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On the properties for modifications of classical orthogonal polynomials of discrete variables

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Abstract

We consider a modification of moment functionals for some classical polynomials of a discrete variable by adding a mass point at $x = 0$. We obtain the resulting orthogonal polynomials, identify them as hypergeometric functions and derive the second-order difference equation which these polynomials satisfy. The corresponding tridiagonal matrices and associated polynomials were also studied.

Keywords: Meixner, Charlier and Kravchuk polynomials; Discrete measures; Hypergeometric functions; Associated polynomials

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1. Introduction

The study of orthogonal polynomials with respect to a modification of a linear functional in the linear space of polynomials with real coefficients via the addition of one or two delta Dirac measures has been performed by several authors. In particular, Chihara [6] has considered some properties of such polynomials in terms of the location of the mass point with respect to the support of a positive measure. More recently Marcellán and Maroni [10] analyzed a more general situation for regular (quasi-definite) linear functionals, i.e., such that the principal submatrices of the corresponding infinite Hankel matrices associated with the moment sequences are nonsingular.

Special emphasis is given to the modifications of classical linear functionals (Hermite, Laguerre, Jacobi and Bessel). Koornwinder [9] considered a system of polynomials orthogonal with respect to the classical weight function for Jacobi polynomials with two extra point masses added at $x = -1$

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and $x = 1$. For generalized Laguerre polynomials $\{L_n^{\alpha, A}(x)\}_{n=0}^{\infty}$ that are orthogonal on $[0, \infty)$ with respect to the linear functional \mathcal{C} on the linear space of polynomials with real coefficients defined as

$$\langle \mathcal{C}, P \rangle = \int_0^{\infty} P(x) x^{\alpha} e^{-x} dx + AP(0), \quad \alpha > -1, \quad A \geq 0,$$

Koekoek and Koekoek [8] found a differential equation of the form

$$A \sum_{i=0}^{\infty} a_i(x) y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny = 0,$$

where the coefficients $a_i(x)$, $i \in \{1, 2, 3, \dots\}$, are independent of n and a_0 depends on n but is independent of x . In the above paper, representation formulas for the new orthogonal polynomial sequences, as well as the second-order differential equation that such polynomials satisfy, were deduced.

In the open problem section of the *Proceedings of the Third International Symposium on Orthogonal Polynomials and their Applications* held in Erice (Italy), Askey [2] raised the following question:

Consider the Meixner Polynomials $M_n^{\gamma, \mu}(x)$, add or subtract a mass point at $x = 0$ and find the resulting polynomials. Identify them as hypergeometric functions and show that these polynomials satisfy a difference equation in x .

In [4] Bavinck and van Haeringen gave the solution to the problem of finding the second-order difference equation for generalized Meixner polynomials, as well as the infinite-order difference equation which these polynomials satisfy. For generalized Charlier polynomials Bavinck and Koekoek [3] found the corresponding infinite-order difference equation.

In [1] we obtained the representation for such generalized Meixner polynomials as a hypergeometric function ${}_3F_2$, as well as the corresponding second-order difference equation. Now we generalize this result for the Kravchuk and Charlier polynomials and continue the algebraic approach presented by Godoy et al. [7] in the framework of a more general theory based on the addition of a delta Dirac measure to a discrete semiclassical linear functional. We analyze the relation between tridiagonal matrices of the *perturbed or generalized* $P_n^A(x)$ and classical $P_n(x)$ polynomials, as well as the associated polynomials corresponding to the sequence $\{P_n^A(x)\}_{n=0}^{\infty}$.

The structure of the paper is as follows. In Section 2, we provide the basic properties of the classical orthogonal polynomials of discrete variables which will be needed, as well as the main data for the Meixner, Kravchuk and Charlier polynomials. In Section 3 we deduce expressions of the generalized Meixner, Kravchuk and Charlier polynomials and its first difference derivatives, as well as their representation as hypergeometric functions in the direction raised by Askey. In Section 4, we find the second-order difference equation which these generalized polynomials satisfy. In Section 5, from the three-term recurrence relation (TTRR) of the classical orthogonal polynomials we find the TTRR which satisfies the perturbed ones. In Section 6, from the relation of the perturbed polynomials $P_n^A(x)$ as a linear combination of the classical ones, we find the tridiagonal matrices associated with the perturbed monic orthogonal polynomial sequence (PMOPS) $\{P_n^A(x)\}_{n=0}^{\infty}$ as a rank-one perturbation of the tridiagonal matrices associated with the classical monic orthogonal

polynomial sequence (CMOPS) $\{P_n(x)\}_{n=0}^\infty$. Finally, in Section 7 we find the associated polynomials $P_n^{(1),A}(x)$ corresponding to $\{P_n^A(x)\}_{n=0}^\infty$ in terms of the associated polynomials $P_n^{(1)}(x)$ corresponding to $\{P_n(x)\}_{n=0}^\infty$ and the classical ones.

2. Some preliminary results

First, we state some formulas for the classical Meixner, Kravchuk and Charlier polynomials which are useful in order to obtain the generalized polynomials orthogonal with respect to the linear functional \mathcal{U} defined as a modification of the first ones through the addition of a mass point. All the formulas and other properties for the classical Meixner, Kravchuk and Charlier polynomials can be found in a number of books (see for instance the excellent monograph *Orthogonal Polynomials in Discrete Variables* by Nikiforov et al. [11, Ch. 2]).

We will use monic polynomials, i.e., polynomials with the leading coefficient equal to 1. The classical orthogonal polynomials of a discrete variable in the uniform lattice $x(s)=s$ are the polynomial solution of a second-order linear difference equation of hypergeometric type

$$\sigma(x) \Delta \nabla P_n(x) + \tau(x) \Delta P_n(x) + \lambda_n P_n(x) = 0, \tag{1}$$

where

$$\nabla f(x) = f(x) - f(x - 1), \quad \Delta f(x) = f(x + 1) - f(x).$$

Here $\sigma(x)$ and $\tau(x)$ are polynomials in x of degree at most 1 and 2, respectively, and λ_n is a constant.

These polynomials are orthogonal with respect to the linear functional \mathcal{L} on the linear space of polynomials with real coefficients defined as

$$\langle \mathcal{L}, P \rangle = \sum_{x \in \mathbb{N}} \rho(x) P(x), \quad \mathbb{N} = \{0, 1, 2, \dots\}, \tag{2}$$

where $\rho(x)$ is some nonnegative function (weight function) supported in a countable set of the real line and such that

$$\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x).$$

The orthogonality relation is

$$\sum_{x \in \mathbb{N}} P_n(x) P_m(x) \rho(x) = \delta_{nm} d_n^2, \tag{3}$$

where d_n^2 denotes the square of the norm of these classical polynomials.

The polynomial solutions of Eq. (1) are uniquely determined, up to a normalizing factor (R_n), by the difference analog of the Rodrigues formula (see [11, p. 24, Eq. (2.2.7)]):

$$P_n(x) = \frac{R_n}{\rho(x)} \nabla^n \left[\rho(x + n) \prod_{k=1}^n \sigma(x + k) \right]. \tag{4}$$

They satisfy a three-term recurrence relation of the form

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 0, \\ P_{-1}(x) &= 0 \quad \text{and} \quad P_0(x) = 1, \end{aligned} \tag{5}$$

and the Christoffel–Darboux formula

$$\sum_{m=0}^{n-1} \frac{P_m(x)P_m(y)}{d_m^2} = \frac{1}{x-y} \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{d_{n-1}^2}, \quad n = 1, 2, 3, \dots \tag{6}$$

We will consider the following three classical monic orthogonal polynomials (CMOPs) which are solutions of the difference equation (1).

I. The Meixner polynomials, orthogonal with respect to the weight function $\rho(x)$ supported on $[0, \infty)$, with

$$\sigma(x) = x, \quad \tau(x) = \gamma\mu - x(1 - \mu), \quad 0 < \mu < 1, \quad \gamma > 0, \quad \lambda_n = n(1 - \mu),$$

and

$$R_n = \frac{1}{(\mu - 1)^n}, \quad \rho(x) = \frac{\mu^x \Gamma(\gamma + x)}{\Gamma(\gamma) \Gamma(1 + x)}, \quad d_n^2 = \frac{n!(\gamma)_n \mu^n}{(1 - \mu)^{n+2n}}.$$

II. The Kravchuk polynomials, orthogonal with respect to the weight function $\rho(x)$ supported on $[0, N]$, with $n \leq N$,

$$\sigma(x) = x, \quad \tau(x) = \frac{Np - x}{1 - p}, \quad 0 < p < 1, \quad \lambda_n = \frac{n}{1 - p},$$

and

$$R_n = (p - 1)^n, \quad \rho(x) = \frac{p^x N! (1 - p)^{N-x}}{\Gamma(N + 1 - x) \Gamma(1 + x)}, \quad d_n^2 = \frac{n! N! p^n (1 - p)^n}{(N - n)!}.$$

III. The Charlier polynomials, orthogonal with respect to the weight function $\rho(x)$ supported on $[0, \infty)$, with

$$\sigma(x) = x, \quad \tau(x) = \mu - x, \quad \mu > 0, \quad \lambda_n = n,$$

and

$$R_n = (-1)^n, \quad \rho(x) = \frac{\mu^x e^{-\mu}}{\Gamma(1 + x)}, \quad d_n^2 = n! \mu^n.$$

They satisfy the so-called structure relations

$$\frac{x}{n} \nabla M_n^{\gamma, \mu}(x) = \frac{\mu(1 - \gamma - n)}{\mu - 1} M_{n-1}^{\gamma, \mu}(x) + M_n^{\gamma, \mu}(x), \tag{7}$$

$$\frac{x}{n} \nabla K_n^p(x) = p(N - n + 1) K_{n-1}^p(x) + K_n^p(x), \tag{8}$$

$$\frac{x}{n} \nabla C_n^\mu(x) = \mu C_{n-1}^\mu(x) + C_n^\mu(x). \tag{9}$$

These classical polynomials can be represented as hypergeometric functions (see [11, p. 49, Section 2.7])

$$M_n^{\gamma, \mu}(x) = (\gamma)_n \frac{\mu^n}{(\mu - 1)^n} {}_2F_1 \left(\begin{matrix} -n, -x \\ \gamma \end{matrix} \middle| 1 - \frac{1}{\mu} \right), \tag{10}$$

$$K_n^p(x) = \frac{(-p)^n N!}{(N - n)!} {}_2F_1 \left(\begin{matrix} -n, -x \\ -N \end{matrix} \middle| \frac{1}{p} \right), \tag{11}$$

$$C_n^\mu(x) = (-\mu)^n {}_2F_0 \left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| -\frac{1}{\mu} \right), \tag{12}$$

where the hypergeometric function is defined by

$${}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

$$(a)_0 := 1, \quad (a)_k := a(a + 1)(a + 2) \cdots (a + k - 1), \quad k = 1, 2, 3, \dots$$

As a consequence of these representations we can deduce

$$M_n^{\gamma, \mu}(0) = \frac{\mu^n}{(\mu - 1)^n} \frac{\Gamma(n + \gamma)}{\Gamma(\gamma)}, \quad K_n^p(0) = \frac{(-p)^n N!}{(N - n)!}, \quad C_n^\mu(0) = (-\mu)^n. \tag{13}$$

We have proved (see [1, formula (26)]) the following property for the kernels of the Meixner polynomials:

$$\text{Ker}_{n-1}^M(x, 0) \equiv \sum_{m=0}^{n-1} \frac{M_m^{\gamma, \mu}(x) M_m^{\gamma, \mu}(0)}{d_m^2} = \frac{(-1)^{n-1} (1 - \mu)^{n+\gamma-1}}{n!} \nabla M_n^{\gamma, \mu}(x). \tag{14}$$

It is straightforward to show that the kernels of $K_n^p(x)$ and $C_n^\mu(x)$ verify the following relations:

$$\text{Ker}_{n-1}^K(x, 0) \equiv \sum_{m=0}^{n-1} \frac{K_m^p(x) K_m^p(0)}{d_m^2} = \frac{(p - 1)^{1-n}}{n!} \nabla K_n^p(x), \tag{15}$$

$$\text{Ker}_{n-1}^C(x, 0) \equiv \sum_{m=0}^{n-1} \frac{C_m^\mu(x) C_m^\mu(0)}{d_m^2} = \frac{(-1)^{n-1}}{n!} \nabla C_n^\mu(x). \tag{16}$$

3. The definition, orthogonal relation and representation as hypergeometric series

Consider the linear functional \mathcal{Q} on the linear space of polynomials with real coefficients defined as

$$\langle \mathcal{Q}, P \rangle = \langle \mathcal{L}, P \rangle + AP(0), \quad x \in \mathbb{N}, \quad A \geq 0, \tag{17}$$

where \mathcal{L} is a classical moment functional (2) associated to some classical polynomials of a discrete variable.

We will determine the monic polynomials $P_n^A(x)$ which are orthogonal with respect to the functional \mathcal{U} and prove that they exist for all positive A (see (22) below).

To obtain this, we can write the Fourier expansion of such generalized polynomials:

$$P_n^A(x) = P_n(x) + \sum_{k=0}^{n-1} a_{n,k} P_k(x), \tag{18}$$

where P_n denotes the classical monic orthogonal polynomial (CMOP) of degree n .

In order to find the unknown coefficients $a_{n,k}$ we will use the orthogonality of the polynomials $P_n^A(x)$ with respect to \mathcal{U} , i.e.,

$$\langle \mathcal{U}, P_n^A(x) P_k(x) \rangle = 0 \quad \forall k < n.$$

Now putting (18) in (17) we find

$$\langle \mathcal{U}, P_n^A(x) P_k(x) \rangle = \langle \mathcal{L}, P_n^A(x) P_k(x) \rangle + AP_n^A(0) P_k(0). \tag{19}$$

If we use the decomposition (18) and take into account the orthogonality of the classical orthogonal polynomials with respect to the linear functional \mathcal{L} , then the coefficients $a_{n,k}$ are given by

$$a_{n,k} = -A \frac{P_n^A(0) P_k(0)}{d_k^2}. \tag{20}$$

Finally, Eq. (18) provides us the expression

$$P_n^A(x) = P_n(x) - AP_n^A(0) \sum_{k=0}^{n-1} \frac{P_k(0) P_k(x)}{d_k^2}. \tag{21}$$

From (21) we can conclude that the representation of $P_n^A(x)$ exists for any positive value of the mass A . To obtain this it is enough to evaluate (21) in $x = 0$,

$$\left(1 + A \sum_{k=0}^{n-1} \frac{(P_k(0))^2}{d_k^2} \right) P_n^A(0) = P_n(0) \neq 0, \tag{22}$$

and use the fact that

$$1 + A \sum_{k=0}^{n-1} \frac{(P_k(0))^2}{d_k^2} > 0, \quad n = 1, 2, 3, \dots$$

From (22) we can deduce the values of $P_n^A(0)$ as follows:

$$P_n^A(0) = \frac{P_n(0)}{1 + A \sum_{k=0}^{n-1} ((P_k(0))^2 / d_k^2)}. \tag{23}$$

Now in order to obtain an explicit expression for these polynomials we need some properties of the kernels of the CMOP. In [1] we solved this problem for the classical Meixner polynomials. In this work we will prove a similar result for Charlier and Kravchuk polynomials.

Doing some algebraic calculations in (21) and taking into account formulas (14)–(16) we obtain the following three expressions for the generalized polynomials:

For Meixner polynomials:

$$M_n^{\gamma, \mu, A}(x) = M_n^{\gamma, \mu}(x) - AM_n^{\gamma, \mu, A}(0) \frac{(-1)^{n-1} (1 - \mu)^{n+\gamma-1}}{n!} \nabla M_n^{\gamma, \mu}(x). \tag{24}$$

For Kravchuk polynomials:

$$K_n^{p, A}(x) = K_n^p(x) - AK_n^{p, A}(0) \frac{(p - 1)^{n-1}}{n!} \nabla K_n^p(x). \tag{25}$$

For Charlier polynomials:

$$C_n^{\mu, A}(x) = C_n^\mu(x) - AC_n^{\mu, A}(0) \frac{(-1)^{n-1}}{n!} \nabla C_n^\mu(x). \tag{26}$$

In the above formula the values of the polynomials in $x = 0$ could be deduced from (23). Then, we obtain the following analytic expression for the perturbed monic orthogonal polynomials (PMOPs) $P_n^A(x)$ for $x \neq 0$ (when $x = 0$ we can use (23)):

$$M_n^{\gamma, \mu, A}(x) = M_n^{\gamma, \mu}(x) + B_n \nabla M_n^{\gamma, \mu}(x) = (I + B_n \nabla) M_n^{\gamma, \mu}(x), \tag{27}$$

$$K_n^{p, A}(x) = K_n^p(x) + A_n \nabla K_n^p(x) = (I + A_n \nabla) K_n^p(x), \tag{28}$$

$$C_n^{\mu, A}(x) = C_n^\mu(x) + D_n \nabla C_n^\mu(x) = (I + D_n \nabla) C_n^\mu(x), \tag{29}$$

where A_n, B_n and D_n are constants given by

$$B_n = A \frac{\mu^n (1 - \mu)^{\gamma-1} (\gamma)_n}{n! (1 + A \text{Ker}_{n-1}^M(0, 0))},$$

$$A_n = A \frac{N!}{n! (N - n)!} \frac{p^n (1 - p)^{1-n}}{(1 + A \text{Ker}_{n-1}^K(0, 0))},$$

$$D_n = A \frac{\mu^n}{n! (1 + A \text{Ker}_{n-1}^C(0, 0))}.$$

Remark. Using the Rodrigues formula (4), some extension of it follows in a straightforward way:

$$M_n^{\gamma, \mu, A}(x) = (I + B_n \nabla) \left[\frac{\mu^n}{(1 - \mu)^n} \frac{\Gamma(x + 1)}{\mu^x \Gamma(\gamma + x)} \nabla^{(n)} \frac{\mu^x \Gamma(x + \gamma + n)}{\Gamma(x + 1)} \right],$$

$$K_n^{p, A}(x) = (I + A_n \nabla) \left[\frac{(-1)^n \Gamma(x + 1) \Gamma(N - x + 1)}{p^x (1 - p)^{-x}} \nabla^{(n)} \frac{p^x (1 - p)^{-x}}{\Gamma(x + 1) \Gamma(N - x - n + 1)} \right],$$

$$C_n^{\mu, A}(x) = (I + D_n \nabla) \left[\frac{(-1)^n \Gamma(x + 1)}{\mu^x} \nabla^{(n)} \frac{\mu^{x+n}}{\Gamma(x + 1)} \right].$$

Now we can establish the following representation as hypergeometric functions for the generalized polynomials:

Proposition 1. *The orthogonal polynomials $M_n^{\gamma, \mu, A}(x)$, $K_n^{p, A}(x)$ and $C_n^{\mu, A}(x)$ are, up to a constant factor, generalized hypergeometric functions. More precisely,*

$$M_n^{\gamma, \mu, A}(x) = (\gamma)_n \frac{\mu^n}{(\mu - 1)^n} {}_3F_2 \left(\begin{matrix} -n, -x, 1 + xB_n^{-1} \\ \gamma, xB_n^{-1} \end{matrix} \middle| 1 - \frac{1}{\mu} \right), \quad (30)$$

$$K_n^{p, A}(x) = n!(-p)^n \frac{N!}{n!(N - n)!} {}_3F_2 \left(\begin{matrix} -n, -x, 1 + xA_n^{-1} \\ -N, xA_n^{-1} \end{matrix} \middle| \frac{1}{p} \right), \quad (31)$$

$$C_n^{\mu, A}(x) = (-\mu)^n {}_3F_1 \left(\begin{matrix} -n, -x, 1 + xD_n^{-1} \\ xD_n^{-1} \end{matrix} \middle| -\frac{1}{\mu} \right). \quad (32)$$

Proof (sketch). The proof of this proposition is similar to the proof for the Meixner case (see [1]). To obtain the desired result we need to put the hypergeometric representation of these polynomials in formulas (27)–(29) and do some algebraic calculations. Here the coefficients xA_n^{-1} , xB_n^{-1} and xD_n^{-1} are real numbers. In the case when they are nonpositive integers we need to take the analytic continuation of the hypergeometric series (30)–(32). \square

It is straightforward to show that for $A = 0$ the hypergeometric functions (30)–(32) yield to classical polynomials (10)–(12).

4. A second-order difference equation

In [1] we proved that the Meixner polynomials satisfy a second-order difference equation. To prove this result we only used that in the difference equation of hypergeometric type for Meixner polynomials the function $\sigma(x)$ is equal to x . Taking into account that, for Charlier and Kravchuk polynomials, $\sigma(x) = x$, then the following theorem holds:

Theorem 2. *The polynomials $M_n^{\gamma, \mu, A}(x)$, $K_n^{p, A}(x)$ and $C_n^{\mu, A}(x)$ satisfy a second-order linear difference equation*

$$\begin{aligned} & [x + C(C\lambda_n + \lambda_n - \tau(x))](x - 1) \Delta \nabla P_n^A(x) + (x - 1)\tau(x) \Delta P_n^A(x) \\ & + C[(\tau(x) - C\lambda_n)(\lambda_n - 1 - \tau(x - 1)) + \lambda_n(\lambda_n + C) + (x + C\lambda_n) \Delta \tau(x)] \Delta P_n^A(x) \\ & + (x - 1)\lambda_n P_n^A(x) + C\lambda_n[\lambda_n - 1 - \tau(x - 1) + C(\Delta \tau(x) + \lambda_n)]P_n^A(x) = 0, \end{aligned} \quad (33)$$

where

$$x = 0, 1, 2, \dots, \quad \nabla f(x) = f(x) - f(x - 1), \quad \Delta f(x) = f(x + 1) - f(x),$$

and by P_n^A we denote the generalized Meixner, Kravchuk or Charlier polynomials and C is the constant B_n , A_n or D_n , respectively, which is a function of n (see Section 3).

The proof of this theorem for the Meixner case was given in [1]. Here we provide the proof for all three cases.

Proof. We will start from the representations (27)–(29) for the generalized polynomials $P_n^A(x) = P_n(x) + C \nabla P_n(x)$. Multiplying this expression by x , using the second-order difference equation that these classical polynomials satisfy

$$x \Delta \nabla P_n(x) + \tau(x) \Delta P_n(x) + \lambda_n P_n(x) = 0, \tag{34}$$

and using the identity $\nabla P_n(x) = \Delta P_n(x) - \Delta \nabla P_n(x)$ we obtain

$$x P_n^A(x) = (x + C\lambda_n)P_n(x) + C(x + \tau(x)) \Delta P_n(x). \tag{35}$$

Now if we apply the operator Δ to (35), from (34) the equation

$$x \Delta P_n^A(x) = [x - C\tau(x)] \Delta P_n(x) - C\lambda_n P_n(x) \tag{36}$$

follows. In the same way if we apply in (36) the operator ∇ and using (34) we find

$$\begin{aligned} x(x-1) \Delta \nabla P_n^A(x) &= -[(x-1)\tau(x) + C\tau(x)(\lambda_n - \tau(x-1) - 1) + Cx(\lambda_n + \Delta\tau(x))] \Delta P_n(x) \\ &\quad - [x-1 + C(\lambda_n - \tau(x-1) - 1)] \lambda_n P_n(x). \end{aligned} \tag{37}$$

Now from (35)–(37) the following determinant vanishes:

$$\begin{vmatrix} x P_n^A(x) & a(x) & b(x) \\ x \Delta P_n^A(x) & c(x) & d(x) \\ x(x-1) \Delta \nabla P_n^A(x) & e(x) & f(x) \end{vmatrix} = 0, \tag{38}$$

where

$$\begin{aligned} a(x) &= (x + C\lambda_n), & b(x) &= C(x + \tau(x)), & c(x) &= -C\lambda_n, \\ d(x) &= x - C\tau(x), & e(x) &= -[x - 1 + C(\lambda_n - 1 - \tau(x - 1))] \lambda_n, \\ f(x) &= -[(x - 1)\tau(x) + C[\tau(x)(\lambda_n - 1 - \tau(x - 1)) + x(\lambda_n + \Delta\tau(x))]]. \end{aligned}$$

Expanding the determinant in (38) by the first column and dividing by x^2 , the theorem follows. \square

The difference equation of the previous theorem (33) takes the form:

Meixner case $M_n^{\gamma, \mu, A}(x)$:

$$\begin{aligned} &\{x + B_n[(1 - \mu)(x + n + nB_n) - \gamma\mu]\}(x - 1) \Delta \nabla M_n^{\gamma, \mu, A}(x) \\ &+ (x - 1)[\gamma\mu - x(1 - \mu)] \Delta M_n^{\gamma, \mu, A}(x) \\ &+ B_n\{(1 - \mu)[\gamma\mu(n + nB_n + 2x - 1) + (1 - \mu)(x + n^2 - (x + nB_n)(x + n)) \\ &+ 2nB_n] - \gamma\mu(1 + \gamma\mu)\} \Delta M_n^{\gamma, \mu, A}(x) + (x - 1)n(1 - \mu)M_n^{\gamma, \mu, A}(x) \\ &+ nB_n(1 - \mu)[(1 - \mu)(x + n + nB_n - B_n - 1) - 1 - \gamma\mu]M_n^{\gamma, \mu, A}(x) = 0. \end{aligned} \tag{39}$$

Kravchuk case $K_n^{p,A}(x)$:

$$\begin{aligned}
 & \left\{ x + A_n \left[\frac{x + n + nA_n}{1-p} - \frac{Np}{1-p} \right] \right\} (x-1) \Delta \nabla K_n^{p,A}(x) \\
 & + (x-1) \left[\frac{Np-x}{1-p} \right] \Delta K_n^{p,A}(x) \\
 & + A_n \left\{ \frac{1}{1-p} \left[\frac{Np}{1-p} (n + nA_n + 2x - 1) + \frac{x + n^2 - (x + nA_n)(x + n)}{1-p} \right] \right. \\
 & \left. + 2nA_n \right\} - \frac{Np}{1-p} \left(1 + \frac{Np}{1-p} \right) \Delta K_n^{p,A}(x) + \frac{n}{1-p} (x-1) K_n^{p,A}(x) \\
 & + \frac{nA_n}{1-p} \left[\frac{x + n + nA_n - A_n - 1 - Np}{1-p} - 1 \right] K_n^{p,A}(x) = 0. \tag{40}
 \end{aligned}$$

Charlier case $C_n^{\mu,A}(x)$:

$$\begin{aligned}
 & [x + D_n(x + n + nD_n - \mu)](x-1) \Delta \nabla C_n^{\mu,A}(x) \\
 & + (x-1)(\mu - x) \Delta C_n^{\mu,A}(x) \\
 & + D_n[n^2 + x - 2\mu + 2nD_n - (n - x - \mu)(x - \mu + nD_n)] \Delta C_n^{\mu,A}(x) \\
 & + (x-1)nC_n^{\mu,A}(x) + nD_n(x + n - \mu - 2 + nD_n - D_n)C_n^{\mu,A}(x) = 0. \tag{41}
 \end{aligned}$$

5. Three-term recurrence relations

The generalized polynomials satisfy a three-term recurrence relation (TTRR) of the form

$$\begin{aligned}
 xP_n^A(x) &= P_{n+1}^A(x) + \beta_n^A P_n^A(x) + \gamma_n^A P_{n-1}^A(x), \quad n \geq 0, \\
 P_{-1}^A(x) &= 0 \quad \text{and} \quad P_0^A(x) = 1.
 \end{aligned} \tag{42}$$

This is a simple consequence of their orthogonality with respect to a positive functional (see [5] or [11]). To obtain the explicit formula for the recurrence coefficients we can compare the coefficients of x^n on the two sides of (42). Let b_n^A be the coefficient of x^{n-1} in the expansion $P_n^A(x) = x^n + b_n^A x^{n-1} + \dots$; then $\beta_n^A = b_n^A - b_{n+1}^A$. To calculate γ_n^A is sufficient to evaluate (42) in $x = 0$ and remark that $P_n^A(0) \neq 0$.

In order to obtain a general expression for the coefficient β_n^A we can use the formulas (27)–(29) for the generalized polynomials $P_n^A(x) = P_n(x) + C \nabla P_n(x)$, where $C = B_n, A_n$ or D_n respectively. Doing some algebraic calculations we find that $b_n^A = b_n + nC$, where b_n denotes the coefficient of the $n-1$ power in the classical polynomials $P_n(x) = x^n + b_n x^{n-1} + \dots$.

Using these formulas and the main data [11] for classical polynomials we obtain for generalized Meixner, Kravchuk and Charlier polynomials the following TTRR coefficients:

I. Meixner polynomials:

$$b_n = \frac{n\mu}{\mu - 1} \left(\gamma + \frac{n-1}{2} \frac{\mu+1}{\mu} \right), \quad \beta_n = \frac{n + \mu(n + \gamma)}{1 - \gamma},$$

$$b_n^A = \frac{n\mu}{\mu - 1} \left(\gamma + \frac{n-1}{2} \frac{\mu+1}{\mu} \right) + A \frac{\mu^n (1 - \mu)^{\gamma-1} (\gamma)_n}{(n-1)! (1 + A \text{Ker}_{n-1}^M(0, 0))},$$

and then

$$\beta_n^A = \frac{n + \mu(n + \gamma)}{1 - \gamma} - A \frac{\mu^n (1 - \mu)^{\gamma-1} (\gamma)_n}{n!} \left[\frac{\mu(\gamma + n)}{1 + A \text{Ker}_n^M(0, 0)} - \frac{n}{1 + A \text{Ker}_{n-1}^M(0, 0)} \right],$$

$$\gamma_n^A = -\frac{M_{n+1}^{\gamma, \mu, A}(0)}{M_{n-1}^{\gamma, \mu, A}(0)} - \beta_n^A \frac{M_n^{\gamma, \mu, A}(0)}{M_{n-1}^{\gamma, \mu, A}(0)}.$$

II. Kravchuk polynomials:

$$b_n = -n[Np + (n-1)(\frac{1}{2} - p)], \quad \beta_n = n + p(N - 2n),$$

$$b_n^A = -n[Np + (n-1)(\frac{1}{2} - p)] + A \frac{N!}{(n-1)!(N-n)!} \frac{p^n (1-p)^{1-n}}{(1 + A \text{Ker}_{n-1}^K(0, 0))},$$

and then

$$\beta_n^A = n + p(N - 2n) + A \frac{N! p^n (1-p)^{-n}}{n!(N-n)!} \left[\frac{n(1-p)}{1 + A \text{Ker}_{n-1}^K(0, 0)} - \frac{p(N-n)}{1 + A \text{Ker}_n^K(0, 0)} \right],$$

$$\gamma_n^A = -\frac{K_{n+1}^{p, A}(0)}{K_{n-1}^{p, A}(0)} - \beta_n^A \frac{K_n^{p, A}(0)}{K_{n-1}^{p, A}(0)}.$$

III. Charlier polynomials:

$$b_n = -n \left(\mu + \frac{n-1}{2} \right), \quad \beta_n = n + \mu,$$

$$b_n^A = -n \left(\mu + \frac{n-1}{2} \right) + A \frac{\mu^n}{(n-1)! (1 + A \text{Ker}_{n-1}^C(0, 0))},$$

and then

$$\beta_n^A = n + \mu + A \frac{\mu^n}{n!} \left[\frac{n}{1 + A \text{Ker}_{n-1}^C(0, 0)} - \frac{\mu}{1 + A \text{Ker}_n^K(0, 0)} \right],$$

$$\gamma_n^A = -\frac{C_{n+1}^{\mu, A}(0)}{C_{n-1}^{\mu, A}(0)} - \beta_n^A \frac{C_n^{\mu, A}(0)}{C_{n-1}^{\mu, A}(0)}.$$

6. Relation between tridiagonal matrices J_{n+1} and J_{n+1}^A

In this section we are interested to find the relation between the tridiagonal matrices corresponding to the CMOP $P_n(x)$ and generalized polynomials $P_n^A(x)$, which we denote by J_{n+1} and J_{n+1}^A , respectively. The crucial role that the tridiagonal matrices play in the numerical study of zeros of orthogonal polynomials defined on the real line is well known, because the zeros are the eigenvalues.

If we rewrite the TTRR (42) for PMOP in the matrix form we obtain

$$x\mathbf{P}_n^A = J_{n+1}^A \mathbf{P}_n^A + P_{n+1}^A(x) \mathbf{e}_{n+1}^{(n+1)}, \tag{43}$$

where the corresponding tridiagonal matrix and the *perturbed polynomial vector* are denoted by J_{n+1}^A and \mathbf{P}_n^A , respectively ($n \geq 0$):

$$J_{n+1}^A = \begin{pmatrix} \beta_0^A & 1 & 0 & 0 & \cdots & 0 & 0 \\ \gamma_1^A & \beta_1^A & 1 & 0 & \cdots & 0 & 0 \\ 0 & \gamma_2^A & \beta_2^A & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_n^A & \beta_n^A \end{pmatrix} \quad \text{and} \quad \mathbf{P}_n^A = \begin{pmatrix} P_0^A(x) \\ P_1^A(x) \\ P_2^A(x) \\ \vdots \\ P_n^A(x) \end{pmatrix},$$

and for a given integer $n \geq 1$ by $\mathbf{e}_j^{(n+1)}$ ($0 \leq j \leq n + 1$), we denote

$$\mathbf{e}_j^{(n+1)} := (0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0)^T \in \mathbb{R}^{n+1}.$$

A similar notation will be used for the tridiagonal matrix J_{n+1} , with the β_i 's and γ_i 's replaced by the corresponding coefficients of the three-term recurrence relation for $P_n(x)$ and for the *polynomial vector* \mathbf{P}_n :

$$J_{n+1} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \gamma_1 & \beta_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \gamma_2 & \beta_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_n & \beta_n \end{pmatrix} \quad \text{and} \quad \mathbf{P}_n = \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \\ P_n(x) \end{pmatrix}.$$

From the relation (21) and using (23), we deduce

$$P_{n+1}^A(x) = P_{n+1}(x) + \sum_{j=0}^n a_{n+1,j} P_j(x), \tag{44}$$

or in the matrix form,

$$\mathbf{P}_n^A = R_{n+1} \mathbf{P}_n, \tag{45}$$

where R_{n+1} denotes the lower triangular matrix with 1 entries in the main diagonal:

$$R_{n+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{1,0} & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{2,0} & a_{2,1} & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} & 1 \end{pmatrix},$$

and $a_{m,j}$ are

$$a_{m,j} = -A \frac{P_m(0) P_j(0)}{1 + A \sum_{k=0}^{m-1} ((P_k(0))^2 / d_k^2) d_j^2}, \quad 0 \leq j \leq m - 1.$$

Now putting (45) in (43) and using (44) we find

$$xR_{n+1} \mathbf{P}_n = J_{n+1}^A R_{n+1} \mathbf{P}_n + \left(P_{n+1}(x) + \sum_{j=0}^n a_{n+1,j} P_j(x) \right) \mathbf{e}_{n+1}^{(n+1)},$$

from where, using the TTRR in the matrix form for the classical polynomials $P_n(x)$, we find

$$J_{n+1} \mathbf{P}_n = R_{n+1}^{-1} J_{n+1}^A R_{n+1} \mathbf{P}_n + \sum_{j=0}^n a_{n+1,j} P_j(x) \mathbf{e}_{n+1}^{(n+1)}.$$

Finally, from this equation we obtain the following relation between tridiagonal matrices J_{n+1} and J_{n+1}^A :

$$J_{n+1}^A = R_{n+1} (J_{n+1} + A_{n+1}) R_{n+1}^{-1}, \tag{46}$$

where A_{n+1} is a rank-one matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{n+1,0} & a_{n+1,1} & a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n-1} & a_{n+1,n} \end{pmatrix}.$$

We conclude that J_{n+1}^A is a rank-one perturbation of J_{n+1} . Finally, we obtain the corresponding elements of the matrices R_{n+1} and A_{n+1} in the three studied cases, i.e., the Meixner, Kravchuk and Charlier perturbed orthogonal polynomials.

Meixner case:

$$a_{m,k}^M = A \frac{(-1)^{m+k+1} \mu^m (\gamma)_m}{k! (1 - \mu)^{m-k-\gamma} (1 + \text{Ker}_{m-1}^M(0,0))}. \tag{47}$$

Kravchuk case:

$$a_{m,k}^K = A \frac{(-1)^{m+k+1} p^m N!}{k! (1 - p)^k (N - m)! (1 + \text{Ker}_{m-1}^K(0,0))}. \tag{48}$$

Charlier case:

$$a_{m,k}^C = A \frac{(-1)^{m+k+1} \mu^m}{k!(1 + \text{Ker}_{m-1}^C(0,0))}. \tag{49}$$

7. Associated polynomials

In this section we study the monic associated polynomials corresponding to PMOPS $\{P_n^A(x)\}_{n=0}^\infty$. They are defined as follows [5]:

$$P_n^{(1),A}(x) = \frac{1}{c_0^A} \mathcal{U} \left\{ \frac{P_{n+1}^A(x) - P_{n+1}^A(y)}{x - y} \right\}, \tag{50}$$

where it is understood that \mathcal{U} operates on y and c_0^A is the first moment of the functional, i.e., $c_0^A = \mathcal{U}(1)$. Taking into account that $\mathcal{U}\{P_n^A(x)\} = \mathcal{L}\{P_n^A(x)\} + AP_n^A(0)$ and using the relation (44) for the generalized polynomials, as well as the linearity of the classical functional \mathcal{L} , we obtain the following relation:

$$P_n^{(1),A}(x) = \frac{c_0}{c_0^A} \left[P_n^{(1)}(x) - AP_{n+1}^A(0) \sum_{k=0}^n \frac{P_k(0)P_{k-1}^{(1)}(x)}{d_k^2} \right] + \frac{A}{c_0^A} \frac{P_{n+1}^A(x) - P_{n+1}^A(0)}{x}. \tag{51}$$

Lemma 3. For all integers $n \geq 1$,

$$(x - y) \sum_{k=0}^n \frac{P_k(x)P_{k-1}^{(1)}(y)}{d_k^2} = \frac{P_{n+1}(x)P_{n-1}^{(1)}(y) - P_n(x)P_n^{(1)}(y)}{d_n^2} + \frac{1}{\gamma_1}.$$

Proof. We will use the recurrence relation (5) for the classical polynomials $P_n(x)$,

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 1,$$

$$P_{-1}(x) = 0 \quad \text{and} \quad P_0(x) = 1,$$

and the corresponding for the associated (see [5, p. 85, Eq. (4.3)]),

$$yP_{n-1}^{(1)}(y) = P_n^{(1)}(y) + \beta_n P_{n-1}^{(1)}(y) + \gamma_n P_{n-2}^{(1)}(y), \quad n \geq 1,$$

$$P_{-1}^{(1)}(x) = 0 \quad \text{and} \quad P_0^{(1)}(x) = 1.$$

If we multiply the first equation by $P_{n-1}^{(1)}(y)$ and the second one by $P_n(x)$, subtract both and divide the resulting expression by d_n^2 , then

$$(x - y) \frac{P_k(x)P_{k-1}^{(1)}(y)}{d_k^2} = \frac{P_{k+1}(x)P_{k-1}^{(1)}(y) - P_k(x)P_k^{(1)}(y)}{d_k^2} - \frac{P_k(x)P_{k-2}^{(1)}(y) - P_{k-1}(x)P_{k-1}^{(1)}(y)}{d_{k-1}^2}.$$

Summing from $k = 2$ to n we find

$$(x - y) \sum_{k=2}^n \frac{P_k(x)P_{k-1}^{(1)}(y)}{d_k^2} = \frac{P_{n+1}(x)P_{n-1}^{(1)}(y) - P_n(x)P_n^{(1)}(y)}{d_n^2} - \frac{P_2(x) - P_1(x)P_1^{(1)}(y)}{d_1^2}.$$

But

$$(x - y) \sum_{k=0}^n \frac{P_k(x)P_{k-1}^{(1)}(y)}{d_k^2} = (x - y) \frac{P_1(x)P_0^{(1)}(y)}{d_1^2} + (x - y) \sum_{k=2}^n \frac{P_k(x)P_{k-1}^{(1)}(y)}{d_k^2}.$$

Using the previous expression for the last sum, the identities $xP_1(x) = P_2(x) + \beta_1P_1(x) + \gamma_1$ and $P_1^{(1)}(y) = y - \beta_1$, and doing some straightforward computations, the lemma follows. \square

Lemma 4. *If for the generalized polynomials the following relation holds,*

$$P_n(x)P_{n+1}(0) - P_{n+1}(x)P_n(0) = xD \nabla P_n(x), \quad n \geq 1,$$

where D is some constant independent of x (but in general it could be a function of n), then

$$P_{n-1}^{(1)}(x)P_{n+1}(0) - P_n^{(1)}(x)P_n(0) = xD \nabla P_{n-1}^{(1)}(x).$$

Proof. From the hypothesis of the lemma, we have, $\forall n \geq 1$,

$$P_n(x)P_{n+1}(0) - P_{n+1}(x)P_n(0) = xD(P_n(x) - P_n(x - 1)),$$

$$P_n(y)P_{n+1}(0) - P_{n+1}(y)P_n(0) = yD(P_n(y) - P_n(y - 1)).$$

Subtracting them, dividing by $x - y$, applying the functional \mathcal{L} and taking into account the identities ($n \geq 1$)

$$\mathcal{L} \left\{ \frac{xP_n(x) - yP_n(y)}{x - y} \right\} = xP_{n-1}^{(1)}(x) \quad \text{and} \quad \mathcal{L} \left\{ \frac{xP_n(x - 1) - yP_n(y - 1)}{x - y} \right\} = xP_{n-1}^{(1)}(x - 1),$$

the lemma follows. \square

Now we can consider the former expression for the monic associated polynomials corresponding to PMOPS $\{P_n^A(x)\}_{n=0}^\infty$. From (51) and using the previous lemmas we find

$$P_n^{(1),A}(x) = \frac{c_0}{c_0^A} \left[P_n^{(1)}(x) - AP_{n+1}^A(0) \left(\frac{P_{n+1}(0)P_{n-1}^{(1)}(x) - P_n(0)P_n^{(1)}(x)}{d_n^2} + \frac{1}{\gamma_1} \right) \right] + \frac{A}{c_0^A} \frac{P_{n+1}^A(x) - P_{n+1}^A(0)}{x}, \tag{52}$$

or

$$P_n^{(1),A}(x) = \frac{c_0}{c_0^A} \left[P_n^{(1)}(x) - AP_{n+1}^A(0)xCd_n^2 \nabla P_{n-1}^{(1)}(x) \right] + \frac{A}{c_0^A} \frac{P_{n+1}^A(x) - P_{n+1}^A(0)}{x}, \tag{53}$$

where $C = B_n$, A_n and D_n (27)–(29) for the Meixner, Kravchuk and Charlier orthogonal polynomials, respectively. These two equations are valid when $x \neq 0$ and $n \geq 1$. In the case $x = 0$ we must make use of the definitions:

$$P_n^{(1),A}(0) = \frac{1}{c_0^A} \mathcal{U} \left\{ \frac{P_n^A(y) - P_{n+1}^A(0)}{y} \right\},$$

or, alternatively, the formula (52) and the limit when $x \rightarrow 0$.

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