**RSME Springer Series** 14

# Antonio García García

# The Use of Frames in Sampling Theory



Española



### **RSME Springer Series**

#### Volume 14

#### **Editor-in-Chief**

Maria A. Hernández Cifre, Departamento de Matemáticas, Universidad de Murcia, Murcia, Spain

#### **Series Editors**

Nicolas Andruskiewitsch, FaMAF - CIEM (CONICET), Universidad Nacional de Córdoba, Córdoba, Argentina

Francisco Marcellán, Departamento de Matemáticas, Universidad Carlos III de Madrid, Leganés, Spain

Pablo Mira, Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Cartagena, Spain

Timothy G. Myers, Centre de Recerca Matemàtica, Barcelona, Spain

Joaquín Pérez, Departamento de Geometría y Topología, Universidad de Granada, Granada, Spain

Marta Sanz-Solé, Department of Mathematics and Computer Science, Barcelona Graduate School of Mathematics (BGSMath), Universitat de Barcelona, Barcelona, Spain

Karl Schwede, Department of Mathematics, University of Utah, Salt Lake City, USA

As of 2015, RSME - Real Sociedad Matemática Española - and Springer cooperate in order to publish works by authors and volume editors under the auspices of a co-branded series of publications including advanced textbooks, Lecture Notes, collections of surveys resulting from international workshops and Summer Schools, SpringerBriefs, monographs as well as contributed volumes and conference proceedings. The works in the series are written in English only, aiming to offer high level research results in the fields of pure and applied mathematics to a global readership of students, researchers, professionals, and policymakers. Antonio García García

# The Use of Frames in Sampling Theory



Real Sociedad Matemática Española



Antonio García García Departments of Mathematics Universidad Carlos III de Madrid Madrid, Spain

ISSN 2509-8888 ISSN 2509-8896 (electronic) RSME Springer Series ISBN 978-3-031-63241-9 ISBN 978-3-031-63242-6 (eBook) https://doi.org/10.1007/978-3-031-63242-6

© The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2024

This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

If disposing of this product, please recycle the paper.

For those who enjoy maths...

#### Preface

When introducing a new book on sampling theory, the author should first justify the need for yet another title in an already long list of books and survey papers published on the subject. In spite of the fact that a rich mathematical literature in sampling already exists, the main reason for this book is to build a new narrative from my own experience in sampling research. Since it is almost impossible to be exhaustive, it is necessary to make a choice of contents giving a justification for this selection. Namely, my choice is guided by the idea that entitles the book: *The Use of Frames in Sampling Theory*.

Roughly speaking, sampling theory deals with the reconstruction of functions f, belonging, in general, to a Hilbert space  $\mathcal{H}$  of continuous functions, through their values (samples)  $\{f(t_n)\}$  at an appropriate sequence of points  $\{t_n\}$  by means of a sampling expansion  $f(t) = \sum_n f(t_n) S_n(t)$  involving these values. Frequently, the functions to be recovered belong to a reproducing kernel Hilbert space where the point values can be expressed in terms of the reproducing kernels  $\{k_{t_n}\}$  as the inner products  $f(t_n) = \langle f, k_{t_n} \rangle$ . This leads us to study the properties of the sequence  $\{k_{t_n}\}$  in the Hilbert space  $\mathcal{H}$ .

As we will see through this book, another similar situation is very frequent: the available data is a sequence  $\{(\mathcal{L}f)(t_n)\}$ , where  $\mathcal{L}$  represents the linear device involved in the sampling process. The above samples can be expressed as a sequence of inner products  $\{\langle x, x_n \rangle\}$  in an auxiliary Hilbert space  $\mathcal{K}$ , where  $x \in \mathcal{K}$  is an element univocally related to the function  $f \in \mathcal{H}$  to be recovered by means of an operator  $\mathcal{T}: \mathcal{K} \to \mathcal{H}$ —an isomorphism in general–, and  $\{x_n\}$  is a fixed sequence in  $\mathcal{K}$ . This auxiliary space  $\mathcal{K}$  will be usually an  $\ell^2$  or  $L^2$ -space.

Thus, the task is to recover x (and consequently f) uniquely from the available data  $\{\langle x, x_n \rangle\}$  in a stable way, which means that the norm  $\|\{\langle x, x_n \rangle\} - \{\langle y, x_n \rangle\}\|_{\ell^2}$  is small if and only if  $\|x - y\|_{\mathcal{K}}$  is small.

This leads us to the concept of *frame* in a separable Hilbert space: the sequence  $\{x_n\}$  should be a frame for the auxiliary space  $\mathcal{K}$ . Given a frame  $\{x_n\}$  for  $\mathcal{K}$ , any  $x \in \mathcal{K}$  can be expanded as the sum  $x = \sum_n \langle x, x_n \rangle y_n$  where the sequence  $\{y_n\}$  is a

dual frame of  $\{x_n\}$ . Finally, the relationship  $f = \mathcal{T}x$  will lead to a sampling formula like  $f = \sum_n (\mathcal{L}f)(t_n) S_n(t)$ , where  $S_n = \mathcal{T}y_n$ .

Whenever the space  $\mathcal{H}$  where the sampling works has a predetermined structure, an appropriate choice, if possible, of the dual frame  $\{y_n\}$  will lead us to a sampling formula sharing the same structure as the functions in  $\mathcal{H}$ . This is the case for shift-invariant spaces.

The main aim of this book is to exhibit how the use of frames, including Riesz and orthonormal bases, in the above sampling schemes comprises a variety of examples: from the classical Shannon sampling theorem up to average sampling in certain subspaces of Hilbert-Schmidt operators. Thus, the book constructs a unified narrative which helps to understand how sampling theory works. Naturally, it does not intend to be exhaustive and, as a consequence, some important topics are not included.

Although the ideas in the book are partially borrowed from some previous research of the author and his collaborators, this does not mean a mere cut and paste from previous work. The author has tried to give a unified and comprehensive treatment on the subject by completing, when necessary, the presentation of previous research.

The book is divided into seven chapters whose contents we now briefly describe:

In Chap. 1, sampling theory is presented in a non-rigorous way including a short historical note on who is considered its father, namely, Claude E. Shannon, without forgetting other important authors, such as J. M. Whittaker, V. Kotel'nikov, K. Ogura or I. Someya. Surely, the vast number of its applications has helped the growth of sampling theory in other related fields, not least in mathematics itself. In the second part of the chapter, an almost self-contained introduction to frame theory is included. Frame theory is the main mathematical axis around which the other chapters are constructed. Of course, frames include, as particular cases, both orthonormal and Riesz bases which probably lead to the more known examples.

Chapter 2 includes a brief introduction to reproducing kernel Hilbert spaces (hereafter RKHSs) leading to the study of an important example of spaces in sampling theory: Paley-Wiener spaces  $PW_{\pi\sigma}$ . Thus, this chapter may be also useful as a mathematical introductory course in sampling theory, avoiding perhaps some specialized topics. The needed mathematical background just consists of a classical course in harmonic analysis and a basic knowledge of Hilbert spaces, functional analysis and complex analysis.

Chapter 3 is a non-standard collection of results concerning regular sampling in shift-invariant subspaces of  $L^2(\mathbb{R}^d)$ . The first part is devoted to the search of Riesz bases giving Shannon-type sampling formulas in shift-invariant subspaces of  $L^2(\mathbb{R}^d)$ . To this end, we follow the main technique in the book. More precisely, in the particular  $L^2(\mathbb{R})$  case by searching Riesz bases expansions in an auxiliary space—here  $L^2(0, 1)$ —having the available (regular) samples as coefficients. Finally, the inverse Fourier transform—the operator  $\mathcal{T}$  in this case—gives the desired sampling results.

The second part of Chap. 3 deals with a topic that does not usually appear in the sampling literature. Very often, in interpolation theory, input data are transformed

by using differences of the given data: thus we are skipping irrelevant information when changes between neighbors are not very significant. The (regular) Shannontype sampling formulas obtained are (infinite) interpolation formulas; in intervals where a function does not change too much, the available data can be reduced by using regular samples with a bigger sampling period along with their differences. New sampling formulas can be obtained in a straightforward way by using the same technique as in Sect. 3.1.

Finally, the third part is devoted to generalizing shift-invariant subspaces: the shift operator can be replaced by another unitary operator U, leading to the concept of U-invariant subspaces. Some results are proven in this new setting that directly apply to the shift-invariant case.

Chapter 4 concerns to the so-called Kramer sampling theorem. The Kramer sampling theorem has been the cornerstone for a significant mathematical literature which has flourished for the past few years on the topic of sampling theorems associated with differential or difference problems. In this chapter, we take a unified and abstract point of view, enlightened with a variety of examples belonging to different areas in mathematics: orthogonal polynomials and moment problems, difference and differential problems, entire functions, reproducing kernel Hilbert spaces, Lagrange and Hermite interpolation, semi-inner-product Banach spaces, or Hilbert spaces of distributions with reproducing distribution. In short, this chapter shows, by means of a unified approach, how sampling theory is transversal to other mathematical fields.

In Chap. 5, a generalized sampling theory is proposed. It is based on dealing with convolution systems on discrete abelian groups as a unifying strategy. On the one hand, working in locally compact abelian groups allows us to gather all the classical groups along with their products covering most of practical situations. On the other hand, we see that most of linear sampling devices can be treated as convolution systems defined on appropriate spaces. Thus a sampling theory is developed by following the master line that guides the proposal of this book. A comprehensive list of examples showcases the richness of the followed approach.

Chapter 6 is devoted to the study of sampling-related frames in finite-dimensional U-invariant subspaces of a separable Hilbert space. In applications, frames in finite-dimensional spaces are required and, as in finite dimension a frame is nothing but a spanning set of vectors, finite frames require control of condition numbers and over the spectrum of certain matrices. Thus, frame theory, firstly considered as a part of applied harmonic analysis, meets matrix analysis and numerical linear algebra. As an important example, the theory developed in this chapter encompasses the problem of periodic extensions of finite signals. The chapter includes also a section devoted to finite sampling associated with a non-abelian group defined as the knit product of two groups.

Finally, Chap. 7 is focused on the average sampling problem in shift-invariantlike subspaces of Hilbert-Schmidt operators in  $L^2(\mathbb{R}^d)$ . This problem is motivated by the recovery of a time-varying system from its diagonal channel samples, a signal processing problem appearing in orthogonal frequency-division multiplexing. Hilbert-Schmidt operators are the simplest examples of time-varying systems whose models are, in general, pseudo-differential operators. Having in mind the properties of certain operators (including the Hilbert-Schmidt ones), we can define shiftinvariant-like subspaces of Hilbert-Schmidt operators and average samples as in the classical  $L^2(\mathbb{R}^d)$  setting. The diagonal channel samples appear as a particular case. Following the general steps outlined in Chap. 5, we obtain the desired sampling results. These results are intimately related with topics belonging to time-frequency analysis.

A list of references closes each chapter of the book. It does not intend to be exhaustive and the author apologizes in advance for any important omission. The list includes, for obvious reasons, several references due to the author and his collaborators.

I would like to mention my sampling students: P. E. Fernández-Moncada, H. R. Fernández-Morales, M. A. Hernández-Medina, G. Pérez-Villalón and A. Portal, and also my sampling coauthors in papers cited in this book: W. N. Everitt, A. Ibort, K. H. Kwon, L. L. Littlejohn, J. Moro, M. J. Muñoz-Bouzo, A. Ortega, F. H. Szafraniec and A. Zayed. Thanks a lot, it was my pleasure to do mathematics with all of you.

I would like also to express here an anonymous acknowledgment to those authors who, over the years, taught me sampling theory, and the most important thing, to enjoy the mathematics that come with it.

Most of the results stated throughout this book are well-known or have been previously published. The author, at this stage, only claims originality in the way they have been set out. He will be satisfied if this contributes to make sampling theory better known to the overall scientific community.

Finally, I would like to thank Prof. Francisco Marcellán for encouraging me to carry out this nice project. Many thanks go also to the anonymous reviewers for their critical reading of the book; their comments and suggestions helped me to improve both the presentation and the contents of the book.

Leganés-Madrid, Spain April 2024 Antonio García García

## Contents

1	What	at Does	Sampling Theory Mean?	1	
	1.1	A Historical Note			
		1.1.1	What's Next?	4	
	1.2	A Brie	ef on the Theory of Frames in a Separable Hilbert Space	6	
		1.2.1	The Basic Frame Theory	6	
		1.2.2	Two Important Particular Cases	13	
		1.2.3	The Case of Finite Frames	19	
	1.3	A Not	e Concerning the Needed Mathematical Background	21	
	References			25	
2	Basic Sampling Theory				
	2.1	RKH	Ss Are Suitable Spaces for Sampling	28	
		2.1.1	The Basic RKHSs Theory	28	
		2.1.2	Some Examples in the Finite Dimensional Setting	31	
	2.2 The Paradigmatic Example: Palev-Wiener Spaces $PW_{\pi\pi}$		aradigmatic Example: Paley-Wiener Spaces $PW_{\pi\sigma}$	33	
		2.2.1	Properties of the Paley-Wiener Space $PW_{\pi}$	34	
		2.2.2	Irregular Sampling: Paley-Wiener-Levinson Theorem	44	
		2.2.3	Sampling Through Another Type of Samples	5(	
		2.2.4	Multidimensional Shannon Sampling Formula	60	
		2.2.5	Recovering Bandlimited Functions in the		
			Distributional Sense	6	
	Refe	erences		67	
3	Sam	nnling i	n Shift-Invariant Subspaces	69	
	3.1	The B	asic Theory: Shannon-Type Sampling Formulas	69	
	0.11	3.1.1	Some Sampling Examples Involving B-Splines	75	
		3.1.2	Sampling Formulas for Two-Dimensional	, .	
			Shift-Invariant Subspaces	77	
	3.2	Samp	ling Formulas Involving Samples Differences	81	
		3.2.1	The Involved Riesz Bases in $L^2(0, 1)$ and Their Dual	0.	
			Bases	83	

		3.2.2	Sampling Results in the One-Dimensional Case	88
		3.2.3	Sampling Results in the Two-Dimensional Case	96
	3.3	From	Classical Shift-Invariant to U-Invariant Subspaces	101
		3.3.1	Stationary Sequences in a Hilbert Space	101
		3.3.2	The Case of Multiple Generators	104
		3.3.3	Generalized U-Average Sampling	109
	Refe	erences		112
4	A R	eview o	on Kramer Sampling Theorem	113
	4.1	Sampl	ling in RKHSs: Kramer Sampling Theorem	114
		4.1.1	The Classical Kramer Sampling Theorem	114
		4.1.2	An Abstract Version of the Kramer Sampling Theorem	115
		4.1.3	Kramer Sampling Theorem and Indeterminate	
			Moment Problems	121
		4.1.4	Positive Matrices and RKHSs	124
		4.1.5	A Converse of the Kramer Theorem	130
		4.1.6	Sampling Through Samples of a Related Function	134
	4.2	Analy	tic Version of the Kramer Sampling Theorem	140
		4.2.1	Analytic Kramer Kernels	142
		4.2.2	Analytic Kramer Kernels and Lagrange-Type	
			Interpolation Series	146
		4.2.3	Sampling Related to the One-Dimensional Dirac	
			Operator	154
	4.3	Other	Versions of the Kramer Sampling Theorem	162
		4.3.1	The Semi-inner-Product Version in Banach Spaces	163
		4.3.2	A Distributional Version of the Kramer Sampling	
			Theorem	172
	Refe	erences		177
5	A G	enerali	zed Sampling Theory	181
•	51	Data S	Samples as a Filtering Process	183
	5.2	Convo	olution Systems on Discrete Abelian Groups	186
	0.2	521	A Brief on Harmonic Analysis on Discrete Abelian	100
		5.2.1	Groups	186
		522	Lattices on $\mathbb{R}^d$	188
		523	The Convolution Systems Theory	190
	53	Regul	ar Sampling in <i>U</i> -Invariant Subspaces	196
	5.5	5 3 1	A General Sampling Result	196
		532	Some Examples from Usual Sampling Settings	203
		533	The Case of Pointwise Samples whenever $\mathcal{H} = I^2(\mathbb{R}^d)$	205
		534	Sampling in a Subgroup	200
	Refe	rences		211

6	Finite Frames Related to Sampling in Finite-DimensionalU-Invariant Subspaces219					
	6.1	Samp	ling in Finite-Dimensional U-Invariant Subspaces	221		
		6.1.1	Structure-Preserving Left-Inverses of			
			Cross-Covariances Matrix	228		
		6.1.2	The Derived Sampling Results	232		
		6.1.3	An Application: Periodic Extensions of Finite Signals	237		
	6.2	A Generalization Involving the Knit Product of Groups		244		
		6.2.1	A Brief on the Knit Product $\mathbf{G} = \mathbf{N} \bowtie \mathbf{H}$ of Groups	245		
		6.2.2	Invariant Subspaces Associated with a Unitary			
			Representation	246		
		6.2.3	Sampling Indexed by N when N is an Abelian Subgroup	248		
		6.2.4	Sampling Indexed by <b>H</b> when <b>H</b> is an Abelian Subgroup	255		
	Refe	erences		259		
7	Sampling in Shift-Invariant-Like Subspaces of					
÷.	Hilk	pert-Scl	hmidt Operators	261		
	7.1	A Mo	tivation: Time-Varying Systems	264		
	7.2	The K	Cohn-Nirenberg and Weyl Transforms	265		
		7.2.1	The Class of Hilbert-Schmidt Operators	265		
		7.2.2	The Kohn-Nirenberg and Weyl Transforms	267		
	7.3	Riesz	Sequences of Translated Operators in $\mathcal{HS}(\mathbb{R}^d)$	271		
		7.3.1	Riesz Sequences of Translated Operators	274		
		7.3.2	A-Shift-Invariant Subspaces	276		
		7.3.3	The Isomorphism $\mathcal{T}_{\mathbf{S}}$ connecting $\ell_{\mu}^{2}(\Lambda)$ with $V_{\mathbf{S}}^{2}$	278		
		7.3.4	An Expression for the Diagonal Channel Samples	279		
		7.3.5	The Associated Convolution System	281		
	7.4 The Sampling Results		ampling Results	283		
		7.4.1	Sampling by using the Diagonal Channel Samples			
			at a Lattice	283		
		7.4.2	The General Case of Average Sampling	288		
		7.4.3	Sampling at a Sub-lattice	291		
	7.5	Some	Closing Remarks	293		
	Refe	erences	-	297		
In	dex	• • • • • • • • •		299		

# List of Symbols

$\mathbb{C}$	The set of complex numbers
$\mathbb{C}^d$	The <i>d</i> -dimensional complex space
ΧΑ	Characteristic function of the set A
$\mathcal{D}(\mathbb{R})$	Vector space of test functions
$\Delta^k_+$	k-th forward difference operator
$\Delta^k$	k-th backward difference operator
$\Delta_0$	Central difference operator
$\delta_{n,m}$	Kronecker delta
$\mathcal{E}(\mathbb{R})$	Space of functions infinitely differentiable on $\mathbb{R}$
$essinf_{t \in I}  f(t) $	The essential infimum of $f$ on the interval $I$
$\ell^2(\mathbb{N})$	The Hilbert space of square summable sequences on $\mathbb N$
$\ell^2_{_N}(G)$	The product Hilbert space $\ell^2(G) \times \cdots \times \ell^2(G)$ ( <i>N</i> times)
span A	Linear space spanned by A
$\operatorname{ess} \sup_{t \in I}  f(t) $	Essential supremum of $f$ on the interval $I$
$  f  _{0}$	The essential infimum of $f$ on the interval $I$
$  \underline{f}  _{\infty}$	Essential supremum of $f$ on the interval $I$
$\mathbb{C}^{\mathbb{Z}}$	Sequences of complex numbers indexed by $\mathbb{Z}$
$\mathbb{C}^{\Omega}$	The set of functions $f: \Omega \longrightarrow \mathbb{C}$
$\mathbb{R}^d$	The <i>d</i> -dimensional Euclidean space
T	The unidimensional torus
$\mathbf{N} \bowtie \mathbf{H}$	The knit product of the groups $N$ and $H$
$\mathcal{D}_{2N}$	Dihedral group of symmetries of the N-gon
$\mathcal{E}'(\mathbb{R})$	The space of compact supported distributions
$\mathcal{F}(f) = \widehat{f}$	Fourier transform of $f$
$\mathcal{F}^{-1}(\widehat{f}) = f$	Inverse Fourier transform of $\widehat{f}$
$\mathcal{F}_s$	Symplectic Fourier transform
$\mathcal{F}_W$	Fourier-Wigner transform
$\mathcal{HS}(\mathbb{R}^d)$	Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$
$\mathcal{S}'(\mathbb{R}^d)$	Tempered distributions in $\mathbb{R}^d$
$\mathcal{S}(\mathbb{R}^d)$	Schwartz space in $\mathbb{R}^d$

$\mathcal{S}_0(\mathbb{R}^d)$	Feichtinger's algebra
$\mathcal{T}^{1}$	Trace class operators
$\mathcal{T}^p$	Schatten <i>p</i> -class
$\mathcal{U}(\mathcal{H})$	Unitary operators on $\mathcal{H}$
$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
<u>span</u> A	Closed linear subspace spanned by A
$\overline{A}$	Closure of the set A
$\overline{B}(0;r)$	Closed ball of radius r centered at 0
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers
supp <i>f</i>	Support of the function $f$
$\mathcal{T}_{s}$	Symmetric coordinate transformation
$\widehat{G}$	Dual group of the group G
$\mathbb{Z}$	The set of integer numbers
$A \otimes B$	Kronecker product of the matrices A and B
$A^{\perp}$	Orthogonal subspace of the set A
B(0; r)	Open ball of radius r centered at 0
$B(L^2(\mathbb{R}^d))$	The Banach space of bounded operators in $L^2(\mathbb{R}^d)$
$D_N$	<i>N</i> -th Dirichlet kernel
$GL(2d,\mathbb{R})$	General linear group of $2d \times 2d$ real matrices under multiplica-
	tion
$H(\mathbb{C})$	Set of entire functions
$H^2(\mathbb{C}^+)$	Hardy space in the upper plane
Hf	Hilbert transform of $f$
$J_{ u}(t)$	Bessel function of order $\nu$
$K \rtimes_{\sigma} H$	Semi-direct product of the groups $K$ and $H$ under the action $\sigma$
$L^2(\mathbb{R})$	The Hilbert space of square integrable functions on $\mathbb{R}$
m(E)	Lebesgue measure of the measurable set $E$
<i>M</i> *	Transpose conjugate of the matrix M
$M^{\dagger}$	Moore-Penrose pseudo-inverse of the matrix M
$N_m$	B-spline of order $m - 1$
O(d)	Orthogonal group of order d
$PW_{\pi}$	Paley-Wiener space
$PW_{\pi\sigma}$	Paley-Wiener spaces with $\sigma > 0$
$T^*$	Adjoint operator of T
Ζφ	Zak transform of $\varphi$
l.c.m. $(N_1, N_2, )$	Least common multiple of $N_1$ and $N_2$
LTI	Linear time-invariant system
MIMO	Multi-input multi-output system
KKBS	Reproducing kernel Banach space
KKHS	Reproducing kernel Hilbert space
s.i.p. RKBS	sem1-inner-product reproducing kernel Banach space