in mind the properties of certain operators (including the Hilbert-Schmidt ones), we can define shift-invariant-like subspaces of Hilbert-Schmidt operators and average samples as in the classical $L^2(\mathbb{R}^d)$ setting. The diagonal channel samples appear as a particular case. Following the general steps outlined in Chap. 5, we obtain the desired sampling results. These results are intimately related with topics belonging to time-frequency analysis.

A list of references closes each chapter of the book. It does not intend to be exhaustive and the author apologizes in advance for any important omission. The list includes, for obvious reasons, several references due to the author and his collaborators.

I would like to mention my sampling students: P. E. Fernández-Moncada, H. R. Fernández-Morales, M. A. Hernández-Medina, G. Pérez-Villalón and A. Portal, and also my sampling coauthors in papers cited in this book: W. N. Everitt, A. Ibort, K. H. Kwon, L. L. Littlejohn, J. Moro, M. J. Muñoz-Bouzo, A. Ortega, F. H. Szafraniec and A. Zayed. Thanks a lot, it was my pleasure to do mathematics with all of you.

I would like also to express here an anonymous acknowledgment to those authors who, over the years, taught me sampling theory, and the most important thing, to enjoy the mathematics that come with it.

Most of the results stated throughout this book are well-known or have been previously published. The author, at this stage, only claims originality in the way they have been set out. He will be satisfied if this contributes to make sampling theory better known to the overall scientific community.

Finally, I would like to thank Prof. Francisco Marcellán for encouraging me to carry out this nice project. Many thanks go also to the anonymous reviewers for their critical reading of the book; their comments and suggestions helped me to improve both the presentation and the contents of the book.

Leganés-Madrid, Spain
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Antonio García García

Contents

List of Symbols

$\mathbb{C}$	The set of complex numbers
$\mathbb{C}^d$	The d-dimensional complex space
χ_A	Characteristic function of the set A
$\mathcal{D}(\mathbb{R})$	Vector space of test functions
Δ_+^k	k-th forward difference operator
Δ_-^k	k-th backward difference operator
Δ_0	Central difference operator
$\delta_{n,m}$	Kronecker delta
$\mathcal{E}(\mathbb{R})$	Space of functions infinitely differentiable on $\mathbb{R}$
$\operatorname{ess\,inf}_{t\in I}\lvert f(t)\rvert$	The essential infimum of f on the interval I
$\ell^2(\mathbb{N})$	The Hilbert space of square summable sequences on $\mathbb{N}$
$\ell^2_N(G)$	The product Hilbert space $\ell^2(G) \times \cdots \times \ell^2(G)$ (N times)
$\operatorname{span} A$	Linear space spanned by A
$\operatorname{ess\,sup}_{t\in I}\lvert f(t)\rvert$	Essential supremum of f on the interval I
$\lvert\lvert f\rvert\rvert_0$	The essential infimum of f on the interval I
$\lvert\lvert f\rvert\rvert_\infty$	Essential supremum of f on the interval I
$\mathbb{C}^{\mathbb{Z}}$	Sequences of complex numbers indexed by $\mathbb{Z}$
$\mathbb{C}^{\Omega}$	The set of functions $f : \Omega \longrightarrow \mathbb{C}$
$\mathbb{R}^d$	The d-dimensional Euclidean space
$\mathbb{T}$	The unidimensional torus
$\mathbf{N} \bowtie \mathbf{H}$	The knit product of the groups **N** and **H**
$\mathcal{D}_{2N}$	Dihedral group of symmetries of the N-gon
$\mathcal{E}'(\mathbb{R})$	The space of compact supported distributions
$\mathcal{F}(f) = \widehat{f}$	Fourier transform of f
$\mathcal{F}^{-1}(\widehat{f}) = f$	Inverse Fourier transform of $\widehat{f}$
$\mathcal{F}_s$	Symplectic Fourier transform
$\mathcal{F}_W$	Fourier-Wigner transform
$\mathcal{HS}(\mathbb{R}^d)$	Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$
$\mathcal{S}'(\mathbb{R}^d)$	Tempered distributions in $\mathbb{R}^d$
$\mathcal{S}(\mathbb{R}^d)$	Schwartz space in $\mathbb{R}^d$

$\mathcal{S}_0(\mathbb{R}^d)$	Feichtinger's algebra
$\mathcal{T}^1$	Trace class operators
$\mathcal{T}^p$	Schatten p-class
$\mathcal{U}(\mathcal{H})$	Unitary operators on $\mathcal{H}$
$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\overline{\text{span}} A$	Closed linear subspace spanned by A
$\overline{A}$	Closure of the set A
$\overline{B}(0; r)$	Closed ball of radius r centered at 0
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers
supp f	Support of the function f
$\mathcal{T}_s$	Symmetric coordinate transformation
$\widehat{G}$	Dual group of the group G
$\mathbb{Z}$	The set of integer numbers
$A \otimes B$	Kronecker product of the matrices A and B
$A^\perp$	Orthogonal subspace of the set A
$B(0; r)$	Open ball of radius r centered at 0
$B\big(L^2(\mathbb{R}^d)\big)$	The Banach space of bounded operators in $L^2(\mathbb{R}^d)$
D_N	N-th Dirichlet kernel
$GL(2d, \mathbb{R})$	General linear group of $2d \times 2d$ real matrices under multiplication
$H(\mathbb{C})$	Set of entire functions
$H^2(\mathbb{C}^+)$	Hardy space in the upper plane
Hf	Hilbert transform of f
$J_\nu(t)$	Bessel function of order ν
$K \rtimes_\sigma H$	Semi-direct product of the groups K and H under the action σ
$L^2(\mathbb{R})$	The Hilbert space of square integrable functions on $\mathbb{R}$
$m(E)$	Lebesgue measure of the measurable set E
M^*	Transpose conjugate of the matrix M
$M^\dagger$	Moore-Penrose pseudo-inverse of the matrix M
N_m	B-spline of order $m-1$
$O(d)$	Orthogonal group of order d
PW_π	Paley-Wiener space
$PW_{\pi\sigma}$	Paley-Wiener spaces with $\sigma > 0$
T^*	Adjoint operator of T
$Z\varphi$	Zak transform of φ
l.c.m. $(N_1, N_2,)$	Least common multiple of N_1 and N_2
LTI	Linear time-invariant system
MIMO	Multi-input multi-output system
RKBS	Reproducing kernel Banach space
RKHS	Reproducing kernel Hilbert space
s.i.p. RKBS	semi-inner-product reproducing kernel Banach space